LOCAL LANGLANDS CONJECTURE FOR p-ADIC GSpin_4, GSpin_6, AND THEIR INNER FORMS

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ABSTRACT. We establish the local Langlands conjecture for small rank general spin groups GSpin_4 and GSpin_6 as well as their inner forms. We construct appropriate L-packets and prove that these L-packets satisfy the properties expected of them to the extent that the corresponding local factors are available. We are also able to determine the exact sizes of the L-packets in many cases.

1. INTRODUCTION

In this article, we construct L-packets for the split general spin groups GSpin_4, GSpin_6, and their inner forms over a p-adic field F of characteristic 0, and more importantly, establish their internal structures in terms of characters of component groups, as predicted by the Local Langlands Conjecture (LLC). This establishes the LLC for the groups in question (cf. Theorems 5.1 and 6.1 and Propositions 5.16 and 6.12). The construction of the L-packets is essentially an exercise in restriction of representations, thanks to the structure, as algebraic groups, of the groups we consider; however, proving the properties of the L-packets requires some deep results of Hiraga-Saito as well as Aubert-Baum-Plymen-Solleveld as we explain below.

Let W_F denote the Weil group of F. In a general setting, if G denotes a connected, reductive, linear, algebraic group over F, then the Local Langlands Conjecture asserts that there is a surjective, finite-to-one map from the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible smooth complex representations of $G(F)$ to the set $\Phi(G)$ of $\widehat{G}$-conjugacy classes of L-parameters, i.e., admissible homomorphisms $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$. Here $\widehat{G} = {}^L G^0$ denotes the connected component of the L-group of G, i.e., the complex dual of G [Bor79]. Given $\varphi \in \Phi(G)$, its fiber $\Pi_\varphi(G)$, which is called an L-packet for G, is expected to be controlled by a certain finite group living in the complex dual group $\widehat{G}$. Furthermore, the map is supposed to preserve certain local factors, such as γ-factors, L-factors, and ϵ-factors.

The LLC is already known for several cases: GL_n [HT01, Hen00, Sch13], SL_n [GK82], U_2 and U_3 [Rog90], F-inner forms of GL_n and SL_n [HS12, ABPS14], GSp_4 [GT11], Sp_4 [GT10], the F-inner form $\mathrm{GSp}_{1,1}$ of GSp_4 [GT14], the F-inner form $\mathrm{Sp}_{1,1}$ of Sp_4 [Cho14c], quasi-split orthogonal and symplectic groups [Art13], unitary groups [Mok15], and non quasi-split inner forms of unitary groups [KMSW14].

We consider the case of $G = \mathrm{GSpin}_4, \mathrm{GSpin}_6$, or one of their non quasi-split F-inner forms. Our approach is based on the study of the restriction of representations from a connected reductive F-group to a closed subgroup having an identical derived group as G itself. This approach originates in the earlier work on the LLC for SL_n [GK82]. Gelbart and Knapp studied the restriction of representations of GL_n to SL_n and established the LLC for SL_n, assuming the LLC for GL_n which was later proved [HT01, Hen00, Sch13]. Given an L-parameter $\varphi \in \Phi(\mathrm{SL}_n)$, the L-packet $\Pi_\varphi(\mathrm{SL}_n)$ is proved to be in bijection with the component group $\mathcal{S}_\varphi(\widehat{\mathrm{SL}_n})$ of the centralizer of the image of φ in $\widehat{\mathrm{SL}_n}$. The multiplicity one property for the restriction from GL_n to SL_n [HS75, Tad92] and the fact that all L-packets of GL_n are singletons [HT01, Hen00, Sch13], are indispensable to establishing the bijection.

Later on, Hiraga and Saito extended the LLC for SL_n and the result of Labesse and Langlands [LL79] for the non-split inner form SL_2' of SL_2 to the non-split inner form SL_n' of SL_n [HS12], except for the representations of SL_n' whose liftings to the split GL_n are not generic. Those cases were dealt with afterwards

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by Aubert, Baum, Plymen, and Solleveld [ABPS14]. The restriction approach is also used in the case of SL'_n, after establishing the LLC for the non-split inner form GL'_n of GL_n by means of the LLC for GL_n and the local Jacquet-Langlands correspondence [JL70, DKV84, Rog83, Bad08]. However, there is some subtlety in applying the restriction technique from GL'_n to SL'_n since, unlike in the case of split SL_n, the multiplicity one property fails in this case. Thus, Hiraga and Saito consider a central extension $\mathcal{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{SL}_n})$ of the connected component group $\mathrm{PGL}_n(\mathbb{C})$ by a certain quotient group $\widehat{Z}_{\varphi,\mathrm{sc}}(\mathrm{SL}_n)$ of the center of $\mathrm{SL}_n(\mathbb{C})$. They then prove that the set of irreducible representations of the finite group $\mathcal{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{SL}_n})$ governs the restriction from GL'_n to SL'_n and parameterizes the L-packets for SL'_n. The central extension approach turns out to also include the case of $G = \mathrm{SL}_n$ and the previous parameterization of L-packets of SL_n.

The LLC for the groups we consider in this paper is related to the LLC for SL_n [GK82] and its F-inner form SL'_n [HS12, ABPS14]. Write G for GSpin_4, GSpin_6, or one of their non quasi-split F-inner forms. It follows from the structure of G as an algebraic group that it is an intermediate group between a product of SL_{m_i} and a product of GL_{m_i} or their F-inner forms with suitable integers m_i. We give an explicit description of each group structure in Section 2.2. We are then able to utilize the LLC for GL_n [HT01, Hen00, Sch13] and GL'_n [HS12]. At the same time, using a theorem of Labesse [Lab85], we are able to define a surjective, finite-to-one map

$$\mathcal{L} : \mathrm{Irr}(G) \longrightarrow \Phi(G), \tag{1.1}$$

and construct L-packets $\Pi_\varphi(G)$ for each $\varphi \in \Phi(G)$.

We next study the internal structure of each L-packet. Based on the work of Hiraga and Saito [HS12], we investigate the central extension $\mathcal{S}_{\varphi,\mathrm{sc}}$ for our case and prove that $\mathcal{S}_{\varphi,\mathrm{sc}}$ is embedded into $\mathcal{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{SL}_n})$. This is where the internal structures of the L-packets for SL_n and SL'_n are needed. We then prove that there is a one-to-one correspondence

$$\Pi_\varphi(G) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G),$$

where $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G)$ denotes the set of irreducible representations of $\mathcal{S}_{\varphi,\mathrm{sc}}$ with central character ζ_G corresponding to the group G via the Kottwitz isomorphism [Kot86] (cf. Theorems 5.1 and 6.1). Moreover, using Galois cohomology, we prove that the possible sizes for the L-packet $\Pi_\varphi(G)$ are 1, 2, and 4 when $p \neq 2$ and 1, 2, 4, and 8 when $p = 2$ (cf. Propositions 5.5 and 6.4). In the case of $G = \mathrm{GSpin}_4$ we are also able to show that only 1, 2, and 4 occur for any p (see Remarks 5.10 and 5.11). We do this using the classification of the group of characters stabilizing representations. Further, we describe the group structure of the central extension $\mathcal{S}_{\varphi,\mathrm{sc}}$, provide all the sizes of L-packets for GSpin_4 and its non quasi-split F-inner forms, and discuss the multiplicity in restriction. In the case of GSpin_6, unlike that of GSpin_4, we do not classify the group of characters for GSpin_6 nor do we discuss the group structure of $\mathcal{S}_{\varphi,\mathrm{sc}}$. This is due to the fact that a full classification of irreducible L-parameters in $\Phi(\mathrm{SL}_4)$ is not currently available. Accordingly, the determination of all the sizes of L-packets for GSpin_6 and its non quasi-split F-inner forms as well as the multiplicity in restriction are not addressed in this paper. These questions will require further study of the finite group $\mathcal{S}_{\varphi,\mathrm{sc}}$ for each L-parameter φ for GSpin_6.

Furthermore, in Sections 5.4 and 6.4 we verify that the Plancherel measure is preserved within L-packets and between inner twisting. Also, the $\mathcal{L}$-map satisfies that an irreducible representation is essentially square integrable (respectively, tempered) if and only if its L-parameter does not factor through any proper Levi subgroup (respectively, the image of its L-parameter is bounded).

When G is the split group GSpin_4 or GSpin_6, we prove that the local L-, ϵ-, and γ-factors are preserved via the $\mathcal{L}$-map in (1.1). Given $\tau \in \mathrm{Irr}(\mathrm{GL}_r)$, $r \geq 1$, and $\sigma \in \mathrm{Irr}(G)$ which is assumed to be either ψ-generic or non-supercuspidal if $r > 1$, we let φ_τ be the L-parameter of τ via the LLC for GL_r and let $\varphi_\sigma = \mathcal{L}(\sigma)$. Thanks to the structure theory detailed in Section 2.2, we are able to use results on the generic Langlands functorial transfer from general spin groups to the general linear groups which are already available [ACS1, AS06, AS14]. As a result, we prove

$$\begin{aligned} L(s, \tau \times \sigma) &= L(s, \varphi_\tau \otimes \varphi_\sigma), \\ \epsilon(s, \tau \times \sigma, \psi) &= \epsilon(s, \varphi_\tau \otimes \varphi_\sigma, \psi), \\ \gamma(s, \tau \times \sigma, \psi) &= \gamma(s, \varphi_\tau \otimes \varphi_\sigma, \psi). \end{aligned} \tag{1.2}$$

Here, the local factors on the left hand side are those attached by Shahidi [Sha90b, Theorem 3.5] initially to generic representations and extended to all non-generic non-supercuspidal representations via the Langlands classification and the multiplicativity of the local factors [Sha90b, §9], and the factors on the right hand side are Artin local factors associated with the given representations of the Weil-Deligne group of F (cf. Sections 5.5 and 6.5).

Another expected property of the L-packets is that they should satisfy the local character identities of the theory of (twisted) endoscopy (see [CG14], for example). It is certainly desirable to study this question for the groups we consider and the L-packets we construct, a task we leave for a future work.

We finally remark that, due to lack of the LLC for the quasi-split special unitary group SU_n, our method is currently limited to split groups GSpin_4, GSpin_6, and their non quasi-split F-inner forms. The case of the quasi-split non-split groups GSpin_4^* and GSpin_6^* will be addressed in a forthcoming work.

The structure of this paper is as follows. In Section 2 we introduce the basic notations and review some background material. We also describe the algebraic group structure, F-points, and L-groups of the groups GSpin_4, GSpin_6, and their non quasi-split F-inner forms, which are the groups under consideration in this paper. Section 3 states the LLC and the conjectural structure of L-packets in a general setting. In Section 4 we review some well-known results on restriction. We then prove our main results: the LLC for GSpin_4 and its non quasi-split F inner forms in Section 5, and the LLC for GSpin_6 and its non quasi-split F-inner forms in Section 6. Furthermore, we describe the possible sizes of L-packets in each case and prove the equality of local factors via the $\mathcal{L}$-map.

2. The Preliminaries

2.1. **Notations and Conventions.** Let p be a prime number. We denote by F a p-adic field of characteristic 0, i.e., a finite extension of $\mathbb{Q}_p$. Let $\bar{F}$ be an algebraic closure of F. Denote by $\mathcal{O}_F$ the ring of integers of F, and by $\mathcal{P}$ the maximal ideal in $\mathcal{O}_F$. Let q denote the cardinality of the residue field $\mathcal{O}_F/\mathcal{P}$.

We denote by W_F the Weil group of F and by Γ the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. Let G be a connected, reductive, linear, algebraic group over F. Fixing Γ-invariant splitting data, we define the L-group of G as a semi-direct product ${}^L G := \widehat{G} \rtimes \Gamma$ (see [Bor79, Section 2]).

For an integer $i \in \mathbb{N}$ and a connected, reductive, algebraic group G over F, we set

$$H^i(F, G) := H^i(\mathrm{Gal}(\bar{F}/F), G(\bar{F})),$$

the Galois cohomology of G. For any topological group H, we denote by $\pi_0(H)$ the group H/H° of connected components of H, where H° denotes the identity component of H. By $Z(H)$ we will denote the center of H. We write H^D for the group $\mathrm{Hom}(H, \mathbb{C}^\times)$ of all continuous characters. Also, H_{der} denotes the derived groups of H. We denote by $\mathbb{1}$ the trivial character. The cardinality of a finite set S is denoted by $|S|$. For two integers x and y, $x|y$ means that y is divisible by x. For any positive integer n, we denote by μ_n the algebraic group such that $\mu_n(R) = \{r \in R : r^n = 1\}$ for any F-algebra R.

Given connected reductive algebraic groups G and G' over F, we say that G and G' are F-*inner forms* with respect to an $\bar{F}$-isomorphism $\varphi : G' \to G$ if $\varphi \circ \tau(\varphi)^{-1}$ is an inner automorphism ($g \mapsto xgx^{-1}$) defined over $\bar{F}$ for all $\tau \in \mathrm{Gal}(\bar{F}/F)$ (see [Bor79, 2.4(3)], [Kot97, p.280]). We often omit the references to F and φ when there is no danger of confusion. We recall that if two F-inner forms G and G' are quasi-split over F, then G and G' are isomorphic over F, [Bor79, Remarks 2.4(3)].

When G and G' are inner forms of each other, we have ${}^L G \cong {}^L G'$ [Bor79, Section 2.4(3)]. In particular, if G' is an inner form of an F-split group G with the action of Γ on $\widehat{G}$ trivial, we write ${}^L G = \widehat{G} \cong {}^L G' = \widehat{G'}$.

For positive integers m, n, and d, we let D be a central division algebra of dimension d^2 over F (possibly $D = F$, in which case $d = 1$). Let $\mathrm{GL}_m(D)$ denote the group of all invertible elements of $m \times m$ matrices

over D. Let $\mathrm{SL}_m(D)$ be the subgroup of elements in $\mathrm{GL}_m(D)$ with reduced norm Nrd equal to 1. Note that there are algebraic groups over F, whose groups of F-points are respectively $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$. By abuse of notation, we shall write $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$ for their algebraic groups over F as well. Note that any F-inner forms of the split general linear group GL_n and the split special linear group SL_n are respectively of the form $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$, where $n = md$ (see [PR94, Sections 2.2 & 2.3]).

2.2. **Group Structures.** In this section we describe the structure of the split groups GSpin_4 and GSpin_6 over F as well as their non quasi-split F-inner forms. These are the groups we work with in this paper. The exact knowledge of the structure of these groups allow us, on the one hand, to use techniques from restriction of representations to construct L-packets and, on the other hand, make use of generic local transfer from the general spin groups to general linear groups in order to prove preservation of local L-, ϵ-, and γ-factors for our L-packets.

2.2.1. *Split Groups.* We first give a description of the split groups GSpin_4 and GSpin_6 in terms of abstract root data. Let
$$X_{2n} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$$
and let
$$X_{2n}^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_{2n}^*$$
be the dual $\mathbb{Z}$-module with the standard $\mathbb{Z}$-pairing between them. We let
$$\Delta_{2n} = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$$
and
$$\Delta_{2n}^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^*\}$$
denote the simple roots and coroots, respectively, and let R_{2n} and $R_{2n}^\vee$ be the roots and coroots they generate. Then
$$\Psi_{2n} = (X_{2n}, R_{2n}, \Delta_{2n}, X_{2n}^\vee, R_{2n}^\vee, \Delta_{2n}^\vee)$$
is a based root datum determining the split, connected, reductive group GSpin_{2n} over F. See [ACS2, AS14] for more details. We are mostly interested in Ψ_4 and Ψ_6 in this paper.

In fact, the derived group of GSpin_{2n} is isomorphic to Spin_{2n} and we have [AS06, Proposition 2.2] the following isomorphism of algebraic groups over F :
$$\mathrm{GSpin}_{2n} \cong (\mathrm{GL}_1 \times \mathrm{Spin}_{2n})/\{(1,1),(-1,c)\}, \tag{2.1}$$
where c denotes the non-trivial element in the center of Spin_{2n} given by
$$c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1) = e_0^*(-1)^{-1} = e_0^*(-1)$$
in our root data notation.

When $n = 2$ or 3 we have the accidental isomorphisms
$$\begin{aligned}\mathrm{Spin}_4 &\cong \mathrm{SL}_2 \times \mathrm{SL}_2 \\ \mathrm{Spin}_6 &\cong \mathrm{SL}_4\end{aligned}$$
as algebraic groups over F. The element c will then be identified with $(-I_2, -I_2) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ and $-I_4 \in \mathrm{SL}_4$, respectively. Therefore, we have the following isomorphisms of algebraic groups over F :
$$\begin{aligned}\mathrm{GSpin}_4 &\cong (\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{SL}_2)/\left\{(1, I_2, I_2),(-1, -I_2, -I_2)\right\}, &(2.2) \\ \mathrm{GSpin}_6 &\cong (\mathrm{GL}_1 \times \mathrm{Spin}_6)/\left\{(1, I_4),(-1, -I_4)\right\}. &(2.3)\end{aligned}$$

For our purposes here, the following is a more convenient description of these two groups.

Proposition 2.1. *As algebraic groups over F we have the following isomorphisms:*
$$\mathrm{GSpin}_4 \cong \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det g_1 = \det g_2\} \tag{2.4}$$
$$\mathrm{GSpin}_6 \cong \{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_4 : g_1^2 = \det g_2\}. \tag{2.5}$$

Proof. We verify these isomorphisms by giving explicit isomorphisms between the respective root data. For (2.4) consider $\mathrm{GL}_2 \times \mathrm{GL}_2$ and let T_1 be its usual maximal split torus, with $T \subset T_1$ denoting the maximal split torus of

$$G_4 = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det g_1 = \det g_2\}.$$

Using the notation f_{ij} and f_{ij}^*, $1 \le i, j \le 2$, for the usual $\mathbb{Z}$-basis of characters and cocharacters of $\mathrm{GL}_2 \times \mathrm{GL}_2$ with the standard $\mathbb{Z}$-pairing between them, the character lattice $X^*(T)$, a quotient of $X^*(T_1)$, and the cocharacter lattice $X_*(T)$, a sublattice of $X_*(T_1)$, can be given by

$$X^*(T) = \frac{\mathbb{Z}\langle f_{11}, f_{12}\rangle \oplus \mathbb{Z}\langle f_{21}, f_{22}\rangle}{\mathbb{Z}\langle f_{11} + f_{12} - f_{21} - f_{22}\rangle}$$

and

$$X_*(T) = \langle f_{11} + f_{12} - f_{21} - f_{22}\rangle^{\perp}.$$

Here $\perp$ means orthogonal complement with respect to the $\mathbb{Z}$-pairing.

