Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation

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Abstract

This paper is concerned with the global well-posedness and inviscid limits of several systems of Boussinesq equations with fractional dissipation. Three main results are proven. The first result assesses the global regularity of two systems of equations close to the critical 2D Boussinesq equations. This is achieved by examining their inviscid limits. The second result relates the global regularity of a general system of d-dimensional Boussinesq equations to that of its formal inviscid limit. The third obtains the global existence, uniqueness and inviscid limit of a system of 2D Boussinesq equations with the Yudovich-type initial data.

Keywords: Boussinesq equations, fractional Laplacian, global regularity, inviscid limits
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1. Introduction

This paper studies the global regularity and inviscid limits of several Boussinesq systems of equations with dissipation given by a fractional Laplacian. The Boussinesq equations concerned here model large-scale atmospheric and oceanic flows and also play important roles in the study of Rayleigh–Bénard convection (see, e.g., [14, 19, 31, 37]). Our goal here is several fold: first, to establish the global regularity of two systems of Boussinesq equations that are close to the 2D Boussinesq equations with critical dissipation through the study of

their inviscid limits; second, to prove a connection between the global regularity of a general system of d-dimensional Boussinesq equations and that of its formal inviscid limit; and third, to obtain the global existence and uniqueness as well as the inviscid limit of a system of 2D Boussinesq equations with the Yudovich-type initial data.

Our first result was mainly motivated by a recent progress on the global regularity issue concerning the 2D Boussinesq equations with fractional Laplacian dissipation or with partial dissipation. Due to their mathematical significance, these 2D equations have attracted considerable attention in the last few years (see, e.g., [1–3, 6, 8–10, 15–18, 20–26, 28–30, 33, 35, 36, 42]). Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier–Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [32], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows. One main pursuit has been to establish the global regularity of the following 2D Boussinesq system with minimal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field denoting the velocity, $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function denoting the temperature in the content of thermal convection and the density in the modelling of geophysical fluids, e_2 is the unit vector in the x_2 direction, $\nu \geqslant 0$ denotes the viscosity, $\kappa \geqslant 0$ denotes the thermal diffusivity, and $\alpha \in [0, 2]$ and $\beta \in [0, 2]$ are real parameters. Here we adopt the convention that $\alpha = 0$ or $\beta = 0$ implies the corresponding dissipative term is set to zero. In addition, $\Lambda = \sqrt{-\Delta}$ denotes the Zygmund operator (see [39]), which can be defined through the Fourier transform,

$$\widehat{\Lambda f}(\xi) = |\xi| \, \widehat{f}(\xi).$$

There are geophysical circumstances in which the Boussinesq equations with fractional Laplacian arise. Flows in the middle atmosphere travelling upwards undergo changes due to the changes in atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modelled using the space fractional Laplacian (see [7, 19]).

Quite a few papers have been devoted to (1.1) and the most recent work targets the critical and the supercritical cases (see, e.g., [10, 15, 16, 20–26, 28, 33, 42]). In two papers [23, 24] Hmidi *et al* were able to show the global regularity of (1.1) for two critical cases: (1.1) with $\nu > 0, \alpha = 1$ and $\kappa = 0$, and (1.1) with $\nu = 0, \kappa > 0$ and $\beta = 1$. Miao and Xue [33] obtained the global regularity for (1.1) with $\nu > 0, \kappa > 0$ and

$$\alpha \in \left(\frac{6 - \sqrt{6}}{4}, 1 \right), \quad \beta \in \left(1 - \alpha, \min \left\{ \frac{7 + 2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1 - \alpha)}{\sqrt{6} - 2\alpha}, 2 - 2\alpha \right\} \right).$$

In addition, Constantin and Vicol [15] verified the global regularity of (1.1) with

$$\alpha \in (0, 2), \quad \beta \in (0, 2), \quad \beta > \frac{2}{2 + \alpha}.$$

The global regularity for the general critical case

$$\alpha \in (0, 1), \quad \beta \in (0, 1), \quad \alpha + \beta = 1$$

appears to be open at this moment. It is worth remarking that success has also been achieved on the global regularity issue beyond the critical case. Hmidi [20] proved the global regularity

for the 2D Boussinesq equations with logarithmically supercritical dissipation while Chae and Wu [10] obtained the global regularity of a generalized Boussinesq equation with the velocity determined by the vorticity via an operator logarithmically more singular than the Biot–Savart law. Our first result assesses the global regularity of two systems of equations close to the critical 2D Boussinesq equations. This is achieved by studying the inviscid limits of these equations and combining with the known global regularity result of the critical Boussinesq equations. The details are given in section 3.

