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Siegel modular forms and representations

Abstract. This paper explicitly describes the procedure of associating an automorphic representation of $PGSp(2n, \mathbb{A})$ with a Siegel modular form of degree n for the full modular group $\Gamma_n = Sp(2n, \mathbb{Z})$, generalizing the well-known procedure for $n = 1$. This will show that the so-called "standard" and "spinor" L-functions associated with such forms are obtained as Langlands L-functions. The theory of Euler products, developed by Langlands, applied to a Levi subgroup of the exceptional group of type F_4, is then used to establish meromorphic continuation for the spinor L-function when $n = 3$.

1. Introduction

Let f be a Siegel modular form of degree n for the full modular group $\Gamma_n = Sp(2n, \mathbb{Z})$. If f is an eigenfunction for the action of the Hecke algebra, then there are two L-functions associated with f. Let $a_0, a_1, \ldots, a_n$ be the Satake parameters of f, and define the *standard L-function*

$$L_1(s, f) = \prod_p \left((1 - p^{-s}) \prod_{i=1}^n (1 - a_i p^{-s})(1 - a_i^{-1} p^{-s}) \right)^{-1}, \qquad (1)$$

and the *spinor L-function*

$$L_2(s, f) = \prod_p \left(\prod_{k=0}^n \prod_{1 \le i_1 < \ldots < i_k \le n} (1 - a_0 a_{i_1} \ldots a_{i_k} p^{-s}) \right)^{-1}. \qquad (2)$$

One goal of this note is to "explain" the definition of these L-functions within the general framework of automorphic representations. To do so, we first associate an automorphic representation with the classical modular form f. We then identify the above L-functions with certain Langlands L-functions coming from two different representations of the dual group. Langlands' theory of Euler products will then imply the following (cf. Sect. 4.6).

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Theorem. *The L-functions $L_1(s, f)$ and $L_2(s, f)$ have meromorphic continuation to all of $\mathbb{C}$ when $n = 3$.*

We should remark that Böcherer has proved stronger results for the standard L-functions (cf. [Bö]). Andrianov also gives all the analytic properties of spinor L-function when $n = 2$ in [An].

The procedure of associating an automorphic representation to a classical modular form is well known in the case of elliptic modular forms, see [Ge] Chapter 3 or [Bu, Sect. 3.6]. There, one associates with an eigenform f an automorphic representation π_f of GL(2, $\mathbb{A}$), where $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. If f has no character, then π_f will have trivial central character, thus descends to a representation of PGL(2, $\mathbb{A}$). In the higher dimensional case we will associate with an eigenform f of degree n an automorphic representation of the group GSp(2n, $\mathbb{A}$) which is isomorphic to GL(2, $\mathbb{A}$) if $n = 1$. Since f will be assumed to have no character, this is really a representation of PGSp(2n, $\mathbb{A}$).

To be more precise, utilizing a strong approximation theorem, we first lift f to a function Φ_f on $G(\mathbb{A})$, where $G = $ GSp(2n). We will assume f to be cuspidal, so that Φ_f lies in the cuspidal subspace $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. (Here we have to translate the classical cusp condition into the group theoretic one.) Then let π be the subrepresentation of this space generated by Φ_f. This π may not be irreducible, but if f is an eigenform, then all the irreducible components of π will turn out to be isomorphic. This isomorphism class is the automorphic representation π_f associated with f. In Theorem 2 we shall describe its local components in terms of Satake parameters (at the finite places) and Harish-Chandra parameters (at the infinite place).

We have to go through some Hecke algebra computations to identify the classical Satake parameters of the eigenform f with the Satake parameters of the local components of π_f which are spherical representations of the local groups GSp(2n, $\mathbb{Q}_p$). The group theoretic Satake parameters can be taken to be in the maximal torus in the dual group of G which is $\hat{G} = $ GSpin(2n + 1, $\mathbb{C}$). Since we have a representation with trivial central character, the Satake parameters will in fact be elements of its derived group Spin(2n + 1, $\mathbb{C}$).

This latter group has two distinguished finite-dimensional representations, namely the projection onto SO(2n + 1, $\mathbb{C}$) which we denote by ϱ_1, and the 2^n-dimensional spin representation ϱ_2, the smallest genuine representation (not factoring through ϱ_1) of this group. We use the weight structure of these finite-dimensional representations to determine the form of the standard Euler factor associated with ϱ_1 and ϱ_2. It turns out that these Euler factors are precisely the same as the ones in (1) and (2). In other words, the classical L-functions identify with standard Langlands L-functions coming from the representations ϱ_1 and ϱ_2 of the dual group (see Theorem 3).

Having established this identification, one can apply results from representation theory to classical modular forms. For modular forms of degree 3, the underlying group GSp(6) can be embedded, as a Levi, in a Chevalley group of type F_4. We then make use of Langlands' method developed in [La] to prove the meromorphic continuation of the spin L-function for $n = 3$ (cf. Corollary 1).

2. Notation

The right group for handling Siegel modular forms (of degree n) in the context of automorphic representations is

$$G = \mathrm{GSp}(2n) = \{g \in \mathrm{GL}(2n) : \exists\, \mu(g) \in \mathrm{GL}(1)\ \ gJ\,{}^tg = \mu(g)J\},$$

where

$$J = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}, \quad \mathbf{1}_n \text{ the } n \times n \text{ identity matrix.}$$

If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the conditions are equivalent to

$$A\,{}^tD - B\,{}^tC = \mu(g)\mathbf{1}, \quad A\,{}^tB = B\,{}^tA, \quad C\,{}^tD = D\,{}^tC.$$

The function μ is called the multiplier homomorphism. Its kernel is the group $\mathrm{Sp}(2n)$ and there is an exact sequence

$$1 \longrightarrow \mathrm{Sp}(2n) \longrightarrow G \longrightarrow \mathrm{GL}(1) \longrightarrow 1.$$

The center Z of G consists of the scalar matrices, and the standard maximal torus is

$$T = \{\mathrm{diag}(u_1, \ldots, u_n, v_1, \ldots, v_n) : u_1 v_1 = \ldots = u_n v_n \neq 0\}.$$

We often write an element $t \in T$ in the form

$$t = \mathrm{diag}(u_1, \ldots, u_n, u_1^{-1}u_0, \ldots, u_n^{-1}u_0), \quad u_i \in \mathrm{GL}(1); \tag{3}$$

then $u_0 = \mu(t)$. We fix the following characters of the maximal torus $T \subset G$. If $t \in T$ is written in the form (3), then let

$$e_i(t) = u_i, \quad i = 0, 1, \ldots, n.$$

These characters are a basis for the character lattice of G,

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n.$$

We also fix the following cocharacters of T:

$$f_0(u) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{u, \ldots, u}_{n}),$$

$$f_1(u) = \mathrm{diag}(\underbrace{u, 1, \ldots, 1}_{n}, \underbrace{u^{-1}, 1, \ldots, 1}_{n}),$$

$$\vdots$$

$$f_n(u) = \mathrm{diag}(\underbrace{1, \ldots, 1, u}_{n}, \underbrace{1, \ldots, 1, u^{-1}}_{n}).$$

Then these elements are a $\mathbb{Z}$-basis for the cocharacter lattice of G,

$$X^\vee = \mathbb{Z} f_0 \oplus \mathbb{Z} f_1 \oplus \ldots \oplus \mathbb{Z} f_n.$$

With the natural pairing $\langle \, , \, \rangle : \ X \times X^\vee \longrightarrow \mathbb{Z}$, we have

$$\langle e_i, f_j \rangle = \delta_{ij}.$$

We now choose the following set of simple roots:

$$\alpha_1(t) = u_{n-1}^{-1} u_n, \quad \ldots \quad , \alpha_{n-1}(t) = u_1^{-1} u_2, \quad a_n(t) = u_1^2 u_0^{-1}; \qquad (4)$$

here t is written in the form (3). In other words,

$$\alpha_1 = e_n - e_{n-1}, \quad \ldots \, , \quad \alpha_{n-1} = e_2 - e_1, \quad \alpha_n = 2e_1 - e_0.$$

The numbering of the simple roots is such that the Dynkin diagram is

The corresponding coroots are

$$\alpha_1^\vee = f_n - f_{n-1}, \quad \ldots \quad \alpha_{n-1}^\vee = f_2 - f_1, \quad \alpha_n^\vee = f_1. \qquad (5)$$

If we let $R = \{\alpha_1, \ldots, \alpha_n\} \subset X$, $R^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset X^\vee$, then

$$(X, R, X^\vee, R^\vee)$$

is the root datum of $G = \mathrm{GSp}(2n)$. The Cartan matrix is

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 & -1 \\ & & & & & & -2 & 2 \end{pmatrix}.$$

With our choice of simple roots, we get the Borel subgroup $B = TN$, where N consists of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$, with $A \in \mathrm{GL}(n)$ lower triangular unipotent and B symmetric. (Note that some authors use a different set of simple roots that results in A being upper triangular as was pointed out by the referee.) The torus T acts on the Lie-algebra $\mathfrak{n}$ of N by the adjoint representation Ad. It is easy to compute the modular factor $\delta_B(t) = \det(\mathrm{Ad}_\mathfrak{n}(t))$. If t is given in the form (3), then

$$\delta_B(t) = u_0^{-n(n+1)/2} u_1^2 u_2^4 \cdot \ldots \cdot u_n^{2n}. \qquad (6)$$

3. Local representations

In Theorem 2 we shall describe the local components of an automorphic representation associated with a classical Siegel modular form. This is done in terms of Satake parameters at the finite places, and in terms of the weight (or Harish-Chandra parameters, if the weight is large enough) at the archimedean place. In the following sections we shall therefore collect all the required facts about local representations.