Let R and Δ denote the roots and simple roots in G_4 and let $R^\vee$ and $\Delta^\vee$ be coroots and simple coroots. An isomorphism of based root data $\Psi_4 \longrightarrow \Psi(G_4) = (X^*(T), R, \Delta, X_*(T), R^\vee, \Delta^\vee)$ amounts to isomorphisms of $\mathbb{Z}$-modules

$$\iota : X_4 \longrightarrow X^*(T) \quad \text{and} \quad \iota^\vee : X_*(T) \longrightarrow X_4^\vee$$

such that

$$\langle \iota(x), y\rangle = \langle x, \iota^\vee(y)\rangle, \qquad x \in X_4, y \in X_*(T),$$
$$\iota(\Delta_4) = \Delta, \quad \text{and} \quad \iota^\vee(\Delta^\vee) = \Delta_4^\vee$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
R_4 & \xrightarrow{\vee} & R_4^\vee \\
\iota \downarrow & & \uparrow \iota^\vee \\
R & \xrightarrow{\vee} & R^\vee
\end{array}
$$

Setting

$$\Delta = \{\beta_1 = f_{11} - f_{12}, \beta_2 = -f_{11} - f_{12} + 2f_{21}\} \,(\mathrm{mod}\, f_{11} + f_{12} - f_{21} - f_{22})$$

and

$$\Delta^\vee = \{\beta_1^\vee = f_{11}^* - f_{12}^*, \beta_2^\vee = f_{21}^* - f_{22}^*\},$$

let S denote the 3×3 matrix of ι with respect to the $\mathbb{Z}$-bases (e_0, e_1, e_2) of X_4 and $(f_{11}, f_{12}, f_{21})(\mathrm{mod}\, f_{11} + f_{12} - f_{21} - f_{22})$ of $X^*(T)$, respectively. Similarly, let $S^\vee$ denote the matrix of $\iota^\vee$ with respect to the $\mathbb{Z}$-bases $(f_{11}^* + f_{22}^*, f_{12}^* + f_{22}^*, -f_{21}^* + f_{22}^*)$ of $X_*(T)$ and (e_0^*, e_1^*, e_2^*) of $X_4^\vee$, respectively. Assuming that $\iota(\alpha_1) = \beta_1$ and $\iota(\alpha_2) = \beta_2$ along with the conditions for root data isomorphisms we detailed above, plus the requirement $\det S = 1$ gives, after some computations, a unique choice for the $\mathbb{Z}$-isomorphisms ι and $\iota^\vee$ given by

$$S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad S^\vee = {}^t S.$$

(Alternatively, we could have assumed that $\det S = -1$ or that $\iota(\alpha_1) = \beta_2$, etc., which would have led to slightly different matrices S and $S^\vee$.) This shows that G_4 is indeed a realization, as an algebraic group over F, for GSpin_4.

In fact, using S and $S^\vee$ above, we can arrive at the following explicit realizations. For

$$t = \left(\begin{bmatrix} a & \\ & b \end{bmatrix}, \begin{bmatrix} c & \\ & d \end{bmatrix}\right) \in T, \quad (\text{with } ab = cd)$$

we have,

$$e_1(t) = c/b, \quad e_2(t) = c/a, \quad e_0(t) = 1/c.$$

Similarly, for $\lambda \in \mathrm{GL}_1$, we have

$$
\begin{aligned}
e_1^*(\lambda) &= \left(\begin{bmatrix} 1 & \\ & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} 1 & \\ & \lambda^{-1} \end{bmatrix} \right) \\
e_2^*(\lambda) &= \left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} \lambda^{-1} & \\ & \lambda \end{bmatrix} \right) \\
e_0^*(\lambda) &= \left(\begin{bmatrix} \lambda^{-1} & \\ & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} \lambda^{-1} & \\ & \lambda^{-1} \end{bmatrix} \right).
\end{aligned}
$$

The proof for the isomorphism between GSpin_6 and $G_6 = \{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_4 : g_1^2 = \det g_2\}$ is similar. With notations f_0 and f_1, f_2, f_3, f_4 and their duals for the characters and cocharacters of T_1, the split torus in $\mathrm{GL}_1 \times \mathrm{GL}_4$, which contains the split torus T of G_4, we can write

$$
X^*(T) = \frac{\mathbb{Z} f_0 \oplus \mathbb{Z}\langle f_1, f_2, f_3, f_4 \rangle}{\mathbb{Z}\langle 2f_0 - f_1 - f_2 - f_3 - f_4 \rangle}
$$

and

$$
X_*(T) = \langle 2f_0 - f_1 - f_2 - f_3 - f_4 \rangle^{\perp}.
$$

Again, we set

$$
\Delta = \{\beta_1 = f_2 - f_3, \beta_2 = f_1 - f_2, \beta_3 = f_3 - f_4\} \,(\mathrm{mod}\, 2f_0 - f_1 - f_2 - f_3 - f_4)
$$

and

$$
\Delta^{\vee} = \{\beta_1^{\vee} = f_2^* - f_3^*, \beta_2^{\vee} = f_1^* - f_2^*, \beta_3^{\vee} = f_3^* - f_4^*\}.
$$

Now, we let S denote the matrix of ι with respect to (e_0, e_1, e_2, e_3) and $(f_0, f_1, f_2, f_3)(\mathrm{mod}\, 2f_0 - f_1 - f_2 - f_3 - f_4)$ and let $S^{\vee}$ denote the matrix of $\iota^{\vee}$ with respect to $(f_0^* + 2f_4^*, f_1^* - f_4^*, f_2^* - f_4^*, f_3^* - f_4^*)$ and $(e_0^*, e_1^*, e_2^*, e_3^*)$. Assuming $\iota(\alpha_i) = \beta_i$ for $1 \leq i \leq 3$ and the conditions for the root data isomorphim along with the requirement $\det S = 1$ we get, after similar computations, that

$$
S = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad S^{\vee} = {}^t S = S.
$$

This shows that G_6 is indeed a realization, as an algebraic group over F, for GSpin_6.

Again, with the above S we have the following explicit realizations. For

$$
t = (z, \mathrm{diag}(a, b, c, d)) \in T \quad (\text{with } z^2 = abcd)
$$

we have

$$
\begin{aligned}
e_1(t) &= abz^{-1} \\
e_2(t) &= acz^{-1} \\
e_3(t) &= bcz^{-1} \\
e_0(t) &= dz^{-1}
\end{aligned}
$$

Similarly, for $\lambda \in \mathrm{GL}_1$, we have

$$
\begin{aligned}
e_1^*(\lambda) &= \left(\lambda^{-1}, \mathrm{diag}(1, 1, \lambda^{-1}, \lambda^{-1}) \right) \\
e_2^*(\lambda) &= \left(\lambda^{-1}, \mathrm{diag}(1, \lambda^{-1}, 1, \lambda^{-1}) \right) \\
e_3^*(\lambda) &= \left(\lambda^{-1}, \mathrm{diag}(\lambda^{-1}, 1, 1, \lambda^{-1}) \right) \\
e_0^*(\lambda) &= \left(\lambda^{-1}, \mathrm{diag}(\lambda^{-1}, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}) \right).
\end{aligned}
$$

$\square$

The proposition implies that we have the following inclusions

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \subset \quad \mathrm{GSpin}_4 \subset \quad \mathrm{GL}_2 \times \mathrm{GL}_2 \tag{2.6}$$

$$\mathrm{SL}_4 \subset \quad \mathrm{GSpin}_6 \subset \quad \mathrm{GL}_1 \times \mathrm{GL}_4. \tag{2.7}$$

Since

$$\mathrm{SL}_2 \times \mathrm{SL}_2 = (\mathrm{GSpin}_4)_{\mathrm{der}} = (\mathrm{GL}_2 \times \mathrm{GL}_2)_{\mathrm{der}}$$

$$\mathrm{SL}_4 = (\mathrm{GSpin}_6)_{\mathrm{der}} = (\mathrm{GL}_1 \times \mathrm{GL}_4)_{\mathrm{der}}$$

the two groups GSpin_4 and GSpin_6 fit in the setting (cf. (4.1))

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G}, \tag{2.8}$$

with

$$G = \mathrm{GSpin}_4, \quad \widetilde{G} = \mathrm{GL}_2 \times \mathrm{GL}_2$$

and

$$G = \mathrm{GSpin}_6, \quad \widetilde{G} = \mathrm{GL}_1 \times \mathrm{GL}_v,$$

respectively.

Moreover, using the surjective map $\mathrm{GL}_2 \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_1$ defined by by $(g_1, g_2) \mapsto (\det g_1)(\det g_2)^{-1}$, the isomorphism (2.4) gives an exact sequence of algebraic groups

$$1 \longrightarrow \mathrm{GSpin}_4 \longrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_1 \longrightarrow 1. \tag{2.9}$$

Likewise, using the surjective map $\mathrm{GL}_1 \times \mathrm{GL}_4 \longrightarrow \mathrm{GL}_1$ defined by $(g_1, g_2) \mapsto g_1^{-2}(\det g_2)$, the isomorphism (2.5) yields the exact sequence

$$1 \longrightarrow \mathrm{GSpin}_6 \longrightarrow \mathrm{GL}_1 \times \mathrm{GL}_4 \longrightarrow \mathrm{GL}_1 \longrightarrow 1. \tag{2.10}$$

2.2.2. *Inner Forms.* Using the Satake classification [Sat71, p.119] for admissible diagrams of the F-inner forms, we only have two (up to isomorphism) non quasi-split F-inner forms of GSpin_4, denoted by $\mathrm{GSpin}_4^{2,1}$ and $\mathrm{GSpin}_4^{1,1}$, whose diagrams are respectively

$$\circ \qquad\qquad \bullet$$
$$\text{and}$$
$$\bullet \qquad\qquad \bullet$$
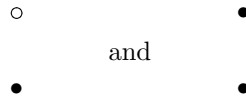
The left diagram gives

$$\mathrm{SL}_2 \times \mathrm{SL}_1(D) \subset \mathrm{GSpin}_4^{2,1} \subset \mathrm{GL}_2 \times \mathrm{GL}_1(D) \tag{2.11}$$

and the right one gives

$$\mathrm{SL}_1(D) \times \mathrm{SL}_1(D) \subset \mathrm{GSpin}_4^{1,1} \subset \mathrm{GL}_1(D) \times \mathrm{GL}_1(D). \tag{2.12}$$

Here, D denotes the quaternion division algebra over F. (Recall from Section 2.1 that we are writing $\mathrm{GL}_m(D)$ and $\mathrm{SL}_m(D)$ for both algebraic groups over F and their F-points, by abuse of notation.)

Similarly, we only have two (up to isomorphism) non quasi-split F-inner forms of GSpin_6, denoted by $\mathrm{GSpin}_6^{2,0}$ and $\mathrm{GSpin}_6^{1,0}$, whose diagrams are respectively
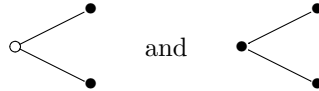
The left diagram gives

$$\mathrm{SL}_2(D) \subset \mathrm{GSpin}_6^{2,0} \subset \mathrm{GL}_1 \times \mathrm{GL}_2(D) \tag{2.13}$$

and the right one gives

$$\mathrm{SL}_1(D_4) \subset \mathrm{GSpin}_6^{1,0} \subset \mathrm{GL}_1 \times \mathrm{GL}_1(D_4). \tag{2.14}$$

Here, D_4 is a division algebra of dimension 16 over F. We note that the two division algebras D_4 and its opposite D_4^{op} of dimension 16 (with invariants $1/4$, $-1/4$ in $\mathbb{Q}/\mathbb{Z}$) have canonically isomorphic multiplicative groups $D_4^{\times}$ and $(D_4^{op})^{\times}$.

Similar to the split forms GSpin_4 and GSpin_6, the F-inner forms of GSpin_4 and GSpin_6 as well as the F-inner forms of $\mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathrm{GL}_2 \times \mathrm{GL}_2$, SL_4, and GL_4 all satisfy the setting (2.8).

The arguments giving (2.9) and (2.10) apply again, with det replaced by the reduced norm Nrd, to show that for F-inner forms $\mathrm{GSpin}_4^{2,1}$, $\mathrm{GSpin}_4^{1,1}$, $\mathrm{GSpin}_6^{2,0}$, and $\mathrm{GSpin}_6^{1,0}$, we have

$$1 \longrightarrow \mathrm{GSpin}_4^{2,1} \longrightarrow \mathrm{GL}_2 \times \mathrm{GL}_1(D) \longrightarrow \mathrm{GL}_1 \longrightarrow 1, \tag{2.15}$$

$$1 \longrightarrow \mathrm{GSpin}_4^{1,1} \longrightarrow \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \longrightarrow \mathrm{GL}_1 \longrightarrow 1, \tag{2.16}$$

$$1 \longrightarrow \mathrm{GSpin}_6^{2,0} \longrightarrow \mathrm{GL}_1 \times \mathrm{GL}_2(D) \longrightarrow \mathrm{GL}_1 \longrightarrow 1, \tag{2.17}$$

$$1 \longrightarrow \mathrm{GSpin}_6^{1,0} \longrightarrow \mathrm{GL}_1 \times \mathrm{GL}_1(D_4) \longrightarrow \mathrm{GL}_1 \longrightarrow 1. \tag{2.18}$$

Using (2.15) – (2.18), we have the following isomorphisms of algebraic groups over F, which are analogues of Proposition 2.1.

$$\begin{aligned}
\mathrm{GSpin}_4^{2,1} &\cong \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_1(D) : \det g_1 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_4^{1,1} &\cong \{(g_1, g_2) \in \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) : \mathrm{Nrd}\, g_1 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_6^{2,0} &\cong \{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_2(D) : g_1^2 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_6^{1,0} &\cong \{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_1(D_4) : g_1^2 = \mathrm{Nrd}\, g_2\}.
\end{aligned}$$

Finally, recall [AS06, Proposition 2.3] that we have

$$\pi_0(Z(\mathrm{GSpin}_4)) \cong \pi_0(Z(\mathrm{GSpin}_6)) \cong \mathbb{Z}/2\mathbb{Z}. \tag{2.19}$$

This also holds for their F-inner forms since F-inner forms have the same center.

2.3. **The F-points.** We now describe the F-rational points of the split groups GSpin_4 and GSpin_6 as well as their non quasi-split inner forms. We start with the following.

Lemma 2.2. *Let G be an F-inner form of GSpin_{2n} with $n \geq 1$ an integer. Then*

$$H^1(F, G) = 1.$$

Proof. We have [AS06, Propositions 2.2 and 2.10]

$$\widehat{\mathrm{GSpin}_{2n}} = \mathrm{GSO}_{2n}(\mathbb{C})$$

and

$$Z(\widehat{\mathrm{GSpin}_{2n}})^\Gamma = Z(\mathrm{GSO}_{2n}(\mathbb{C})) \cong \mathbb{C}^\times.$$

Here, $\widehat{}$ denotes the connected component of the L-group. Applying the Kottwitz isomorphism [Kot86, Theorem 1.2], we can conclude

$$H^1(F, \mathrm{GSpin}_{2n}) \cong \pi_0(Z(\widehat{\mathrm{GSpin}_{2n}})^\Gamma)^D = \pi_0(\mathbb{C}^\times)^D = 1.$$

Since the Kottwitz isomorphism is valid for any connected reductive algebraic group over a p-adic field of characteristic 0, and since F-inner forms of algebraic groups share the same L-groups (cf. Section 2.1), the proof is complete. $\square$

Also, recall that $H^1(F, \mathrm{GL}_n) = 1$ and $H^1(F, \mathrm{Spin}_n) = 1$ for $n \geq 1$ (see [PR94, Lemma 2.8 and Theorem 6.4], for example). It follows from (2.1) that the group $\mathrm{GSpin}_4(F)$ can be described via

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^\times \times \mathrm{Spin}_4(F) \longrightarrow \mathrm{GSpin}_4(F) \longrightarrow H^1(F, \{\pm 1\}) \longrightarrow 1$$

or

$$1 \longrightarrow (F^\times \times \mathrm{Spin}_4(F))/\{\pm 1\} \longrightarrow \mathrm{GSpin}_4(F) \longrightarrow F^\times/(F^\times)^2 \longrightarrow 1.$$

Likewise, the group $\mathrm{GSpin}_6(F)$ can be described via

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^\times \times \mathrm{Spin}_6(F) \longrightarrow \mathrm{GSpin}_6(F) \longrightarrow H^1(F, \{\pm 1\}) \longrightarrow 1$$

or

$$1 \longrightarrow (F^\times \times \mathrm{Spin}_6(F))/\{\pm 1\} \longrightarrow \mathrm{GSpin}_6(F) \longrightarrow F^\times/(F^\times)^2 \longrightarrow 1.$$

Using Lemma 2.2, we apply Galois cohomology to (2.9) to obtain

$$1 \longrightarrow \mathrm{GSpin}_4(F) \longrightarrow \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \longrightarrow F^\times \longrightarrow 1. \tag{2.20}$$

Thus, we have

$$\mathrm{GSpin}_4(F) \cong \{(g_1, g_2) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) : \det g_1 = \det g_2\}. \tag{2.21}$$

Likewise, using Lemma 2.2 and (2.10) we obtain

$$1 \longrightarrow \mathrm{GSpin}_6(F) \longrightarrow \mathrm{GL}_1(F) \times \mathrm{GL}_4(F) \longrightarrow F^\times \longrightarrow 1. \tag{2.22}$$

Thus, we have

$$\mathrm{GSpin}_6(F) \cong \{(g_1, g_2) \in \mathrm{GL}_1(F) \times \mathrm{GL}_2(F) : (g_1)^{-2} = \det g_2\}. \tag{2.23}$$

Similarly, using (2.15) – (2.18) we get

$$
\begin{aligned}
1 &\longrightarrow \mathrm{GSpin}_4^{2,1}(F) \longrightarrow \mathrm{GL}_2(F) \times \mathrm{GL}_1(D) \longrightarrow F^\times \longrightarrow 1, \\
1 &\longrightarrow \mathrm{GSpin}_4^{1,1}(F) \longrightarrow \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \longrightarrow F^\times \longrightarrow 1, \\
1 &\longrightarrow \mathrm{GSpin}_6^{2,0}(F) \longrightarrow F^\times \times \mathrm{GL}_2(D) \longrightarrow F^\times \longrightarrow 1, \\
1 &\longrightarrow \mathrm{GSpin}_6^{1,0}(F) \longrightarrow F^\times \times \mathrm{GL}_1(D_4) \longrightarrow F^\times \longrightarrow 1.
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\mathrm{GSpin}_4^{2,1}(F) &\cong \{(g_1, g_2) \in \mathrm{GL}_2(F) \times \mathrm{GL}_1(D) : \det g_1 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_4^{1,1}(F) &\cong \{(g_1, g_2) \in \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) : \mathrm{Nrd}\, g_1 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_6^{2,0}(F) &\cong \{(g_1, g_2) \in F^\times \times \mathrm{GL}_2(D) : g_1^2 = \mathrm{Nrd}\, g_2\}, \\
\mathrm{GSpin}_6^{1,0}(F) &\cong \{(g_1, g_2) \in F^\times \times \mathrm{GL}_1(D_4) : g_1^2 = \mathrm{Nrd}\, g_2\}.
\end{aligned}
$$

2.4. **L-groups.** We recall the following descriptions of the L-groups of (the split groups) GSpin_4 and GSpin_6 from [AS06, Proposition 2.2] and [GT11, Sections 3 and 4]:

$$^L\mathrm{GSpin}_4 = \widehat{\mathrm{GSpin}}_4 = \mathrm{GSO}_4(\mathbb{C}) \cong (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))/\{(z^{-1}, z) : z \in \mathbb{C}^\times\}, \tag{2.24}$$

$$^L\mathrm{GSpin}_6 = \widehat{\mathrm{GSpin}}_6 = \mathrm{GSO}_6(\mathbb{C}) \cong (\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}))/\{(z^{-2}, z) : z \in \mathbb{C}^\times\}. \tag{2.25}$$

This immediately gives

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \xrightarrow{pr_4} \widehat{\mathrm{GSpin}}_4 \longrightarrow 1 \tag{2.26}$$

and

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \xrightarrow{pr_6} \widehat{\mathrm{GSpin}}_6 \longrightarrow 1. \tag{2.27}$$

Here, $\mathbb{C}^\times$ is considered as $\widehat{\mathrm{GL}}_1$ in (2.9) and (2.10). Further, the injection $\mathbb{C}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ is given by $g \longmapsto (g^{-1}I_2, gI_2)$ and the injection $\mathbb{C}^\times \hookrightarrow \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$ is given by $g \longmapsto (g^{-2}, gI_4)$.

