The Orbit Method for Reductive Lie Groups

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April 2022

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A (highly abridged) timeline:

- Connected compact groups (Weyl, 1920s).
- $SL_2(\mathbb{R})$ (Bargmann, 1947).
- $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $GL_n(\mathbb{H})$ (Vogan, 1986).
- Complex classical groups (Barbasch, 1989).
- Some other low-rank groups.
- Atlas (ongoing).

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- A *real co-adjoint cover* is a homogeneous G-space $\widetilde{\mathbb{O}}$ equipped with a finite G-equivariant map $\widetilde{\mathbb{O}} \to \mathbb{O}$. Write $\mathrm{Cov}^{i\mathbb{R}}(G)$ for the set of isomorphism classes of real co-adjoint covers.

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- Write $\mathrm{Orb}_n^{i\mathbb{R}}(G)$ (resp. $\mathrm{Cov}_n^{i\mathbb{R}}(G)$) for the nilpotent orbits (resp. covers).

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Conjecture (Vogan)

There is a set $\mathrm{Cov}^{i\mathbb{R}}_{int}(G)$ of *integral co-adjoint covers*

$$\mathrm{Cov}^{i\mathbb{R}}_{n}(G) \subset \mathrm{Cov}^{i\mathbb{R}}_{int}(G) \subset \mathrm{Cov}^{i\mathbb{R}}(G)$$

For each $\widetilde{\mathbb{O}} \in \mathrm{Cov}^{i\mathbb{R}}_{int}(G)$, there is an associated finite set

$$\Pi_{\widetilde{\mathbb{O}}}(G) \subset \widehat{G}_u$$

called a *Kirillov packet*. The union should exhaust most of $\widehat{G}_u$.

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Then

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We will always work on the algebraic side, i.e. we will define

$$\Pi_{\widetilde{\mathbb{O}}}(G) \subset \{\text{irreducible } G\text{-equivariant Harish-Chandra } U(\mathfrak{g})\text{-bimodules}\}$$

Outline

1. Parameterize $\mathrm{Cov}^{i\mathbb{R}}(G)$ and define $\mathrm{Cov}^{i\mathbb{R}}_{int}(G)$.
2. Construct Kirillov packets $\Pi_{\widetilde{\mathbb{O}}}(G)$ for nilpotent covers (unipotent representations).
3. Construct Kirillov packets $\Pi_{\widetilde{\mathbb{O}}}(G)$ for arbitrary covers.
4. Explain relation with Arthur packets.
5. Sketch generalization for arbitrary G.

Real to complex

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- Write $\mathrm{Orb}_n(G)$ (resp. $\mathrm{Cov}_n(G)$) for the nilpotent orbits (resp. covers).

Real to complex

Trivial but important lemma

If $\mu \in \mathfrak{g}^*$, define $\iota(\mu) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, i\mathbb{R})$ by

$$(\iota(\mu))(X) = \mathrm{Im}(\mu(X)).$$

Then $\mu \mapsto \iota(\mu)$ defines a G-equivariant isomorphism of real vector spaces

$$\iota : \mathfrak{g}^* \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, i\mathbb{R}).$$

The isomorphism ι induces a bijection (also denoted by ι)

$$\iota : \mathrm{Orb}(G) \xrightarrow{\sim} \mathrm{Orb}^{i\mathbb{R}}(G)$$

which lifts to a bijection (still denoted by ι)

$$\iota : \mathrm{Cov}(G) \xrightarrow{\sim} \mathrm{Cov}^{i\mathbb{R}}(G)$$

Birational induction

Definition

A *birational induction datum* is a triple

$$(L, \widetilde{\mathbb{O}}_L, \mu)$$

consisting of

- a Levi subgroup $L \subset G$,
- a complex nilpotent cover $\widetilde{\mathbb{O}}_L \in \mathrm{Cov}_n(L)$, and
- an element $\mu \in \mathfrak{z}(\mathfrak{l})^*$.

The group G acts by conjugation on the set of birational induction data. Write $\Omega(G)$ for the set of G-conjugacy classes.