Notation. In Sects. 3.1 through 3.4, F is a $\mathfrak{p}$-adic field, $\mathcal{O}$ its ring of integers, $\omega \in \mathcal{O}$ a generator of the maximal ideal, and $q = \sharp \mathcal{O}/\omega \mathcal{O}$. The symbols $G, T, \ldots$ denote F-points of the underlying algebraic groups. We let $K = \mathrm{GSp}(2n, \mathcal{O})$.

 Section 3.5 deals with the archimedean place, i.e., the underlying local field is $\mathbb{R}$.

3.1. The Satake isomorphism for $\mathrm{GSp}(2n)$

The Satake isomorphism for a reductive $\mathfrak{p}$-adic group is nicely described in [Ca]. Let $\mathcal{H}(G, K)$ be the unramified Hecke algebra of G, consisting of compactly supported functions $f : G \to \mathbb{C}$ which are left and right K-invariant. The product in $\mathcal{H}(G, K)$ is given by convolution

$$(f * g)(x) = \int_G f(xy) g(y^{-1})\, dy.$$

Let $^\circ T := T(\mathcal{O})$. Then we also have the Hecke algebra $\mathcal{H}(T, {}^\circ T)$. Special elements in this Hecke algebra are

$$X_0 := \mathrm{char}\big(\mathrm{diag}(\mathcal{O}^*, \ldots, \mathcal{O}^*, \omega \mathcal{O}^*, \ldots, \omega \mathcal{O}^*)\big),$$
$$X_1 := \mathrm{char}\big(\mathrm{diag}(\omega \mathcal{O}^*, \mathcal{O}^*, \ldots, \mathcal{O}^*, \omega^{-1} \mathcal{O}^*, \mathcal{O}^*, \ldots, \mathcal{O}^*)\big),$$
$$\vdots$$
$$X_n := \mathrm{char}\big(\mathrm{diag}(\mathcal{O}^*, \ldots, \mathcal{O}^*, \omega \mathcal{O}^*, \mathcal{O}^*, \ldots, \mathcal{O}^*, \omega^{-1} \mathcal{O}^*)\big),$$

where "char" stands for characteristic function. It is easily seen that

$$X_0^k = \mathrm{char}\big(\mathrm{diag}(\mathcal{O}^*, \ldots, \mathcal{O}^*, \omega^k \mathcal{O}^*, \ldots, \omega^k \mathcal{O}^*)\big), \quad k \in \mathbb{Z},$$

and similarly for the other X_i. It is then clear that

$$\mathcal{H}(T, {}^\circ T) = \mathbb{C}[X_0^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}].$$

Now, for an element $f \in \mathcal{H}(G, K)$, the Satake transform is defined by

$$(Sf)(t) = |\delta_B(t)|^{1/2} \int_N f(tn)\, dn = |\delta_B(t)|^{-1/2} \int_N f(nt)\, dn.$$

Sf is an element of $\mathcal{H}(T, {}^\circ T)$, and in fact S defines an isomorphism

$$S: \ \mathcal{H}(G, K) \xrightarrow{\ \widetilde{\ }\ } \mathcal{H}(T, {}^\circ T)^W.$$

Here W denotes the Weyl group of G, which acts naturally on the torus and on $\mathcal{H}(T, {}^\circ T)$, and we take invariant elements.

Assume now that $f = \mathrm{char}(KMK)$ with $KMK = \coprod M_i K$. Because of the Iwasawa decomposition $G = BK$ we may assume that

$$M_i = \begin{pmatrix} A_i & B_i \\ 0 & \omega^{d_{i0}}\, {}^t A_i^{-1} \end{pmatrix} \quad \text{with} \quad A_i = \begin{pmatrix} \omega^{d_{i1}} & & 0 \\ & \ddots & \\ * & & \omega^{d_{in}} \end{pmatrix}. \tag{7}$$

Here ω is a prime element and the d_{ij} are integers. Note that d_{i0} does not depend on i, since it equals the valuation of $\mu(M)$.

Lemma 1. *With f as above, we have*

$$Sf = q^{\delta n(n+1)/4}\, X_0^\delta \sum_i \prod_{j=1}^n (q^{-j} X_j)^{d_{ij}}.$$

where δ is the valuation of $\mu(M)$.

Proof. We have to compute

$$(Sf)(t) = |\delta_B(t)|^{1/2} \sum_i \int_N \mathbf{1}_{M_i K}(tn)\, dn,$$

where we may assume $t = \mathrm{diag}(\omega^{k_1}, \dots, \omega^{k_n}, \omega^{-k_1+k_0}, \dots, \omega^{-k_n+k_0})$. Consider $\int_N \mathbf{1}_{M_i K}(tn)\, dn$ for fixed i. We have $tn \in M_i K$ if and only if $n \in t^{-1} M_i K$. It is clear that we can find such an n only if $t^{-1} M_i$ has units on the diagonal, i.e., if $k_j = d_{ij}$ for all j. Assuming this is the case, then with $n' := t^{-1} M_i \in N$ our integral equals

$$\int_N \mathbf{1}_{n' K}(n)\, dn = \int_N \mathbf{1}_K(n)\, dn = 1.$$

We have proved that

$$\int_N \mathbf{1}_{M_i K}(tn)\, dn = \begin{cases} 1 & \text{if } k_j = d_{ij} \text{ for all } j = 0, 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

With the characteristic functions X_j defined above, this may be written as

$$\left(t \longmapsto \int_N \mathbf{1}_{M_i K}(tn)\, dn \right) = X_0^{d_{i0}} X_1^{d_{i1}} \cdot \dots \cdot X_n^{d_{in}}. \tag{8}$$

It follows from (6) that

$$|\delta_B(t)|^{1/2} = q^{k_0 n(n+1)/4} q^{-k_1 - 2k_2 - \dots - nk_n}.$$

Multiplying this by the function (8), we may replace k_j by d_{ij}, and the assertion follows. $\square$

3.2. Spherical representations

An irreducible admissible representation of G is called *spherical* if it contains a non-zero vector fixed by K. All the spherical representations of G are obtained as follows. Let $\chi_0, \dots, \chi_n$ be unramified characters of F^* (i.e., homomorphisms $F^* \to \mathbb{C}^*$ which are trivial on $\mathcal{O}^*$). They define an unramified character of the Borel subgroup $B = TN$ which is trivial on N and which, on T, is given by

$$t \longmapsto \chi_0(u_0)\chi_1(u_1) \cdot \ldots \cdot \chi_n(u_n), \tag{9}$$

with $t \in T$ of the form (3). Normalized induction to G yields a representation which has a unique spherical constituent. We denote this spherical representation by

$$\pi(\chi_0, \chi_1, \dots, \chi_n).$$

The isomorphism class of this representation depends only on the unramified characters modulo the action of the Weyl group. It is further known that each spherical representation is obtained in this way. Thus there is a bijection between unramified characters of T modulo the action of the Weyl group, and isomorphism classes of spherical representations of G.

Each unramified character of F^* is determined by its value on a prime element $\omega \in F$. This value may be any non-zero complex number. With the *Satake parameters* $b_i := \chi_i(\omega)$, the character (9) is thus also determined by the vector $(b_0, b_1, \dots, b_n) \in (\mathbb{C}^*)^{n+1}$. The Weyl group acts on this complex torus, and we see that unramified representations of G are parameterized by the orbit space $(\mathbb{C}^*)^{n+1}/W$. In fact, the Satake isomorphism

$$S: \ \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[X_0^{\pm 1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}]^W$$

identifies the Hecke algebra with the coordinate ring of $(\mathbb{C}^*)^{n+1}/W$. Each point $(b_0, \dots, b_n) \in (\mathbb{C}^*)^{n+1}/W$ determines a character, i.e. an algebra homomorphism $\mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^W \to \mathbb{C}$, by mapping X_i to b_i. Via S this also defines a character of $\mathcal{H}(G, K)$, which is nothing but the action of $\mathcal{H}(G, K)$ on the one-dimensional space of spherical vectors in $\pi(\chi_0, \dots, \chi_n)$.

These well-known facts may be summarized in the following commutative diagram, in which all the maps are bijections:

$$
\begin{array}{ccc}
\{\text{spherical representations}\} & \longrightarrow & \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}(G, K), \mathbb{C}) \\
\uparrow & & \downarrow \\
\{\text{unramified characters}\}/W & \longleftarrow & (\mathbb{C}^*)^{n+1}/W
\end{array}
$$

The left arrow is induction and then taking the spherical constituent, the top arrow is the action of $\mathcal{H}(G, K)$ on the space of spherical vectors, the map on the right comes from the identification $\mathcal{H}(G, K) \simeq \mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^W$, and the bottom arrow assigns to the Satake parameters $(b_0, \dots, b_n)$ the unramified character with $\chi_i(\omega) = b_i$.

Lemma 1 can now be restated in the following way. Let v be a spherical vector in $\pi := \pi(\chi_0, \ldots, \chi_n)$, and let $f = \mathbf{1}_{KMK}$ be the element from Lemma 1. Then

$$\pi(f)v = q^{\delta n(n+1)/4} b_0^\delta \sum_i \prod_{j=1}^n (q^{-j} b_j)^{d_{ij}} v, \quad \text{where } b_j = \chi_j(\omega). \tag{10}$$

Clearly the representation induced from the character (9) has central character $\chi_0^2 \chi_1 \ldots \chi_n$. This character is trivial if and only if the Satake parameters satisfy

$$b_0^2 b_1 \cdot \ldots \cdot b_n = 1. \tag{11}$$

Thus, exactly these parameters give spherical representations of the group $\mathrm{PGSp}(2n, F)$.