The inclusions in (2.6) and (2.7) provide the following surjective maps:

$$\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \xrightarrow{pr_4} \widehat{\mathrm{GSpin}}_4 \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) = \widehat{\mathrm{SL}_2 \times \mathrm{SL}_2}$$

$$\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \xrightarrow{pr_6} \widehat{\mathrm{GSpin}}_6 \twoheadrightarrow \mathrm{PGL}_4(\mathbb{C}) = \widehat{\mathrm{SL}_4}.$$

Since F-inner forms of algebraic groups share the same L-groups (cf. Section 2.1), we have

$$^L\mathrm{GSpin}_4 = {}^L\mathrm{GSpin}_4^{2,1} = {}^L\mathrm{GSpin}_4^{1,1}$$
$$^L\mathrm{GSpin}_6 = {}^L\mathrm{GSpin}_6^{2,0} = {}^L\mathrm{GSpin}_6^{1,0}.$$

3. Local Langlands Conjecture in a General Setting

In this section we quickly review some generalities about the Local Langlands Conjecture (LLC). Let G be a connected, reductive, algebraic group over F. Write $G(F)$ for the group of F-points of G. Let $\mathrm{Irr}(G)$ denote the set of equivalence classes of irreducible, smooth, complex representations of $G(F)$. By abuse of notation, we identify an equivalence class with its representatives. We write $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{temp}}(G)$ for the subsets of $\mathrm{Irr}(G)$ consisting, respectively, of discrete series and tempered representations. Moreover, we write $\Pi_{\mathrm{scusp}}(G)$ for the subset in $\mathrm{Irr}(G)$ consisting of supercuspidal ones. Furthermore, we write

$$\Pi_{\mathrm{scusp,unit}}(G) = \Pi_{\mathrm{scusp}}(G) \cap \Pi_{\mathrm{disc}}(G).$$

Note that we have

$$\Pi_{\mathrm{scusp,unit}}(G) \subset \Pi_{\mathrm{disc}}(G) \subset \Pi_{\mathrm{temp}}(G) \subset \mathrm{Irr}(G). \tag{3.1}$$

Let $\Phi(G)$ denote the set of $\widehat{G}$-conjugacy classes of L-parameters, i.e., admissible homomorphisms

$$\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G,$$

(see [Bor79, Section 8.2]). We denote the centralizer of the image of φ in $\widehat{G}$ by C_φ. The center of ${}^L G$ is the Γ-invariant group $Z(\widehat{G})^\Gamma$. Note that $C_\varphi \supset Z(\widehat{G})^\Gamma$. We say that φ is elliptic if the quotient group $C_\varphi / Z(\widehat{G})^\Gamma$ is finite, and φ is tempered if $\varphi(W_F)$ is bounded. We denote by $\Phi_{\mathrm{ell}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ the subset of $\Phi(G)$ which consist, respectively, of elliptic and tempered L-parameters of G. We set

$$\Phi_{\mathrm{disc}}(G) = \Phi_{\mathrm{ell}}(G) \cap \Phi_{\mathrm{temp}}(G).$$

Moreover, we write $\Phi_{\mathrm{sim}}(G)$ for the subset in $\Phi(G)$ consisting of irreducible ones. Furthermore, let

$$\Phi_{\mathrm{sim,disc}}(G) = \Phi_{\mathrm{sim}}(G) \cap \Phi_{\mathrm{disc}}(G).$$

We then have, in parallel to (3.1),

$$\Phi_{\mathrm{sim,disc}}(G) \subset \Phi_{\mathrm{disc}}(G) \subset \Phi_{\mathrm{temp}}(G) \subset \Phi(G). \tag{3.2}$$

For any connected reductive group G over F, the Local Langlands Conjecture predicts that there is a surjective finite-to-one map

$$\mathrm{Irr}(G) \longrightarrow \Phi(G).$$

This map is supposed to satisfy a number of natural properties. For instance, it preserves certain γ-facotrs, L-factors, and ϵ-factors, which one can attach to both sides. Moreover, considering the fibers of the map, one can partition $\mathrm{Irr}(G)$ into a disjoint union of finite subsets, called the L-packets. Each packet is conjectured to be characterized by component groups in the L-group, which, for groups we are considering in this paper, are discussed in Sections 5.2 and 6.2. It is also expected that $\Phi_{\mathrm{disc}}(G)$ and $\Phi_{\mathrm{temp}}(G)$ parameterize $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{temp}}(G)$, respectively.

Denote by $\widehat{G}_{\mathrm{sc}}$ the simply connected cover of the derived group $\widehat{G}_{\mathrm{der}}$ of $\widehat{G}$, and by $\widehat{G}_{\mathrm{ad}}$ the quotient group $\widehat{G}/Z(\widehat{G})$. We consider

$$S_\varphi := C_\varphi / Z(\widehat{G})^\Gamma \subset \widehat{G}_{\mathrm{ad}}.$$

Write $S_{\varphi,\mathrm{sc}}$ for the full pre-image of S_φ in $\widehat{G}_{\mathrm{sc}}$. We then have an exact sequence

$$1 \longrightarrow Z(\widehat{G}_{\mathrm{sc}}) \longrightarrow S_{\varphi,\mathrm{sc}} \longrightarrow S_\varphi \longrightarrow 1. \tag{3.3}$$

We let

$$\begin{aligned}
\mathcal{S}_\varphi &:= \pi_0(S_\varphi) \\
\mathcal{S}_{\varphi,\mathrm{sc}} &:= \pi_0(S_{\varphi,\mathrm{sc}}) \\
\widehat{Z}_{\varphi,\mathrm{sc}} &:= Z(\widehat{G}_{\mathrm{sc}})/(Z(\widehat{G}_{\mathrm{sc}}) \cap S_{\varphi,\mathrm{sc}}^\circ).
\end{aligned}$$

We then have (see [Art13, (9.2.2)]) a central extension

$$1 \longrightarrow \widehat{Z}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_\varphi \longrightarrow 1. \tag{3.4}$$

Next, let G' be an F-inner form of G. Fix a character $\zeta_{G'}$ of $Z(\widehat{G}_{\mathrm{sc}})$ whose restriction to $Z(\widehat{G}_{\mathrm{sc}})^\Gamma$ corresponds to the class of the F-inner form G' via the Kottwitz isomorphism [Kot86, Theorem 1.2]. We denote by $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_{G'})$ the set of irreducible representations of $\mathcal{S}_{\varphi,\mathrm{sc}}$ with central character $\zeta_{G'}$ on $Z(\widehat{G}_{\mathrm{sc}})$. It is expected that, given an L-parameter φ for G', there is a bijection between the L-packet $\Pi_\varphi(G')$ associated to φ and the set $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_{G'})$ [Art06, Section 3]. We note that when $G' = G$ the character $\zeta_{G'}$ is the trivial character $\mathbb{1}$ so that

$$\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \mathbb{1}) = \mathrm{Irr}(\mathcal{S}_\varphi).$$

In particular, if φ is elliptic, then we have $S_\varphi = \mathcal{S}_\varphi$ and $\widehat{Z}_{\varphi,\mathrm{sc}} = Z(\widehat{G}_{\mathrm{sc}})$ since $C_\phi/Z(\widehat{G})^\Gamma$ is finite and $Z(\widehat{G})^\Gamma$ contains S_ϕ° [Kot84, Lemma 10.3.1]. Thus the exact sequence (3.4) turns out to be the same as (3.3).

4. REVIEW OF RESULTS ON RESTRICTION

In this section, we review several results about restriction. We refer to [GK82, Lab85, Tad92, HS12] for details.

4.1. Results of Gelbart-Knapp, Tadić, and Hiraga-Saito. For this section, we let G and $\widetilde{G}$ be connected, reductive, algebraic groups over F satisfying the property that

$$G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}} \subseteq G \subseteq \widetilde{G}, \tag{4.1}$$

where the subscript "der" stands for the derived group. Given $\sigma \in \mathrm{Irr}(G)$, by [GK82, Lemma 2.3] and [Tad92, Proposition 2.2], there exists $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ such that

$$\sigma \hookrightarrow \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma}). \tag{4.2}$$

In other words, σ is an irreducible constituent in the restriction $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma})$ of $\widetilde{\sigma}$ from $\widetilde{G}(F)$ to $G(F)$. It turns out, [Tad92, Proposition 2.4 & Corollary 2.5] and [GK82, Lemma 2.1], that $\Pi_\sigma(G)$ is finite and independent of the choice of the lifting $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$. We write $\Pi_\sigma(G) = \Pi_{\widetilde{\sigma}}(G)$ for the set of equivalence classes of all irreducible constituents of $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma})$. It is clear that for any irreducible constituents σ_1 and σ_2 in $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma})$, we have $\Pi_{\sigma_1}(G) = \Pi_{\sigma_2}(G)$.

Remark 4.1. A member (equivalently all members) of $\Pi_{\widetilde{\sigma}}(G)$ is supercuspidal, essentially square-integrable, or essentially tempered if and only if $\widetilde{\sigma}$ is (see [Tad92, Proposition 2.7]).

We recall that the stabilizer of σ in $\widetilde{G}$ is defined as

$$\widetilde{G}_\sigma := \left\{ \tilde{g} \in \widetilde{G}(F) : {}^{\tilde{g}}\sigma \cong \sigma \right\}.$$

The quotient of $\widetilde{G}(F)/\widetilde{G}_\sigma$ acts by conjugation on the set $\Pi_{\widetilde{\sigma}}(G)$ simply transitively (see [GK82, Lemma 2.1(c)]) and there is a bijection between $\widetilde{G}(F)/\widetilde{G}_\sigma$ and $\Pi_{\widetilde{\sigma}}(G)$.

We also recall the following useful result.

Proposition 4.2. ([GK82, Lemma 2.4], [Tad92, Corollary 2.5], *and* [HS12, Lemma 2.2]) *Let* $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2 \in \mathrm{Irr}(\widetilde{G})$. *The following statements are equivalent:*

(1) There exists a character $\chi \in (\widetilde{G}(F)/G(F))^D$ *such that* $\widetilde{\sigma}_1 \cong \widetilde{\sigma}_2 \otimes \chi$;
(2) $\Pi_{\widetilde{\sigma}_1}(G) \cap \Pi_{\widetilde{\sigma}_2}(G) \neq \emptyset$;
(3) $\Pi_{\widetilde{\sigma}_1}(G) = \Pi_{\widetilde{\sigma}_2}(G)$.

Since $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma})$ is completely reducible by [GK82, Lemma 2.1] and [Tad92, Lemma 2.1], we have the decomposition

$$\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma}) = m \bigoplus_{\tau \in \Pi_{\widetilde{\sigma}}(G)} \tau \tag{4.3}$$

(see [HS12, Chapter 2]), where the positive integer m denotes the common multiplicity over $\tau \in \Pi_\sigma(G)$ [GK82, Lemma 2.1(b)]. Given $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$, we define

$$I(\widetilde{\sigma}) := \left\{ \chi \in (\widetilde{G}(F)/G(F))^D : \widetilde{\sigma} \otimes \chi \cong \widetilde{\sigma} \right\}. \tag{4.4}$$

Considering the dimension of the $\mathbb{C}$-vector space $\mathrm{End}_G(\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\sigma}))$, we have (see [Cho14b, Proposition 3.1]) the equality

$$m^2 \cdot |\Pi_\sigma(G)| = I(\widetilde{\sigma}). \tag{4.5}$$

Following [HS12, Chapter 2], since $\widetilde{\sigma} \cong \widetilde{\sigma} \otimes \chi$ for $\chi \in I(\widetilde{\sigma})$, we have a non-zero endomorphism $I_\chi \in \mathrm{Aut}_{\mathbb{C}}(V_{\widetilde{\sigma}})$ such that

$$I_\chi \circ (\widetilde{\sigma} \otimes \chi) = \widetilde{\sigma} \circ I_\chi.$$

For each $z \in \mathbb{C}^\times$, we denote by $z \cdot \mathrm{id}_{V_{\widetilde{\sigma}}}$ the scalar endomorphism $\tilde{v} \mapsto z \cdot \tilde{v}$ for $v \in V_{\widetilde{\sigma}}$. We identify $\mathbb{C}^\times$ with the subgroup of $\mathrm{Aut}_{\mathbb{C}}(V_{\widetilde{\sigma}})$ consisting of $z \cdot \mathrm{id}_{V_{\widetilde{\sigma}}}$. Define $\mathcal{A}(\widetilde{\sigma})$ as the subgroup of $\mathrm{Aut}_{\mathbb{C}}(V_{\widetilde{\sigma}})$ generated by $\{I_\chi : \chi \in I(\widetilde{\sigma})\}$ and $\mathbb{C}^\times$. Then the map $I_\chi \mapsto \chi$ induces the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathcal{A}(\widetilde{\sigma}) \longrightarrow I(\widetilde{\sigma}) \longrightarrow 1. \tag{4.6}$$

We denote by $\mathrm{Irr}(\mathcal{A}(\widetilde{\sigma}), \mathrm{id})$ the set of isomorphism classes of irreducible smooth representations of the group $\mathcal{A}(\widetilde{\sigma})$ such that $z \cdot \mathrm{id}_{V_{\widetilde{\sigma}}} \in \mathbb{C}^\times$ acts as the scalar z. By [HS12, Corollary 2.10], we then have a decomposition

$$V_{\widetilde{\sigma}} \cong \bigoplus_{\xi \in \mathrm{Irr}(\mathcal{A}(\widetilde{\sigma}), \mathrm{id})} \xi \boxtimes \sigma_\xi \tag{4.7}$$

as representations of the direct product $\mathcal{A}(\widetilde{\sigma}) \times G(F)$. It follows that there is a one-to-one correspondence

$$\mathrm{Irr}(\mathcal{A}(\widetilde{\sigma}), \mathrm{id}) \cong \Pi_{\widetilde{\sigma}}(G), \tag{4.8}$$

sending $\xi \mapsto \sigma_\xi$. We denote by ξ_σ the preimage of σ via the correspondence (4.8).

4.2. **A Theorem of Labesse.** We recall a theorem of Labesse in [Lab85] which verifies the existence of a lifting of a given L-parameter in the following setting. Let G and $\widetilde{G}$ be connected, reductive, algebraic groups over F with an exact sequence of connected components of L-groups

$$1 \longrightarrow \widehat{S} \longrightarrow \widehat{\widetilde{G}} \xrightarrow{pr} \widehat{G} \longrightarrow 1,$$

where $\widehat{S}$ is a central torus in $\widehat{\widetilde{G}}$, and the surjective homomorphism pr is compatible with Γ-actions on $\widehat{\widetilde{G}}$ and $\widehat{G}$. Then, Labesse proves in [Lab85, Théorèm 8.1] that for any $\varphi \in \Pi(G)$, there exists $\widetilde{\varphi} \in \Pi(\widetilde{G})$ such that

$$\varphi = \widetilde{\varphi} \circ pr.$$

We note that the analogous result has been proved in [Wei74, Hen80, GT10] for the case $G = SL_n$ and $\widetilde{G} = GL_n$.

4.3. **L-packets for Inner Forms of SL_n.** In this section we recall some results about the LLC for $\mathrm{SL}_m(D)$ in [HS12, Chapter 12] and [ABPS14, Section 3]. Throughout this section, let $G(F) = \mathrm{SL}_m(D)$ and $G^*(F) = \mathrm{SL}_n(F)$, where D be a central division algebra of dimension d^2 over F with $n = md$ (possibly $D = F$, in which case $d = 1$).