The second main result relates the global regularity of a general d-dimensional Boussinesq equations to that of its formal inviscid limit. A special case of this result states that if the 3D inviscid Boussinesq equations have a classical solution on $[0, T]$, then any 3D dissipative Boussinesq equations with viscosity or thermal diffusivity in a suitable range also possess a classical solution on $[0, T]$. This result extends the work of Constantin on the Euler and the Navier–Stokes equations [13]. We defer the precise statement and the proof to section 4.

The last part of this paper examines the inviscid limit of the following 2D Boussinesq equations with Yudovich-type initial data,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta\, e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.2}$$

where $\nu > 0$, $\alpha \in (0, 1]$ and $\beta \in (1, 2]$. Here the Yudovich-type initial data refer to $\theta_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ and the initial vorticity $\omega_0 = \nabla \times u_0 \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, where $q > \frac{2}{\beta-1}$. Previously Danchin and Paicu [16] studied the global well-posedness of (1.2) with the Yudovich-type data for the case when $\nu = 0$ and $\beta = 2$. In addition, Hmidi and Zerguine [25] studied the global regularity of (1.2) with $\nu = 0$ and $\beta \in (1, 2]$, but with an initial data (u_0, θ_0) in a more regular functional setting. As our first step, we establish the global existence and uniqueness of solutions to (1.2) with either $\nu > 0$ or $\nu = 0$. In particular, the solutions are shown to obey global bounds independent of ν in the functional setting

$$\omega \in L^\infty([0, T]; L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)),$$

$$\theta \in L^\infty([0, T]; L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)) \cap L^1([0, T]; B_{r,1}^{1+\frac{2}{r}}(\mathbb{R}^2)),$$

where $T > 0$ is arbitrarily fixed and $r \in (2, q)$. Combining these global bounds with the Yudovich approach allows us to show that the difference between a solution $(u^{(\nu)}, \theta^{(\nu)})$ of (1.2) with $\nu > 0$ and the corresponding solution of (u, θ) of (1.2) with $\nu = 0$ satisfies

$$\|(u^{(\nu)}, \theta^{(\nu)})(\cdot, t) - (u, \theta)(\cdot, t)\|_{L^2} \leqslant C(T)\,(\nu t)^{e^{-b(t)}} \tag{1.3}$$

for any $T > 0$ and $t \leqslant T$, where $C(T)$ is a constant depending on T and the initial norm only, and

$$b(t) = C \int_0^t \left(\|\omega(\cdot, \tau)\|_{L^\infty} + \|\theta(\cdot, \tau)\|_{B_{r,1}^{1+\frac{2}{r}}} \right) d\tau.$$

Clearly the convergence rate in (1.3) deteriorates as time evolves and may not be improved when the initial data is of the Yudovich type. The precise statement and detailed proof of these results are provided in section 5.

In addition to the sections containing the main results, section 2 presents some preliminary facts and estimates to be used in the subsequent sections. In addition, an appendix on some of the functional spaces used in this paper and the Osgood inequality is provided.

2. Preliminary estimates

This section contains an upper bound for the solution of an ordinary differential equation (ODE) and a regularity criteria for a general Boussinesq equations. These results will be used in the subsequent sections.

The following lemma gives a global upper bound for the solution of an ODE involving a small parameter. It slightly extends a previous result of Constantin (see [13, p 315]).