3.3. Dual groups

In Sect. 2 we described the root datum

$$(X, R, X^\vee, R^\vee)$$

of $G = \mathrm{GSp}(2n)$. By definition, the dual group $\hat{G} = \mathrm{GSpin}(2n+1, \mathbb{C})$ has root datum

$$(X', R', X'^\vee, R'^\vee) = (X^\vee, R^\vee, X, R).$$

Let $e'_0, \ldots, e'_n$ be a basis of X' such that

$$e'_i = f_i$$

under the identification $X' = X^\vee$, and let $f'_0, \ldots, f'_n$ be a basis of $X'^\vee$ such that

$$f'_i = e_i$$

under the identification $X'^\vee = X$. The complex torus $(\mathbb{C}^*)^{n+1}$ of the previous section may be viewed as the maximal torus $\hat{T} \subset \hat{G}$. Elements of $\hat{T}/W$ are in one-one correspondence with semisimple conjugacy classes in $\hat{G}$. By the previous section, we have a correspondence between spherical representations of G and those semisimple conjugacy classes. If χ is an unramified character of T defining a spherical representation, and $\hat{t} \in \hat{T}$ is the associated parameter, then χ and $\hat{t}$ are related by the general relation

$$\chi(\varphi(\omega)) = \varphi(\hat{t}) \quad \text{for all } \varphi \in X^\vee = X' \tag{12}$$

(see [GS] p. 26). It is clear that each $\hat{t} \in \hat{T}$ can be written uniquely as

$$\hat{t} = \prod_{i=0}^n f'_i(t_i), \quad t_i \in \mathbb{C}^*. \tag{13}$$

We have the pairing $\langle e'_i, f'_j \rangle = \delta_{ij}$, and thus $t_i = e'_i(\hat{t})$. Therefore putting $\varphi = e'_i = f_i$ in (12) yields $b_i = t_i$. We proved the following.

Lemma 2. *Let $\pi = \pi(\chi)$ be a spherical principal series representation of $G = \mathrm{GSp}(2n, F)$. Let $b_0, b_1, \ldots, b_n$ be the Satake parameters of π, as in 3.2. Then the associated semisimple conjugacy class in $\mathrm{GSpin}(2n + 1, \mathbb{C})$ is represented by*

$$\hat{t} = \prod_{i=0}^{n} f_i'(b_i).$$

Set $\overline{G} := \mathrm{PGSp}(2n, F) = G/C$, where C is the center of G. The dual group of $\overline{G}$ is

$$\hat{\overline{G}} = \mathrm{Spin}(2n + 1, \mathbb{C}) \subset \mathrm{GSpin}(2n + 1, \mathbb{C}),$$

which is also the derived group of $\hat{G}$. The element $t \in \hat{T}$ will correspond to a representation of G with *trivial* central character, i.e., to a representation of $\overline{G}$, if and only if it lies in

$$\hat{\overline{T}} := \hat{T} \cap \mathrm{Spin}(2n + 1, \mathbb{C}),$$

the maximal torus of $\mathrm{Spin}(2n + 1, \mathbb{C})$.

3.4. Local Euler factors

For a spherical representation π of $\overline{G}$ with parameter $\hat{t} \in \hat{\overline{T}}$ and a finite-dimensional representation

$$\varrho : \ \mathrm{Spin}(2n + 1, \mathbb{C}) \longrightarrow \mathrm{GL}(m, \mathbb{C}),$$

Langlands defines the local Euler factor $L(s, \pi, \varrho) = \det\left(1 - \varrho(\hat{t})q^{-s}\right)^{-1}$. In this section we are going to compute this factor for the "projection representation"

$$\varrho_1 : \ \mathrm{Spin}(2n + 1, \mathbb{C}) \longrightarrow \mathrm{SO}(2n + 1, \mathbb{C}), \tag{14}$$

and for the 2^n-dimensional spin representation ϱ_2. Similar computations can also be found in [As].

The projection representation. One can find the weight structure of the representation ϱ_1 in [FH]. All the weight spaces of this $(2n + 1)$-dimensional representation are one-dimensional. In the notation of the previous section the weights are

$$e_1', \ \ldots, \ e_n', \ 0, \ -e_1', \ \ldots, \ -e_n'.$$

The eigenvalues of an operator $\rho(\hat{t})$ are therefore 1 and

$$e_i'(\hat{t})^{\pm 1}, \quad i = 1, \ldots, n.$$

With $\hat{t}$ as in Lemma 2, we get $e_i'(\hat{t}) = b_i$. This proves the following.

Lemma 3. *If π is a spherical representation of $\overline{G}$ with Satake parameters $b_0, b_1, \ldots, b_n$ as in 3.2, then*

$$L(s, \pi, \varrho_1)^{-1} = (1 - q^{-s}) \prod_{i=1}^{n} (1 - b_i q^{-s})(1 - b_i^{-1} q^{-s}).$$

The spin representation. Again from [FH], the 2^n-dimensional spin representation ϱ_2 of $\mathrm{Spin}(2n+1,\mathbb{C})$ has highest weight $\frac{1}{2}(e_1' + \ldots + e_n')$. Indeed, all the weight spaces are one-dimensional, and the weights of ϱ_2 are

$$\frac{c_1 e_1' + \ldots + c_n e_n'}{2}, \quad c_i \in \{\pm 1\}. \tag{15}$$

Let such a weight be given. Define a character η of $\hat{T}$ by

$$\eta(\hat{t}) = t_0 \prod_{\substack{i=1 \\ c_i=1}}^{n} t_i, \quad \text{if } \hat{t} = \prod_{i=0}^{n} f_i'(t_i).$$

We claim that η coincides with the character (15) on $\hat{\hat{T}}$, the maximal torus of $\mathrm{Spin}(2n+1,\mathbb{C})$. An element $\hat{t}$ as above lies in this maximal torus if and only if $t_0^2 t_1 \cdot \ldots \cdot t_n = 1$. Assuming this relation, we have

$$(c_1 e_1' + \ldots + c_n e_n')(\hat{t}) = \prod_{i=1}^{n} \left(e_i'(\hat{t})\right)^{c_i} = \prod_{i=1}^{n} t_i^{c_i}$$

$$= t_0^2 \prod_{i=1}^{n} t_i^{c_i+1} = t_0^2 \prod_{\substack{i=1 \\ c_i=1}}^{n} t_i^2 = (2\eta)(\hat{t}),$$

and it follows that indeed $\frac{1}{2}(c_1 e_1' + \ldots + c_n e_n') = \eta$ on $\hat{\hat{T}}$. We can thus easily compute the eigenvalues of $\varrho_2(\hat{t})$, and get the following lemma.

Lemma 4. *If π is a spherical representation of $\bar{G}$ with Satake parameters $b_0, b_1, \ldots, b_n$ as in 3.2, then*

$$L(s, \pi, \varrho_2)^{-1} = \prod_{k=0}^{n} \prod_{1 \le i_1 < \ldots < i_k \le n} (1 - b_0 b_{i_1} \ldots b_{i_k} q^{-s}).$$

3.5. Lowest weight representations for $\mathrm{PGSp}(2n, \mathbb{R})$

In this section we shall construct a class of lowest weight representations of $\mathrm{PGSp}(2n, \mathbb{R})$, which appear as the infinite component in the automorphic representation attached to a holomorphic Siegel modular form.

The Lie algebra of $\mathrm{Sp}(2n, \mathbb{R})$ is explicitly given by

$$\mathfrak{g} = \left\{ X \in M(2n, \mathbb{R}) : XJ + J\,{}^t X = 0 \right\}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(2n, \mathbb{R}) : B = {}^t B, \ C = {}^t C, \ A = -{}^t D \right\}. \tag{16}$$

The standard maximal compact subgroup K_∞ of $\mathrm{Sp}(2n, \mathbb{R})$ is

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}) : A\,{}^t A + B\,{}^t B = \mathbf{1}, \ A\,{}^t B = B\,{}^t A \right\}. \tag{17}$$

We have $K_\infty \simeq U(n)$ via $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB$. The Lie algebra of K_∞ is

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M(2n, \mathbb{R}) : \ A = -{}^tA, \ B = {}^tB \right\}.$$

This is also the 1-eigenspace of the Cartan involution $\theta X = -{}^tX$. The (-1)-eigenspace is

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M(2n, \mathbb{R}) : \ A = {}^tA, \ B = {}^tB \right\},$$

so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It is easy to see that $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{p}_\mathbb{C}^-$, with

$$\mathfrak{p}_\mathbb{C}^\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \in M(2n, \mathbb{C}) : \ A = {}^tA \right\}. \tag{18}$$

Each of $\mathfrak{p}_\mathbb{C}^\pm$ is an abelian subalgebra of $\mathfrak{g}_\mathbb{C}$, we have $[\mathfrak{k}_\mathbb{C}, \mathfrak{p}_\mathbb{C}^\pm] \subset \mathfrak{p}_\mathbb{C}^\pm$, and

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}^-.$$

In the complexified Lie-algebra $\mathfrak{k}_\mathbb{C}$ of K let

$$T_i = -i \begin{pmatrix} 0 & D_i \\ -D_i & 0 \end{pmatrix}, \tag{19}$$

where D_i is the diagonal matrix with entry 1 at position (i, i), and zeros elsewhere. Then $\mathfrak{h} = \mathbb{R}T_1 + \ldots + \mathbb{R}T_n$ is a compact Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let e_i be the linear form on $\mathfrak{h}_\mathbb{C}$ which sends T_i to 1, and T_j to 0 for $j \neq i$. Then the following is a system of positive roots for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$:

$$\begin{aligned} &2e_j, &&1 \le j \le n, \\ &e_j + e_k, &&1 \le j < k \le n, \\ &e_j - e_k, &&1 \le j < k \le n \end{aligned} \tag{20}$$

(the simple roots are $e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n$). In fact, $\mathfrak{p}_\mathbb{C}^+$ is spanned by the root spaces for the first two types of roots. If D_j is as above, then

$$\begin{pmatrix} D_j & iD_j \\ iD_j & -D_j \end{pmatrix}, \quad 1 \le j \le n,$$

spans the root space for the root $2e_j$, and if E_{jk} is the matrix with entry 1 at positions (j, k) and (k, j), and zeros elsewhere, then

$$\begin{pmatrix} E_{jk} & iE_{jk} \\ iE_{jk} & -E_{jk} \end{pmatrix}, \quad 1 \le j < k \le n,$$

spans the root space for the root $e_j + e_k$. These roots are the *non-compact* positive roots. The other positive roots are *compact*; the root space of $e_j - e_k$ is spanned by the element

$$\begin{pmatrix} F_{jk} & -iE_{jk} \\ iE_{jk} & F_{jk} \end{pmatrix}, \quad 1 \le j < k \le n,$$

where F_{jk} is the matrix with entry 1 at position (j, k), entry -1 at position (k, j), and zeros elsewhere.

