Braids, links and cluster algebras

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Plan

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   - The Laurent phenomenon
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I. Cluster algebras

Cluster algebras

Cluster algebras were defined by Fomin and Zelevinsky around the year 2000. A cluster algebra $\mathcal{A}$ is a subalgebra of the field $\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_m)$ of rational functions in $n+m$-variables, generated by a collection of sets of cardinality $n+m$ called **clusters**. All clusters contain the variables $y_1, \dots, y_m$ (called **frozen variables**) and, starting from the **initial cluster**

$$\mathbf{x} := \{x_1, \dots, x_n, y_1, \dots, y_m\}$$

one may reach all other clusters by iterating a combinatorial rule called **mutation**. The mutation is encoded by a **quiver** with n **mutable vertices** and m **frozen vertices**, and one is allowed to mutate only at the mutable vertices.

Mutation

Consider the initial cluster $\mathbf{x} = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and a quiver Q with $m+n$ vertices, numbered $1, \ldots, n+m$. The vertices $n+1, \ldots, n+m$ are frozen. For $k = 1, \ldots, n$, the mutation of the pair $(\mathbf{x}, Q)$ is another pair $(\mu_k(\mathbf{x}), \mu_k(Q))$ constructed as follows:

- $\mu_k(\mathbf{x}) = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$ where

$$x_k x'_k = \prod_{x \to x_k} x + \prod_{x_k \to x} x$$

- $\mu_k(Q)$ is obtained by following the 3-step procedure:
  - Reversing all the arrows incident with k.
  - For each path $i \to k \to j$ in Q, add a new arrow $i \to j$.
  - The previous steps may have created arrows between frozens, as well as two-cycles. Delete these.

Examples

$$(x \to y) \overset{\mu_1}{\longleftrightarrow} \left(\frac{1+y}{x} \leftarrow y\right), \qquad \mathcal{A}(\bullet \to \square) = \mathbb{C}\left[x, \frac{1+y}{x}, y^{\pm 1}\right].$$

$$[x_1 \to x_2 \to y] \overset{\mu_1}{\longleftrightarrow} \left[\boxed{\frac{1+x_2}{x_1}} \leftarrow x_2 \to y\right] \overset{\mu_2}{\longleftrightarrow} \left[\frac{1+x_2}{x_1} \to \frac{y+yx_2+x_1}{x_1x_2} \text{ — } y\right]$$

$$\Updownarrow \mu_1$$

$$\begin{tikzcd} \left[\frac{y+x_1}{x_2} \arrow[r] & x_1 \arrow[r] & y \arrow[ll, bend right]\right] \end{tikzcd} \overset{\mu_2}{\longleftrightarrow} \left[\left(\frac{y+yx_2+x_1+x_1x_2}{x_1x_2}\right)\left(\frac{x_1}{1+x_2}\right) \leftarrow \frac{y+yx_2+x_1}{x_1x_2} \leftarrow y\right]$$

$$\left(\frac{y+yx_2+x_1+x_1x_2}{x_1x_2}\right)\left(\frac{x_1}{1+x_2}\right) = \frac{y+x_1}{x_2}$$

$$\Updownarrow \mu_1$$

$$\begin{tikzcd} x_2 \arrow[rr, bend left] & x_1 \arrow[l] & y \end{tikzcd}$$

$$\mathcal{A}(\bullet \to \bullet \to \square) = \mathbb{C}\left[x_1, x_2, \frac{1+x_2}{x_1}, \frac{y+yx_2+x_1}{x_1x_2}, \frac{y+x_1}{x_2}, y^{\pm 1}\right]$$

The Laurent Phenomenon

$$\mathcal{A} \subseteq \bigcap_{\tilde{x} \text{ cluster}} \mathbb{C}[\tilde{x}^{\pm 1}]$$

Theorem (The Laurent Phenomenon, Fomin-Zelevinsky 2000)

Let $\mathcal{A}$ be a cluster algebra, and let $\tilde{\mathbf{x}}$ be any cluster (not necessarily the initial one). Then,

$$\boxed{\mathcal{A} \subseteq \mathbb{C}[\tilde{\mathbf{x}}^{\pm 1}].}$$

Mutation may get incredibly complicated, but we can always cancel things so that the end-result is a *Laurent* polynomial! Observe that we have $\tilde{\mathbf{x}} \subseteq \mathcal{A}$ so localizing we get an equality:

$$\mathcal{A}[\tilde{\mathbf{x}}^{\pm 1}] = \mathbb{C}[\tilde{\mathbf{x}}^{\pm 1}]$$

$$X \supseteq \{\tilde{x}_i \neq 0 \ \forall \tilde{x}_i \in \tilde{x}\} \cong (\mathbb{C}^\times)^{m+n}$$