Since Γ acts on $\widehat{G}$ trivially, we shall use $\widehat{G} = {}^L G^0$ instead of ${}^L G = \widehat{G} \times \Gamma$. Note that $\widehat{G} = \widehat{G^*} = \mathrm{PGL}(\mathbb{C})$. Let

$$\varphi : W_F \times SL_2(\mathbb{C}) \to \widehat{G}$$

be an L–parameter. Note that

$$Z(\widehat{G}_{\mathrm{sc}}) = \mu_{\mathbf{n}}(\mathbb{C}) \quad \text{and} \quad Z(\widehat{G})^\Gamma = 1.$$

With notation as in Section 3, we have the exact sequence

$$1 \longrightarrow \widehat{Z}_{\varphi, \mathrm{sc}} \longrightarrow \mathcal{S}_{\varphi, \mathrm{sc}} \longrightarrow \mathcal{S}_\varphi \longrightarrow 1. \tag{4.9}$$

In the case at hand, $\widehat{Z}_{\varphi, \mathrm{sc}} = \mu_{\mathbf{n}}(\mathbb{C})/(\mu_{\mathbf{n}}(\mathbb{C}) \cap S_{\varphi, \mathrm{sc}}^\circ)$. In particular, when φ is elliptic, since both $S_{\varphi, \mathrm{sc}}$ in $\mathrm{SL}_n(\mathbb{C})$ and S_φ in $\mathrm{PGL}_n(\mathbb{C})$ are finite, the exact sequence (4.9) becomes

$$1 \longrightarrow \mu_{\mathbf{n}}(\mathbb{C}) \longrightarrow S_{\varphi, \mathrm{sc}} \longrightarrow S_\varphi \longrightarrow 1.$$

Since G is an F–inner form of G^*, we can fix a character ζ_G of $Z(\widehat{G}_{\mathrm{sc}})$ which corresponds to the inner form G of G^* via the Kottwitz isomorphism [Kot86, Theorem 1.2]. When $D = F$ (i.e., $G = G^*$) we have $\zeta_G = \mathbb{1}$.

Consider the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_n(\mathbb{C}) \xrightarrow{pr} \mathrm{PGL}_n(\mathbb{C}) = \widehat{G} \longrightarrow 1.$$

By the argument in Section 4.2, we have an L-parameter $\widetilde{\varphi}$ for $\widetilde{G}$

$$\widetilde{\varphi} : W_F \times \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

such that $pr \circ \widetilde{\varphi} = \varphi$ (also see [Wei74, Hen80]). By the LLC for $\widetilde{G}$ [HS12], we have a unique irreducible representation $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G})$ associated to the L-parameter $\widetilde{\varphi}$ and the L-packet $\Pi_\varphi(G)$ equals the set $\Pi_{\widetilde{\sigma}}(G)$ defined in Section 4.1. By [HS12, Lemma 12.6] and [ABPS14, Section 3], there is a homomorphism $\Lambda_{SL_n} : \mathcal{S}_{\varphi,\mathrm{sc}} \to \mathcal{A}(\widetilde{\sigma})$ (unique up to one-dimensional characters of $\mathcal{S}_\varphi$) making the following diagram commute:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
  &   & \downarrow{\scriptstyle \zeta_G} & & \downarrow{\scriptstyle \Lambda_{SL_n}} & & \downarrow{\scriptstyle \cong} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{A}(\widetilde{\sigma}) & \longrightarrow & I(\widetilde{\sigma}) & \longrightarrow & 1.
\end{array}
\tag{4.10}
$$

Combining (4.7) and (4.10), we have the following decomposition

$$V_{\widetilde{\sigma}} \cong \bigoplus_{\sigma \in \Pi_\varphi(G)} \rho_\sigma \boxtimes \sigma \quad = \quad \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G)} \rho \boxtimes \sigma_\rho \tag{4.11}$$

as representations of $\mathcal{S}_{\varphi,\mathrm{sc}} \times G(F)$. Here, $\rho_\sigma \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G)$ is given by $\xi_\sigma \circ \Lambda_{SL_n}$ with $\xi_\sigma \in \mathrm{Irr}(\mathcal{A}(\widetilde{\sigma}), \mathrm{id})$, and $\sigma_\rho \in \Pi_\varphi(G)$ denotes the image of $\rho \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G)$ via the bijection between $\Pi_\varphi(G)$ and $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_G)$.

5. Local Langlands Correspondence for GSpin_4 and its inner forms

In this section we establish the LLC for GSpin_4 and all its non quasi-split F-inner forms.

5.1. Construction of L-packets of GSpin_4 and Its Inner Forms.
It follows from the arguments in Section 4 on restriction and the group structure (2.6) that given $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_4)$, there is a lifting $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ such that

$$\sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_2 \times \mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma}).$$

By the LLC for GL_n [HT01, Hen00, Sch13], we have a unique $\widetilde{\varphi}_{\widetilde{\sigma}} \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$ corresponding to the representation $\widetilde{\sigma}$. We now define a map

$$
\begin{array}{cccc}
\mathcal{L}_4 : \mathrm{Irr}(\mathrm{GSpin}_4) & \longrightarrow & \Phi(\mathrm{GSpin}_4) \\
\sigma & \longmapsto & pr_4 \circ \widetilde{\varphi}_{\widetilde{\sigma}}.
\end{array}
\tag{5.1}
$$

Note that $\mathcal{L}_4$ does not depend on the choice of the lifting $\widetilde{\sigma}$. Indeed, if $\widetilde{\sigma}' \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ is another lifting, it follows from Proposition 4.2 and (2.20) that $\widetilde{\sigma}' \cong \widetilde{\sigma} \otimes \chi$ for some quasi-character χ on

$$(\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))/\mathrm{GSpin}_4(F) \cong F^\times.$$

Morover,

$$F^\times \cong H^1(F, \mathbb{C}^\times),$$

where $\mathbb{C}^\times$ is as in (2.26). The LLC for $\mathrm{GL}_2 \times \mathrm{GL}_2$ maps $\widetilde{\sigma}'$ to $\widetilde{\varphi}_{\widetilde{\sigma}} \otimes \chi$ (by abuse of notation, employing χ for both the quasi-character and its parameter via Local Class Field Theory). Since $pr_4(\widetilde{\varphi}_{\widetilde{\sigma}} \otimes \chi) = pr_4(\widetilde{\varphi}_{\widetilde{\sigma}})$ by (2.26), the map $\mathcal{L}_4$ is well-defined.

Moreover, we note that $\mathcal{L}_4$ is surjective. Indeed, by Labesse's Theorem in Section 4.2, $\varphi \in \Phi(\mathrm{GSpin}_4)$ can be lifted to some $\widetilde{\varphi} \in \Phi(GL_2 \times \mathrm{GL}_2)$. We then obtain $\widetilde{\sigma} \in \mathrm{Irr}(GL_2 \times \mathrm{GL}_2)$ via the LLC for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Thus, any irreducible constituent in the restriction $\mathrm{Res}^{\mathrm{GL}_2 \times \mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma})$ has the image φ via the map $\mathcal{L}_4$.

As expected, for each $\varphi \in \Phi(\mathrm{GSpin}_4)$, we define the L-packet $\Pi_\varphi(\mathrm{GSpin}_4)$ as the set of all inequivalent irreducible constituents of $\widetilde{\sigma}$

$$\Pi_\varphi(\mathrm{GSpin}_4) := \Pi_{\widetilde{\sigma}}(\mathrm{GSpin}_4) = \left\{ \sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_2 \times \mathrm{GL}_2}_{\mathrm{GSpin}_4}(\widetilde{\sigma}) \right\} / \cong, \tag{5.2}$$

where $\widetilde{\sigma}$ is the unique member in $\Pi_{\widetilde{\varphi}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ and $\widetilde{\varphi} \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$ is such that $pr_4 \circ \widetilde{\varphi} = \varphi$. By the LLC for GL_2 and Proposition 4.2, the fiber does not depends on the choice of $\widetilde{\varphi}$.

We define the L-packets for the non quasi-split inner forms similarly. Using the group structure described in Section 2.2, given $\sigma_4^{2,1} \in \mathrm{Irr}(\mathrm{GSpin}_4^{2,1})$, there is a lifting $\widetilde{\sigma}_4^{2,1} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_1(D))$ such that

$$\sigma_4^{2,1} \hookrightarrow \mathrm{Res}_{\mathrm{GSpin}_4^{2,1}}^{\mathrm{GL}_2 \times \mathrm{GL}_1(D)}(\widetilde{\sigma}_4^{2,1}).$$

Combining the LLC for GL_2 and $\mathrm{GL}_1(D)$ [HS12], we have a unique $\widetilde{\varphi}_{\widetilde{\sigma}_4^{2,1}} \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_1(D))$ corresponding to the representation $\widetilde{\sigma}_4^{2,1}$. We thus define the map

$$\begin{aligned} \mathcal{L}_4^{2,1} : \mathrm{Irr}(\mathrm{GSpin}_4^{2,1}) &\longrightarrow \Phi(\mathrm{GSpin}_4^{2,1}) \\ \sigma_4^{2,1} &\longmapsto pr_4 \circ \widetilde{\varphi}_{\widetilde{\sigma}_4^{2,1}}. \end{aligned} \tag{5.3}$$

Again, it follows from the LLC for GL_2 and $\mathrm{GL}_1(D)$ that this map is well-defined and surjective.

Likewise, for the other F-inner form $\mathrm{GSpin}_4^{1,1}$ of GSpin_4, we have a well-defined and surjective map

$$\begin{aligned} \mathcal{L}_4^{1,1} : \mathrm{Irr}(\mathrm{GSpin}_4^{1,1}) &\longrightarrow \Phi(\mathrm{GSpin}_4^{1,1}) \\ \sigma_4^{1,1} &\longmapsto pr_4 \circ \widetilde{\varphi}_{\widetilde{\sigma}_4^{1,1}}. \end{aligned} \tag{5.4}$$

We similarly define L-packets

$$\Pi_\varphi(\mathrm{GSpin}_4^{2,1}) = \Pi_{\widetilde{\sigma}_4^{2,1}}(\mathrm{GSpin}_4^{2,1}), \quad \varphi \in \Phi(\mathrm{GSpin}_4^{2,1}) \tag{5.5}$$

and

$$\Pi_\varphi(\mathrm{GSpin}_4^{1,1}) = \Pi_{\widetilde{\sigma}_4^{1,1}}(\mathrm{GSpin}_4^{1,1}), \quad \varphi \in \Phi(\mathrm{GSpin}_4^{1,1}). \tag{5.6}$$

Again, these L-packet do not depend on the choice of $\widetilde{\varphi}$ for similar reasons.

5.2. Internal Structure of L-packets of GSpin_4 and Its Inner Forms.

In this section we continue to employ the notation in Section 3. For simplicity of notation, we shall write $\mathrm{GSpin}_\sharp$ for the split GSpin_4, and its non quasi-split F-inner forms $\mathrm{GSpin}_4^{2,1}$ and $\mathrm{GSpin}_4^{1,1}$. Likewise, we shall write $\mathrm{SL}_\sharp$ and $\mathrm{GL}_\sharp$ for the corresponding groups in (2.6), (2.11), and (2.12) so that we have

$$\mathrm{SL}_\sharp \subset \mathrm{GSpin}_\sharp \subset \mathrm{GL}_\sharp \tag{5.7}$$

in all cases. From Section 2.4, we recall that

$$(\widehat{\mathrm{GSpin}_\sharp})_{\mathrm{ad}} = \mathrm{PSO}_4(\mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}),$$

$$(\widehat{\mathrm{GSpin}_\sharp})_{\mathrm{sc}} = \mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}),$$

$$Z((\widehat{\mathrm{GSpin}_\sharp})_{\mathrm{sc}}) = Z((\widehat{\mathrm{GSpin}_\sharp})_{\mathrm{sc}})^\Gamma \cong \mu_2(\mathbb{C}) \times \mu_2(\mathbb{C}).$$

Let $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$ be given. We fix a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\sharp)$ via the surjective map $\widehat{\mathrm{GL}_\sharp} \longrightarrow \widehat{\mathrm{GSpin}_\sharp}$ (cf. Theorem 4.2). With notation as in Section 3, we have

$$S_\varphi \subset \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}),$$

$$S_{\varphi,\mathrm{sc}} \subset \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}).$$

We then have (cf. (3.4)) a central extension

$$1 \longrightarrow \widehat{Z}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_\varphi \longrightarrow 1. \tag{5.8}$$

Let ζ_4, $\zeta_4^{2,1}$, and $\zeta_4^{1,1}$ be characters on $Z((\widehat{\mathrm{GSpin}_\sharp})_{\mathrm{sc}})$ which respectively correspond to GSpin_4, $\mathrm{GSpin}_4^{2,1}$, and $\mathrm{GSpin}_4^{1,1}$ via the Kottwitz isomorphism [Kot86, Theorem 1.2]. Note that

$$\zeta_4 = \mathbb{1}, \quad \zeta_4^{2,1} = \mathbb{1} \times \mathrm{sgn}, \quad \text{and} \quad \zeta_4^{1,1} = \mathrm{sgn} \times \mathrm{sgn},$$

where sgn is the non-trivial character of $\mu_2(\mathbb{C})$.

Theorem 5.1. *Given an L-parameter $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$, there is a one-to-one bijection*

$$\Pi_\varphi(\mathrm{GSpin}_\sharp) \quad \overset{1-1}{\longleftrightarrow} \quad \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\sharp),$$
$$\sigma \quad \mapsto \quad \rho_\sigma,$$

such that we have the following decomposition

$$V_{\widetilde{\sigma}} \cong \bigoplus_{\sigma \in \Pi_\varphi(\mathrm{GSpin}_\sharp)} \rho_\sigma \boxtimes \sigma$$

as representations of the direct product $\mathcal{S}_{\varphi,\mathrm{sc}} \times \mathrm{GSpin}_\sharp(F)$, where $\widetilde{\sigma} \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_\sharp)$ is an extension of $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\sharp)$ to $\mathrm{GL}_\sharp(F)$ as in Section 4 and $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\sharp)$ is a lifting of $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$. Here, $\zeta_\sharp \in \left\{ \zeta_4, \zeta_4^{2,1}, \zeta_4^{1,1} \right\}$ according to which inner form $\mathrm{GSpin}_\sharp$ is.

Proof. We follow the ideas in Section 4.3 and [CL14, Theorem 5.4.1]. Given $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$, we choose a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\sharp)$ and obtain the projection $\bar{\varphi} \in \Phi(\mathrm{SL}_\sharp)$ in the following commutative diagram

(5.9)

$$\widehat{\mathrm{GL}_\sharp} = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$$
$$\downarrow pr_4$$
$$W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\ \varphi\ } \widehat{\mathrm{GSpin}_\sharp} = \mathrm{GSO}_4(\mathbb{C})$$
$$\downarrow \bar{pr}$$
$$\widehat{\mathrm{SL}_\sharp} = \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) .$$

We then have $\widetilde{\sigma} \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_\sharp)$ which is an extension of $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\sharp)$. In addition to (2.26), we also have

$$1 \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \longrightarrow \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\bar{pr} \circ pr_4} \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1$$

Considering the kernels of the projections pr_4 and $\bar{pr} \circ pr_4$, we set

$$X^{\mathrm{GSpin}_\sharp}(\widetilde{\varphi}) := \{ a \in H^1(W_F, \mathbb{C}^\times) : a\widetilde{\varphi} \cong \widetilde{\varphi} \}$$
$$X^{\mathrm{SL}_\sharp}(\widetilde{\varphi}) := \{ a \in H^1(W_F, \mathbb{C}^\times \times \mathbb{C}^\times) : a\widetilde{\varphi} \cong \widetilde{\varphi} \}.$$

Moreover, by (2.9) and its analogues for the two non quasi-split F-inner forms, we have

$$\mathrm{GL}_\sharp(F)/\mathrm{GSpin}_\sharp(F) \cong F^\times.$$

As an easy consequence of Galois cohomology, we also have

$$\mathrm{GL}_\sharp(F)/\mathrm{SL}_\sharp(F) \cong F^\times \times F^\times.$$

Set

$$I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) := \{ \chi \in (F^\times)^D \cong (\mathrm{GL}_\sharp(F)/\mathrm{GSpin}_\sharp(F))^D : \widetilde{\sigma}\chi \cong \widetilde{\sigma} \}$$
$$I^{\mathrm{SL}_\sharp}(\widetilde{\sigma}) := \{ \chi \in (F^\times)^D \times (F^\times)^D \cong (\mathrm{GL}_\sharp(F)/\mathrm{SL}_\sharp(F))^D : \widetilde{\sigma}\chi \cong \widetilde{\sigma} \}.$$

Remark 5.2. Since any character on $\mathrm{GL}_n(F)$ (respectively, $\mathrm{GL}_m(D)$) is of the form $\chi \circ \det$ (respectively, $\chi \circ \mathrm{Nrd}$) for some character χ on $F^\times$ (see [BH06, Section 53.5]), we often make no distinction between χ and $\chi \circ \det$ (respectively, $\chi \circ \mathrm{Nrd}$). Moreover, we note that $\chi \in (\mathrm{GL}_\sharp(F))^D$ can be written as $\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2$, where $\widetilde{\chi}_i$ with $i = 1, 2$, is a character of $\mathrm{GL}_2(F)$ or $D^\times$ depending on the type of $\mathrm{GL}_\sharp$.