Let k be a positive integer (it will be the weight of a Siegel modular form in our case). We shall construct a lowest weight representation of G with lowest weight $k(e_1 + \ldots + e_n)$. Let Ξ_k be the character of K given by

$$\Xi_k(\kappa) = j(\kappa, I)^{-k},$$

where $j(g, Z) = \det(CZ + D)$ is the classical automorphic factor, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, and Z is in the Siegel upper half plane (Ξ is a character because K is the stabilizer of $I = \mathrm{diag}(i, \ldots, i)$ under linear fractional transformations). Let $\xi_k = d\Xi_k$ be the differential, complexified to a linear form $\mathfrak{k}_{\mathbb{C}} \to \mathbb{C}$. Explicitly,

$$\xi_k(T_j) = k, \qquad j = 1, \ldots, n,$$
$$\xi_k(X_{e_j - e_l}) = 0 \qquad \text{for all } j \neq l.$$

Because of $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}^-] \subset \mathfrak{p}_{\mathbb{C}}^-$, the character ξ_k can be extended to a character of $\mathfrak{k}_C + \mathfrak{p}_{\mathbb{C}}^-$. Consider

$$V_k = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}^-)} \mathbb{C}_{\xi_k}.$$

Here $\mathbb{C}_{\xi_k} = \mathbb{C}$ with the action of $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}^-$ given by ξ_k. If v_0 denotes the special vector $1 \otimes 1 \in V_k$, then $T_j v_0 = k v_0$, and v_0 is annihilated by all negative roots. Thus we have a representation of $\mathfrak{g}_{\mathbb{C}}$ of lowest weight $k(e_1 + \ldots + e_n)$. It is also clear that

$$V_k = \mathcal{U}(\mathfrak{p}_{\mathbb{C}}^+) v_0.$$

V_k is in fact a $(\mathfrak{g}_{\mathbb{C}}, K)$-module, and globalizes to a unitary representation of $\mathrm{Sp}(2n, \mathbb{R})$ which in the following we shall denote by π_k (it depends only on the positive integer k). The center of $\mathrm{Sp}(2n, \mathbb{R})$ consists of only two elements, and it is easy to see that the central character of π_k is given by $(-1)^{nk}$. Thus, if nk is even, π_k descends to a representation of $\mathrm{PSp}(2n, \mathbb{R})$. Assuming this is the case, we get a representation of $\mathrm{PGSp}(2n, \mathbb{R})$ by inducing π_k from the subgroup $\mathrm{PSp}(2n, \mathbb{R})$ (of index 2). This new representation shall also be denoted by π_k. It has a lowest weight vector of weight $k(e_1 + \ldots + e_n)$, and a highest weight vector of weight $-k(e_1 + \ldots + e_n)$.

By Harish-Chandra, the discrete series representations of $\mathrm{Sp}(2n, \mathbb{R})$ are parameterized by elements in the weight lattice which do not lie on a wall, modulo the action of the Weyl group of K (see, for instance, [Kna] Theorem 9.20). If λ is such a Harish-Chandra parameter, then the lowest K-type of the corresponding discrete series representation is given by the Blattner parameter

$$\Lambda = \lambda + \delta_{\mathrm{nc}} - \delta_{\mathrm{c}},$$

where δ_{nc} (resp. δ_{c}) is half of the sum of the non-compact (resp. compact) positive roots; here "positive" means with respect to the Weyl chamber in which λ lies.

The choice (20) of positive roots leads to the *holomorphic* discrete series representations. The corresponding Harish-Chandra parameters are of the form

$$\lambda = a_1 e_1 + \ldots + a_n e_n, \qquad a_i \in \mathbb{Z}, \ a_1 > \ldots > a_n > 0.$$

The relation with the Blattner parameter is

$$\Lambda = \lambda + \sum_{j=1}^{n} j e_j.$$

We shall be particularly interested in the discrete series representation with Harish-Chandra parameter

$$(k-1)e_1 + \ldots (k-n)e_n,$$

where $k > n$ is an integer; it has the lowest K-type $k e_1 + \ldots + k e_n$, and must thus coincide with the representation π_k constructed above. If $k = n$, then the Harish-Chandra parameter comes to lie on a wall and this identifies our π_k, $k = n$, as a limit of discrete series. If $k < n$, then π_k is neither a discrete series representation, nor a limit of them.

4. Siegel modular forms

For basic facts about classical Siegel modular forms we refer to [AZ], [Fr1], or [Kl].

4.1. Lifting of Siegel modular forms

Siegel modular forms of degree n are certain holomorphic functions on the Siegel upper half plane $\mathbb{H}_n$, which by definition is the complex manifold consisting of complex symmetric $n \times n$ matrices with positive definite imaginary part. Strong approximation allows one to regard Siegel modular forms of degree n as functions on $G(\mathbb{A})$, where $G = \mathrm{GSp}(2n)$, and where the ground field is $\mathbb{Q}$.

Let f be a Siegel modular form of weight k and degree n. We shall assume that f is a modular form with respect to the full modular group $\Gamma_n = \mathrm{Sp}(2n, \mathbb{Z})$, i.e., $f|_k \gamma = f$ for all $\gamma \in \Gamma_n$, where

$$(f|_k h)(Z) = \mu(h)^{nk/2} j(h, Z)^{-k} f(h\langle Z \rangle) \quad \text{for } h \in G_\infty^+, \ Z \in \mathbb{H}_n$$

(here μ is the multiplier, $j(h, Z) = \det(CZ + D)$ for $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $h\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$). We remark that this operation differs from the classical one used in [An] by a factor. We do so to make the center of $G(\mathbb{A})$ act trivially. More precisely, the classical operation is

$$(f \|_k h)(Z) = \det(h)^{k-(n+1)/2} j(h, Z)^{-k} f(h\langle Z \rangle).$$

The relation between the two operations is

$$f \parallel_k h = \det(h)^{(k-n-1)/2} f|_k h \quad \text{for all } h \in G_\infty^+. \tag{21}$$

To f is associated a function $\Phi_f : G(\mathbb{A}) \to \mathbb{C}$ as follows. One uses strong approximation for $\mathrm{Sp}(2n)$ (cf. [Kne]) to show that

$$G(\mathbb{A}) = G(\mathbb{Q}) \, G(\mathbb{R})^+ \prod_{p<\infty} G(\mathbb{Z}_p),$$

where $G(\mathbb{R})^+$ denotes those elements of $G(\mathbb{R})$ which have positive multiplier. Write an element $g \in G(\mathbb{A})$ as

$$g = g_{\mathbb{Q}} g_\infty k_0 \quad \text{with } g_{\mathbb{Q}} \in G(\mathbb{Q}), \ g_\infty \in G_\infty^+, \ k_0 \in K_0, \tag{22}$$

where $K_0 = \prod_{p<\infty} K_p$ with $K_p = G(\mathbb{Z}_p)$. Then we define

$$\Phi_f(g) = (f|_k g_\infty)(I), \tag{23}$$

where $I = \mathrm{diag}(i, \dots, i) \in \mathbb{H}_n$. This is well-defined because of the transformation properties of f.

The map $f \mapsto \Phi_f$ injects the space of modular forms of weight k into a space of functions Φ on $G_\mathbb{A}$ having the following properties:

i) $\Phi(\varrho g) = \Phi(g)$ for $\varrho \in G(\mathbb{Q})$,
ii) $\Phi(g k_0) = \Phi(g)$ for $k_0 \in K_0$,
iii) $\Phi(g k_\infty) = \Phi(g) \, j(k_\infty, I)^{-k}$ for $k_\infty \in K_\infty$,
iv) $\Phi(g z) = \Phi(g)$ for $z \in Z(\mathbb{A})$.

Here $Z \simeq \mathrm{GL}(1)$ is the center of $\mathrm{GSp}(2n)$, and $K_\infty \simeq U(n)$ is the standard maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ (and is equal to the stabilizer of I under linear fractional transformations; see also the next section).

Lemma 5. *If $f \in S_k(\Gamma_n)$, then the automorphic form Φ_f is cuspidal, i.e.,*

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Phi_f(ng) \, dn = 0 \qquad \text{for all } g \in G(\mathbb{A})$$

for each unipotent radical N of each proper parabolic subgroup of G.