↑ Cluster tori

Cluster varieties

Definition

Let X be an affine algebraic variety. We say that X is a **cluster variety** if there exists a cluster algebra $\mathcal{A}$ such that

$$X = \mathrm{Spec}(\mathcal{A}).$$

Examples of cluster varieties include:

- The basic affine space G/U (Fomin-Zelevinsky).
- The affine cone over the Grassmannian $\mathrm{Gr}(k, n)$ (Scott, Postnikov, Oh, Speyer...)
- The affine cone over parabolic flag varieties (Geiss-Leclerc-Schroer)
- Double Bruhat cells (Berenstein-Fomin-Zelevinsky)
- Positroid varieties (Galashin-Lam, Serhiyenko-Sherman-Bennett-Williams)

Why?

Why would we like to have a cluster structure on an affine variety X?

- Notions of *positive* part of X (Fomin-Zelevinsky, Schiffler-Lee, Gross-Hacking-Keel-Kontsevich)
- (In nice cases) An explicit basis of $\mathbb{C}[X]$ (Gross-Hacking-Keel-Kontsevich)
- (In nice cases) Mirror symmetry for X (Fock-Goncharov, Gross-Hacking-Keel-Kontsevich)
- (In nice cases) Information about the cohomology of X.

Theorem (Lam-Speyer)

For nice enough Q, the cluster variety X is a smooth affine algebraic variety. Moreover, the mixed Hodge structure on the cohomology $H^*(X, \mathbb{Q})$ is of mixed Tate type, and it is split over $\mathbb{Q}$ (in particular, it is a direct sum of pure Hodge structures).

II. Braid varieties (type A)

The symmetric group

Fix $n > 0$. We recall the Coxeter presentation of the symmetric group.

Definition

The symmetric group S_n is the group with generators s_i, $i = 1, \ldots,$ $n-1$ and relations

Braid rel

$\parallel$ $(i, i+1)$

- $s_i s_j = s_j s_i$ if $|i - j| > 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.
- $s_i^2 = 1$.

In terms of the usual definition of the symmetric group, $s_i = (i, i+1)$.

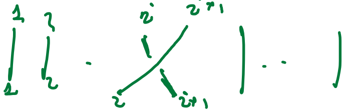
Definition

The *length* of an element $w \in S_n$ is the minimum number of s_i's we need to write w. Equivalently,

$$\ell(w) = \{(i, j) | 1 \leq i < j \leq n, w(j) < w(i)\}$$

Note that S_n has a unique element of maximal length, that we denote by w_0. $= [n, n-1, \ldots, 2, 1)$

Positive braid monoid

$\sigma_i =$ 

Fix $n > 0$.

Definition

The positive braid monoid B_n^+ is the group with generators σ_i, $i = 1, \dots, n-1$ and relations

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

We will call elements of B_n^+ *positive braids.*

We have a surjection $\varpi : B_n^+ \to S_n$, given by $\sigma_i \mapsto s_i$.

Definition

The length of an element $\beta \in B_n^+$ is the minimum number of σ_i's we need to write it. If $w \in S_n$, we denote by $\beta(w) \in B_n$ its unique lift of minimal length. Note that $\ell(w) = \ell(\beta(w))$.

Demazure products

$\delta(\sigma_1\sigma_1\sigma_2\sigma_2) = s_1 s_2$

Let $\beta = \sigma_{i_1}\cdots\sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a *positive* braid in n strands. The *Demazure* (aka *greedy*, aka *0-Hecke*) product of β, $\delta(\beta) \in S_n$ is defined inductively on $\ell = \ell(\beta)$ as follows:

$$\delta(1) = 1 \in S_n$$

$$\delta(\beta\sigma_i) = \begin{cases} \delta(\beta)s_i & \text{if } \ell(\delta(\beta)) < \ell(\delta(\beta)s_i) \\ \delta(\beta) & \text{else.} \end{cases}$$

It is easy to check that this does not depend on the chosen braid word of β.

