Last time:

① Sheaves on X

$$U \subseteq X \rightsquigarrow F(U)$$

② Čech cohomology:

$$\check{H}^*(X, F)$$

Thm: If $F = \mathbb{Q}_X$ constant sheaf

then $\check{H}^*(X, \mathbb{Q}_X) = H_*(X, \mathbb{Q})$

usual homology

Question: Is there a sheaf F on X s.t.

$$\check{H}^*(X, F) = IH_*^{\bar{m}}(X) \,?$$

Algebra of sheaves.

Sheaves: F, G sheaves on X.

① homomorphisms: $F \xrightarrow{\varphi} G$

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi(U)} & G(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\varphi(V)} & G(V) \end{array}$$

commutes with restriction

$\rightarrow$ $\varphi_x : F_x \rightarrow G_x$ hom on stalks

② $\ker\varphi(U) := \ker(\varphi(U))$ is a sheaf

③ $\operatorname{im}\varphi(U) := \operatorname{im}(\varphi(U))^{+}$ is a ~~pre~~sheaf

④ $F \subseteq G$ subsheaf if $F(U) \subseteq G(U)$
is a sheaf.

⑤ G/F quotient sheaf is $G/F(U) = G(U)/F(U)$ $+$

is ~~pre~~sheaf

⑥ Let $\phi : X \to Y$ continuous map.

$F \leadsto X$, $G \leadsto Y$ sheaves.

ⓐ $\phi_* F$ on Y is a sheaf

$$\phi_* F(U) = F(\phi^{-1}(U))$$

ⓑ $\phi^* G$ on X is a ~~pre~~sheaf

$$\phi^* G(U) = \left(\lim_{\substack{\phi(U) \subseteq V}} G(V)\right)^{\#?}$$

V open in Y

Adjoint:

$$\mathrm{Hom}_{sh(X)}(\phi^* G, F) \simeq \mathrm{Hom}_{sh(Y)}(G, \phi_* F)$$

Complexes of sheaves:

A complex of sheaves on X is a sequence $F_\bullet = (F_i)_{i \in \mathbb{Z}}$

$$\cdots \to F_{i-1} \to F_i \xrightarrow{\partial_i} F_{i+1} \to \cdots$$

s.t. $\partial_{i+1} \circ \partial_i = 0 \quad \forall i$

Hyper cohomology

Given $F_\bullet$ on X.

$\Rightarrow C^p(X, F_q)$ Čech cochains

Two boundary operators:

$$\begin{array}{ccc} C^p(X, F_q) & \xrightarrow{\partial_2} & C^p(X, F_{q+1}) \\ \partial_1 \downarrow & & \\ C^{p+1}(X, F_q) & & \end{array}$$

Let $K^n := \bigoplus_{p+q=n} C^p(X, \mathcal{F}_q)$

and

$\tilde{\partial} : K^n \to K^{n+1}$ by $\tilde{\partial} = \partial_1 + (-1)^p \partial_2$

Claim $\tilde{\partial}^2 = 0$ $(\partial_1 \partial_2 = \partial_2 \partial_1)$

Def: $\mathbb{H}^n(X, \mathcal{F}_\bullet) := \ker \tilde{\partial}_n / \operatorname{im} \tilde{\partial}_{n-1}$

Prop: $\mathcal{F}_\bullet = \Omega_X[n]$

$$\cdots 0 \to 0 \to \underset{\substack{\uparrow \\ -n}}{\Omega_X} \to 0 \to 0 \cdots$$

Then:

$$\mathbb{H}^*(X, \Omega_X[n]) = \check{H}^*(X, \Omega_X)$$

Goal: Find a sheaf complex "$\underline{\underline{IC}}_X$" s.t.

$$\mathbb{H}^*(X, \underline{\underline{IC}}_X) = IH^*_{\bar{p}}(X).$$

Borel-Moore constructions

Recall $C_i(X)$ i-chains of X

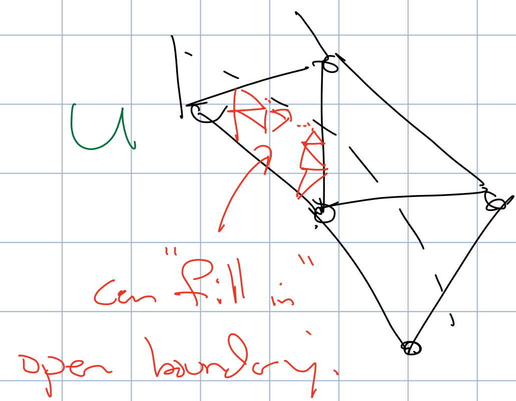
$U \subseteq X \rightsquigarrow C_i(U)$ i-chains on X.