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- Choose a parabolic $P = LN \subset G$. Consider the twisted generalized Springer map

$$\rho : G \times_P (\mu + \overline{\mathbb{O}}_L + \mathfrak{p}^\perp) \to \mathfrak{g}^*$$

Image of ρ is closure of co-adjoint orbit $\mathrm{Ind}(L, \mathbb{O}_L, \mu) \in \mathrm{Orb}(G)$.

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- Form $\widetilde{X}_L = \mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}_L])$. Consider

$$\widetilde{\rho} : G \times_P (\{\mu\} \times \widetilde{X}_L \times \mathfrak{p}^\perp) \to G \times_P (\mu + \overline{\mathbb{O}}_L + \mathfrak{p}^\perp) \to \mathfrak{g}^*$$

  Image of $\widetilde{\rho}$ is closure of $\mathrm{Ind}(L, \mathbb{O}_L, \mu)$ and preimage is cover $\mathrm{Bind}(L, \widetilde{\mathbb{O}}_L, \mu) \in \mathrm{Cov}(G)$.

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Construction in previous slide defines a map

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$$\mathrm{Bind}(L, \widetilde{\mathbb{O}}_L, \mu) = \widetilde{\mathbb{O}} \implies L = G.$$

Write $\Omega_m(G)$ for the set of birational induction data $(L, \widetilde{\mathbb{O}}_L, \mu)$ such that $\widetilde{\mathbb{O}}_L$ is birationally rigid.

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Proposition (Losev-MB-Matvieievskyi)

There is a bijection

$$\mathrm{Bind} : \Omega_m(G) \xrightarrow{\sim} \mathrm{Cov}(G)$$

Example

Let $G = \mathrm{SL}(2,\mathbb{C})$. Then

$$\Omega_m(G) = \{(T, \{0\}, \mu) \mid \mu \in \mathfrak{t}^*\} \cup \{(G, \{0\}, 0), (G, \widetilde{\mathbb{O}}_{reg}, 0)\}$$

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We have

- $\mathrm{Bind}(T, \{0\}, \mu) = G\mu$ (for $\mu \neq 0$)
- $\mathrm{Bind}(T, \{0\}, 0) = \mathbb{O}_{reg}$
- $\mathrm{Bind}(G, \{0\}, 0) = \{0\}$
- $\mathrm{Bind}(G, \widetilde{\mathbb{O}}_{reg}, 0) = \widetilde{\mathbb{O}}_{reg}$

Integral covers

Applying our trivial lemma to $\mathfrak{z}(\mathfrak{l})$, we get a bijection

$$\iota : \mathfrak{z}(\mathfrak{l})^* \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(\mathfrak{z}(\mathfrak{l}), i\mathbb{R})$$

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Image is the set

$$\{\mu \in \mathfrak{z}(\mathfrak{l})^* \mid \frac{1}{2}(\iota(\mu) + \overline{\iota(\mu)}) \in X^*(L)\}$$

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Definition

- A birational induction datum $(L, \widetilde{\mathbb{O}}_L, \mu) \in \Omega(G)$ is *integral* if $\mu \in \widehat{L}_{1,u}$. Write $\Omega_{m,int}(G)$ for the set of G-conjugacy classes of integral minimal birational induction data.

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- A real co-adjoint cover is *integral* if it lies in the image of $\mathrm{Cov}_{int}(G)$. Write $\mathrm{Cov}_{int}^{i\mathbb{R}}(G)$ for the set of integral real co-adjoint covers.

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- A real co-adjoint cover is *integral* if it lies in the image of $\mathrm{Cov}_{int}(G)$. Write $\mathrm{Cov}_{int}^{i\mathbb{R}}(G)$ for the set of integral real co-adjoint covers.

Goal: attach a finite Kirillov packet $\Pi_{\widetilde{\mathbb{O}}}(G)$ to each real integral co-adjoint cover $\widetilde{\mathbb{O}} \in \mathrm{Cov}_{int}^{i\mathbb{R}}(G)$.

Let $\widetilde{\mathbb{O}} \in \mathrm{Cov}_n(G)$ and let $A = \mathbb{C}[\widetilde{\mathbb{O}}]$.

Filtered quantizations of nilpotent covers

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Let $X = \mathrm{Spec}(A)$. By (iii), X is a normal affine irreducible variety.

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(iii) A is finitely-generated.