Proof. It is enough to verify the cusp condition for the standard maximal parabolics. If $P = MN$ is one of those, then by the Iwasawa decomposition we may assume $g \in M(\mathbb{A})$. Using strong approximation for $M(\mathbb{A})$, we may further assume that $g \in M_\infty^+ := M(\mathbb{R}) \cap G_\infty^+$. Let V be the intersection of the unipotent radicals of all standard maximal parabolics. Clearly it is enough to show that

$$\int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \Phi_f(ng) \, dn = 0.$$

Now V consists of all elements of the form $\begin{pmatrix} \mathbf{1} & v \\ & \mathbf{1} \end{pmatrix}$ with $v \in V'$, where V' is the set of symmetric $n \times n$ matrices with non-zero entries only in last row and last column (thus $V \simeq V' \simeq \mathbb{G}_a^n$). This is easy to see from the fact that a root space X_α belongs to V if and only if $\alpha = c_1\alpha_1 + \ldots + c_n\alpha_n$ with all $c_i > 0$ (and α_i as in (4)). With $Z := g\langle I \rangle$ one gets

$$\int\limits_{V(\mathbb{Q})\backslash V(\mathbb{A})} \Phi_f(ng)\, dn = \int\limits_{V(\mathbb{Z})\backslash V(\mathbb{R})} \Phi_f(ng)\, dn$$

$$= \int\limits_{V(\mathbb{Z})\backslash V(\mathbb{R})} f(ng\langle I\rangle) j(ng, I)^{-k}\, dn = \int\limits_{V'(\mathbb{Z})\backslash V'(\mathbb{R})} f(Z+v) j(g, I)^{-k}\, dv.$$

At this point we use the Fourier expansion of f:

$$f(Z) = \sum_R c_R \exp\left(2\pi i \operatorname{tr}(RZ)\right), \quad Z \in \mathbb{H}_n,$$

where R runs over semi-integral, positive definite matrices (it is here where we use the classical cusp condition). Thus

$$\int\limits_{V(\mathbb{Q})\backslash V(\mathbb{A})} \Phi_f(ng)\, dn = \sum_R c_R \exp\left(2\pi i \operatorname{tr}(RZ)\right) j(g, I)^{-k}$$

$$\cdot \int\limits_{V'(\mathbb{Z})\backslash V'(\mathbb{R})} \exp\left(2\pi i \operatorname{tr}(Rv)\right) dv.$$

Using the fact that R is non-degenerate, one checks that the map

$$v \mapsto \exp\left(2\pi i \operatorname{tr}(Rv)\right)$$

is a non-trivial character of V'. Thus our integral is zero. □

The lemma shows that the map $f \mapsto \Phi_f$ gives a one-one correspondence between the space $S_k(\Gamma_n)$ of cusp forms of weight k, and a subspace of $L_0^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, the space of cuspidal functions in

$$L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})).$$

Also note that

$$\Gamma_n\backslash\mathbb{H}_n \simeq Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K. \tag{24}$$

To see this, map an element $g = g_{\mathbb{Q}}g_\infty k \in G(\mathbb{A})$ to $g_\infty\langle I\rangle$. Also, every point of the left hand side of the isomorphism can be written as $g_\infty\langle I\rangle$ for some $g_\infty \in G_\infty^+$ which can in turn be mapped to the image of g_∞ in the right hand side.

If we start with a Haar measure on $G(\mathbb{A})$ and take the induced measure on the right side of (24), then the corresponding measure on the left side is induced from (a suitable multiple of) the usual invariant volume element d^*Z on $\mathbb{H}_n$. In other

words, if a Γ_n-invariant function f on $\mathbb{H}_n$ and a $G(\mathbb{Q})$-invariant function Φ on $Z(\mathbb{A})\backslash G(\mathbb{A})/K$ are related by

$$f(g_\infty\langle I\rangle) = \Phi(g_\infty) \quad \text{for all } g_\infty \in G_\infty^+,$$

then

$$\int\limits_{\Gamma_n\backslash\mathbb{H}_n} f(Z)\,d^*Z = \int\limits_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K} \Phi(g)\,dg = \int\limits_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \Phi(g)\,dg$$

(integrations taken over fundamental domains). We apply this to the function

$$\Phi_{f_1}(g)\,\overline{\Phi_{f_2}(g)} = f_1(g\langle I\rangle)\,\overline{f_2(g\langle I\rangle)}\,\mu(g)^{nk}\,|j(g, I)|^{-2k}$$
$$= f_1(Z)\,\overline{f_2(Z)}\,\det(Y)^k,$$

where $f_1, f_2 \in S_k(\Gamma_n)$, and where $g\langle I\rangle = X + iY$ for $g \in G_\infty^+$. We obtain

$$\int\limits_{\Gamma_n\backslash\mathbb{H}_n} f_1(Z)\,\overline{f_2(Z)}\,\det(Y)^k\,d^*Z = \int\limits_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \Phi_{f_1}(g)\,\overline{\Phi_{f_2}(g)}\,dg. \qquad (25)$$

On the left, we have the classical Petersson scalar product, and the ordinary L^2 scalar product on the right. This proves the following.

Lemma 6. *Upon suitable normalization of measures, the map $f \mapsto \Phi_f$ from $S_k(\Gamma_n)$ into $L_0^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an isometry.*

4.2. Holomorphy and differential operators

We are going to express the holomorphy of the function f on $\mathbb{H}_n$ in terms of the annihilation of Φ_f by certain differential operators. If Φ is a function on $G(\mathbb{A})$ which is smooth as a function of $G(\mathbb{R})$, and if $X \in \mathfrak{g}$, the real Lie-algebra of $\mathrm{Sp}(2n, \mathbb{R})$, then we define as usual

$$(X\Phi)(g) := \frac{d}{dt}\bigg|_0 \Phi(g\exp(tX)), \quad g \in G(\mathbb{A}).$$

This action of $\mathfrak{g}$ on smooth functions $G(\mathbb{A}) \to \mathbb{C}$ extends to an action of $\mathfrak{g}_\mathbb{C}$ by linearity. We have identified $\mathfrak{g}$ with a space of matrices in (16).

The stabilizer of $I \in \mathbb{H}_n$ (under linear fractional transformations) is the maximal compact subgroup $K_\infty \simeq U(n)$ of $\mathrm{Sp}(2n, \mathbb{R})$, given by (17). As in 3.5 we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{p}$ is the (-1)-eigenspace under the Cartan involution. Consider the projection

$$\rho : \mathrm{Sp}(2n, \mathbb{R}) \longrightarrow \mathbb{H}_n,$$
$$g \longmapsto g\langle I\rangle,$$

which induces the homeomorphism $\mathrm{Sp}(2n, \mathbb{R})/K_\infty \simeq \mathbb{H}_n$. Its differential at the identity

$$d\rho : \mathfrak{g} \longrightarrow T_I \mathbb{H}_n$$

has kernel $\mathfrak{k}$, and therefore induces an isomorphism $\mathfrak{p} \xrightarrow{\sim} T_I \mathbb{H}_n$. If $T_I \mathbb{H}_n$ is naturally identified with the space of symmetric complex $n \times n$ matrices, then a small calculation shows that

$$d\rho(X) = \frac{d}{dt}\Big|_0 \exp(tX)\langle I\rangle = 2(B + IA) \quad \text{for } X = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{p}.$$

Let J be the complex structure (multiplication with i) on $T_I \mathbb{H}_n$. Carried over to $\mathfrak{p}$, it is given by

$$J\begin{pmatrix} A & B \\ B & -A \end{pmatrix} = \begin{pmatrix} B & -A \\ -A & -B \end{pmatrix}.$$

Now we have $\mathfrak{p}_\mathbb{C} = \mathfrak{p} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{p}_\mathbb{C}^-$, where $\mathfrak{p}_\mathbb{C}^\pm$ is the $(\pm i)$-eigenspace of J. Thus the elements of $\mathfrak{p}_\mathbb{C}^+$ (resp. $\mathfrak{p}_\mathbb{C}^-$) correspond to linear combinations of differential operators

$$\frac{d}{dz_j} \in T_I \mathbb{H}_n \otimes_\mathbb{R} \mathbb{C} \quad \left(\text{resp. } \frac{d}{d\bar{z}_j}\right), \tag{26}$$

where $z_j = x_j + iy_j$ are coordinates on $\mathbb{H}_n$ about the point I. Explicitly, we have

$$\mathfrak{p}_\mathbb{C}^\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \in M(2n, \mathbb{C}) : A = {}^t A \right\}, \tag{27}$$

thus these spaces $\mathfrak{p}_\mathbb{C}^\pm$ coincide with the ones already defined in (18).

Lemma 7. *Let f be a smooth function on $\mathbb{H}_n$ transforming like a modular form, and let the function Φ_f on $G(\mathbb{A})$ be defined by (23). Then f is holomorphic if and only if $\mathfrak{p}_\mathbb{C}^-.\Phi_f = 0$.*

Proof. By the transformation properties of Φ, it is enough to prove the following. Let f be a smooth function on $\mathbb{H}_n$, and define $\Phi : \mathrm{Sp}(2n, \mathbb{R}) \to \mathbb{C}$ by

$$\Phi(g) = f(g\langle I\rangle)j(g, I)^{-k}, \quad g \in \mathrm{Sp}(2n, \mathbb{R}).$$

Then f is holomorphic if and only if $\mathfrak{p}_\mathbb{C}^-.\Phi = 0$.

Let $\tilde{f}(g) = f(g\langle I\rangle)$ and $\tilde{j}(g) = j(g, I)^{-k}$. We are going to prove that

$$(X\Phi)(g) = (X\tilde{f})(g)j(g, I)^{-k} \quad \text{for } X \in \mathfrak{p}_\mathbb{C}^-. \tag{28}$$

By considering the function $f'(Z) := f(h\langle Z\rangle)j(h, Z)^{-k}$ instead of f, it is enough to do this for $g = \mathbf{1}$, i.e., we will show that

$$(X\Phi)(\mathbf{1}) = (X\tilde{f})(\mathbf{1}) \quad \text{for } X \in \mathfrak{p}_\mathbb{C}^-. \tag{29}$$

By the product rule, we have

$$(X\Phi)(\mathbf{1}) = (X\tilde{f})(\mathbf{1}) + f(I)(X\tilde{j})(\mathbf{1}) \quad \text{for } X \in \mathfrak{g}_{\mathbb{C}}$$

(this is immediate for $X \in \mathfrak{g}$, and also holds for $X \in \mathfrak{g}_{\mathbb{C}}$ by linearity). Using (27), one gets $(X\tilde{j})(\mathbf{1}) = 0$ for $X \in \mathfrak{p}_{\mathbb{C}}^-$. This proves (29), and therefore also (28). Thus Φ is annihilated by $\mathfrak{p}_{\mathbb{C}}^-$ if and only $\tilde{f}$ is. But by the definition of $\mathfrak{p}_{\mathbb{C}}^-$ we have $\mathfrak{p}_{\mathbb{C}}^-.\tilde{f} = 0$ if and only if f is holomorphic (see (26)).   □

4.3. Hecke operators

We now fix a prime number p. Let $\mathcal{H}_p^{\text{class}}$ be the p-component of the classical Hecke algebra of $G = \text{GSp}(2n)$. It is spanned by double cosets

$$\Gamma M \Gamma \quad \text{with } M \in G(\mathbb{Z}[p^{-1}])^+,$$

where $\mathbb{Z}[p^{-1}]$ is the ring of rational numbers with only p-powers in the denominator.

