Last time:

- $X = X_n \supseteq \cdots \supseteq X_0$ stratified pseudo-manifold

- $\bar{p} : \{2, \dots, n\} \to \mathbb{Z}$ perversity

Defn: $\xi \in C_i^T(X)$ is $(\bar{p}, i)$-allowable if

$$\dim(|\xi| \cap X_{n-k}) \le \underbrace{i-k}_{(\text{proper})} + p(k)$$

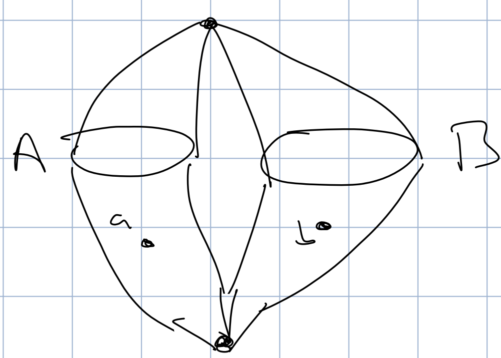
$IC_i^{\bar{p}}(X) = \{ \xi \in C_i(X) \mid$

① ξ is $(\bar{p}, i)$-allowable

② $\partial\xi$ is $(\bar{p}, i-1)$-allowable. $\}$

$\to IH_i^{\bar{p}}(X)$ intersection homology groups.

Ex) $X = \text{susp}(S^1 \sqcup S^1)$



$H_0 = \mathbb{Q}$
$H_1 = \mathbb{Q}$
$H_2 = \mathbb{Q} \oplus \mathbb{Q}$.

$X_2 = X$, $X_1 = X_0 = 2$ pts.

$\bar{p} = (0)$

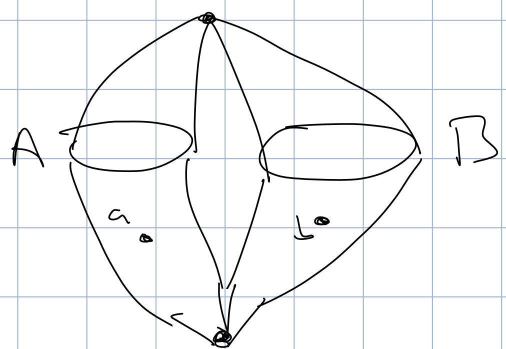
0-chains: miss X_0
1-chains: miss X_0
2-chains: nbhd of boundary misses X_0

$IH_0^{(0)}(X) = \mathbb{Q} \oplus \mathbb{Q}$

$IH_1^{(0)}(X) = 0$ ⇝

$IH_2^{(0)}(X) = \mathbb{Q} \oplus \mathbb{Q}$.

Ex) $X = \operatorname{susp}(S^1 \cup S^1)$



$\bar{p} = (1)$

0 - chains miss X_0
1 - chains ok if ∂ miss X_0
2 - chains ok.

$IH_0^{(1)}(X) = \mathbb{Q}$

$IH_1^{(1)}(X) = \mathbb{Q}$

$IH_2^{(1)}(X) = \mathbb{Q} \oplus \mathbb{Q}.$

$\bar{p} = (-1)$ all cycles miss X_0

$IH_0^{(-1)}(X) = \mathbb{Q} \oplus \mathbb{Q}$

$IH_1^{(-1)}(X) = \mathbb{Q} \oplus \mathbb{Q}$

$IH_2^{(-1)}(X) = 0$

Ex) X manifold $\quad X \supseteq \emptyset \supseteq \cdots \supseteq \emptyset.$

$$IH^{\bar{p}}_i(X) = H_i(X) \text{ for any } \bar{p}.$$

Q: Does $IH^{\bar{p}}_i(X)$ depend on stratification?

<u>Yes</u>.

Ex) $X =$ 

$X_2 \supseteq \emptyset \supseteq \emptyset$

$X_2 \supseteq pt \supseteq pt$

$\bar{p} = (-1)$

$$H^{(-1)}_2(X) = H_2(X) = \mathbb{Z}$$

$$H^{(-1)}_2(X) = H_2(X \setminus pt) = 0.$$

Thm (Goresky-MacPherson)

If $p(2)=0$ and

$p(k) \le p(k+1) \le p(k)+1 \quad \forall k \ge n$,

then $IH^{\bar{p}}(X)$ is a top. invariant

(proof uses sheaves)

G-M perversity range:

$\bar{0}$ $\qquad\qquad\qquad\qquad$ $\bar{t}$

$(0,0,\ldots,0) \rightsquigarrow (0,1,2,3,4,\ldots,n-2)$

zero perversity $\qquad\qquad$ top perversity

$\bar{m}$

$(0,0,1,1,2,2,3,3,\ldots)$

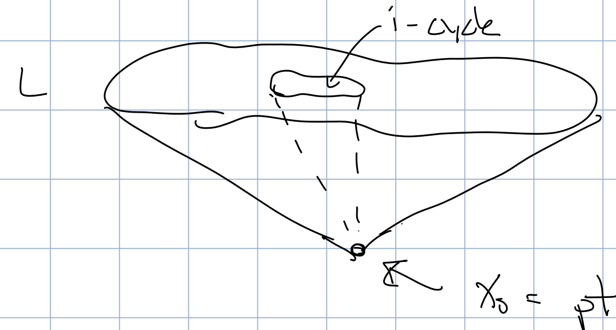
middle perversity.