It follows from the definitions that

$$X^{\mathrm{GSpin}_\sharp}(\widetilde{\varphi}) \subset X^{\mathrm{SL}_\sharp}(\widetilde{\varphi}) \quad \text{and} \quad I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) \subset I^{\mathrm{SL}_\sharp}(\widetilde{\sigma}). \tag{5.10}$$

We have $(F^\times)^D \cong H^1(W_F, \mathbb{C}^\times)$ by the local class field theory. It is also immediate from the LLC for GL_n that

$$X^{\mathrm{GSpin}_\sharp}(\widetilde{\varphi}) \cong I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) \quad \text{and} \quad X^{\mathrm{SL}_\sharp}(\widetilde{\varphi}) \cong I^{\mathrm{SL}_\sharp}(\widetilde{\sigma})$$

as groups of characters. Now, with the component group notation $\mathcal{S}$ of Section 3, we claim that

$$I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) \cong \mathcal{S}_\varphi. \tag{5.11}$$

Indeed, by [CL14, Lemma 5.3.4] and the above arguments, this claim follows from

$$\mathcal{S}_\varphi \cong X^{\mathrm{GSpin}_\sharp}(\widetilde{\varphi}),$$

since $\mathcal{S}_{\widetilde{\varphi}}$ is always trivial by the LLC for $\mathrm{GL}_\sharp$. Notice that $\widetilde{\varphi}$ and φ here are respectively ϕ and $\phi^\sharp$ in [CL14, Lemma 5.3.4]. Thus, due to (4.10), (5.10), and (5.11) we have

$$\mathcal{S}_\varphi \subset \mathcal{S}_{\bar{\varphi}}. \tag{5.12}$$

Since the centralizer $C_{\bar{\varphi}}$ (in $\widehat{\mathrm{SL}}_\sharp$) is equal to the image of the disjoint union

$$\coprod_{\nu \in \mathrm{Hom}(W_F, \mathbb{C}^\times)} \left\{ h \in \mathrm{GSO}_4(\mathbb{C}) : h\varphi(w)h^{-1}\varphi(w)^{-1} = \nu(w) \right\}$$

from the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GSO}_4(\mathbb{C}) \xrightarrow{\bar{pr}} \widehat{\mathrm{SL}}_\sharp = \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1,$$

we have

$$S_\varphi \subset C_{\bar{\varphi}} = S_{\bar{\varphi}}.$$

So, the pre-images in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ via the isogeny $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ also satisfy

$$S_{\varphi,\mathrm{sc}} \subset S_{\bar{\varphi},\mathrm{sc}} \subset \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}).$$

This provides the inclusion of the identity components

$$S_{\varphi,\mathrm{sc}}^\circ \subset S_{\bar{\varphi},\mathrm{sc}}^\circ, \tag{5.13}$$

and by the definition $\widehat{Z}_{\varphi,\mathrm{sc}} := Z(\widehat{G}_{\mathrm{sc}})/(Z(\widehat{G}_{\mathrm{sc}}) \cap S_{\varphi,\mathrm{sc}}^\circ)$ of Section 3 we have the surjection

$$\widehat{Z}_{\varphi,\mathrm{sc}} \twoheadrightarrow \widehat{Z}_{\bar{\varphi},\mathrm{sc}}. \tag{5.14}$$

Combining (5.12), (5.13), and (5.14), we have the following commutative diagram of component groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\twoheadrightarrow} & & \downarrow{\scriptstyle\cap} & & \downarrow{\scriptstyle\cap} & & \\
1 & \longrightarrow & \widehat{Z}_{\bar{\varphi},\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\bar{\varphi},\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\bar{\varphi}} & \longrightarrow & 1.
\end{array} \tag{5.15}$$

We apply the Hiraga-Saito homomorphism Λ_{SL_n} of (4.10) in the case of our $\mathrm{SL}_\sharp$, which we denote by $\Lambda_{\mathrm{SL}_2 \times \mathrm{SL}_2}$. The restriction

$$\Lambda_\sharp := \Lambda_{\mathrm{SL}_2 \times \mathrm{SL}_2}|_{\mathcal{S}_{\varphi,\mathrm{sc}}}$$

then gives the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\zeta_\sharp} & & \downarrow{\scriptstyle\Lambda_\sharp} & & \downarrow{\scriptstyle\cap} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{A}(\widetilde{\sigma}) & \longrightarrow & I^{\mathrm{SL}_\sharp}(\widetilde{\sigma}) & \longrightarrow & 1.
\end{array} \tag{5.16}$$

Note that $\zeta_\sharp$ is identified with ζ_G in (4.10) as the character on $\mu_2(\mathbb{C}) \times \mu_2(\mathbb{C})$ since both are determined according to $\mathrm{SL}_\sharp$.

Similar to $\mathcal{A}(\widetilde{\sigma})$ (cf. Section 4), we write $\mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma})$ for the subgroup of $\mathrm{Aut}_\mathbb{C}(V_{\widetilde{\sigma}})$ generated by $\mathbb{C}^\times$ and $\left\{ I_\chi : \chi \in I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) \right\}$. Hence, $\mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) \subset \mathcal{A}(\widetilde{\sigma})$. By the definition of $\Lambda_{\mathrm{SL}_2 \times \mathrm{SL}_2}$ in (4.10) and the

commutative diagram (5.15), it is immediate that the image of $\Lambda_\sharp$ is $\mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma})$. We thus have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \zeta_\sharp} & & \downarrow{\scriptstyle \Lambda_\sharp} & & \downarrow{\scriptstyle \cong} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) & \longrightarrow & I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}) & \longrightarrow & 1.
\end{array}
\tag{5.17}
$$

The representation $\rho_\sigma \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\sharp)$ is defined by $\xi_\sigma \circ \Lambda_\sharp$, where $\xi_\sigma \in \mathrm{Irr}(\mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}), \mathrm{id})$ is the character as in the decomposition (4.7). Therefore, arguments of Section 4, our construction of L-packets $\Pi_\varphi(\mathrm{GSpin}_\sharp)$ in Section 5.1 and diagram (5.17) above give

$$
\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\sharp) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{A}^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}), \mathrm{id}) \overset{1-1}{\longleftrightarrow} \Pi_\varphi(\mathrm{GSpin}_\sharp).
\tag{5.18}
$$

We also have the following decomposition

$$
V_{\widetilde{\sigma}} \cong \bigoplus_{\sigma \in \Pi_\varphi(\mathrm{GSpin}_\sharp)} \rho_\sigma \boxtimes \sigma = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\sharp)} \rho \boxtimes \sigma_\rho,
$$

where σ_ρ denotes the image of ρ via the bijection between $\Pi_\varphi(\mathrm{GSpin}_\sharp)$ and $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\sharp)$. Hence, the proof of Theorem 5.1 is complete. $\qquad\square$

Remark 5.3. Similar to $\Lambda_{\mathrm{SL}_2 \times \mathrm{SL}_2}$ in Section 4, since $\Lambda_\sharp$ is unique up to $\mathrm{Hom}(I^{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}), \mathbb{C}^\times) \cong \mathrm{Hom}(\mathcal{S}_\varphi, \mathbb{C}^\times)$, the same is true for the bijection (5.18).

Remark 5.4. Given $\varphi \in \Phi(\mathrm{GSpin}_4)$, by Theorem 5.1, we have a one-to-one bijection

$$
\Pi_\varphi(\mathrm{GSpin}_4) \cup \Pi_\varphi(\mathrm{GSpin}_4^{2,1}) \cup \Pi_\varphi(\mathrm{GSpin}_4^{1,2}) \cup \Pi_\varphi(\mathrm{GSpin}_4^{1,1}) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}),
$$

where $\mathrm{GSpin}_4^{1,2} \cong \{(g_1, g_2) \in (\mathrm{GL}_1(D) \times \mathrm{GL}_2) : \mathrm{Nrd}(g_1) = \det g_2\}$, which is isomorphic to $\mathrm{GSpin}_4^{2,1}$.

5.3. **L-packet Sizes for GSpin_4 and Its Inner Forms.** In this subsection, we describe the group $I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$ for $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ case by case, discuss the group structure of $\mathcal{S}_{\varphi,\mathrm{sc}}$, and provide all the sizes of L-packets for GSpin_4, $\mathrm{GSpin}_4^{2,1}$, and $\mathrm{GSpin}_4^{1,1}$ using Theorem 5.1. Let us first give the possible cardinalities for the L-packets of GSpin_4 and its inner forms using Galois cohomology.

Proposition 5.5. *Let $\Pi_\varphi(\mathrm{GSpin}_\sharp)$ be an L-packet associated to $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$. Then we have*

$$
\left| \Pi_\varphi(\mathrm{GSpin}_\sharp) \right| \,\Big|\, \left| F^\times / (F^\times)^2 \right|,
$$

which yields the possible cardinalities

$$
\left| \Pi_\varphi(\mathrm{GSpin}_\sharp) \right| = \left\{ \begin{array}{ll} 1, \ 2, \ 4, & \text{if } p \neq 2, \\ 1, \ 2, \ 4, \ 8, & \text{if } p = 2. \end{array} \right.
$$

Proof. We follow the idea of a similar result in the case of Sp_4 in [CG15]. The exact sequence of algebraic groups

$$
1 \longrightarrow Z(\mathrm{GSpin}_\sharp) \longrightarrow Z(\mathrm{GL}_\sharp) \times \mathrm{GSpin}_\sharp \longrightarrow \mathrm{GL}_\sharp \longrightarrow 1,
$$

where the second map is given by multiplication (considering $\mathrm{GSpin}_\sharp \subset \mathrm{GL}_\sharp$), gives a long exact sequence

$$
\cdots \longrightarrow (F^\times \times F^\times) \times \mathrm{GSpin}_\sharp(F) \longrightarrow \mathrm{GL}_\sharp(F) \longrightarrow H^1(F, Z(\mathrm{GSpin}_\sharp)) \longrightarrow H^1(F, Z(\mathrm{GL}_\sharp) \times \mathrm{GSpin}_\sharp) \longrightarrow 1.
$$

Since $H^1(F, Z(\mathrm{GL}_\sharp) \times \mathrm{GSpin}_\sharp) = 1$ by Lemma 2.2 and [PR94, Lemmas 2.8], we have

$$
\mathrm{GL}_\sharp(F) / \left((F^\times \times F^\times) \times \mathrm{GSpin}_\sharp(F) \right) \hookrightarrow H^1(F, Z(\mathrm{GSpin}_\sharp)).
$$

Also, the exact sequence

$$
1 \longrightarrow Z(\mathrm{GSpin}_\sharp)^\circ \longrightarrow Z(\mathrm{GSpin}_\sharp) \longrightarrow \pi_0(Z(\mathrm{GSpin}_\sharp)) \longrightarrow 1
$$

gives

$$
H^1(F, Z(\mathrm{GSpin}_\sharp)) \hookrightarrow H^1(F, \pi_0(Z(\mathrm{GSpin}_\sharp))),
$$

since, by [AS06, Proposition 2.3], $Z(\mathrm{GSpin}_\sharp)^\circ \cong \mathrm{GL}_1$ and $H^1(F, Z(\mathrm{GSpin}_\sharp)^\circ) = 1$. Combining the above with (2.19), we have

$$\mathrm{GL}_\sharp(F)/\left((F^\times \times F^\times) \times \mathrm{GSpin}_\sharp(F)\right) \hookrightarrow H^1(F, \pi_0(Z(\mathrm{GSpin}_\sharp))) \cong H^1(F, \mathbb{Z}/2\mathbb{Z}) \cong F^\times/(F^\times)^2. \qquad (5.19)$$

On the other hand, we know from Section 4 that

$$\Pi_\varphi(\mathrm{GSpin}_\sharp) \overset{1-1}{\longleftrightarrow} \mathrm{GL}_\sharp(F)/\mathrm{GL}_\sharp(F)_{\widetilde{\sigma}} \hookrightarrow \mathrm{GL}_\sharp(F)/\left((F^\times \times F^\times) \times \mathrm{GSpin}_\sharp(F)\right),$$

where $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_\sharp)$ corresponds to a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\sharp)$, via the LLC for GL_n and its inner forms, of $\varphi \in \Phi(\mathrm{GSpin}_\sharp)$. Thus the proof is complete. $\qquad\square$

In what follows, we give an explicit description of the group $I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$ for $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$ case by case, and show that among the possible cardinalities in Proposition 5.5, only 1, 2, and 4 do indeed occur (see Remarks 5.9 - 5.11 and (5.21) - (5.22)). To this end, we verify that $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong 1, \mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$, which yields $\mathcal{S}_\varphi \cong 1$, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$ by (5.11). Thus, due to Theorem 5.1, the size of an L-packet for GSpin_4 is either 1, 2, or 4. Moreover, we show that the central extension $\mathcal{S}_{\varphi,\mathrm{sc}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^3$, an abelian subgroup of order 16 in $Q_8 \times Q_8$, or $Q_8 \times \mathbb{Z}/2\mathbb{Z}$, for irreducible $\varphi \in \Phi(\mathrm{GSpin}_4)$, and is always abelian for reducible $\varphi \in \Phi(\mathrm{GSpin}_4)$. Here Q_8 is the quaternion group of order 8. Due to Theorem 5.1, these facts about $\mathcal{S}_{\varphi,\mathrm{sc}}$ imply that the size of an L-packet for non-split inner forms $\mathrm{GSpin}_4^{2,1}$ and $\mathrm{GSpin}_4^{1,1}$ is either 1, 2, or 4.

To describe the group $I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$, given $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_2 \times \mathrm{GL}_2)$, we set $\widetilde{\sigma} = \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2$ with $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathrm{Irr}(\mathrm{GL}_2)$. Due to Remark 5.2, we note that $\chi \in (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))^D$ is decomposed into $\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2$, where $\widetilde{\chi}_i$ with $i = 1, 2$ is a character of $\mathrm{GL}_2(F)$, and we identify $\widetilde{\chi}_i$ and $\widetilde{\chi}_i \circ \det$, since any character on $\mathrm{GL}_n(F)$ is of the form $\widetilde{\chi} \circ \det$ for some character $\widetilde{\chi}$ on $F^\times$.

Lemma 5.6. *Let $\chi \in I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$ be given. Then, χ is of the form*

$$\widetilde{\chi} \boxtimes \widetilde{\chi}^{-1},$$

where $\widetilde{\chi} \in (F^\times)^D$.

Proof. Since $\chi = \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2$ as above and χ is trivial on $\mathrm{GSpin}_4(F)$, by the structure of $\mathrm{GSpin}_4(F)$ in (2.21), we have

$$\chi(g_1, g_2) = \widetilde{\chi}_1(\det g_1) \boxtimes \widetilde{\chi}_2(\det g_2) = \widetilde{\chi}_1(\det g_1)\widetilde{\chi}_2(\det g_2) = 1$$

for all $(g_1, g_2) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ with $\det g_1 = \det g_2$. Since the determinant map $\det : \mathrm{GL}_2(F) \to F^\times$ is surjective, $\widetilde{\chi}_1\widetilde{\chi}_2(x)$ must be trivial for all $x \in F^\times$. Thus, $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ are inverse each other. $\qquad\square$

Proposition 5.7. *We have*

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) = \begin{cases} I^{\mathrm{SL}_2}(\widetilde{\sigma}_1), & \text{if } \widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta} \text{ for some } \widetilde{\eta} \in (F^\times)^D; \\ I^{\mathrm{SL}_2}(\widetilde{\sigma}_1) \cap I^{\mathrm{SL}_2}(\widetilde{\sigma}_2), & \text{if } \widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta} \text{ for any } \widetilde{\eta} \in (F^\times)^D. \end{cases}$$

Proof. Since $\widetilde{\sigma} = \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2$ with $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathrm{Irr}(\mathrm{GL}_2)$, and by Lemma 5.6, we have

$$\widetilde{\sigma}\chi \cong \widetilde{\sigma} \iff \widetilde{\sigma}_1\widetilde{\chi} \boxtimes \widetilde{\sigma}_2\widetilde{\chi}^{-1} \cong \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2 \iff \widetilde{\sigma}_i\widetilde{\chi} \cong \widetilde{\sigma}_i, \ i = 1, 2. \qquad (5.20)$$

This shows that $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) = I^{\mathrm{SL}_2}(\widetilde{\sigma}_1) \cap I^{\mathrm{SL}_2}(\widetilde{\sigma}_2)$. In particular, if $\widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}$ for some $\widetilde{\eta} \in (F^\times)^D$, then $I^{\mathrm{SL}_2}(\widetilde{\sigma}_1) = I^{\mathrm{SL}_2}(\widetilde{\sigma}_2)$. Thus we have, by (5.20), that $I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) = I^{\mathrm{SL}_2}(\widetilde{\sigma}_1)$. $\qquad\square$

Remark 5.8. By the LLC for SL_2 ([GK82]), we have $I^{\mathrm{SL}_2}(\widetilde{\sigma}_i) \cong \pi_0(C_{\varphi_i})$ for $\varphi_i \in \Phi(\mathrm{SL}_2)$ corresponding to $\sigma_i \subset \mathrm{Res}^{\mathrm{GL}_2}_{\mathrm{SL}_2}(\widetilde{\sigma}_i)$. Recalling that $\mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$, it then follows from [GP92, Corollary 6.6] that $I^{\mathrm{SL}_2}(\widetilde{\sigma}_1)$ and $I^{\mathrm{SL}_2}(\widetilde{\sigma}_2)$ consist of quadratic characters. Then, $I^{\mathrm{GSpin}_4}(\widetilde{\sigma})$ consists of quadratic characters of $F^\times$.

Given $\varphi \in \Phi(\mathrm{GSpin}_4)$, fix the lift

$$\widetilde{\varphi} = \widetilde{\varphi}_1 \otimes \widetilde{\varphi}_2 \in \Phi(\mathrm{GL}_2 \times \mathrm{GL}_2)$$

with $\widetilde{\varphi}_i \in \Phi(\mathrm{GL}_2)$ such that $\varphi = pr_4 \circ \widetilde{\varphi}$, as in Section 5.1. Let

$$\widetilde{\sigma} = \widetilde{\sigma}_1 \boxtimes \widetilde{\sigma}_2 \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_2 \times \mathrm{GL}_2)$$

be the unique member such that $\{\widetilde{\sigma}_i\} = \Pi_{\widetilde{\varphi}_i}(\mathrm{GL}_2)$. Recall (see [GT10, Section 5]) that an irreducible L-parameter $\phi \in \Phi(\mathrm{GL}_2)$ is called *primitive* if the restriction $\phi|_{W_F}$ is not of the form $\mathrm{Ind}_{W_E}^{W_F}\theta$ for some finite extension E over F. Moreover, ϕ is called *dihedral* with respect to (w.r.t.) a quadratic extension E over F if $\phi|_{W_F} \cong \mathrm{Ind}_{W_E}^{W_F}\theta$ or equivalently if $(\phi|_{W_F}) \otimes \omega_{E/F} \cong (\phi|_{W_F})$ ($\Leftrightarrow \phi \otimes \omega_{E/F} \cong \phi$) for a quadratic character $\omega_{E/F}$ corresponding to quadratic E/F via the local class field theory. A primitive representation exists only when p divides $\dim \phi$ ([Koc77]). We can now make the following remarks.

Remark 5.9. If $\widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}$ for some $\widetilde{\eta} \in (F^\times)^D$, we know that $\widetilde{\varphi}_1$ is dihedral w.r.t. one quadratic character (respectively, primitive, dihedral w.r.t. three quadratic characters) if and only if $\widetilde{\varphi}_2$ is dihedral (respectively, primitive, dihedral w.r.t. three quadratic characters).