Lemma 8. *There is an isomorphism*

$$\mathcal{H}_p^{\text{class}} \xrightarrow{\sim} \mathcal{H}(G_p, K_p).$$

The double coset $\Gamma M \Gamma$ corresponds under this isomorphism to the characteristic function of $K_p M K_p$.

Proof. It is easy to see that there are bijections

$$\Gamma \backslash G(\mathbb{Z}[p^{-1}])^+/\Gamma \xleftrightarrow{\sim} G(\mathbb{Z}) \backslash G(\mathbb{Z}[p^{-1}])/G(\mathbb{Z}) \xleftrightarrow{\sim} K_p \backslash G_p/K_p,$$

induced by the inclusions of the groups. Thus the Hecke algebras are isomorphic as vector spaces. There are similar bijections with cosets instead of double cosets. This fact is used to check that the classical multiplication of double cosets coincides with the convolution product on the p-adic Hecke algebra.   □

From now on we may identify the two Hecke algebras. We recall how these algebras act on modular forms. If f is a classical modular form of weight k and degree n, and if $T = \Gamma M \Gamma \in \mathcal{H}_p^{\text{class}}$, then

$$f|_k T := \sum_i f|_k M_i, \quad \text{where } \Gamma M \Gamma = \coprod_i \Gamma M_i,$$

is again a modular form. This defines a right action of $\mathcal{H}_p^{\text{class}}$ on the space of modular forms of weight k and degree n. As before, this action differs slightly from Andrianov's action, which is defined with $\|_k$ instead of $|_k$. It follows from (21) that

$$f \|_k T = \det(M)^{(k-n-1)/2} f|_k T \quad \text{for } T = \Gamma M \Gamma. \tag{30}$$

There is also a left action of $\mathcal{H}(G_p, K_p)$ on adelic functions given by

$$(T\Phi)(g) = \int_{G_p} T(h)\Phi(gh)\,dh, \quad T \in \mathcal{H}(G_p, K_p), \ g \in G(\mathbb{A}).$$

If T is the characteristic function of $K_p M K_p = \coprod M_i K_p$, and if Φ is right K_p-invariant, then

$$(T\Phi)(g) = \sum \Phi(g M_i).$$

In the proof of the following lemma we use the notation $M^* = \mu(M)M^{-1}$, which is an anti-involution of G. Note that

$$\Gamma M \Gamma = \Gamma M^* \Gamma,$$

since M may be chosen to be diagonal ($M \mapsto M^*$ is not the identity on diagonal matrices, but operates as conjugation with J, which is an element of Γ).

Lemma 9. *The lifting $f \longmapsto \Phi_f$ defined by (23) is Hecke-equivariant, i.e.,*

$$T\Phi_f = \Phi_{f|T} \quad \text{for each } T \in \mathcal{H}(G_p, K_p)$$

(here we identified the p-adic and the classical Hecke algebra according to Lemma 8).

Proof. It is enough to check that both sides are equal when evaluated at an element $g \in G_\infty^+$. We may further assume that T is a double coset,

$$T = \Gamma M \Gamma = \coprod_i \Gamma M_i \quad \text{with } M_i \in G(\mathbb{Z}[p^{-1}])^+.$$

By the above considerations, we have $K_p M K_p = \coprod K_p M_i$, and therefore

$$K_p M K_p = K_p M^* K_p = \coprod_i M_i^* K_p.$$

For a matrix $M \in G(\mathbb{Q})$ we write M_∞ for the matrix considered as an element of G_∞, and M_p considered as an element of G_p. With these notations, we have

$$(T\Phi_f)(g) = \sum_i \Phi_f(g(M_i^*)_p) = \sum_i \Phi_f((M_i^*)_{\mathbb{Q}}^{-1} g(M_i^*)_p)$$

$$= \sum_i \Phi_f((M_i^*)_\infty^{-1} g) = \sum_i \left(f|(M_i^*)^{-1}g\right)(I)$$

$$= \sum_i \left(f|M_i g\right)(I) = ((f|T)|g)(I) = \Phi_{f|T}(g).\ \square$$

It follows from this lemma and (30) that

$$T\Phi_f = \det(M)^{(n+1-k)/2} \Phi_{f\|T} \quad \text{for } T = \Gamma M \Gamma. \tag{31}$$

The following theorem summarizes the results of the previous sections.

Theorem 1. *The mapping $f \mapsto \Phi_f$ defined by (23) maps the space $S_k(\Gamma_n)$ of classical Siegel cuspforms of degree n and weight k isometrically (see Lemma 6) and in a Hecke-equivariant way (see Lemma 9) into a subspace of $L_0^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ consisting of continuous functions Φ on $G(\mathbb{A})$ with the following properties:*

i) $\Phi(\varrho g) = \Phi(g)$ *for $\varrho \in G(\mathbb{Q})$,*
ii) $\Phi(g k_0) = \Phi(g)$ *for $k_0 \in K_0 = \prod_{p < \infty} G(\mathbb{Z}_p)$,*
iii) $\Phi(g k_\infty) = \Phi(g) j(k_\infty, I)^{-k}$ *for $k_\infty \in K_\infty \simeq U(n)$,*
iv) $\Phi(g z) = \Phi(g)$ *for $z \in Z(\mathbb{A})$, the center of $G(\mathbb{A})$,*
v) Φ *is smooth as a function of $G(\mathbb{R})^+$ (fixed finite components), and is annihilated by $\mathfrak{p}_{\mathbb{C}}^-$ (see Lemma 7),*
vi) Φ *is cuspidal (see Lemma 5).*

4.4. The associated representation

We now associate an automorphic representation of $\mathrm{PGSp}(2n)$ with a Hecke eigenform of degree n.

Let f be a cuspidal Hecke eigenform of degree n and weight k. Let Φ_f be the associated function on $G(\mathbb{A})$, defined by (23). Then Φ_f is an automorphic form on $G(\mathbb{A})$ which lies in $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, and in fact in the cuspidal subspace $L_0^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ by Lemma 5. Denote by V_f the subspace of this L_0^2-space spanned by all right translates of Φ_f. Let π be any irreducible constituent of this unitary representation (it is well known that $L_0^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes discretely into irreducible components). Then π is an automorphic representation of $G(\mathbb{A})$ which is trivial on $Z(\mathbb{A})$. We may thus consider π as an automorphic representation of $\mathrm{PGSp}(2n, \mathbb{A})$.

Let

$$\pi = \bigotimes_p \pi_p \tag{32}$$

be the decomposition of π into irreducible representations π_p of the local groups $G_p = G(\mathbb{Q}_p)$ (restricted tensor product). Because Φ_f is right $K_p = G(\mathbb{Z}_p)$-invariant at every finite place p, the representation π_p is spherical for every such p. As discussed in 3.2, it is of the form $\pi(\chi_0, \ldots, \chi_n)$ for certain unramified characters χ_i of $\mathbb{Q}_p^*$, with Satake parameters $\beta_i := \chi_i(p)$.

We also have the *classical* Satake parameters of the eigenform f, defined as follows (see [An]). There is a character $\lambda \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}_p^{\mathrm{class}}, \mathbb{C})$, such that

$$f \parallel_k T = \lambda(T) f.$$

It is known that there are non-zero complex numbers $a_0, \ldots, a_n$ such that

$$\lambda(\Gamma M \Gamma) = a_0^\delta \sum_i \prod_{j=1}^n (a_j p^{-j})^{d_{ij}},$$

where $\Gamma M \Gamma = \coprod_i \Gamma M_i$ and

$$M_i = \begin{pmatrix} \omega^\delta {}^t A_i^{-1} & B_i \\ 0 & A_i \end{pmatrix}, \quad \text{with} \quad A_i = \begin{pmatrix} \omega^{d_{i1}} & & * \\ & \ddots & \\ 0 & & \omega^{d_{in}} \end{pmatrix}. \tag{33}$$

The $a_0, \ldots, a_n$ are the *classical Satake p-parameters* of the eigenform f.

Lemma 10. *If $a_0, a_1, \ldots, a_n$ are the classical Satake p-parameters of f, then*

$$p^{n(n+1)/4 - nk/2} a_0, \ a_1, \ \ldots, \ a_n$$

are the Satake parameters of the spherical representation π_p, the local component of π at p.