Remark: there are other $\bar{p}$'s for which

$IH^{\bar{p}}$ is a top. invariant.

Remark: $IH^{\bar p}$ is not a homotopy invariant

Ex) $X = c(\underbrace{L}_{n-1}) = L\times[0,1) \big/ L\times\{0\}$



ξ is a boundary iff $i \geq n-1-p(n)$

$$\to \quad IH^{\bar p}_i(X) = \begin{cases} 0 & \text{if } i \geq n-1-p(n) \\ IH^{\bar p}_i(L) & \text{if } i < n-1-p(n) \end{cases}$$

Remark! $IH^{\bar p}$ is not functorial: $\phi : X \to Y$

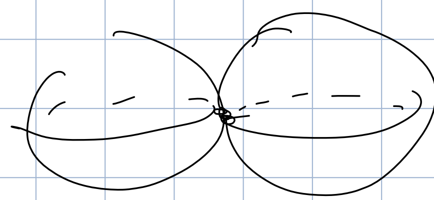
$\rightsquigarrow$ ϕ^* DNE

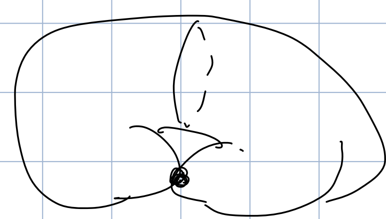
Normal spaces

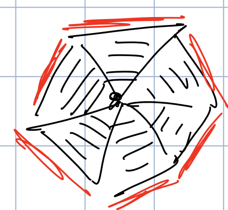
X is normal if $\forall x \in X$,

$\exists$ a nbd U s.t. $U \setminus \underbrace{\Sigma}_{\text{singular locus}}$ is connected

Ex: X manifold is normal.

Ex:  not normal.

Ex:  not normal.

Ex) 

normal: ← $(n-1)$-simplices "around" pts are connected.

Alg geom: local rings of X are integrally closed.

Top: $\forall x \in X$, $H_n(X, X-x) = \mathbb{Q}$.

Thm (G-M) If X is normal, then

① $IH_i^{\bar{t}}(X) \cong H_i(X)$

② $IH_i^{\bar{0}}(X) \cong H^{2n-i}(X)$

Thm: Let $\tilde{X}$ be the normalization of X

then $IH_i^{\bar{p}}(\tilde{X}) = IH_i^{\bar{p}}(X)$ for any

G-M perversity $\bar{p}$.

If X is an elliptic curve, then $\tilde{X}$ is nonsingular

$$\to IH_*^{\bar{p}}(X) = IH_*^{\bar{p}}(\tilde{X}) = H_*(\tilde{X}).$$

Intersection product + Poincaré Duality

$$\begin{array}{ccccc} IH_i^{\bar{p}}(X) & \times & IH_j^{\bar{q}}(X) & \to & IH_{i+j-n}^{\overline{p+q}}(X) \\ \cup & & \cup & & \\ C & & D & & \end{array}$$

1. $\dim C \cap X_{n-k} \leq i - k + p(k)$

2. $\dim D \cap X_{n-k} \leq j - k + q(k)$

Suppose $\overbrace{C \cap D}^{(i+j-n)\text{-chain}} \cap X_{n-k}$ proper

$\to$ $\dim \leq i-k+p(k)+j-k+q(k)-(n-k)$

$= (i+j-n) - k + (p(k)+q(k))$

$\Rightarrow$ $C \cap D$ is $(\bar{p}+\bar{q}, i+j-n)$-allowable.

<u>Defn:</u> C, D are dim. transverse $(C \pitchfork D)$ if

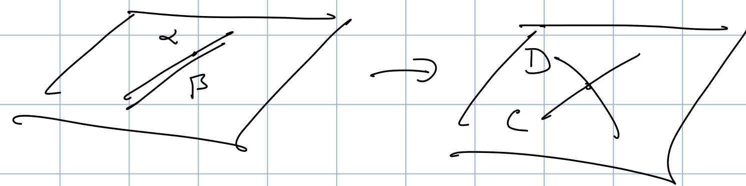
① $C \cap D$ is proper (dim $= i+j-n$.)

② $C \cap D$ is $(\bar{p}+\bar{q})$ allowable.

<u>Moving Lemma</u> (McCrory):

Given $\alpha \in IH^{\bar{p}}_i(X)$, $\beta \in IH^{\bar{q}}_j(X)$,

$\exists\, C, D$ s.t. $\alpha = [C]$, $\beta = [D]$ and $C \pitchfork D$.