Let $X = \mathrm{Spec}(A)$. By (iii), X is a normal affine irreducible variety. Also

(iv) X has symplectic singularities in the sense of (Beauville, 1999).

Filtered quantizations of nilpotent covers

A *filtered quantization* of A is a pair $(\mathcal{A}, \theta)$ consisting of a filtered algebra $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ such that

$$[\mathcal{A}_m, \mathcal{A}_n] \subseteq \mathcal{A}_{m+n-2},$$

and an isomorphism of graded Poisson algebras

$$\theta : A \xrightarrow{\sim} \mathrm{gr}(\mathcal{A}).$$

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Thanks to (iv), one can classify (isomorphism classes of) filtered quantizations of A.

Theorem (Losev, 2016)

There is a vector space $\mathfrak{h}_X$ and a finite reflection group $W_X \curvearrowright \mathfrak{h}_X$ such that

$$\mathfrak{h}_X / W_X \xrightarrow{\sim} \{\text{filtered quants of } A\}, \qquad \lambda \mapsto \mathcal{A}_\lambda.$$

Example

Let $P \subset G$ be a parabolic subgroup and let

$$\widetilde{\mathbb{O}} = \text{open } G\text{-orbit on } T^*(G/P)$$

Then

$$\mathfrak{h}_X = (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^* = \text{chars of } \mathfrak{p}.$$

Each $\lambda \in \mathfrak{h}_X$ determines a TDO $\mathcal{D}_{G/P}^{\lambda+\rho(\mathfrak{u})}$ on G/P. Let

$$\mathcal{A}_\lambda = \Gamma(G/P, \mathcal{D}_{G/P}^{\lambda+\rho(\mathfrak{u})})$$

Then $\mathcal{A}_\lambda$ is a filtered quantization of $\mathbb{C}[\widetilde{\mathbb{O}}]$.
Special case: $P = B$.

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$$\varphi : S(\mathfrak{g}) \to A$$

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- There is a G-equivariant co-moment map

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Both structures lift (uniquely) to $\mathcal{A}_0$, i.e.

- G on A by filtered algebra automorphisms.
- There is a G-equivariant co-moment map

$$\Phi : U(\mathfrak{g}) \to \mathcal{A}_0.$$

Unipotent ideals

Definition (Losev-MB-Matvieievskyi)

The unipotent ideal attached to $\widetilde{\mathbb{O}}$ is the two-sided ideal

$$I(\widetilde{\mathbb{O}}) := \ker\left(\Phi : U(\mathfrak{g}) \to \mathcal{A}_0\right)$$

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Proposition (Losev-MB-Matvieievskyi, MB-Matvieievskyi)

The unipotent ideal $I(\widetilde{\mathbb{O}})$ has the following properties:

- $I(\widetilde{\mathbb{O}})$ is primitive.
- $I(\widetilde{\mathbb{O}})$ is maximal.
- $I(\widetilde{\mathbb{O}})$ is completely prime.
- $V(I(\widetilde{\mathbb{O}})) = \overline{\mathbb{O}}$.

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Theorem (Losev-MB-Matvieievskyi, MB-Matvieievskyi)

Can compute $\lambda(\widetilde{\mathbb{O}})$ in all cases ('compute' means: combinatorial formulas in classical types and tables in exceptional types).

Examples of unipotent ideals

Example: 0-orbit

If $\widetilde{\mathbb{O}} = \{0\}$, then

$$I(\widetilde{\mathbb{O}}) = \mathfrak{g}U(\mathfrak{g}) = \text{max ideal of infl char } \rho.$$

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Example: principal orbit

If $\widetilde{\mathbb{O}}$ is the principal orbit, then

$$I(\widetilde{\mathbb{O}}) = \mathrm{Ann}_{U(\mathfrak{g})}(\Delta(-\rho)) = \text{max ideal of infl char } 0.$$