Lemma 2.1. *Let $\gamma > 0$ and $G > 0$ be parameters. Let $T > 0$. Let F_1 and F_2 be nonnegative continuous functions on $[0, T]$. Consider the ODE*

$$
\begin{cases}
\dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} \leqslant \gamma\, F_1 + F_2\, Y + G\, Y^2, \\
Y(0) = 0.
\end{cases}
\tag{2.1}
$$

If we set

$$
\gamma_0 = \frac{1}{8T G \int_0^T F_1(\tau)\mathrm{e}^{\int_\tau^T F_2(s)\,\mathrm{d}s}\,\mathrm{d}\tau},
$$

then, for any $\gamma \in (0,\ \gamma_0)$ and $t \in [0, T]$, any solution to (2.1) obeys

$$
Y(t) \leqslant \min\left\{ \frac{3\mathrm{e}^{-\int_t^T F_2(\tau)\,\mathrm{d}\tau}}{2T G},\ 12\gamma \int_0^T F_1(\tau)\mathrm{e}^{\int_\tau^t F_2(s)\,\mathrm{d}s}\,\mathrm{d}\tau \right\}.
\tag{2.2}
$$

Proof. Obviously (2.1) is equivalent to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(Y\, \mathrm{e}^{-\int_0^t F_2(\tau)\,\mathrm{d}\tau} \right) \leqslant (\gamma\, F_1 + GY^2)\mathrm{e}^{-\int_0^t F_2(\tau)\,\mathrm{d}\tau}.
$$

Or, in terms of $U = Y\, \mathrm{e}^{-\int_0^t F_2(\tau)\,\mathrm{d}\tau}$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}U \leqslant \gamma\, F_1\, \mathrm{e}^{-\int_0^t F_2(\tau)\,\mathrm{d}\tau} + G\, \mathrm{e}^{\int_0^t F_2(\tau)\,\mathrm{d}\tau} U^2.
$$

By lemma 1.3 of [13], if we set

$$
\gamma_0 = \left(8T G\mathrm{e}^{\int_0^T F_2(\tau)\,\mathrm{d}\tau} \int_0^T F_1(\tau)\mathrm{e}^{-\int_0^\tau F_2(s)\,\mathrm{d}s}\,\mathrm{d}\tau \right)^{-1},
$$

then

$$
U(t) \leqslant \min\left\{ \frac{3\mathrm{e}^{-\int_0^T F_2(\tau)\,\mathrm{d}\tau}}{2T G},\ 12\gamma \int_0^T F_1(\tau)\mathrm{e}^{-\int_0^\tau F_2(s)\,\mathrm{d}s}\,\mathrm{d}\tau \right\}.
$$

Therefore, $Y(t) = \mathrm{e}^{\int_0^t F_2(\tau)\,\mathrm{d}\tau} U(t)$ obeys (2.2). This completes the proof of lemma 2.1. $\square$

For the convenience of later applications, we state a local existence and regularity criterion for the following general d-dimensional Boussinesq equations:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_d, \\
\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, \\
\nabla \cdot u = 0, \\
u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{cases}
\tag{2.3}
$$

where $\nu \geqslant 0$, $\kappa \geqslant 0$, $\alpha > 0$ and $\beta > 0$ are real parameters, and e_d is the unit vector in the direction of the last coordinate axis.

Lemma 2.2. *Let $\nu \geqslant 0$, $\kappa \geqslant 0$, $\alpha > 0$ and $\beta > 0$ be real parameters. Assume that $(u_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$. Then there exists $T = T(\|(u_0, \theta_0)\|_{H^s}) > 0$ such that (2.3) has a unique solution (u, θ) on $[0, T]$ satisfying $(u, \theta) \in C([0, T]; H^s)$. In addition, if we further know that*

$$\int_0^{T_0} \|\nabla u(\tau)\|_{L^\infty} \, d\tau < \infty \tag{2.4}$$

for $T_0 > T$, then the solution (u, θ) can be extended to $[0, T_0]$.