Example

$\delta(\beta) = w_0$ if and only if β contains $\beta(w_0)$ as a (not necessarily consecutive!) subword.

The flag variety

$F^I = 0 \subseteq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \dots \subsetneq \mathbb{C}^n$

std flag $\overset{\shortparallel}{F^B}$ $\forall$ upper triang matrix

We will consider the flag variety

$$\mathcal{F}_n := \{F_\bullet = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^n) \mid \dim F_i = i\}$$

If $A \in \mathrm{GL}_n$, we denote by F^A the flag

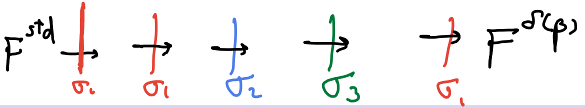
$$F_i^A := \mathrm{span}\{\text{first } i \text{ columns of } A\}$$

This gives us the usual identification $\mathcal{F}_n = \mathrm{GL}_n / B$. Note that we also get a natural action of S_n on $\mathcal{F}_n$.

Definition

The *standard flag* is $F^{\mathrm{std}} = F^I$. If $w \in S_n$, we have $F^w \in \mathcal{F}_n$.

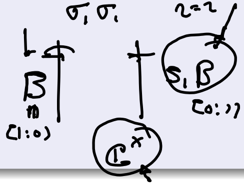
Brick varieties



Definition

Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n^+$. We define the *open brick variety* to be the subvariety $\mathrm{brick}^\circ(\beta) \subseteq \underline{\mathcal{F}\ell_n^{\ell+1}}$ consisting of tuples $(\mathcal{F}^0, \dots, \mathcal{F}^\ell)$ satisfying:

- $\mathcal{F}^0 = \mathcal{F}^{std}$.
- $\mathcal{F}^j_{i_{j+1}} \neq \mathcal{F}^{j+1}_{i_{j+1}}$, $\mathcal{F}^j_i = \mathcal{F}^{j+1}_i$ for $i \neq i_{j+1}$.
- $\mathcal{F}^\ell = \delta \mathcal{F}^{std}$.



Remark

The (closed) brick variety is defined by relaxing the condition that two consecutive flags *must* differ: they are allowed to be the same. Note that $\boxed{\mathrm{brick}^\circ(\beta)}$ does not depend on the chosen braid word of β, but $\mathrm{brick}(\beta)$ may.

Braid varieties

For $i = 1, \ldots, n-1$ and $z \in \mathbb{C}$, we denote by $B_i(z)$ the matrix that is the identity everywhere except at the i and $i+1$-st row and columns, where it is the matrix

$$\begin{array}{cc} \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}. & \begin{array}{c} i \\ i+1 \end{array} \\ \begin{array}{cc} i & i+1 \end{array} & \end{array}$$

Definition

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid with Demazure product δ. We define the *braid variety* $X(\beta) \subseteq \mathbb{C}^\ell$ to be:

$$X(\beta) := \{\underbrace{(z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \mid}_{\text{vanishing of } \binom{n}{2} \text{ equations}} B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell)\delta^{-1} \text{ is upper triangular}\}$$

Braid varieties = Open Brick varieties

Note that, if $B \in \mathrm{GL}_n$, then the flags that differ from F^B at precisely the i-th subspace are precisely those of the form $F^{BB_i^{-1}(z)}$ for $z \in \mathbb{C}$.

Lemma

$$X(\sigma_{i_1} \cdots \sigma_{i_\ell}) \cong \mathrm{brick}^\circ(\sigma_{i_\ell} \cdots \sigma_{i_1})$$

Remark

- The variety $X(\beta)$ does *not* depend on the braid word chosen for β. Actually, $B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$.
- If $\beta = \beta(w)$ for some $w \in S_n$, then $X(\beta) = \mathrm{pt}$.
- From now on, we will assume that $\delta(\beta) = w_0$. Indeed, it is easy to see that $X(\beta) \cong X(\beta \cdot \beta(\delta^{-1}w_0))$.