Unipotent ideals for $G = \mathrm{Sp}(8)$

$\tilde{\mathbb{O}}$	$\lambda(\tilde{\mathbb{O}})$	$\tilde{\mathbb{O}}$	$\lambda(\tilde{\mathbb{O}})$	$\tilde{\mathbb{O}}$	$\lambda(\tilde{\mathbb{O}})$
(8)	$(0,0,0,0)$	$(42^2)_2$	$(1,1,\frac{1}{2},0)$	$(3^22)_2$	$(\frac{3}{2},1,\frac{1}{2},\frac{1}{2})$
$(8)_2$	$(\frac{1}{2},0,0,0)$	$(42^2)_2$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	(3^21^2)	$(2,1,\frac{1}{2},\frac{1}{2})$
(62)	$(\frac{1}{2},\frac{1}{2},0,0)$	$(42^2)_2$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},0)$	(2^4)	$(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_2$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$	$(42^2)_4$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(2^4)_2$	$(2,1,1,0)$
$(62)_2$	$(1,0,0,0)$	(421^2)	$(2,1,0,0)$	(2^31^2)	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_2$	$(1,\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,0,0)$	$(2^31^2)_2$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$
$(62)_4$	$(1,\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,0,0)$	(2^21^4)	$(3,2,1,0)$
(61^2)	$(\frac{3}{2},\frac{1}{2},0,0)$	$(421^2)_2$	$(2,1,\frac{1}{2},0)$	$(2^21^4)_2$	$(3,2,1,0)$
$(61^2)_2$	$(\frac{3}{2},\frac{1}{2},0,0)$	$(421^2)_4$	$(2,1,\frac{1}{2},0)$	(21^6)	$(\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$
(4^2)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	(41^4)	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},0)$	$(21^6)_2$	$(\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$
$(4^2)_2$	$(1,\frac{1}{2},\frac{1}{2},0)$	$(41^4)_2$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},0)$	(1^8)	$(4,3,2,1)$
(42^2)	$(1,1,0,0)$	(3^22)	$(1,1,1,0)$		

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Classification of unipotent ideals

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Suppose

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is a morphism of covers. There is an induced finite G-equivariant morphism of varieties

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Classification of unipotent ideals

Sometimes $I(\widetilde{\mathbb{O}}_1) = I(\widetilde{\mathbb{O}}_2)$ for $\widetilde{\mathbb{O}}_1 \neq \widetilde{\mathbb{O}}_2$. We can describe exactly when this happens.
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This induced morphism is étale over the open subset

$$\widetilde{\mathbb{O}}_2 \subset \mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}_2]).$$

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Definition (Losev-MB-Matvieievskyi)

- A finite G-equivariant morphism $\widetilde{X}_1 \to \widetilde{X}_2$ is *almost étale* if it is étale over the open subset

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Theorem (Losev-MB-Matvieievskyi)

$I(\widetilde{\mathbb{O}}_1) = I(\widetilde{\mathbb{O}}_2)$ iff $\widetilde{\mathbb{O}}_1 \sim \widetilde{\mathbb{O}}_2$.

Description of $U(\mathfrak{g})/I(\widetilde{\mathbb{O}})$

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- Every equivalence class of covers $[\widetilde{\mathbb{O}}]$ contains a unique maximal element $\widetilde{\mathbb{O}}_{\max}$. Write $\mathcal{A}_0^{\max}$ for its canonical quantization and $\Gamma = \mathrm{Aut}_G(\widetilde{\mathbb{O}}_{\max}, \mathbb{O})$.

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- Γ acts on $\mathcal{A}_0^{\max}$ by filtered algebra automorphisms.
- There is an isomorphism

$$U(\mathfrak{g})/I(\widetilde{\mathbb{O}}) \simeq (\mathcal{A}_0^{\max})^{\Gamma}.$$

Unipotent representations

Definition (Losev-MB-Matvieievskyi)

Let $\widetilde{\mathbb{O}} \in \mathsf{Cov}_n(G)$. Then

$$\mathrm{Unip}_{\widetilde{\mathbb{O}}}(G) := \{\text{irreducible } G\text{-equivariant Harish-Chandra } U(\mathfrak{g})/I(\widetilde{\mathbb{O}})\text{-bimodules}\}$$

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Theorem (Losev-MB-Matvieievskyi)

There is a bijection

$$\{\text{irreducible } \Gamma\text{-reps}\} \xrightarrow{\sim} \mathrm{Unip}_{\widetilde{\mathbb{O}}}(G), \qquad \sigma \mapsto \mathrm{Hom}_{\Gamma}(\sigma, \mathcal{A}_0^{\max}).$$