Proof. For the self-containedness, we briefly explain the lines of proof. The local well-posedness of (3.1) can be established through a standard procedure such as the Picard-type theorem (see, e.g., [32]). To prove the regularity criterion, we obtain by standard energy estimates that

$$\frac{d}{dt} \left(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2 \right) + C \|u\|_{H^{s+\frac{\alpha}{2}}}^2 + C \|\theta\|_{H^{s+\frac{\beta}{2}}}^2$$
$$\leqslant C(1 + \|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \left(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2 \right), \tag{2.5}$$

where C's are constants. In addition, it follows from the θ-equation that, for any $t > 0$,

$$\|\nabla \theta(t)\|_{L^\infty} \leqslant \|\nabla \theta_0\|_{L^\infty} \exp \left(\int_0^t \|\nabla u\|_{L^\infty} \, d\tau \right).$$

Therefore, (2.4) implies that $\nabla \theta \in L^\infty([0, T_0]; L^\infty(\mathbb{R}^d))$. Gronwall's inequality applied to (2.5) then leads to a bound for $\|(u, \theta)(\cdot, T_0)\|_{H^s}$ and thus the desired extension. This completes the proof of lemma 2.2. □

3. 2D Boussinesq equations close to the critical equations

This section studies the global well-posedness and the inviscid limits of two systems of equations close to the critical Boussinesq equations. First we consider the initial-value problem (IVP)

$$\begin{cases} \partial_t u^{(\nu)} + u^{(\nu)} \cdot \nabla u^{(\nu)} + \nu \Lambda^\alpha u^{(\nu)} = -\nabla p^{(\nu)} + \theta^{(\nu)} e_2, \\ \partial_t \theta^{(\nu)} + u^{(\nu)} \cdot \nabla \theta^{(\nu)} + \Lambda \theta^{(\nu)} = 0, \\ \nabla \cdot u^{(\nu)} = 0, \\ u^{(\nu)}(x, 0) = u_0(x), \quad \theta^{(\nu)}(x, 0) = \theta_0(x), \end{cases} \tag{3.1}$$

where $0 < \alpha \leqslant 1$ and $\nu > 0$ are real parameters. When $\nu = 0$, (3.1) formally reduces to the IVP for the 2D Boussinesq equations with critical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{3.2}$$

The local well-posedness of (3.1) and (3.2) follows from Kato [27]. We show that the solutions of (3.1) converge to the corresponding ones of (3.2) with an explicit rate as $\nu \to 0$. More precisely, we have the following theorem.

Theorem 3.1. *Let $\nu > 0$ and $\alpha \in (0, 1]$. Let $\sigma > 3$. Consider (3.1) with $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$. Let $T > 0$. Then there exists $\nu_0 = \nu_0(T) > 0$ such that, for $0 < \nu \leqslant \nu_0$, (3.1) has a unique global solution satisfying $(u^{(\nu)}, \theta^{(\nu)}) \in C([0, T]; H^\sigma(\mathbb{R}^2))$. In addition, for any*

$0 \leqslant s \leqslant \sigma - 1$ and $0 < \nu \leqslant \nu_0$, the difference between $(u^{(\nu)}, \theta^{(\nu)})$ and the corresponding solution (u, θ) of (3.2) satisfies

$$\|(u^{(\nu)}, \theta^{(\nu)}) - (u, \theta)\|_{H^s} \leqslant C(T)\,\nu,$$

where $C = C(T)$ is a constant dependent on T and $\|(u, \theta)\|_{L^\infty([0,T]; H^{s+1})}$ only.

Similar results can also be established for the 2D Boussinesq equations

$$\begin{cases} \partial_t u^{(\kappa)} + u^{(\kappa)} \cdot \nabla u^{(\kappa)} + \Lambda u^{(\kappa)} = -\nabla p^{(\kappa)} + \theta^{(\kappa)} e_2, \\ \partial_t \theta^{(\kappa)} + u^{(\kappa)} \cdot \nabla \theta^{(\kappa)} + \kappa \Lambda^\beta \theta^{(\kappa)} = 0, \\ \nabla \cdot u^{(\kappa)} = 0, \\ u^{(\kappa)}(x, 0) = u_0(x), \quad \theta^{(\kappa)}(x, 0) = \theta_0(x), \end{cases} \tag{3.3}$$

where $\kappa > 0$ and $0 < \beta \leqslant 1$ are real parameters. When $\kappa = 0$, (3.3) formally reduces to the IVP for the critical Boussinesq–Navier–Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \Lambda u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{3.4}$$

We show that the solutions of (3.3) converge to the corresponding ones of (3.4) with an explicit rate as $\kappa \to 0$. More precisely, we have the following theorem.