Proof. With $T = \Gamma M \Gamma$, by (31) we have

$$T \Phi_f = \det(M)^{(n+1-k)/2} \lambda(T) \Phi_f = \mu(M)^{n(n+1-k)/2} \lambda(T) \Phi_f$$

$$= p^{\delta n(n+1-k)/2} a_0^\delta \sum_i \prod_{j=1}^n (a_j p^{-j})^{d_{ij}} \Phi_f.$$

Thus $T \Phi_f = \tilde\lambda(T) \Phi_f$ with

$$\tilde\lambda(T) = p^{\delta n(n+1)/4} b_0^\delta \sum_i \prod_{j=1}^n (b_j p^{-j})^{d_{ij}},$$

where $b_0 = p^{n(n+1)/4 - nk/2} a_0$ and $b_j = a_j$ for $j = 1, \ldots, n$. It follows from (10) that $b_0, \ldots, b_n$ are the Satake parameters of π_p (note that if M_i is of the form (33), then M_i^* is of the form (7)). $\square$

It follows from (11) that for the classical Satake parameters we have

$$a_0^2 a_1 \cdot \ldots \cdot a_n = p^{kn - n(n+1)/2},$$

a relation also found in [AK].

We shall now describe the local component π_∞ in the tensor product decomposition (32) of $\pi = \pi_f$. This component will only depend on the weight k of the modular form f, which is a positive integer. Recall the definition (19) of the torus elements $T_i \in \mathfrak{k}_{\mathbb{C}}$.

Lemma 11. *The representation π_∞ of $G(\mathbb{R})$ contains a smooth vector v_∞ with the following properties:*

i) $\pi(k_\infty) v_\infty = j(k_\infty, I)^{-k} v_\infty$ *for all $k_\infty \in K_\infty$.*
ii) $T_i v_\infty = k v_\infty$ *for all $i = 1, \ldots, n$.*
iii) $\mathfrak{p}_{\mathbb{C}}^- v_\infty = 0$.

Proof. In π the vector $\Phi_f \in V_f$ projects to a pure tensor $v = \otimes_{p \leq \infty} v_p$ with $v_p \in \pi_p$ a spherical vector for finite p, and $v_\infty \in \pi_\infty$ inheriting the analytic properties of Φ_f. Thus i) follows from Theorem 1 iii). Then ii) follows by a straightforward computation. Finally iii) follows from Theorem 1 v). □

By this lemma, π_∞ is an irreducible unitary representation of $\mathrm{PGSp}(2n, \mathbb{R})$ with a lowest weight vector of weight $k(e_1 + \ldots + e_n)$. There is only one such representation, namely the representation π_k constructed in 3.5. To summarize,

Theorem 2. *Let π be the automorphic representation of $\mathrm{PGSp}(2n, \mathbb{A})$ associated with $f \in S_k(\Gamma_n)$, as described at the beginning of this section. Then the local components π_p of π are given as follows:*

i) *At the archimedean place, π_∞ is the lowest weight representation π_k constructed in 3.5 (it is a holomorphic discrete series representation exactly for $k > n$, and a limit of discrete series for $k = n$).*

ii) *At a finite place, π_p is the spherical principal series representation (see Sect. 3.2) of $\mathrm{PGSp}(2n, \mathbb{Q}_p)$ with Satake parameters $p^{n(n+1)/4 - nk/2} a_0, a_1, \ldots, a_n$, where $a_1, \ldots, a_n$ are the classical Satake parameters of the modular form f.*

4.5. Vector valued modular forms

For completeness we shall indicate in this section how to extend the lifting procedure described in the previous sections to vector valued Siegel modular forms. A good source for vector valued modular forms is [Fr2]. The lifting procedure for these forms is briefly described in [We].

Let (ρ, W) be a finite-dimensional rational representation of $\mathrm{GL}(n, \mathbb{C})$. Rational means that there is an integer k such that the representation $A \mapsto \det(A)^{-k} \rho(A)$ is polynomial (holomorphic). A vector valued modular form of type ρ and degree n is a holomorphic function $f : \mathbb{H}_n \to W$ with the property that

$$f(\gamma \langle Z \rangle) = \rho(CZ + D) f(Z) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n, \ Z \in \mathbb{H}_n. \quad (34)$$

For the one-dimensional representation $\rho(A) = \det(A)^k$ we recover the usual scalar valued modular forms. Cuspidality is defined by properties of Fourier coefficients, as in the scalar case. The space of all cusp forms of type ρ and degree n shall be denoted by $S_\rho(\Gamma_n)$.

Assuming ρ is irreducible, we now associate a function Φ on the adele group to a modular form f, as follows. Let $m \in \mathbb{R}$ be the number such that $\rho(s) = s^{2m} \operatorname{id}_W$, for each scalar matrix $s = \operatorname{diag}(s, \ldots, s) \in \mathrm{GL}(n, \mathbb{C})$, $s > 0$ (if $\rho = \det^k$, then $m = nk/2$). Then we define

$$\tilde{\Phi}(g) = \mu(g_\infty)^m \rho(CI + D)^{-1} f(g\langle I \rangle), \quad (35)$$

with $g = g_\mathbb{Q} g_\infty k_0 \in G(\mathbb{A})$ as in (22) and $g_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

This function is well-defined in view of (34). The factor $\mu(g_\infty)^m$ ensures that $\tilde{\Phi}$ descends to a function on $\mathrm{PGSp}(2n, \mathbb{A})$. We further have $\tilde{\Phi}(\gamma g k_0) = \tilde{\Phi}(g)$ for all $\gamma \in G(\mathbb{Q})$ and $k_0 \in K_0$, and

$$\tilde{\Phi}(g k_\infty) = \rho(k_\infty)^{-1} \tilde{\Phi}(g) \quad \text{for all } k_\infty \in K_\infty \simeq U(n), \tag{36}$$

where the identification $K_\infty \simeq U(n)$ is given by $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A - iB$. Just as in Lemma 7 one proves that the holomorphy of f is equivalent to $\mathfrak{p}_{\mathbb{C}}^-.\tilde{\Phi} = 0$.

On the group one would like to have scalar valued functions lying in the usual L^2-space, instead of the vector valued $\tilde{\Phi}$. Therefore let L be any non-zero linear form on W, the space of ρ, and define

$$\Phi(g) = L(\tilde{\Phi}(g)), \quad g \in G(\mathbb{A}).$$

We will eventually consider the space generated by all right translates of Φ, and therefore the choice of L is irrelevant.

The Petersson scalar product on $S_\rho(\Gamma_n)$ is given by

$$\langle f_1, f_2 \rangle = \int_{\Gamma_n \backslash \mathbb{H}_n} \langle \rho(Y^{1/2}) f_1(Z), \rho(Y^{1/2}) f_2(Z) \rangle d^* Z.$$

Here the inner product $\langle \, , \rangle$ is a $U(n)$-invariant hermitian form on W (unique up to scalars), and $Y^{1/2}$ denotes the unique positive definite square root of the positive definite matrix Y. Note that this is a generalization of the Petersson scalar product in (25).

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ and $Z = g\langle I \rangle$ we have $Y = (C\,{}^tC + D\,{}^tD)^{-1} = (M\,{}^t\bar{M})^{-1}$, where $M = CI + D$. Write $M = pu$ with a symmetric positive p and a unitary u. Then $p^{-1} = Y^{1/2}$, and there exists a constant $c > 0$ such that

$$\|\rho(Y^{1/2}) f(Z)\|^2 = \|\rho(M)^{-1} f(Z)\|^2 = \|\tilde{\Phi}(g)\|^2$$
$$= c \int_{K_\infty} \left| L(\rho(k)^{-1} \tilde{\Phi}(g)) \right|^2 dk = c \int_{K_\infty} |\Phi(gk)|^2 dk.$$

Now integration yields

$$\int_{\Gamma_n \backslash \mathbb{H}_n} \|\rho(Y^{1/2}) f(Z)\|^2 d^* Z = c \int_{Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}) / K} \int_{K_\infty} |\Phi(gk)|^2 dk \, dg$$

$$= c \int_{Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A})} |\Phi(g)|^2 dk \, dg.$$

This shows that, after suitable normalization of measures, the map $f \mapsto \Phi$ is a norm-preserving map of Hilbert spaces from $S_\rho(\Gamma_n)$ to

$$L^2(Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A})).$$

The image is contained in the space of cuspidal functions, cf. Lemma 5.

Assuming that f is a Hecke eigenform, one can now associate an automorphic representation π_f of $\mathrm{PGSp}(2n, \mathbb{A})$ with f as in the previous section. The space of π_f will be an irreducible subspace of the $G(\mathbb{A})$-space generated by Φ. For each place p one can characterize the local components π_p of π_f as in Theorem 2. We shall indicate what is different now at the archimedean place $p = \infty$.

It follows from (36) that the $U(n)$-module τ generated by $U(n)$-translates of Φ is isomorphic to the *contragredient* of $\rho\big|_{U(n)}$ (note that $\rho\big|_{U(n)}$ remains irreducible by the unitary trick). Furthermore, Φ has the lowest weight property because f is holomorphic. Consequently, the local component π_∞ is the *lowest weight representation of $G(\mathbb{R})$ with minimal K_∞-type τ* (more precisely, since $G(\mathbb{R})$ is disconnected, π_∞ combines a lowest and a highest weight representation).

The irreducible representations of $K_\infty \simeq U(n)$ are parameterized by elements in the weight lattice $\Lambda = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_n$, modulo the action of the real Weyl group W_r. Since $W_r \simeq S_n$ acts by permuting the coefficients of the e_i, the irreducible representations of K_∞ are in 1-1 correspondence with the weights

$$\lambda = r_1 e_1 + \ldots + r_n e_n, \quad r_i \in \mathbb{Z}, \; r_1 \geq \ldots \geq r_n.$$

We denote the finite-dimensional irreducible representation corresponding to this λ by τ_λ or $\tau_{r_1, \ldots, r_n}$. The correspondence is that τ_λ contains a vector v_0 that is annihilated by the compact positive root vectors, and such that $T_j v_0 = r_j v_0$ (see (19)). Thus v_0 is a *highest weight vector*.

The irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$ are also parameterized by integers $r_1 \geq \ldots \geq r_n$. If $\rho_{r_1, \ldots, r_n}$ denotes the corresponding representation, parameterized as in [Fr2] appendix to I.6, then one checks that $\rho_{r_1, \ldots, r_n}\big|_{U(n)} = \tau_{-r_n, \ldots, -r_1}$, assuming the identification $K_\infty \simeq U(n)$ fixed above. In other words, *the contragredient of $\rho_{r_1, \ldots, r_n}\big|_{U(n)}$ is $\tau_{r_1, \ldots, r_n}$.*

It follows that if f is a cusp form of type $\rho_{r_1, \ldots, r_n}$ in the sense of [Fr2], then π_∞ is the representation of $G(\mathbb{R})$ with minimal K_∞-type $\tau_{r_1, \ldots, r_n}$. In the scalar valued case we have $r_1 = \ldots = r_n = k$. If $r_n > n$, then π_∞ is a holomorphic discrete series representation with Harish-Chandra parameter $(r_1 - 1)e_1 + \ldots + (r_n - n)e_n$, cf. Sect. 3.5.

4.6. L-functions

Let f be a cuspidal Siegel eigenform, and let π_f be the automorphic representation of $\mathrm{PGSp}(2n, \mathbb{A})$ associated with f as in the previous section. The dual group of $\mathrm{PGSp}(2n)$ is $\mathrm{Spin}(2n + 1)$. In Sect. 3.4 we considered two finite-dimensional representations of $\mathrm{Spin}(2n + 1, \mathbb{C})$, the "projection representation" ϱ_1, and the 2^n-dimensional spin representation ϱ_2.

Corresponding to these two representations of the connected component of the L-group, we have two Langlands L-functions associated with π_f. According to our results in Sect. 3.4, they are given by

$$L(s, \pi_f, \varrho_1) = \prod_p \left((1 - p^{-s}) \prod_{i=1}^n (1 - b_i p^{-s})(1 - b_i^{-1} p^{-s}) \right)^{-1},$$

and

$$L(s, \pi_f, \varrho_2) = \prod_p \left(\prod_{k=0}^{n} \prod_{1 \leq i_1 < \ldots < i_k \leq n} (1 - b_0 b_{i_1} \ldots b_{i_k} p^{-s}) \right)^{-1},$$

where $b_0, b_1, \ldots, b_n$ are the Satake parameters of the local component $\pi_{f,p}$. But by Lemma 10, these Satake parameters are essentially the same as the classical Satake parameters of the eigenform f. In view of the definitions (1) and (2) of the classical L-functions, we obtain the following result.

Theorem 3. *Let π_f be the cuspidal automorphic representation of the group* $\mathrm{PGSp}(2n, \mathbb{A})$ *associated with the cuspidal Siegel eigenform f of degree n and weight k. Then for the standard L-function of f we have the identity*

$$L_1(s, f) = L(s, \pi_f, \varrho_1),$$

where ϱ_1 is the projection representation (14) of $\mathrm{Spin}(2n + 1, \mathbb{C})$, *and for the spin L-function of f we have*

$$L_2(s', f) = L(s, \pi_f, \varrho_2), \quad s' = s - n(n - 1)/4 + nk/2,$$

where ϱ_2 is the 2^n-dimensional spin representation of $\mathrm{Spin}(2n + 1, \mathbb{C})$.

We also remark that in the case $n = 2$ there is an isomorphism $\mathrm{Spin}(5, \mathbb{C}) \simeq \mathrm{Sp}(4, \mathbb{C})$, and the spin representation becomes the standard representation of $\mathrm{Sp}(4, \mathbb{C})$.

This result allows us to carry over general group theoretic results to classical L-functions. As an example, we prove the meromorphic continuation of the spin L-functions for cuspforms in $S_k(\Gamma_3)$ using Langlands's theory of Euler products.

Let $\mathbf{M}$ be a maximal standard Levi subgroup in a connected reductive Chevalley group $\mathbf{G}$. Consider $\mathbf{G}$ as a group over $\mathbb{Q}$. (However, this theory is available in more generality, cf. [Sh], for example.) Let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic in $\mathbf{G}$. Denote by $\hat{P} = \hat{M}\hat{N}$ the parabolic in $\hat{G}$, the complex dual of $\mathbf{G}$, corresponding to $\mathbf{P}$ (cf. [Bo]). Let r denote the adjoint action of $\hat{M}$ on the Lie algebra of $\hat{N}$ and write $r = \oplus_{i=1}^{m} r_i$, with r_i's the irreducible constituents of r.

Let $\pi = \otimes_p \pi_p$ be a cuspform on $M = \mathbf{M}(\mathbb{A})$. Let

$$f = \otimes_p f_p \in I(s, \pi) = \otimes_p I(s, \pi_p),$$

with the same notation as in Sect. 2 of [Sh]. Let S be a finite set of places for which every π_p with $\pi \notin S$ is unramified. The theory of Euler products developed by Langlands (cf. [La]) then implies that

$$M(s, \pi)f = (\prod_{p \in S} A(s, \pi_p) f_p) \otimes (\otimes_{p \notin S} \tilde{f}_p) \cdot \prod_{i=1}^{m} \frac{L_S(is, \pi, \tilde{r}_i)}{L_S(1 + is, \pi, \tilde{r}_i)} \qquad (37)$$

has a meromorphic continuation to all of $\mathbb{C}$. For our purposes we may assume $S = \{\infty\}$. The intertwining operator $A(s, \pi_\infty)$ is non-vanishing and has meromorphic continuation to all of $\mathbb{C}$ (cf. Section 17 of [KS]). This is true even if S contains finite places (cf. [Sh]).

Now assume $m = 1$. From the above discussion and (37), it follows that

$$F(s) = \frac{L(s, \pi, \tilde{r}_i)}{L(s + 1, \pi, \tilde{r}_i)}$$
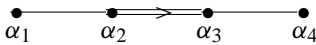
is meromorphic. Writing this as

$$L(s, \pi, \tilde{r}_i) = F(s)L(s + 1, \pi, \tilde{r}_i)$$

and noting that $L(s, \pi, \tilde{r}_i)$ is analytic if $\mathrm{Re}(s)$ is sufficiently large, one concludes by induction that $L(s, \pi, \tilde{r}_i)$ has a meromorphic continuation to all of $\mathbb{C}$. In particular, we get the following.

Theorem 4. *The L-function $L(s, \pi_f, \varrho_1)$ of Theorem 3 has a meromorphic continuation to all of $\mathbb{C}$.*

Proof. Consider $\mathbf{M} = \mathrm{GL}(1) \times \mathrm{Sp}(6)$ as a standard Levi subgroup in $\mathbf{G} = \mathrm{Sp}(8)$. The dual of $\mathbf{M}$ is $\mathbb{C}^\times \times \mathrm{SO}(7, \mathbb{C})$ with $m = 1$ and $r = r_1$ is seven dimensional (cf. Case (C_n) of Sect. 4 of [Sh]). Now, if we put the representation $\sigma = \mathbf{1} \otimes \pi_f$ on $M = \mathbf{M}(\mathbb{A})$, then, since r_1 is self-contragredient, $L(s, \pi_f, \varrho_1) = L(s, \sigma, r_1)$, which has meromorphic continuation to all of $\mathbb{C}$ by our previous argument. $\quad\square$

To prove a similar result for the other L-function consider the following. Let $\mathbf{G}$ be a Chevalley group of type F_4. This is a split (as well as adjoint) simply connected simple algebraic group with Dynkin diagram

Consider the standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$, where $\mathbf{M}$ is the standard Levi subgroup corresponding to $\{\alpha_2, \alpha_3, \alpha_4\}$. One can then show that $\mathbf{M} \simeq \mathrm{GSp}(6)$ (cf. [As]). Note that the complex dual of $\mathbf{G}$ is again of type F_4 and contains the dual parabolic $\hat{P} = \hat{M}\hat{N}$ (see [Bo]). In fact,

$$\hat{M} = \mathrm{GSpin}(7, \mathbb{C})$$

is the dual of $\mathbf{M}$. This is the case (xxii) of Sect. 6 of [La] with $m = 2$ and we have $r = r_1 \oplus r_2$ with r_1 an eight dimensional and r_2 a seven dimensional representation of $\hat{M}$. Indeed, if r is restricted to $\mathrm{Spin}(7, \mathbb{C}) \subset \hat{M}$, then r_1 is the eight dimensional spin representation while r_2 is what we called the projection representation in Sect. 3.4. Note that these representations are self-dual, and therefore we do not have to care about contragredients in the following.

Put a cuspform π on $M = \mathbf{M}(\mathbb{A})$. Now (37) and its subsequent argument again imply that

$$G(s) = \frac{L(s, \pi, r_1)}{L(s + 1, \pi, r_1)} \cdot \frac{L(2s, \pi, r_2)}{L(2s + 1, \pi, r_2)}$$

has a meromorphic continuation to all of $\mathbb{C}$. Write this as

$$L(s, \pi, r_1) = G(s) \cdot \frac{L(2s + 1, \pi, r_2)}{L(2s, \pi, r_2)} \cdot L(s + 1, \pi, r_1).$$

Now if we let π be the representation on $M = \mathrm{GSp}(6, \mathbb{A})$ lifted up from π_f on $\mathrm{PGSp}(6, \mathbb{A})$, then Theorem 4 and an induction similar to its proof imply the following.

Theorem 5. *The L-function $L(s, \pi_f, \varrho_2)$ of Theorem 3 has a meromorphic continuation to all of $\mathbb{C}$ if $n = 3$.*

Hence, in view of Theorem 3, we get

Corollary 1. *Let $f \in S_k(\Gamma_3)$ be a cuspform of degree 3, and let $L_2(s, f)$ be the spin L-function attached to f. Then $L_2(s, f)$ has meromorphic continuation to all of $\mathbb{C}$.*

We remark that the same proof works for vector valued modular forms.

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