Example

Let $G = \mathrm{SL}(2)$ and let $\widetilde{\mathbb{O}}$ be the 2-fold cover of the principal nilpotent orbit. There are G-equivariant isomorphisms

$$\widetilde{\mathbb{O}} = \mathbb{C}^2 \setminus \{0\}, \qquad \mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}]) = \mathbb{C}^2.$$

There is a unique filtered quantization of $\mathbb{C}^2$, namely the Weyl algebra $W(\mathbb{C}^2)$, and $\Gamma = \{\pm 1\}$. Easy exercise:

$$W(\mathbb{C}^2)^\Gamma \simeq U(\mathfrak{g})/I, \qquad I = \text{max ideal with infl char } \frac{1}{2}$$

Hint: surjective map $U(\mathfrak{g}) \to W(\mathbb{C}^2)^\Gamma$ given by

$$e \mapsto \frac{1}{2}x^2 \quad f \mapsto -\frac{1}{2}\partial x^2 \quad h \mapsto x\partial x + \frac{1}{2}$$

Kernel is I.

Example (cont'd)

There are two irreducible $U(\mathfrak{g})/I$-bimodules, namely

$$X_{\mathrm{triv}} := \mathrm{Hom}_{\Gamma}(\mathrm{triv}, W(\mathbb{C}^2)), \qquad X_{\mathrm{sgn}} := \mathrm{Hom}_{\Gamma}(\mathrm{sgn}, W(\mathbb{C}^2)).$$

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These representations are *not* special unipotent in the sense of Arthur-Barbasch-Vogan.

Unitarity

Theorem (Losev-MB-Matvieievskyi)

Suppose G is classical and $\widetilde{\mathbb{O}} \in \mathsf{Cov}_n(G)$. Then $\mathrm{Unip}_{\widetilde{\mathbb{O}}}(G)$ consists of unitary representations.

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- Use classification result to prove exhaustion.
- Show that inducing representations are unitary using Barbasch's classification of unitary representations of complex classical groups.

Defining the orbit method

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$$\widetilde{\mathbb{O}} = \mathrm{Bind}(L, \widetilde{\mathbb{O}}_L, \mu).$$
- Define
$$\Pi_{\widetilde{\mathbb{O}}}(G) := \{X \in \widehat{G} \,|X \text{ is a summand in } \mathrm{Ind}_P^G(\mu \otimes X_L) \text{ for some } X_L \in \mathrm{Unip}_{\widetilde{\mathbb{O}}_L}(L)\}$$

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- All Arthur packets are Kirillov packets.

Arthur packets

An *Arthur parameter* for G is a continuous homomorphism

$$\psi : \mathbb{C}^\times \times \mathrm{SL}(2, \mathbb{C}) \to G^\vee$$

such that

- $\psi|_{\mathrm{SL}(2,\mathbb{C})}$ is algebraic.
- $\psi(\mathbb{C}^\times)$ is bounded.

Let $\Psi(G^\vee)$ denote the set of $G^\vee$-conjugacy classes of Arthur parameters.

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Let $\Psi(G^\vee)$ denote the set of $G^\vee$-conjugacy classes of Arthur parameters.

Theorem (Adams-Barbasch-Vogan)

For each $\psi \in \Psi(G^\vee)$, there is a finite set

$$\Pi_\psi^{\mathrm{Art}}(G) \subset \widehat{G}$$

called an *Arthur packet*. These packets/representations satisfy various properties (endoscopy, stability,...).

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Theorem (MB)

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(iii) $\mathrm{D}(\psi)$ is nilpotent if and only if ψ is unipotent.

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Definition (MB)

Let $\widetilde{\mathbb{O}}$ be a birationally rigid nilpotent cover for $G^{\mathbb{C}}$ and let $\mathbb{O}_\theta$ be a $K^{\mathbb{C}}$-orbit on $\mathbb{O} \cap \mathfrak{p}^*$. A *unipotent representation* attached to $(\widetilde{\mathbb{O}}, \mathbb{O}_\theta)$ is an irreducible $(\mathfrak{g}, K^{\mathbb{C}})$-module X such that

(i) $\mathrm{Ann}(X) = I(\widetilde{\mathbb{O}})$, and

(ii) $V(X) = \overline{\mathbb{O}}_\theta$.