Theorem 3.2. *Let $\kappa > 0$ and $\beta \in (0, 1]$. Let $\sigma > 3$. Consider (3.3) with $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$. Let $T > 0$. Then there exists $\kappa_0 = \kappa_0(T) > 0$ such that, for $0 < \kappa \leqslant \kappa_0$, (3.3) has a unique global solution satisfying $(u^{(\kappa)}, \theta^{(\kappa)}) \in C([0, T]; H^\sigma(\mathbb{R}^2))$. In addition, for any $0 \leqslant s \leqslant \sigma - 1$ and $0 < \kappa \leqslant \kappa_0$, the difference between $(u^{(\kappa)}, \theta^{(\kappa)})$ and the corresponding solution (u, θ) of (3.4) satisfies*

$$\|(u^{(\kappa)}, \theta^{(\kappa)}) - (u, \theta)\|_{H^s} \leqslant C(T)\,\kappa,$$

where $C = C(T)$ is a constant dependent on T and $\|(u, \theta)\|_{L^\infty([0,T]; H^{s+1})}$ only.

To prove theorem 3.1, we need a lemma assessing the global existence of classical solutions to (3.2). It is obtained by combining the work by Hmidi *et al* [24] with the propagation of regularity.

Lemma 3.3. *Assume that $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s \in (2, \infty)$. Then (3.2) has a unique global solution (u, θ) satisfying, for any $T > 0$,*

$$(u, \theta) \in C([0, T]; H^s(\mathbb{R}^2)). \tag{3.5}$$

Proof of lemma 3.3. Since $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, we have, for any $q \in (2, \infty)$,

$$u_0 \in B_{\infty,1}^1(\mathbb{R}^2) \cap \dot{W}^{1,q}(\mathbb{R}^2), \qquad \theta_0 \in B_{\infty,1}^0(\mathbb{R}^2) \cap L^q(\mathbb{R}^2),$$

where $B_{\infty,1}^1$ and $B_{\infty,1}^0$ denote inhomogeneous Besov spaces as defined in the appendix, and $\dot{W}^{1,q}$ denotes a homogeneous Sobolev space. By theorem 1.1 of [24, p 422], (3.2) has a unique global solution (u, θ) satisfying, for any $T > 0$,

$$u \in L^\infty([0, T]; B_{\infty,1}^1 \cap \dot{W}^{1,q}), \qquad \theta \in L^\infty([0, T]; B_{\infty,1}^0 \cap L^q) \cap \widetilde{L}^1([0, T]; B_{q,\infty}^1),$$

where $\widetilde{L}^1([0, T]; B_{q,\infty}^1)$ is defined in the appendix. Since $B_{\infty,1}^1 \hookrightarrow L^\infty$, we have

$$\int_0^T \|\nabla u\|_{L^\infty}\, \mathrm{d}t < \infty.$$

The desired regularity (3.5) then follows from lemma 2.2. This completes the proof of lemma 3.3.　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

Proof of theorem 3.1. Since $(u_0^{(\nu)}, \theta_0^{(\nu)}) \in H^\sigma(\mathbb{R}^2)$ with $\sigma > 3$, the local solution is guaranteed by lemma 2.2. To prove the global well-posedness, it suffices to obtain a global *a priori* bound for $\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^\sigma}$. This is achieved through two steps. The first step is to compare $(u^{(\nu)}, \theta^{(\nu)})$ with a solution (u, θ) of (3.2) to obtain a bound for $\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^s}$ for any $s \leqslant \sigma - 1$. The details will be provided below. Since $\sigma - 1 > 2$, the bound in the first step especially implies that

$$\int_0^T \|\nabla u^{(\nu)}\|_{L^\infty} \, \mathrm{d}t < \infty.$$

With this bound at our disposal, the second step is to use the regularity criterion in lemma 2.2 to establish the global bound for $\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^\sigma}$.