Remark 5.10. Let $\varphi \in \Phi(\mathrm{GSpin}_4)$ be irreducible. Then $\widetilde{\varphi}$, $\widetilde{\varphi}_1$, and $\widetilde{\varphi}_2$ are all irreducible. Combining Proposition 5.7, Remarks 5.8 and 5.9, and [GT10, Porposition 6.3], we conclude the following. When $\widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}$ for some $\widetilde{\eta} \in (F^\times)^D$, we have

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \begin{cases} \{1\}, & \text{if } \widetilde{\varphi}_1 \text{ is primitive or non-trivial on } \mathrm{SL}_2(\mathbb{C}) \text{ (hence so is } \widetilde{\varphi}_2); \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \widetilde{\varphi}_1 \text{ is dihedral w.r.t. one quadratic extension (hence so is } \widetilde{\varphi}_2); \\ (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } \widetilde{\varphi}_1 \text{ is dihedral w.r.t. three quadratic extensions (hence so is } \widetilde{\varphi}_2). \end{cases}$$

When $\widetilde{\sigma}_2 \not\cong \widetilde{\sigma}_1\widetilde{\eta}$ for any $\widetilde{\eta} \in (F^\times)^D$, an analogous assertion holds, but it would depend on the individual parameters and not just whether they are primitive or dihedral. In that case, we have

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\}, \ \mathbb{Z}/2\mathbb{Z}, \ \text{or } (\mathbb{Z}/2\mathbb{Z})^2.$$

Remark 5.11. When $\widetilde{\varphi}_i$ is reducible, $\widetilde{\sigma}_i$ is either the Steinberg representation twisted by a character or an irreducibly induced representation from the Borel subgroup of GL_2. Since the number of irreducible constituents in $\mathrm{Res}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(\widetilde{\sigma}_i)$ is ≤ 2, we have $I^{\mathrm{SL}_2}(\widetilde{\sigma}_i) \cong \{1\}$, or $\mathbb{Z}/2\mathbb{Z}$. This implies, by Proposition 5.7, that

$$I^{\mathrm{GSpin}_4}(\widetilde{\sigma}) \cong \{1\}, \ \text{or } \mathbb{Z}/2\mathbb{Z}.$$

We now describe the central extension $\mathcal{S}_{\varphi,\mathrm{sc}}$ for GSpin_4. For irreducible $\varphi \in \Phi(\mathrm{GSpin}_4)$, from (5.11) and Remark 5.10, we have $\mathcal{S}_\varphi \cong 1$, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$. Since φ is elliptic and $\widehat{Z}_{\varphi,\mathrm{sc}} = Z(\mathrm{Spin}_4(\mathbb{C})) = \mu_2(\mathbb{C}) \times \mu_2(\mathbb{C})$, the exact sequence (5.8) becomes

$$1 \longrightarrow \mu_2(\mathbb{C}) \times \mu_2(\mathbb{C}) \longrightarrow \mathcal{S}_{\varphi,\mathrm{sc}} \subset \mathrm{Spin}_4(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathcal{S}_\varphi \subset \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}) \longrightarrow 1.$$

When $\mathcal{S}_\varphi \cong 1$, it is obvious that $\mathcal{S}_{\varphi,\mathrm{sc}} \cong (\mathbb{Z}/2\mathbb{Z})^2$. When $\mathcal{S}_\varphi \cong \mathbb{Z}/2\mathbb{Z}$, since $(\mathbb{Z}/2\mathbb{Z})^2$ is embedded in $Z(\mathcal{S}_{\varphi,\mathrm{sc}})$, $\mathcal{S}_{\varphi,\mathrm{sc}}$ is an abelian group of order 8. Since $\mathcal{S}_{\varphi,\mathrm{sc}} \subset \mathcal{S}_{\bar{\varphi},\mathrm{sc}}$ from (5.15), using the fact that the group $\mathcal{S}_{\bar{\varphi},\mathrm{sc}} \cong (\mathbb{Z}/2\mathbb{Z})^4$ (see the proof of [Cho14b, Proposition 6.5]), we have $\mathcal{S}_{\varphi,\mathrm{sc}} \cong (\mathbb{Z}/2\mathbb{Z})^3$. When $\mathcal{S}_\varphi \cong (\mathbb{Z}/2\mathbb{Z})^2$, since $\mathcal{S}_{\bar{\varphi},\mathrm{sc}} \cong Q_8 \times Q_8$ from *loc. cit.*, $\mathcal{S}_{\varphi,\mathrm{sc}}$ is a subgroup of order 16 in $Q_8 \times Q_8$. If $\mathcal{S}_{\varphi,\mathrm{sc}}$ is non-abelian, since $Z(\mathcal{S}_{\varphi,\mathrm{sc}}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, we have $\mathcal{S}_{\varphi,\mathrm{sc}} \cong Q_8 \times \mathbb{Z}/2\mathbb{Z}$. It thus follows that $\mathcal{S}_{\varphi,\mathrm{sc}}$ is an abelian subgroup in $Q_8 \times Q_8$ of order 16, or $Q_8 \times \mathbb{Z}/2\mathbb{Z}$. For reducible $\varphi \in \Phi(\mathrm{GSpin}_4)$, (5.11) and Remark 5.11 provide $\mathcal{S}_\varphi \cong 1$ or $\mathbb{Z}/2\mathbb{Z}$. Since $(\mathbb{Z}/2\mathbb{Z})^2$ is embedded in $Z(\mathcal{S}_{\varphi,\mathrm{sc}})$, $\mathcal{S}_{\varphi,\mathrm{sc}}$ must be abelian of order 8.

Using Theorem 5.1, we describe the sizes of L-packets for GSpin_4, $\mathrm{GSpin}_4^{2,1}$, and $\mathrm{GSpin}_4^{1,1}$. Form the previous paragraph, we note that $\mathcal{S}_{\varphi,\mathrm{sc}}$ is either abelian or isomorphic to $Q_8 \times \mathbb{Z}/2\mathbb{Z}$. Thus, we have

$$|\Pi_\varphi(\mathrm{GSpin}_4)| = \left|\Pi_\varphi(\mathrm{GSpin}_4^{2,1})\right| = \left|\Pi_\varphi(\mathrm{GSpin}_4^{1,1})\right| = |\mathcal{S}_\varphi| = 1, \ 2, \text{or } 4, \tag{5.21}$$

according to whether $\mathcal{S}_\varphi \cong \{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$, except when $\mathcal{S}_{\varphi,\mathrm{sc}} \cong Q_8 \times \mathbb{Z}/2\mathbb{Z}$, in which case we have

$$|\Pi_\varphi(\mathrm{GSpin}_4)| = 4, \ \left|\Pi_\varphi(\mathrm{GSpin}_4^{2,1})\right| = \left|\Pi_\varphi(\mathrm{GSpin}_4^{1,1})\right| = 1. \tag{5.22}$$

Furthermore, using (4.5), the multiplicity in restriction from $\mathrm{GL}_\sharp$ to $\mathrm{GSpin}_\sharp$ is 1 except when $\mathcal{S}_{\varphi,\mathrm{sc}} \cong Q_8 \times \mathbb{Z}/2\mathbb{Z}$, in which case the multiplicity is 2.

We should mention that, due to Remarks 5.10 and 5.11, the case of $\mathcal{S}_{\varphi,\mathrm{sc}} \cong Q_8 \times \mathbb{Z}/2\mathbb{Z}$ occurs only if two conditions hold : $\widetilde{\sigma}_2 \cong \widetilde{\sigma}_1\widetilde{\eta}$ for some $\widetilde{\eta} \in (F^\times)^D$, and $\widetilde{\varphi}_1$ is dihedral w.r.t. three quadratic extensions (hence so is $\widetilde{\varphi}_2$).

5.4. **Properties of $\mathcal{L}$-maps for GSpin_4 and its inner forms.** The $\mathcal{L}$-maps defined in Section 5.1 satisfy a number of natural and expected properties which we now verify. In what follows, let $\mathcal{L}_\sharp \in \left\{ \mathcal{L}_4, \mathcal{L}_4^{2,1}, \mathcal{L}_4^{1,1} \right\}$ as the case may be.

Proposition 5.12. *A representation $\sigma_\sharp \in \mathrm{Irr}(\mathrm{GSpin}_\sharp)$ is essentially square integrable if and only if its L-parameter $\varphi_{\sigma_\sharp} := \mathcal{L}_\sharp(\sigma_\sharp)$ does not factor through any proper Levi subgroup of $\mathrm{GSO}_4(\mathbb{C})$.*

Proof. By the definition of $\mathcal{L}_\sharp$, the representation $\sigma_\sharp$ is an irreducible constituent of the restriction $\mathrm{Res}^{\mathrm{GL}_\sharp}_{\mathrm{GSpin}_\sharp}(\widetilde{\sigma}_\sharp)$ for some $\widetilde{\sigma}_\sharp \in \mathrm{Irr}(\mathrm{GL}_\sharp)$. As recalled in Remark 4.1, $\sigma_\sharp$ is essentially square integrable representation if and only if $\widetilde{\sigma}_\sharp$ is so. By the LLC for $\mathrm{GL}_\sharp$ and its inner forms [HT01, Hen00, Sch13, HS12], this is the case if and only if its parameter $\widetilde{\varphi}_{\sigma_\sharp} := \mathcal{L}(\widetilde{\sigma}_\sharp)$ does not factor through any proper Levi subgroup of $\mathrm{GL}_\sharp(\mathbb{C})$. Finally, this is the case if and only if $\varphi_{\sigma_\sharp}$ does not because the projection pr_4 in (2.26) respects Levi subgroups. $\square$

Remark 5.13. In the same way as in the proof of Proposition 5.12, we have that a given $\sigma_\sharp \in \mathrm{Irr}(\mathrm{GSpin}_\sharp)$ is tempered if and only if the image of its L-parameter $\varphi_{\sigma_\sharp} := \mathcal{L}_\sharp(\sigma_\sharp)$ in $\mathrm{GSO}_4(\mathbb{C})$ is bounded.

Due to the fact that restriction of representations preserves the intertwining operator and the Plancherel measure (see [Cho14a, Section 2.2]), our construction of the L-packets in Section 5.1 gives the following result.

Proposition 5.14. *Let $\varphi \in \Phi_{\mathrm{disc}}(\mathrm{GSpin}_\sharp)$ be given. For any $\sigma_1, \sigma_2 \in \Pi_\varphi(\mathrm{GSpin}_\sharp)$, we have the equality of the Plancherel measures*

$$\mu_M(\nu, \tau \boxtimes \sigma_1, w) = \mu_M(\nu, \tau \boxtimes \sigma_2, w), \tag{5.23}$$

where M is an F-Levi subgroup of an F-inner form of GSpin_{2n} of the form of the product of $\mathrm{GSpin}_\sharp$ and copies of F-inner forms of GL_{m_i}, $\tau \boxtimes \sigma_1, \tau \boxtimes \sigma_2 \in \Pi_{\mathrm{disc}}(M)$, $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$, and $w \in W_M$ with $^w M = M$. Further, it is a consequence of the equality of the Plancherel measures that the Plancherel measure is also preserved between F-inner forms in the following sense. Let $\mathrm{GSpin}'_\sharp$ be an F-inner form of $\mathrm{GSpin}_\sharp$. Given $\varphi \in \Phi_{\mathrm{disc}}(\mathrm{GSpin}_\sharp)$, for any $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\sharp)$ and $\sigma' \in \Pi_\varphi(\mathrm{GSpin}'_\sharp)$, we have

$$\mu_M(\nu, \tau \boxtimes \sigma, w) = \mu_{M'}(\nu, \tau' \boxtimes \sigma', w), \tag{5.24}$$

where M' is an F-inner form of M, $\tau \boxtimes \sigma \in \Pi_{\mathrm{disc}}(M)$, $\tau' \boxtimes \sigma' \in \Pi_{\mathrm{disc}}(M')$, and τ and τ' have the same L-parameter.

Proof. Since $\sigma_1, \sigma_2 \in \Pi_\varphi(\mathrm{GSpin}_\sharp)$ are in the same restriction from an irreducible representation from $\mathrm{GL}_\sharp(F)$ to $\mathrm{GSpin}_\sharp(F)$, by [Cho14a, Proposition 2.4], we have (5.23). Similarly, we note that σ and σ' have liftings $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ in $\mathrm{GL}_\sharp(F)$ and $\mathrm{GL}'_\sharp(F)$, respectively. Since $\tau \boxtimes \widetilde{\sigma}$ and $\tau' \boxtimes \widetilde{\sigma}'$ have the same Plancherel measures by [AC89, Lemma 2.1], again by [Cho14a, Proposition 2.4], we have (5.24). $\square$

Finally, we make a remark about the generic representations in L-packets.

Remark 5.15. Consider the case when $\mathrm{GSpin}_\sharp$ is the split GSpin_4. Since $\zeta_\sharp = \mathbb{1}$ now, Theorem 5.1 implies that we have

$$\Pi_\varphi(\mathrm{GSpin}_4) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{S}_\varphi) \cong \mathrm{Irr}(I^{\mathrm{GSpin}_4}(\widetilde{\sigma})). \tag{5.25}$$

Suppose that there is a generic representation in $\Pi_\varphi(\mathrm{GSpin}_4)$ with respect to a given Whittaker data for GSpin_4. Then, by [HS12, Chapter 3], we can normalize the bijection (5.25) so that the trivial character $\mathbb{1} \in \mathrm{Irr}(\mathcal{S}_\varphi)$ maps to the generic representation in $\Pi_\varphi(\mathrm{GSpin}_6)$.

5.5. **Preservation of Local Factors via $\mathcal{L}_4$.** We verify that in the case of the split form GSpin_4 the map $\mathcal{L}_4$ preserves the local L-, ϵ-, and γ-factors when these local factors are defined. On the representation side, these factors are defined via the Langlands-Shahidi method when the representations are generic (or non-supercuspidal induced from generic via the Langlands classification). On the parameter side, these factors are Artin factors associated to the representations of the Weil-Deligne group of F.

Since the Langlands-Shahidi method is available for generic data, we do not yet have a counterpart for the local factors for the non quasi-split F-inner forms of GSpin_4. This is the reason why we are limiting ourselves to the case of the split form below; however, cf. Remark 5.17 below.

Proposition 5.16. *Let τ be an irreducible admissible representation of $\mathrm{GL}_r(F)$, $r \geq 1$, and let σ be an irreducible admissible representation of $\mathrm{GSpin}_4(F)$, which we assume to be either ψ-generic or non-supercuspidal if $r > 1$. Here, generic is defined with respect to a non-trivial additive character ψ of F in the usual way.*

Let φ_τ be the L-parameter of τ via the LLC for $\mathrm{GL}_r(F)$ and let $\varphi_\sigma = \mathcal{L}_4(\sigma)$. Then,

$$
\begin{aligned}
\gamma(s, \tau \times \sigma, \psi) &= \gamma(s, \varphi_\tau \otimes \varphi_\sigma, \psi), \\
L(s, \tau \times \sigma) &= L(s, \varphi_\tau \otimes \varphi_\sigma), \\
\epsilon(s, \tau \times \sigma, \psi) &= \epsilon(s, \varphi_\tau \otimes \varphi_\sigma, \psi).
\end{aligned}
$$

The local factors on the left hand side are those attached by Shahidi [Sha90b, Theorem 3.5] to the ψ-generic representations of the standard Levi subgroup $\mathrm{GL}_r(F) \times \mathrm{GSpin}_4(F)$ in $\mathrm{GSpin}_{2r+4}(F)$ and the standard representation of the dual Levi $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GSO}(4, \mathbb{C})$, and extended to all non-generic non-supercuspidal representations via the Langlands classification and the multiplicativity of the local factors [Sha90b, §9]. The factors on the right hand side are Artin local factors associated to the given representations of the Weil-Deligne group of F.

Proof. The proof relies on two key properties of the local factors defined by Shahidi, namely that they are multiplicative (with respect to parabolic induction) and that they are preserved under taking irreducible constituents upon restriction.

To be more precise, let (G, M) and $(\widetilde{G}, \widetilde{M})$ be a pair of ambient groups and standard Levi subgroups as in the Langlands-Shahidi machinery. Assume that G and $\widetilde{G}$ satisfy (4.1). Moreover, assume that $M = G \cap \widetilde{M}$. If σ and $\widetilde{\sigma}$ are ψ-generic representations of $M(F)$ and $\widetilde{M}(F)$, respectively, such that

$$
\sigma \hookrightarrow \mathrm{Res}^{\widetilde{M}(F)}_{M(F)}(\widetilde{\sigma}),
$$

then

$$
\gamma(s, \sigma, r, \psi) = \gamma(s, \widetilde{\sigma}, \widetilde{r}, \psi),
$$

and similarly for the L- and ϵ-factors. Here, by r we denote any of the irreducible constituents of the adjoint action of the complex dual of M on the Lie algebra of the dual of the unipotent radical of the standard parabolic of G having M as a Levi. Also, $\widetilde{r}$ denotes the corresponding irreducible constituent in the complex dual of $\widetilde{G}$. Below we only need the case where r and $\widetilde{r}$ are standard representations.

For a precise description of the multiplicativity property in general we refer to [Sha90a] and to [Asg02, §5] for the specific case of GSpin groups.

Getting back to the proof, if $\tau \boxtimes \sigma$ is non-generic and non-supercuspidal, then it is a quotient of a standard module of an induced representation $\mathrm{Ind}_P^{\mathrm{GL}_4 \times \mathrm{GSpin}_4}(\pi)$, where π is an essentially tempered representation of the standard Levi subgroup of P. However, such a standard Levi only involves GL factors in our case and hence π is generic. As a result, we may define the local factors associated with $\tau \boxtimes \sigma$ via multiplicativity.

By the above arguments we are reduced to the case where τ and σ are generic supercuspidal. The proof now follows essentially from the LLC for the general linear groups and (5.9) because

$$
\begin{aligned}
\gamma(s, \tau \times \sigma, \psi) &= \gamma(s, \tau \times \widetilde{\sigma}, \psi) \\
&= \gamma(s, \varphi_\tau \otimes \widetilde{\varphi}_{\widetilde{\sigma}}, \psi) \\
&= \gamma(s, \varphi_\tau \otimes (pr_4 \circ \widetilde{\varphi}_{\widetilde{\sigma}}), \psi) \\
&= \gamma(s, \varphi_\tau \otimes \varphi_\sigma, \psi),
\end{aligned}
$$

and similarly for the L- and ϵ-factors. $\square$

Remark 5.17. While we can not make any statement regarding the case of general (non-generic) representations of $\mathrm{GSpin}_4(F)$ or other inner forms due to the current lack of a general satisfactory theory of local factors in this generality. Our L-packets would satisfy properties analogous to the above if such a satisfactory local theory becomes available. By satisfactory we refer to local factors that satisfy the natural properties expected of them such as the so-called "Ten Commandments" for the γ-factors as in [LR05, Theorem 4].

6. LOCAL LANGLANDS CORRESPONDENCE FOR GSpin_6 AND ITS INNER FORMS

In this final section, we establish the LLC for GSpin_6 and its non quasi-split F-inner forms. The method is similar to the case of GSpin_4 we presented in Section 5, but with somewhat weaker results on cardinalities of L-packets.