Example: 2-stranded braids

[1 : 0] [0 : 1]

Let us consider the braid $\sigma^4 \in B_2^+$. We have

W_0 w w w

$$
\begin{aligned}
B_{\sigma^4}(z_1, \ldots, z_4) &= \begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & z_2 \\ z_1 & 1 + z_1 z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix} \\
&= \begin{pmatrix} z_2 & 1 + z_2 z_3 \\ 1 + z_1 z_2 & z_1 + z_3 + z_1 z_2 z_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix} \\
&= \begin{pmatrix} 1 + z_2 z_3 & z_2 + z_4 + z_2 z_3 z_4 \\ \boxed{z_1 + z_3 + z_1 z_2 z_3} & \boxed{1 + z_1 z_2 + z_1 z_4 + z_3 z_4 + z_1 z_2 z_3 z_4} \end{pmatrix}
\end{aligned}
$$

$z_1 + z_3 + z_1 z_2 z_3 \neq 0$

$z_4 = \dfrac{-1 - z_1 z_2}{z_1 + z_3 + z_1 z_2 z_3}$

We can see that

$$X(\sigma^4) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_3 + z_1 z_2 z_3 \neq 0\}.$$

Open Richardson varieties

Let $u, v \in S_n$ with $u \leq v$. The open Richardson variety $R^\circ(u,v)$ is the intersection of the open Schubert cell $C_v^\circ \subseteq \mathcal{F}_n$ with the opposite open Schubert cell $C_\circ^u \subseteq \mathcal{F}_n$,

$$R(u,v) = C_v^\circ \cap C_\circ^u.$$

Let us denote by $\beta(v) \in B_n^+$ a positive lift of minimal length, and similarly for $\beta(u^{-1}w_0)$.

Proposition (Casals-Gorsky-Gorsky-S.)

We have

$$R^\circ(u,v) \cong X(\beta(v)\beta(u^{-1}w_0))$$

The isomorphism simply sends an element $(z_1, \dots, z_\ell) \in X(\beta)$ to the flag $F^{B^{-1}_{\beta(v)}(z_1,\dots,z_{\ell(v)})}$.

Properties of braid varieties

#variables

Theorem (Escobar, 2016)

The braid variety $X(\beta)$ is a smooth algebraic variety of dimension $\ell(\beta) - \ell(\delta(\beta))$.

$\delta(\beta) = w_0$ $\quad \ell(\beta) - \binom{n}{2}$

#equations

Theorem (Casals-Gorsky-Gorsky-Le-Shen-S. 2022)

The braid variety $X(\beta)$ is a cluster variety.

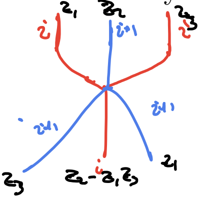
Remark

Both of these theorems are still valid when G is an algebraic group of simply laced type.

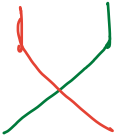
III. Algebraic Weaves

Algebraic Weaves, I

To study the varieties $X(\beta)$ we define correspondences between them that can be encoded by a graphical calculus that we call *algebraic weaves*.



$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$$



$B_i(z)B_j(w) = B_j(w)B_i(z)$ if $|i - j| > 1$.

Algebraic weaves

$z \neq 0$

$B_i(z)B_i(w)$

$U_i(z)L_i(z)B_i(w) = U_i(z)B_i(w+z^{-1})$

- If U is an upper triangular matrix and $z \in \mathbb{C}$, then we can find $z' \in \mathbb{C}$ and U' another upper triangular matrix such that

$$B_i(z)U = U'B_i(z')$$

Colloquially, we can "slide upper triangular matrices to the left, at the cost of a change of variables."

- If $z \neq 0$ then we can factor $B_i(z)$ as $U_i(z)L_i(z)$ where

$$U_i(z) = \begin{pmatrix} -z^{-1} & 1 \\ 0 & z \end{pmatrix}, \qquad L_i(z) = \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix}.$$

Moreover

$$L_i(z)B_i(w) = B_i(w+z^{-1}).$$

Algebraic weaves

- Furthermore, $B_i(0)B_i(w) = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$.