To implement the first step, we consider the difference

$$\bar{u} = u^{(\nu)} - u, \quad \bar{\theta} = \theta^{(\nu)} - \theta, \quad \bar{p} = p^{(\nu)} - p,$$

which satisfy

$$\begin{cases} \partial_t \bar{u} + u \cdot \nabla \bar{u} + \bar{u} \cdot \nabla(u + \bar{u}) + \nu \Lambda^\alpha(u + \bar{u}) = -\nabla \bar{p} + \bar{\theta} e_2, \\ \partial_t \bar{\theta} + u \cdot \nabla \bar{\theta} + \bar{u} \cdot \nabla(\theta + \bar{\theta}) + \Lambda \bar{\theta} = 0, \\ \nabla \cdot \bar{u} = 0, \\ \bar{u}(x, 0) = 0, \quad \bar{\theta}(x, 0) = 0. \end{cases} \tag{3.6}$$

According to lemma 3.3, $(u, \theta) \in C([0, T]; H^\sigma)$. Now let $1 < s \leqslant \sigma - 1$. To estimate the H^s-norm of $(\bar{u}, \bar{\theta})$, we apply $J^s \equiv (Id - \Delta)^{\frac{s}{2}}$ to (3.6) and then take the inner product with $(J^s \bar{u}, J^s \bar{\theta})$ to obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\bar{u}\|_{H^s}^2 + \nu \|\Lambda^{\alpha/2} \bar{u}\|_{H^s}^2 = K_{11} + K_{12} + K_{13} + K_{14} + K_{15}, \tag{3.7}$$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\bar{\theta}\|_{H^s}^2 + \|\Lambda^{1/2} \bar{\theta}\|_{H^s}^2 = K_{21} + K_{22} + K_{23}, \tag{3.8}$$

where

$$K_{11} = -\int J^s(u \cdot \nabla \bar{u}) \, J^s \bar{u} \, \mathrm{d}x, \qquad K_{12} = -\int J^s(\bar{u} \cdot \nabla u) \, J^s \bar{u} \, \mathrm{d}x,$$

$$K_{13} = -\int J^s(\bar{u} \cdot \nabla \bar{u}) \, J^s \bar{u} \, \mathrm{d}x, \qquad K_{14} = \int J^s(\bar{\theta} e_2) \, J^s \bar{u} \, \mathrm{d}x,$$

$$K_{15} = -\nu \int \Lambda^\alpha J^s u \, J^s \bar{u} \, \mathrm{d}x, \qquad K_{21} = -\int J^s(u \cdot \nabla \bar{\theta}) \, J^s \bar{\theta} \, \mathrm{d}x,$$

$$K_{22} = -\int J^s(\bar{u} \cdot \nabla \theta) \, J^s \bar{\theta} \, \mathrm{d}x, \qquad K_{23} = -\int J^s(\bar{u} \cdot \nabla \bar{\theta}) \, J^s \bar{\theta} \, \mathrm{d}x.$$

Thanks to $\nabla \cdot u = 0$, Hölder's inequality, a commutator estimate and Sobolev embedding,

$$|K_{11}| = \left| \int \left(J^s(u \cdot \nabla \bar{u}) - u \cdot \nabla J^s \bar{u} \right) J^s \bar{u} \, \mathrm{d}x \right|$$

$$\leqslant \|J^s(u \cdot \nabla \bar{u}) - u \cdot \nabla J^s \bar{u}\|_{L^2} \|J^s \bar{u}\|_{L^2}$$

$$\leqslant C \left(\|\nabla \bar{u}\|_{L^\infty} \|J^s u\|_{L^2} + \|\nabla u\|_{L^\infty} \|J^s \bar{u}\|_{L^2} \right) \|J^s \bar{u}\|_{L^2}$$