6.1. Construction of L-packets of GSpin_6 and its inner forms.

The discussions on restriction in Section 4 along with the description of the structure of GSpin_6 in (2.7) imply that given $\sigma \in \mathrm{Irr}(\mathrm{GSpin}_6)$, there is a lifting $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_1 \times \mathrm{GL}_4)$ such that

$$\sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_1 \times \mathrm{GL}_4}_{\mathrm{GSpin}_6}(\widetilde{\sigma}).$$

By the LLC for GL_n [HT01, Hen00, Sch13], we have a unique $\widetilde{\varphi}_{\widetilde{\sigma}} \in \Phi(\mathrm{GL}_1 \times \mathrm{GL}_4)$ corresponding to the representation $\widetilde{\sigma}$.

We now define a map

$$
\begin{aligned}
\mathcal{L}_6 : \mathrm{Irr}(\mathrm{GSpin}_6) &\longrightarrow \Phi(\mathrm{GSpin}_6) \\
\sigma &\longmapsto pr_6 \circ \widetilde{\varphi}_{\widetilde{\sigma}}.
\end{aligned}
\tag{6.1}
$$

Note that $\mathcal{L}_4$ does not depend on the choice of the lifting $\widetilde{\sigma}$ because if $\widetilde{\sigma}' \in \mathrm{Irr}(\mathrm{GL}_1 \times \mathrm{GL}_4)$ is another lifting, we have $\widetilde{\sigma}' \cong \widetilde{\sigma} \otimes \chi$ for some quasi-character χ on

$$(\mathrm{GL}_1(F) \times \mathrm{GL}_4(F))/\mathrm{GSpin}_6(F) \cong F^\times.$$

As before,

$$F^\times \cong H^1(F, \mathbb{C}^\times),$$

where $\mathbb{C}^\times$ is as in (2.27). The LLC for $\mathrm{GL}_1 \times \mathrm{GL}_4$ maps $\widetilde{\sigma}'$ to $\widetilde{\varphi}_{\widetilde{\sigma}} \otimes \chi$, again employing χ for both the quasi-character and its parameter via Local Class Field Theory. Since $pr_6(\widetilde{\varphi}_{\widetilde{\sigma}} \otimes \chi) = pr_6(\widetilde{\varphi}_{\widetilde{\sigma}})$ by (2.27), the map $\mathcal{L}_6$ turns out to be well-defined.

As before, $\mathcal{L}_6$ is surjective because by Labesse's Theorem in Section 4.2, $\varphi \in \Phi(\mathrm{GSpin}_6)$ can be lifted to some $\widetilde{\varphi} \in \Phi(GL_1 \times \mathrm{GL}_4)$. We then obtain $\widetilde{\sigma} \in \mathrm{Irr}(GL_1 \times \mathrm{GL}_4)$ via the LLC for $\mathrm{GL}_1 \times \mathrm{GL}_4$. Thus, any irreducible constituent in the restriction $\mathrm{Res}^{\mathrm{GL}_1 \times \mathrm{GL}_4}_{\mathrm{GSpin}_6}(\widetilde{\sigma})$ has the image φ via the map $\mathcal{L}_6$.

For each $\varphi \in \Phi(\mathrm{GSpin}_6)$, we define the L-packet $\Pi_\varphi(\mathrm{GSpin}_6)$ as the set of all inequivalent irreducible constituents of $\widetilde{\sigma}$

$$\Pi_\varphi(\mathrm{GSpin}_6) := \Pi_{\widetilde{\sigma}}(\mathrm{GSpin}_6) = \left\{ \sigma \hookrightarrow \mathrm{Res}^{\mathrm{GL}_1 \times \mathrm{GL}_4}_{\mathrm{GSpin}_6}(\widetilde{\sigma}) \right\} / \cong, \tag{6.2}$$

where $\widetilde{\sigma}$ is the unique member in $\Pi_{\widetilde{\varphi}}(\mathrm{GL1} \times \mathrm{GL}_4)$ and $\widetilde{\varphi} \in \Phi(\mathrm{GL}_1 \times \mathrm{GL}_4)$ is such that $pr_6 \circ \widetilde{\varphi} = \varphi$. By the LLC for GL_4 and Proposition 4.2, the fiber does not depends on the choice of $\widetilde{\varphi}$.

We define the L-packets for the non quasi-split inner forms similarly. Using the group structure described in Section 2.2, given $\sigma_6^{2,0} \in \mathrm{Irr}(\mathrm{GSpin}_6^{2,0})$, there is a lifting $\widetilde{\sigma}_6^{2,0} \in \mathrm{Irr}(\mathrm{GL}_1 \times \mathrm{GL}_2(D))$ such that

$$\sigma_6^{2,0} \hookrightarrow \mathrm{Res}^{\mathrm{GL}_1 \times \mathrm{GL}_2(D)}_{\mathrm{GSpin}_6^{2,0}}(\widetilde{\sigma}_6^{2,0}).$$

Again, combining the LLC for GL_4 and $\mathrm{GL}_2(D)$ [HS12], we have a unique $\widetilde{\varphi}_{\widetilde{\sigma}_6^{2,0}} \in \Phi(\mathrm{GL}_1 \times \mathrm{GL}_2(D))$ corresponding to the representation $\widetilde{\sigma}_6^{2,0}$. We thus define the following map

$$
\begin{aligned}
\mathcal{L}_6^{2,0} : \mathrm{Irr}(\mathrm{GSpin}_6^{2,0}) &\longrightarrow \Phi(\mathrm{GSpin}_6^{2,0}) \\
\sigma_6^{2,0} &\longmapsto pr_6 \circ \widetilde{\varphi}_{\widetilde{\sigma}_6^{2,0}}.
\end{aligned}
\tag{6.3}
$$

Again, it follows from the LLC for GL_1 and $\mathrm{GL}_2(D)$ that this map is well-defined and surjective.

Likewise, for the other F-inner form $\mathrm{GSpin}_6^{1,0}$ of GSpin_6, we have a well-defined and surjective map

$$
\begin{aligned}
\mathcal{L}_6^{1,0} : \mathrm{Irr}(\mathrm{GSpin}_6^{1,0}) &\longrightarrow \Phi(\mathrm{GSpin}_6^{1,0}) \\
\sigma_6^{1,0} &\longmapsto pr_6 \circ \widetilde{\varphi}_{\widetilde{\sigma}_6^{1,0}}.
\end{aligned}
\tag{6.4}
$$

Again, we similarly define L-packets

$$\Pi_\varphi(\mathrm{GSpin}_6^{2,0}) = \Pi_{\widetilde{\sigma}_6^{2,0}}(\mathrm{GSpin}_6^{2,0}), \quad \varphi \in \Phi(\mathrm{GSpin}_6^{2,0}) \tag{6.5}$$

and

$$\Pi_\varphi(\mathrm{GSpin}_6^{1,0}) = \Pi_{\widetilde{\sigma}_6^{1,0}}(\mathrm{GSpin}_6^{1,0}), \quad \varphi \in \Phi(\mathrm{GSpin}_6^{1,0}). \tag{6.6}$$

These L-packet do not depend on the choice of $\widetilde{\varphi}$ for similar reasons.

6.2. Internal structure of L-packets of GSpin_6 and its inner forms.

We continue to employ the notation of Section 3 in this section. For simplicity of notation, we shall write $\mathrm{GSpin}_\flat$ for the split GSpin_6, and its non quasi-split F-inner forms $\mathrm{GSpin}_6^{2,0}$ and $\mathrm{GSpin}_6^{1,0}$. Likewise, we shall write $\mathrm{SL}_\flat$ and $\mathrm{GL}_\flat$ for corresponding groups in (2.7), (2.13), and (2.14) so that we have

$$\mathrm{SL}_\flat \subset \mathrm{GSpin}_\flat \subset \mathrm{GL}_\flat \tag{6.7}$$

in all cases. Recall from Section 2.4 that

$$(\widehat{\mathrm{GSpin}_\flat})_{\mathrm{ad}} = \mathrm{PSO}_6(\mathbb{C}) \cong \mathrm{PGL}_4(\mathbb{C}),$$

$$(\widehat{\mathrm{GSpin}_\flat})_{\mathrm{sc}} = \mathrm{Spin}_6(\mathbb{C}) \cong \mathrm{SL}_4(\mathbb{C}),$$

$$Z((\widehat{\mathrm{GSpin}_\flat})_{\mathrm{sc}}) = Z((\widehat{\mathrm{GSpin}_\flat})_{\mathrm{sc}})^\Gamma \cong \mu_4(\mathbb{C}).$$

Let $\varphi \in \Phi(\mathrm{GSpin}_\flat)$ be given. We fix a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\flat)$ via the surjective map $\widehat{\mathrm{GL}_\flat} \longrightarrow \widehat{\mathrm{GSpin}_\flat}$ (cf. Theorem 4.2). We have

$$S_\varphi \subset \mathrm{PSL}_4(\mathbb{C}),$$

$$S_{\varphi,\mathrm{sc}} \subset \mathrm{SL}_4(\mathbb{C}).$$

We then have (again by (3.4)) a central extension

$$1 \longrightarrow \widehat{Z}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_{\varphi,\mathrm{sc}} \longrightarrow \mathcal{S}_\varphi \longrightarrow 1. \tag{6.8}$$

Let ζ_6, $\zeta_6^{2,0}$, and $\zeta_6^{1,0}$ be characters on $Z((\widehat{\mathrm{GSpin}_\flat})_{\mathrm{sc}})$ which respectively correspond to GSpin_6, $\mathrm{GSpin}_6^{2,0}$, and $\mathrm{GSpin}_6^{1,0}$ via the Kottwitz isomorphism [Kot86, Theorem 1.2]. Note that

$$\zeta_6 = \mathbb{1}, \quad \zeta_6^{2,0} = \mathrm{sgn}, \quad \text{and} \quad \zeta_6^{1,0} = \hat{\zeta}_4,$$

where sgn is the non-trivial character of order 2 on $\mu_4(\mathbb{C})$ and $\hat{\zeta}_4$ is the non-trivial character of order 4 on $\mu_4(\mathbb{C})$ whose restriction to μ_2 equals sgn.

Theorem 6.1. *Given an L-parameter $\varphi \in \Phi(\mathrm{GSpin}_\flat)$, there is a one-one bijection*

$$\Pi_\varphi(\mathrm{GSpin}_\flat) \quad \overset{1-1}{\longleftrightarrow} \quad \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\flat),$$

$$\sigma \quad \mapsto \quad \rho_\sigma,$$

such that we have the following decomposition

$$V_{\widetilde{\sigma}} \cong \bigoplus_{\sigma \in \Pi_\varphi(\mathrm{GSpin}_\flat)} \rho_\sigma \boxtimes \sigma$$

as representations of the direct product $\mathcal{S}_{\varphi,\mathrm{sc}} \times \mathrm{GSpin}_\flat(F)$, where $\widetilde{\sigma} \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_\flat)$ is an extension of $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\flat)$ to $\mathrm{GL}_\flat(F)$ as in Section 4 and $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\flat)$ is a lifting of $\varphi \in \Phi(\mathrm{GSpin}_\flat)$. Here, $\zeta_\flat \in \left\{\zeta_6, \zeta_6^{2,0}, \zeta_6^{1,0}\right\}$ according to which inner form $\mathrm{GSpin}_\flat$ is.

Proof. The idea of the proof is as in the proof of Theorem 5.1. Given $\varphi \in \Phi(\mathrm{GSpin}_\flat)$, we choose a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\flat)$ and obtain the projection $\bar{\varphi} \in \Phi(\mathrm{SL}_\flat)$ in the following commutative diagram

$$
\begin{array}{c}
\widehat{\mathrm{GL}_\flat} = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \qquad\qquad (6.9) \\
\downarrow {\scriptstyle pr_6} \\
W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\ \varphi\ } \widehat{\mathrm{GSpin}_\flat} = \mathrm{GSO}_6(\mathbb{C}) \\
\downarrow {\scriptstyle \bar{pr}} \\
\widehat{\mathrm{SL}_\flat} = \mathrm{PGL}_4(\mathbb{C}) \ .
\end{array}
$$

where the arrows $\widetilde{\varphi}$ and $\bar{\varphi}$ emanate from $W_F \times \mathrm{SL}_2(\mathbb{C})$.

We then have $\widetilde{\sigma} \in \Pi_{\widetilde{\varphi}}(\mathrm{GL}_\flat)$ which is an extension of $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\flat)$. In addition to (2.27), we also have

$$1 \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \longrightarrow \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \xrightarrow{\bar{pr}\circ pr_6} \mathrm{PGL}_4(\mathbb{C}) \longrightarrow 1$$

Considering the kernels of the projections pr_6 and $\bar{pr} \circ pr_6$, we set

$$X^{\mathrm{GSpin}_\flat}(\widetilde{\varphi}) := \{a \in H^1(W_F, \mathbb{C}^\times) : a\widetilde{\varphi} \cong \widetilde{\varphi}\}$$

$$X^{\mathrm{SL}_\flat}(\widetilde{\varphi}) := \{a \in H^1(W_F, \mathbb{C}^\times \times \mathbb{C}^\times) : a\widetilde{\varphi} \cong \widetilde{\varphi}\}.$$

Moreover, by (2.10) and its analogs for the two non quasi-split F-inner forms, we have

$$\mathrm{GL}_\flat(F)/\mathrm{GSpin}_\flat(F) \cong F^\times.$$

As an easy consequence of Galois cohomology, we also have

$$\mathrm{GL}_\flat(F)/\mathrm{SL}_\flat(F) \cong F^\times \times F^\times.$$

Set

$$I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) := \{\chi \in (F^\times)^D \cong (\mathrm{GL}_\flat(F)/\mathrm{GSpin}_\flat(F))^D : \widetilde{\sigma}\chi \cong \widetilde{\sigma}\}$$

$$I^{\mathrm{SL}_\flat}(\widetilde{\sigma}) := \{\chi \in (F^\times)^D \times (F^\times)^D \cong (\mathrm{GL}_\flat(F)/\mathrm{SL}_\flat(F))^D : \widetilde{\sigma}\chi \cong \widetilde{\sigma}\}.$$

As in Remark 5.2, we often make no distinction between χ and $\chi \circ \det$ (respectively, $\chi \circ \mathrm{Nrd}$). It follows from the definitions that

$$X^{\mathrm{GSpin}_\flat}(\widetilde{\varphi}) \subset X^{\mathrm{SL}_\flat}(\widetilde{\varphi}) \quad \text{and} \quad I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) \subset I^{\mathrm{SL}_\flat}(\widetilde{\sigma}). \qquad (6.10)$$

Recalling $(F^\times)^D \cong H^1(W_F, \mathbb{C}^\times)$ by the local class field theory, it is immediate from the LLC for GL_n that

$$X^{\mathrm{GSpin}_\flat}(\widetilde{\varphi}) \cong I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) \quad \text{and} \quad X^{\mathrm{SL}_\flat}(\widetilde{\varphi}) \cong I^{\mathrm{SL}_\flat}(\widetilde{\sigma})$$

as groups of characters. Now, with the component group notation $\mathcal{S}$ of Section 3, we claim that

$$I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) \cong \mathcal{S}_\varphi. \qquad (6.11)$$

Again, by [CL14, Lemma 5.3.4] and above arguments, this claim follows from

$$\mathcal{S}_\varphi \cong X^{\mathrm{GSpin}_\flat}(\widetilde{\varphi}),$$

since $\mathcal{S}_{\widetilde{\varphi}}$ is always trivial by the LLC for $\mathrm{GL}_\flat$. Again note that $\widetilde{\varphi}$ and φ here are respectively ϕ and $\phi^\sharp$ in [CL14, Lemma 5.3.4]. Thus, due to (4.10), (6.10), and (6.11) we have

$$\mathcal{S}_\varphi \subset \mathcal{S}_{\bar{\varphi}}. \qquad (6.12)$$

Since the centralizer $C_{\bar{\varphi}}(\widehat{\mathrm{SL}_\flat})$ is equal to the image of the disjoint union

$$\coprod_{\nu \in \mathrm{Hom}(W_F, \mathbb{C}^\times)} \{h \in \mathrm{GSO}_6(\mathbb{C}) : h\varphi(w)h^{-1}\varphi(w)^{-1} = \nu(w)\}$$

from the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GSO}_6(\mathbb{C}) \xrightarrow{\bar{pr}} \widehat{\mathrm{SL}_\flat} = \mathrm{PGL}_4 \longrightarrow 1,$$

we have

$$S_\varphi \subset C_{\bar{\varphi}} = S_{\bar{\varphi}}.$$

So, the pre-images in $\mathrm{SL}_4(\mathbb{C})$ via the isogeny $\mathrm{SL}_4(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_4(\mathbb{C})$ also satisfy

$$S_{\varphi,\mathrm{sc}} \subset S_{\bar{\varphi},\mathrm{sc}} \subset \mathrm{SL}_4(\mathbb{C}).$$

This provides the inclusion of the identity components

$$S^\circ_{\varphi,\mathrm{sc}} \subset S^\circ_{\bar{\varphi},\mathrm{sc}}, \tag{6.13}$$

and by the definition of $\widehat{Z}_{\varphi,\mathrm{sc}}(G)$ of Section 3, we again have the following surjection

$$\widehat{Z}_{\varphi,\mathrm{sc}} \twoheadrightarrow \widehat{Z}_{\varphi,\mathrm{sc}}(\mathrm{SL}_\flat). \tag{6.14}$$

Combining (6.12), (6.13), and (6.14), we again have the commutative diagram of component groups:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\twoheadrightarrow} & & \downarrow{\scriptstyle\cap} & & \downarrow{\scriptstyle\cap} & & \\
1 & \longrightarrow & \widehat{Z}_{\bar{\varphi},\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\bar{\varphi},\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\bar{\varphi}} & \longrightarrow & 1.
\end{array}
\tag{6.15}
$$

Now apply Hiraga-Saito's homomorphism Λ_{SL_n} of (4.10) in the case of our $\mathrm{SL}_\flat$, which we denote by Λ_{SL_4}. The restriction

$$\Lambda_\flat := \Lambda_{\mathrm{SL}_4}|_{\mathcal{S}_{\varphi,\mathrm{sc}}}$$

then gives the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\zeta_\flat} & & \downarrow{\scriptstyle\Lambda_\flat} & & \downarrow{\scriptstyle\cap} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{A}(\widetilde{\sigma}) & \longrightarrow & I^{\mathrm{SL}_\flat}(\widetilde{\sigma}) & \longrightarrow & 1.
\end{array}
\tag{6.16}
$$

Note that $\zeta_\flat$ is identified with ζ_G in (4.10) as the character on $\mu_4(\mathbb{C})$, since both are determined according to $\mathrm{SL}_\flat$.