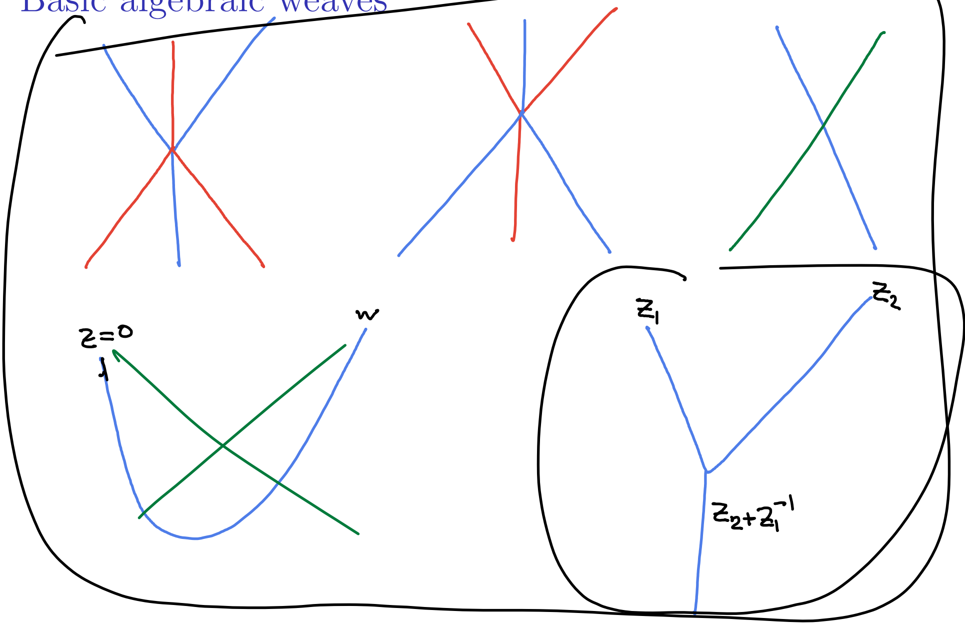
More precisely, we have:

Lemma

Let $\beta = \beta_1 \overset{z}{\sigma_i}\overset{w}{\sigma_i}\beta_2$, and let z, w be the variables corresponding to the first and second σ_i in the middle pair, respectively. Then,

- *The locus $\{z \neq 0\}$ in $X(\beta)$ is isomorphic to $\mathbb{C}_z^\times \times X(\beta_1 \overset{w+z^{-1}}{\sigma_i}\beta_2)$.*
- *The locus $\{z = 0\}$ in $X(\beta)$ is nonempty if and only if $\delta(\beta) = \delta(\beta_1\beta_2)$. In this case, the locus is isomorphic to $\mathbb{C}_w \times X(\beta_1\beta_2)$.*

Basic algebraic weaves



Weaves and tori

We denote $\mathfrak{w} : \beta_1 \to \beta_2$ a weave whose colors all the way north read β_1 and all the way south read β_2. If $\delta(\beta_1) = \delta(\beta_2)$, a weave $\mathfrak{w} : \beta_1 \to \beta_2$ defines a locally closed set in $X(\beta_1)$, isomorphic to

$$(\mathbb{C}^\times)^{\#\text{trivalent vertices}} \times \mathbb{C}^{\#\text{cups}} \times X(\beta_2)$$

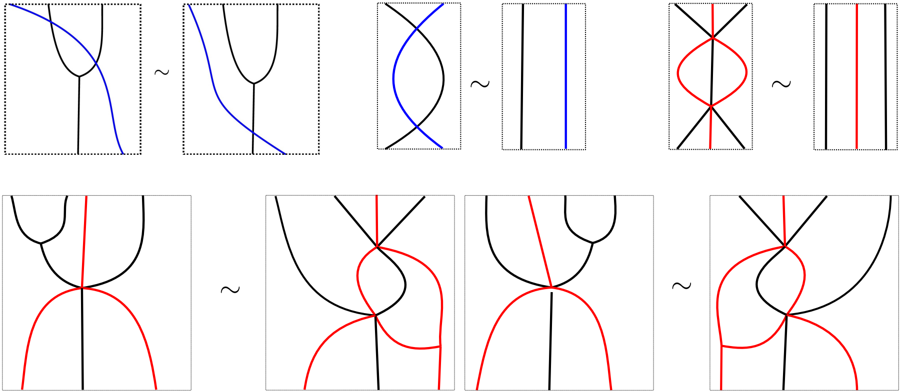
In particular,

Remark

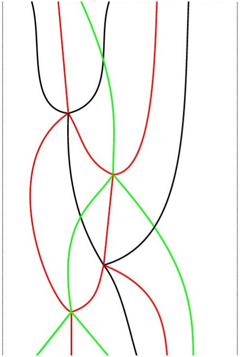
A weave without cups $\mathfrak{w} : \beta \to \delta(\beta)$ defines an open torus in $X(\beta)$, isomorphic to $(\mathbb{C}^\times)^{\ell(\beta)-\ell(\delta(\beta))}$.