$$\leqslant C \|u\|_{H^s} \|\bar{u}\|_{H^s}^2.$$

Similarly,

$$|K_{13}| \leqslant C \|\bar{u}\|_{H^s}^3.$$

Since $H^s(\mathbb{R}^2)$ with $s > 1$ is an algebra,

$$|K_{12}| \leqslant \|J^s(\bar{u} \cdot \nabla u)\|_{L^2} \|J^s\bar{u}\|_{L^2}$$
$$\leqslant \|\bar{u}\|_{H^s} \|\nabla u\|_{H^s} \|J^s\bar{u}\|_{L^2}$$
$$= \|u\|_{H^{s+1}} \|\bar{u}\|_{H^s}^2.$$

By Hölder's inequality,

$$|K_{14}| \leqslant \|\bar{\theta}\|_{H^s} \|\bar{u}\|_{H^s}, \qquad |K_{15}| \leqslant \nu\|u\|_{H^{\alpha+s}} \|\bar{u}\|_{H^s}.$$

Inserting the estimates above in (3.7), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\bar{u}\|_{H^s} \leqslant C \|\bar{u}\|_{H^s}^2 + C \|u\|_{H^{s+1}} \|\bar{u}\|_{H^s} + \nu\|u\|_{H^{\alpha+s}} + \|\bar{\theta}\|_{H^s}. \qquad (3.9)$$

K_{21}, K_{22} and K_{23} obey similar bounds as K_{11}, K_{12} and K_{13}, respectively. That is,

$$|K_{21}| \leqslant C \|u\|_{H^s} \|\bar{\theta}\|_{H^s}^2, \quad |K_{22}| \leqslant C \|\theta\|_{H^{s+1}} \|\bar{u}\|_{H^s} \|\bar{\theta}\|_{H^s}, \quad |K_{23}| \leqslant C \|\bar{u}\|_{H^s} \|\bar{\theta}\|_{H^s}^2.$$

Inserting these estimates in (3.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\bar{\theta}\|_{H^s} \leqslant C \|\bar{u}\|_{H^s} \|\bar{\theta}\|_{H^s} + C \|u\|_{H^s} \|\bar{\theta}\|_{H^s} + C \|\theta\|_{H^{s+1}} \|\bar{u}\|_{H^s}. \qquad (3.10)$$

Adding (3.9) and (3.10) and setting $Y(t) \equiv \|\bar{u}(t)\|_{H^s} + \|\bar{\theta}(t)\|_{H^s}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} Y \leqslant \nu \|u\|_{H^{\alpha+s}} + C_1 (1 + \|u\|_{H^{s+1}} + \|\theta\|_{H^{s+1}})Y + C_2 Y^2,$$

where C_1 and C_2 are constants independent of ν and α. Since $\alpha \in (0, 1]$, $s \leqslant \sigma - 1$ and $(u, \theta) \in C([0, T]; H^\sigma)$, we apply lemma 2.1 to conclude that, if

$$\nu_0 = \frac{1}{8C_2 \int_0^T \|u(\tau)\|_{H^\sigma} \, \mathrm{e}^{C_1 \int_\tau^T (1+\|u(\varrho)\|_{H^{s+1}}+\|\theta(\varrho)\|_{H^{s+1}}) \, \mathrm{d}\varrho} \, \mathrm{d}\tau}$$

then, for $0 < \nu \leqslant \nu_0$ and $0 \leqslant t \leqslant T$,

$$Y(t) \leqslant 12\nu \int_0^T \|u(\tau)\|_{H^\sigma} \, \mathrm{e}^{C_1 \int_\tau^T (1+\|u(\varrho)\|_{H^{s+1}}+\|\theta(\varrho)\|_{H^{s+1}}) \, \mathrm{d}\varrho} \, \mathrm{d}\tau.$$

This completes the proof of theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We remark that the proof of theorem 3.2 is similar and is thus omitted.