Idea for moving lemma

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$$

① Each $X_i - X_{i-1}$ is smooth

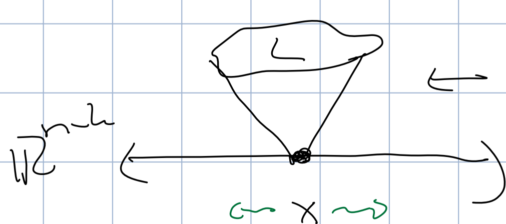
smooth locus

move $|\alpha| \cap |\beta| \rightsquigarrow C \pitchfork D$ in $X - X_{n-1}$

② Suppose $C \pitchfork D$ on $X - X_{n-k}$

For $x \in X_{n-k} - X_{n-k-1}$

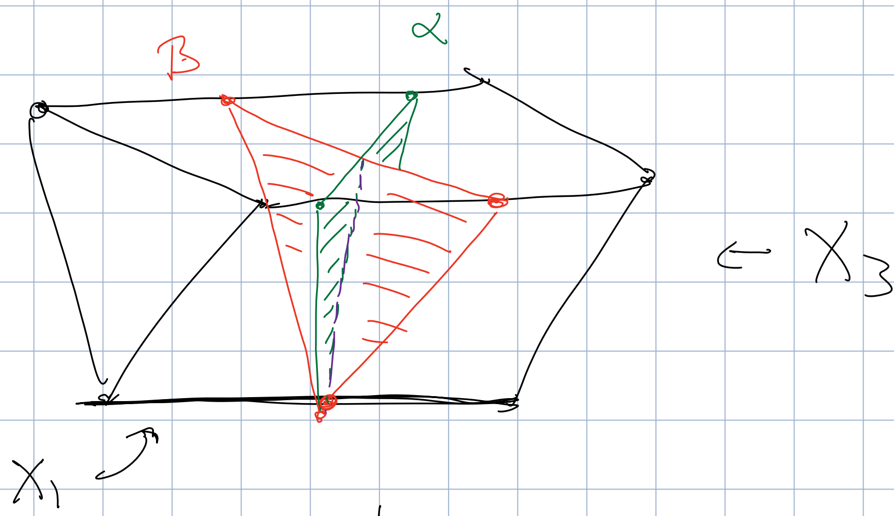
$$N_x = \mathbb{R}^{n-k} \times c(L)$$



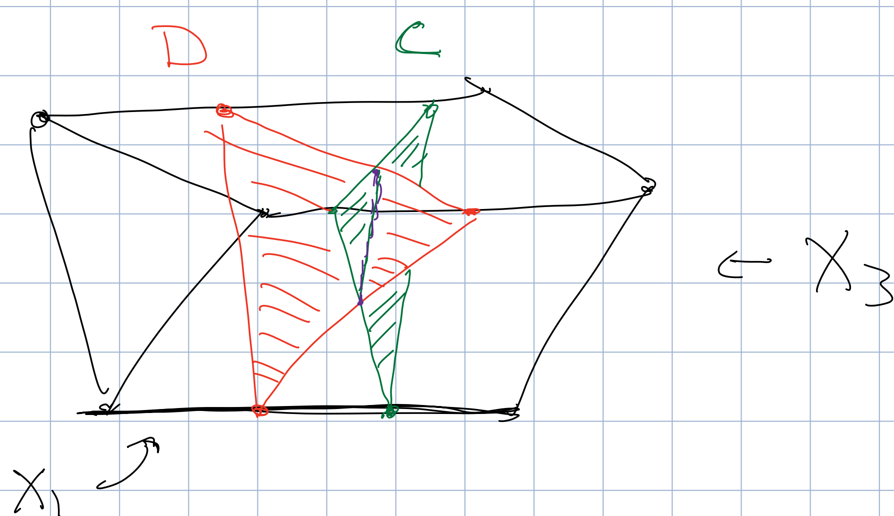
← normal to X^{n-k} in X.

can move here preserving "higher" dim transversality.

Picture from G-M:



move!



Poincaré Duality:

If $\bar{p} + \bar{q} = \bar{t}$ $\quad (p(k) + q(k) = k)$
and
$i + j = n$

Then $IH_i^{\bar{p}}(X) \times IH_{n-i}^{\bar{q}}(X) \to IH_0^{\bar{t}}(X) = \mathbb{Q}$
is non-degenerate.

Cor: $\dim H_i^{\bar{p}} = \dim H_{n-i}^{\bar{q}}$,

If $\bar{p} = \bar{0}$, $\bar{q} = \bar{t}$, and X normal, then

$$H_i(X) \times H^i(X) \to \mathbb{Q}$$

If $\bar{p} = \bar{q} = \bar{m}$, then $\bar{m} + \bar{m} = \bar{t}$, (complex)

$$IH_i^{\bar{m}}(X) \times IH_{n-i}^{\bar{m}}(X) \to \mathbb{Q}$$

Ex)  $\bar{m} = (0)$

H_0	$\mathbb{Q}$		IH_0	$\mathbb{Q} \oplus \mathbb{Q}$
H_1	0	$\to$	IH_1	0
H_2	$\mathbb{Q} \oplus \mathbb{Q}$		IH_2	$\mathbb{Q} \oplus \mathbb{Q}$

Next time: <u>Sheaves</u>