Sometimes two different weaves give the same open torus...

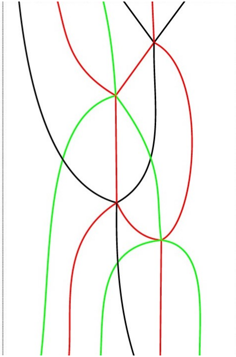
Relations



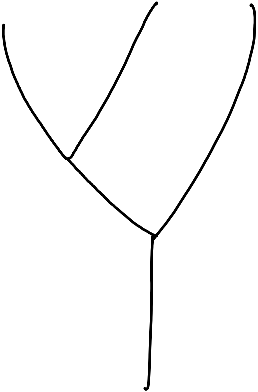
Relations



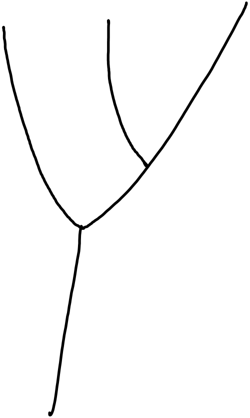
$\sim$



Mutations

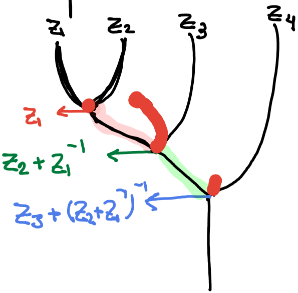


$\neq$



Example

$\beta = \sigma^4 \in B_2$



Torus with coordinates

$$\boxed{\begin{array}{c} s_1 = z_1, \quad s_2 = z_2 + z_1^{-1} \\ s_3 = z_3 + (z_2 + z_1^{-1})^{-1} \end{array}}$$

Not regular functions!!!

$s_1 = z_1 \qquad s_1 s_2 = 1 + z_1 z_2 \qquad s_1 s_2 s_3 = \underline{z_1 + z_3 + z_1 z_2 z_3}$

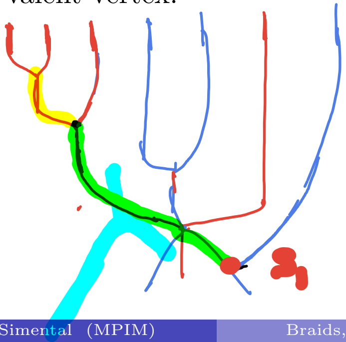
$$z_1 \longrightarrow -1 - z_1 z_2 \longrightarrow \boxed{-z_1 - z_3 - z_1 z_2 z_3}$$

$$\frac{(-1 - z_1 z_2) + 1}{z_1} = z_2 \qquad \frac{-z_1 - z_3 - z_1 z_2 z_3 + z_1}{-1 - z_1 z_2} = z_3$$

Inductive weaves

Let $\beta \in \mathrm{Br}_n^+$. We define the inductive weave of β, $\mathfrak{w}(\beta) : \beta \to \delta(\beta)$ as follows:

- $\mathfrak{w}(1)$ is the empty weave.
- If $\delta(\beta\sigma_i) = \delta(\beta)s_i$, then $\mathfrak{w}(\beta\sigma_i)$ is $\mathfrak{w}(\beta)$ with a disjoint i-colored strand to its right.
- If $\delta(\beta\sigma_i) = \delta(\beta)$, then $\mathfrak{w}(\beta\sigma_i)$ is $\mathfrak{w}(\beta)$ followed by an i-colored 3-valent vertex.

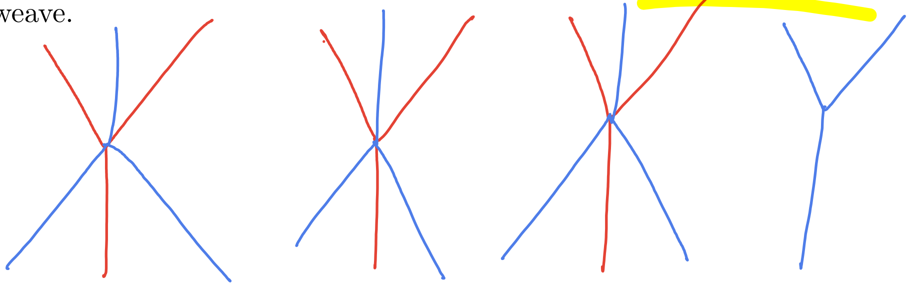


Cycles in $\mathfrak{w}(\beta)$

We define a collection of cycles (= paths in the weave) in $\mathfrak{w}(\beta)$ as follows. For every 3-valent vertex of $\mathfrak{w}(\beta)$:

- Start from the 3-valent vertex and go down.
- If we approach a hexavalent vertex from the left or right, go through.
- If we approach a hexavalent vertex from the middle, branch.
- If we hit another trivalent vertex, stop.