Problem: $U \subseteq X$ gives $C_i(U) \hookrightarrow C_i(X)$

wrong way for restriction

B-M chains: $\sum_{\sigma} c_\sigma \cdot \sigma_i$

possibly infinite sums (but locally finite)



$$C_i^{BM}(U)$$

If $U \subseteq V$, we get restrictions

$$C_i^{BM}(V) \to C_i^{BM}(U).$$

Get a sheaf on X

$$\underline{\underline{C}}_X^{-i} : U \rightsquigarrow C_i^{BM}(U)$$

Subsheaf:

$$\underline{I}\underline{\underline{C}}_X^{-i} : U \rightsquigarrow IC_i^{BM}(U)$$

middle perversity

Boundary maps: $C_i(U) \to C_{i-1}(U)$

$\to$ Complexes: $\underline{\underline{C}}_X^{\bullet}$, $\underline{I}\underline{\underline{C}}_X^{\bullet}$

Thm: ① $\mathbb{H}^k(X, \underline{\mathbb{C}}_X^\bullet) = H_k^{BM}(X)$

② $\mathbb{H}^k(X, \underline{IC}_X^\bullet) = IH_k^{BM}(X)$

If X is compact, then

$$H_k^{BM}(X) = H_k(X)$$

Note: $\mathbb{H}^k(X, \underline{\mathbb{C}}_X^\bullet) = \mathbb{H}^k(X, \underline{\mathbb{Q}}_X)$

constant sheaf

Question: Is there something like $\underline{\mathbb{Q}}_X$ for $\underline{IC}_X^\bullet$?

Quasi-isomorphisms and cohomology sheaves

Given $F_\bullet$ sheaf complex on X,

i^{th}-cohomology sheaf:

$$H^i(F_\bullet) = {}^{\ker \partial_i}/_{\operatorname{im} \partial_{i+1}}$$

Note: This is a family of sheaves!

$$U \subseteq X \rightsquigarrow H^j(F_\bullet)(U).$$

Defn: $\phi : F_\bullet \to G_\bullet$ is a quasi-isomorphism of sheaf complexes if

$$H^i(F_\bullet) \xrightarrow[\sim]{\phi} H^i(G_\bullet) \quad \forall i$$

isom of cohom. sheaves

Thm: If $F_\bullet$, $G_\bullet$ are quasi-isom

then the hyper cohomology:

$$\mathbb{H}^*(X, F_\bullet) \cong \mathbb{H}^*(X, G_\bullet).$$

Thm: If X mfd.

$$\begin{tikzcd}
\underline{\Omega}_X[n]: & 0 \arrow[r] \arrow[d] & \underline{\Omega}_X^{-n} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\underline{\underline{C}}_X^\bullet: & 0 \arrow[r] & \underline{\underline{C}}_X^{-n} \arrow[r] & \underline{\underline{C}}_X^{-n+1} \arrow[r] & \underline{\underline{C}}_X^{-n+2} \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism

exact sequence!

Deligne's construction of $\underline{IC}^\bullet_X$

(quasi-iso)

① <u>Right derived functor</u>:

$X \xrightarrow{\phi} Y$, $F^\bullet$ sheaf complex on X.

$\to \phi_*(F^\bullet)$ complex on Y.

<u>Problem</u>: ϕ^*, ϕ_* are not adjoint on complexes.

$\to$ replace $\phi_*(F^\bullet)$ with "$R\phi_*(F^\bullet)$"

If $F^\bullet \to I^\bullet$ injective resolution on X.

$$R\phi_*(F^\bullet) := \phi_*(I^\bullet)$$

② <u>Truncation</u> $\quad F^\bullet$, $m \in \mathbb{Z}$.

$$F^\bullet: \to F_{m-2} \to F_{m-1} \xrightarrow{d_{m-1}} F_m \to F_{m+1} \to \cdots$$

$$\downarrow$$

$\tau_{\leq m}(F^\bullet)$:

$$\to F_{m-2} \to F_{m-1} \to \ker d_{m-1} \to 0 \to \cdots$$

Stratification:

$$X = X_n \supseteq X_{n-2} \supseteq \cdots \supseteq X_0$$

$\to U_k = X - X_{n-k}$ open sets

$$U_2 \xhookrightarrow{i_2^\circ} U_4 \xhookrightarrow{i_4} \cdots \hookrightarrow U_n \xhookrightarrow{i_n} X$$

smooth locus

Let $\underline{\mathbb{Q}}_{U_2}[n]$ constant sheaf on U_2

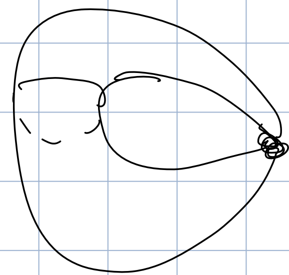
Thm $IC_X^\bullet$ is quasi-isom to

$$\tau_{\le \frac{n-2}{2}} R i_{n*} \cdots \tau_{\le 1} R i_{4*} \tau_{\le 0} R i^\circ_{2*} \underline{\mathbb{Q}}_{U_2}[n]$$

perversities

$\mathbb{R}^k \times C(L)$

Proof use stalks + distinguished nbhds on strata.

Ex) $X =$  A $\qquad U = X \setminus \{A\}$

$$U \overset{i}{\hookrightarrow} X$$

$$\mathbb{H}^k\left(\tau_{\leq 0} R i_* \underline{\mathbb{Q}}_U\right) = \begin{cases} \mathbb{Q} & \text{if } k = 0, 2 \\ 0 & \text{if } k = 1 \end{cases}$$

$$\tau_{\leq 0} R i_* \mathbb{Q}_U \Big|_A \simeq \mathbb{Q}^2$$

stalk