Again, similar to $\mathcal{A}(\widetilde{\sigma})$ (cf. Section 4), we write $\mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma})$ for the subgroup of $\mathrm{Aut}_\mathbb{C}(V_{\widetilde{\sigma}})$ generated by $\mathbb{C}^\times$ and $\left\{ I_\chi : \chi \in I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) \right\}$. Hence, $\mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) \subset \mathcal{A}(\widetilde{\sigma})$. By the definition of Λ_{SL_4} in (4.10) and the commutative diagram (6.15), it is immediate that the image of $\Lambda_\flat$ is $\mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma})$. We thus have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{Z}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_{\varphi,\mathrm{sc}} & \longrightarrow & \mathcal{S}_\varphi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\zeta_\flat} & & \downarrow{\scriptstyle\Lambda_\flat} & & \downarrow{\scriptstyle\cong} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) & \longrightarrow & I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}) & \longrightarrow & 1.
\end{array}
\tag{6.17}
$$

The representation $\rho_\sigma \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\flat)$ is defined by $\xi_\sigma \circ \Lambda_\flat$, where $\xi_\sigma \in \mathrm{Irr}(\mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}), \mathrm{id})$ is the character as in the decomposition (4.7). Now, arguments of Section 4, our construction of L-packets $\Pi_\pi(\mathrm{GSpin}_\flat)$ in Section 6.1 and diagram (6.17) above give

$$\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\flat) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{A}^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}), id) \overset{1-1}{\longleftrightarrow} \Pi_\varphi(\mathrm{GSpin}_\flat). \tag{6.18}$$

We also have the following decomposition

$$V_{\widetilde{\sigma}} \cong \bigoplus_{\sigma \in \Pi_\varphi(\mathrm{GSpin}_\flat)} \rho_\sigma \boxtimes \sigma = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\flat)} \rho \boxtimes \sigma_\rho,$$

where σ_ρ denotes the image of ρ via the bijection between $\Pi_\varphi(\mathrm{GSpin}_\flat)$ and $\mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}, \zeta_\flat)$. Hence, the proof of Theorem 6.1 is complete. $\qquad\square$

Remark 6.2. As before, since $\Lambda_\flat$ is unique up to $\mathrm{Hom}(I^{\mathrm{GSpin}_\flat}(\widetilde{\sigma}), \mathbb{C}^\times) \cong \mathrm{Hom}(\mathcal{S}_\varphi, \mathbb{C}^\times)$, the same is true for the bijection (6.18).

Remark 6.3. Given $\varphi \in \phi(\mathrm{GSpin}_6)$, by Theorem 6.1, we have a one-to-one bijection between

$$\Pi_\varphi(\mathrm{GSpin}_6) \cup \Pi_\varphi(\mathrm{GSpin}_6^{2,0}) \cup \Pi_\varphi(\mathrm{GSpin}_6^{1,0}) \cup \Pi_\varphi(\mathrm{GSpin}_6^{0,1}) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{S}_{\varphi,\mathrm{sc}}),$$

where $\mathrm{GSpin}_6^{0,1}$ is a group isomorphic to $\mathrm{GSpin}_6^{1,0}$ as can be seen by taking the canonical isomorphism $D_4 \cong D_4^{op}$, as discussed in (2.14).

6.3. *L*-packet Sizes for GSpin_6 and Its Inner Forms.
Just as in the case of GSpin_4, we have the following possible cardinalities for the *L*-packets of GSpin_6 and its inner forms.

Proposition 6.4. *Let* $\Pi_\varphi(\mathrm{GSpin}_\flat)$ *be an L-packet associated to* $\varphi \in \Phi(\mathrm{GSpin}_\flat)$. *Then we have*

$$|\Pi_\varphi(\mathrm{GSpin}_\flat)| \ \Big| \ \left|F^\times/(F^\times)^2\right|,$$

which yields the possible cardinalities

$$|\Pi_\varphi(\mathrm{GSpin}_\flat)| = \left\{ \begin{array}{ll} 1, \ 2, \ 4, & \text{if } p \neq 2, \\ 1, \ 2, \ 4, \ 8, & \text{if } p = 2. \end{array} \right.$$

Proof. We proceed similarly as in the proof of Proposition 5.5, again making use of Galois Cohomology. The exact sequence of algebraic groups

$$1 \longrightarrow Z(\mathrm{GSpin}_\flat) \longrightarrow Z(\mathrm{GL}_\flat) \times \mathrm{GSpin}_\flat \longrightarrow \mathrm{GL}_\flat \longrightarrow 1$$

gives a long exact sequence

$$\cdots \longrightarrow (F^\times \times F^\times) \times \mathrm{GSpin}_\flat(F) \longrightarrow \mathrm{GL}_\flat(F) \longrightarrow H^1(F, Z(\mathrm{GSpin}_\flat)) \longrightarrow H^1(F, Z(\mathrm{GL}_\flat) \times \mathrm{GSpin}_\flat) \longrightarrow 1.$$

Since $H^1(F, Z(\mathrm{GL}_\flat) \times \mathrm{GSpin}_\flat) = 1$ by Lemma 2.2 and [PR94, Lemmas 2.8], we have

$$\mathrm{GL}_\flat(F)/\left((F^\times \times F^\times) \times \mathrm{GSpin}_\flat(F)\right) \hookrightarrow H^1(F, Z(\mathrm{GSpin}_\flat)).$$

Also, the exact sequence

$$1 \longrightarrow Z(\mathrm{GSpin}_\flat)^\circ \longrightarrow Z(\mathrm{GSpin}_\flat) \longrightarrow \pi_0(Z(\mathrm{GSpin}_\flat)) \longrightarrow 1,$$

we have

$$H^1(F, Z(\mathrm{GSpin}_\flat)) \hookrightarrow H^1(F, \pi_0(Z(\mathrm{GSpin}_\flat))),$$

since, by [AS06, Proposition 2.3], $Z(\mathrm{GSpin}_\flat)^\circ \cong \mathrm{GL}_1$ and $H^1(F, Z(\mathrm{GSpin}_\flat)^\circ) = 1$. Combining the above with (2.19), we have

$$\mathrm{GL}_\flat(F)/\left((F^\times \times F^\times) \times \mathrm{GSpin}_\flat(F)\right) \hookrightarrow H^1(F, \pi_0(Z(\mathrm{GSpin}_\flat))) \cong H^1(F, \mathbb{Z}/2\mathbb{Z}) \cong F^\times/(F^\times)^2. \qquad (6.19)$$

On the other hand, we know from Section 4 that

$$\Pi_\varphi(\mathrm{GSpin}_\flat) \overset{1-1}{\longleftrightarrow} \mathrm{GL}_\flat(F)/\mathrm{GL}_\flat(F)_{\widetilde{\sigma}} \hookrightarrow \mathrm{GL}_\flat(F)/\left((F^\times \times F^\times) \times \mathrm{GSpin}_\flat(F)\right),$$

where $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_\flat)$ corresponds to a lifting $\widetilde{\varphi} \in \Phi(\mathrm{GL}_\flat)$, via the LLC for GL_n and its inner forms, of $\varphi \in \Phi(\mathrm{GSpin}_\flat)$. This completes the proof. $\qquad \square$

Next, we give a description of the group $I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ for $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_1 \times \mathrm{GL}_4)$. Unlike the case of GSpin_4 we are unable to give a case by case classification (cf. Remark 6.7).

Given $\widetilde{\sigma} \in \mathrm{Irr}(\mathrm{GL}_1 \times \mathrm{GL}_4)$, we set $\widetilde{\sigma} = \widetilde{\eta} \boxtimes \widetilde{\sigma}_0$ with $\widetilde{\eta} \in (\mathrm{GL}_1(F))^D, \widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_4)$. By Remark 5.2, we note that $\chi \in (\mathrm{GL}_\flat(F))^D$ is decomposed into $\widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2$, where $\widetilde{\chi}_1 \in (\mathrm{GL}_1(F))^D$ and $\widetilde{\chi}_2 \in (\mathrm{GL}_\flat(F))^D$. Moreover, we identify $\widetilde{\chi}_2 \in (\mathrm{GL}_4(F))^D$ and $\widetilde{\chi}_2 \circ \det$, since any character on $\mathrm{GL}_n(F)$ is of the form $\widetilde{\chi} \circ \det$ for some character $\widetilde{\chi}$ on $F^\times$.

Lemma 6.5. *Any* $\chi \in I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ *is of the form*

$$\widetilde{\chi}^{-2} \boxtimes \widetilde{\chi},$$

for some $\widetilde{\chi} \in (F^\times)^D$.

Proof. Since $\chi = \widetilde{\chi}_1 \boxtimes \widetilde{\chi}_2$ as above and χ is trivial on $\mathrm{GSpin}_6(F)$, by the structure of $\mathrm{GSpin}_6(F)$ in (2.23), we have

$$\chi((g_1, g_2)) = \widetilde{\chi}_1(g_1) \boxtimes \widetilde{\chi}_2(\det g_2) = \widetilde{\chi}_1(g_1)\widetilde{\chi}_2(\det g_2) = 1$$

for all $(g_1, g_2) \in \mathrm{GL}_1(F) \times \mathrm{GL}_4(F)$ with $(g_1)^2 = \det g_2$. Since the determinant map $\det : \mathrm{GL}_4(F) \to F^\times$ is surjective, $\widetilde{\chi}_1(\widetilde{\chi}_2)^2(x)$ must be trivial for all $x \in F^\times$. Thus, we have $\widetilde{\chi}_1 = (\widetilde{\chi}_2)^{-2}$. $\qquad\square$

Proposition 6.6. *We have*

$$I^{\mathrm{GSpin}_6}(\widetilde{\sigma}) = \{\chi \in I^{\mathrm{SL}_4}(\widetilde{\sigma}_0) : \chi^2 = \mathbb{1}\}.$$

Proof. Since $\widetilde{\sigma} = \widetilde{\eta} \boxtimes \widetilde{\sigma}_0$ with $\widetilde{\eta} \in (\mathrm{GL}_1(F))^D$ and $\widetilde{\sigma}_0 \in \mathrm{Irr}(\mathrm{GL}_4)$, by Lemma 6.5, we have

$$\widetilde{\sigma}\chi \cong \widetilde{\sigma} \iff \widetilde{\eta}\widetilde{\chi}^{-2} \boxtimes \widetilde{\sigma}_0\widetilde{\chi} \cong \widetilde{\eta} \boxtimes \widetilde{\sigma}_0 \iff \widetilde{\sigma}_0\widetilde{\chi} \cong \widetilde{\sigma}_0 \text{ and } \widetilde{\chi}^2 = \mathbb{1}.$$

This completes the proof. $\qquad\square$

Remark 6.7. Propositions 6.4 and 6.6 imply that $I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ is of the form $(\mathbb{Z}/2\mathbb{Z})^r$ with $r = 0, 1, 2$ if $p \neq 2$ and with $r = 0, 1, 2, 3$ if $p = 2$. Unlike the case of SL_2 (see [GT10, Porposition 6.3]), a full classification of irreducible L-parameters in $\Phi(\mathrm{SL}_4)$ is not currently available. Thus, unlike the case of GSpin_4 in Section 5.3 (cf. Remarks 5.10 and 5.11), we do not classify the group $I^{\mathrm{GSpin}_6}(\widetilde{\sigma})$ case by case, nor discuss the group structure of $\mathcal{S}_{\varphi,\mathrm{sc}}$, and accordingly, the description of all the sizes of L-packets for GSpin_6, $\mathrm{GSpin}_6^{2,0}$, and $\mathrm{GSpin}_6^{1,0}$ is not available here.

6.4. **Proerties of $\mathcal{L}$-maps for GSpin_6 and its inner forms.** The $\mathcal{L}$-maps defined in Section 6.1 again satisfy some natural properties similar to the case of GSpin_4. We now verify those properties. In what follows, let $\mathcal{L}_\flat \in \left\{\mathcal{L}_6, \mathcal{L}_6^{2,0}, \mathcal{L}_4^{1,0}\right\}$ as the case may be.

Proposition 6.8. *A representation $\sigma_\flat \in \mathrm{Irr}(\mathrm{GSpin}_\flat)$ is essentially square integrable if and only if its L-parameter $\varphi_{\sigma_\flat} := \mathcal{L}_\flat(\sigma_\flat)$ does not factor through any proper Levi subgroup of $\mathrm{GSO}_6(\mathbb{C})$.*

Proof. The proof is similar to that of Proposition 5.12 so we omit the details. $\qquad\square$

Remark 6.9. We similarly have that a given $\sigma_\flat \in \mathrm{Irr}(\mathrm{GSpin}_\flat)$ is tempered if and only if the image of its L-parameter $\varphi_{\sigma_\flat} := \mathcal{L}_\flat(\sigma_\flat)$ in $\mathrm{GSO}_6(\mathbb{C})$ is bounded.

Again because the restriction of representations preserves the intertwining operator and the Plancherel measure (see [Cho14a, Section 2.2]), our construction of the L-packets in Section 6.1, gives the following result.

Proposition 6.10. *Let $\varphi \in \Phi_{\mathrm{disc}}(\mathrm{GSpin}_\flat)$ be given. For any $\sigma_1, \sigma_2 \in \Pi_\varphi(\mathrm{GSpin}_\flat)$, we have the equality of the Plancherel measures*

$$\mu_M(\nu, \tau \boxtimes \sigma_1, w) = \mu_M(\nu, \tau \boxtimes \sigma_2, w),$$

where M is an F-Levi subgroup of an F-inner form of GSpin_{2n} of the form of the product of $\mathrm{GSpin}_\flat$ and copies of F-inner forms of GL_{m_i}, $\tau \boxtimes \sigma_1, \tau \boxtimes \sigma_2 \in \Pi_{\mathrm{disc}}(M)$, $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$, and $w \in W_M$ with $^w M = M$. Further, it is a consequence of the equality of the Plancherel measures that the Plancherel measure is also preserved between F-inner forms in the following sense. Let $\mathrm{GSpin}_\flat'$ be an F-inner form of $\mathrm{GSpin}_\flat$. Given $\varphi \in \Phi_{\mathrm{disc}}(\mathrm{GSpin}_\flat)$, for any $\sigma \in \Pi_\varphi(\mathrm{GSpin}_\flat)$ and $\sigma' \in \Pi_\varphi(\mathrm{GSpin}_\flat')$ we have

$$\mu_M(\nu, \tau \boxtimes \sigma, w) = \mu_{M'}(\nu, \tau' \boxtimes \sigma', w),$$

where M' is an F-inner form of M, $\tau \boxtimes \sigma \in \Pi_{\mathrm{disc}}(M)$, $\tau' \boxtimes \sigma' \in \Pi_{\mathrm{disc}}(M')$, and τ and τ' have the same L-parameter.

Proof. The proof is similar to that of Proposition 5.14 so we omit the details. $\qquad\square$

Remark 6.11. Similar to Remark 5.15, in the case where $\mathrm{GSpin}_\flat$ is the split group GSpin_6, we have $\zeta_\flat = \mathbb{1}$ and Theorem 6.1 implies

$$\Pi_\varphi(\mathrm{GSpin}_6) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{S}_\varphi(\widehat{\mathrm{GSpin}_6})) \cong \mathrm{Irr}(I^{\mathrm{GSpin}_6}(\widetilde{\sigma})). \tag{6.20}$$

Suppose that there is a generic representation in $\Pi_\varphi(\mathrm{GSpin}_6)$ with respect to a given Whittaker data for GSpin_6. Then, by [HS12, Chapter 3], we can normalize the bijection (6.20) above, so that the trivial character $\mathbb{1} \in \mathrm{Irr}(\mathcal{S}_\varphi)$ maps to the generic representation in $\Pi_\varphi(\mathrm{GSpin}_6)$.

6.5. **Preservation of Local Factors via $\mathcal{L}_6$.** As in Section 5.5, we can again verify that in the case of the split form GSpin_6 the map $\mathcal{L}_6$ preserves the local L-, ϵ-, and γ-factors when these local factors are defined. We recall that on the representation side, these factors are defined via the Langlands-Shahidi method when the representations are generic (or non-supercuspidal induced from generic via the Langlands classification) and on the parameter side, they are Artin factors associated to the representations of the Weil-Deligne group of F.

Since the Langlands-Shahidi method is available for generic data, we do not yet have a counterpart for the local factors for the non quasi-split F-inner forms of GSpin_6. This is the reason why we are limiting ourselves to the case of the split form below; however, a remark similar to Remark 5.17 applies again.

Proposition 6.12. *Let τ be an irreducible admissible representation of $\mathrm{GL}_r(F)$, $r \geq 1$, and let σ be an irreducible admissible representation of $\mathrm{GSpin}_6(F)$, which we assume to be either ψ-generic or non-supercuspidal if $r > 1$. Here, generic is defined with respect to a non-trivial additive character ψ of F in the usual way.*

Let φ_τ be the L-parameter of τ via the LLC for $\mathrm{GL}_r(F)$ and let $\varphi_\sigma = \mathcal{L}_6(\sigma)$. Then,

$$
\begin{array}{rcl}
\gamma(s, \tau \times \sigma, \psi) & = & \gamma(s, \varphi_\tau \otimes \varphi_\sigma, \psi), \\
L(s, \tau \times \sigma) & = & L(s, \varphi_\tau \otimes \varphi_\sigma), \\
\epsilon(s, \tau \times \sigma, \psi) & = & \epsilon(s, \varphi_\tau \otimes \varphi_\sigma, \psi).
\end{array}
$$

The local factors on the left hand side are those attached by Shahidi [Sha90b, Theorem 3.5] to the ψ-generic representations of the standard Levi subgroup $\mathrm{GL}_r(F) \times \mathrm{GSpin}_6(F)$ in $\mathrm{GSpin}_{2r+6}(F)$ and the standard representation of the dual Levi $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GSO}(6, \mathbb{C})$, and extended to all non-generic non-supercuspidal representations via the Langlands classification and the multiplicativity of the local factors [Sha90b, §9]. The factors on the right hand side are Artin local factors associated to the given representations of the Weil-Deligne group of F.

Proof. The proof is similar to that of Proposition 5.16 and we will not repeat it. $\square$

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