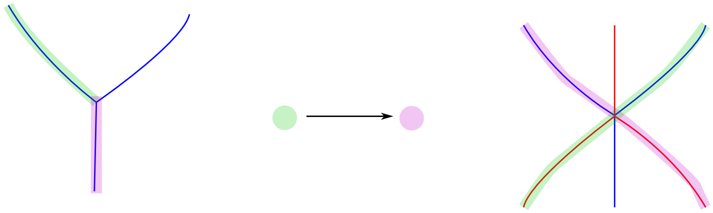
We say that such a cycle is *unbounded* if it falls all the way down the weave.



Intersection quiver

We form a quiver from the weave $\mathfrak{w}(\beta)$ as follows.

- Vertices = cycles in the weave = trivalent vertices in the weave.
  - Frozen vertices = unbounded cycles.
  - Mutable vertices = bounded cycles.
- Arrows: Given by the following rules (we may need to delete 2-cycles afterwards):



Cluster variables

Now we define a basis of the torus given by the inductive weave, by performing an upper-triangular change of basis from the s-basis. Let v be a trivalent vertex in $\mathfrak{w}(\beta)$. We say that another trivalent vertex v' covers v if the cycle starting at v' touches v. Then define inductively:

$$c_v = \pm s_v \prod_{v' \text{ covers } v} c_{v'}.$$

Lemma (Casals-Gorsky-Gorsky-Le-Shen-S. '22)

The functions c_v are regular functions on $X(\beta)$. If v is such that the cycle starting at v is unbounded, then c_v is nowhere vanishing on $X(\beta)$.

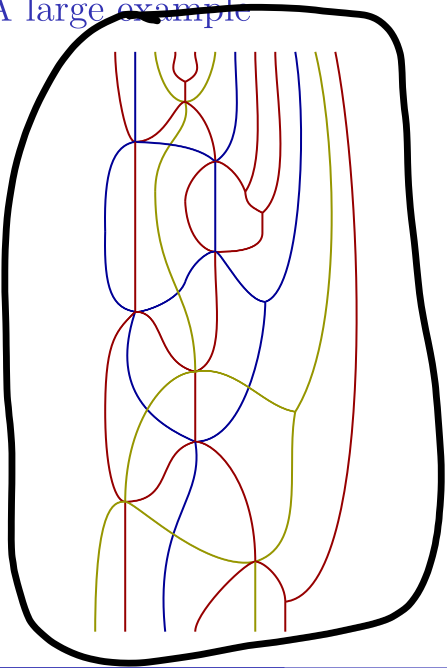
Theorem (Casals-Gorsky-Gorsky-Le-Shen-S. '22)

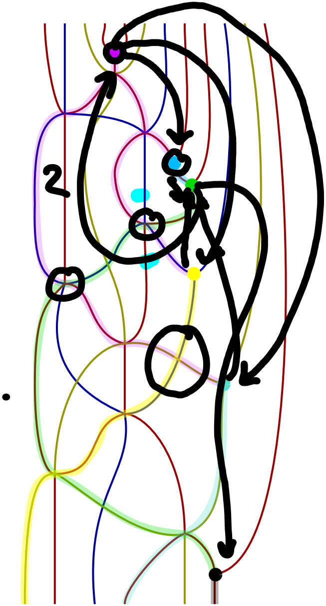
Let β be any positive braid. Then $X(\beta)$ is a cluster variety, with the torus given by $\mathfrak{w}(\beta)$ being a cluster torus. The quiver and cluster variables for the initial seed are given as above, with frozen variables corresponding to unbounded cycles.

Remarks

- Note that $\mathfrak{w}(\beta)$ depends on the braid word for β and not just on β. Different braid words of β give potentially different tori in the same cluster structure, and the quivers Q, Q' are mutation equivalent.
- It would be great to give a similar procedure for *any* weave.
- Related work by B. Hwang-A. Knutson.

A large example





Thanks for your attention!