4. Inviscid limits of general Boussinesq equations

This section is concerned with the global regularity and inviscid limits of a general d-dimensional (d-D) Boussinesq equations with dissipation given by a fractional Laplacian. Consider the IVP for the d-D Boussinesq equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu\Lambda^\alpha u = -\nabla p + \theta e_d, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa_0 \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \qquad (4.1)$$

where $\nu > 0$, $\alpha > 0$, $\kappa_0 \geqslant 0$ and $\beta > 0$ are real parameters, and e_d is the unit vector in the direction of the last coordinate axis. $\kappa_0 \geqslant 0$ is fixed and we study the limit as $\nu \to 0$. When we set $\nu = 0$, (4.1) formally reduces to the IVP for

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \theta e_d, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa_0 \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \qquad (4.2)$$

We establish that, if (4.2) has a classical solution on the time interval $[0, T]$, then (4.1) with small $\nu > 0$ also has a classical solution on $[0, T]$ and the solution approaches the corresponding solution of (4.2) as $\nu \to 0$. More precisely, we have the following theorem.

Theorem 4.1. *Assume that* $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^d)$ *with* $\sigma > 2 + \frac{d}{2}$. *Let* $T > 0$. *Assume that* (u, θ) *is the corresponding solution of (4.2) satisfying* $(u, \theta) \in C([0, T]; H^\sigma(\mathbb{R}^d))$. *Then there exists* $\nu_0 = \nu_0(T, \|(u, \theta)\|_{C([0,T];H^\sigma)}) > 0$ *such that (4.1) with* $0 < \nu \leqslant \nu_0$ *and the same initial data have a unique global solution* $(u^{(\nu)}, \theta^{(\nu)}) \in C([0, T]; H^\sigma)$. *In addition, for* $0 \leqslant s \leqslant \sigma - 1$, $0 < \nu \leqslant \nu_0$ *and* $0 < t \leqslant T$,

$$\|(u^{(\nu)}, \theta^{(\nu)}) - (u, \theta)\|_{H^s(\mathbb{R}^d)} \leqslant C(T)\, \nu,$$

where $C(T)$ *depends on* T *and* $\|(u, \theta)\|_{C([0,T];H^\sigma)}$ *only.*

Proof. It suffices to establish a global *a priori* bound for $\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^\sigma}$. The main step is to show there exists $\nu_0 > 0$ such that, for $0 < \nu \leqslant \nu_0$,

$$\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^s} \leqslant C(T) \quad \text{for any } s \leqslant \sigma - 1. \tag{4.3}$$

This bound especially implies

$$\int_0^T \|\nabla u^{(\nu)}\|_{L^\infty} \, \mathrm{d}t < \infty.$$

Lemma 2.2 then implies a global bound for $\|(u^{(\nu)}, \theta^{(\nu)})\|_{H^\sigma}$. To show (4.3), we consider the differences

$$\widetilde{u} = u^{(\nu)} - u, \quad \widetilde{\theta} = u^{(\nu)} - u, \quad \widetilde{p} = p^{(\nu)} - p$$

which satisfy

$$\partial_t \widetilde{u} + (u + \widetilde{u}) \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u + \nu \Lambda^\alpha \widetilde{u} = -\nabla \widetilde{p} + \widetilde{\theta}\, e_d - \nu \Lambda^\alpha u,$$
$$\partial_t \widetilde{\theta} + (u + \widetilde{u}) \cdot \nabla \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta + \kappa_0 \Lambda^\beta \widetilde{\theta} = 0.$$

Following similar estimates as in the proof of theorem 3.1, we find that $Y(t) \equiv \|\widetilde{u}\|_{H^s} + \|\widetilde{\theta}\|_{H^s}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) \leqslant \nu \|u\|_{H^{s+\alpha}} + C\,(1 + \|u\|_{H^{s+1}} + \|\theta\|_{H^{s+1}})\, Y + C\, Y^2.$$

Applying lemma 2.1 yields the desired bound. This completes the proof of theorem 4.1. $\quad\square$

5. 2D Boussinseq equations with Yudovich-type data

When the initial data are not smooth such as in the Yudovich class, the convergence rates of the inviscid limits may not reach the optimal rate, namely the order $O(\nu)$, as the dissipative coefficient $\nu \to 0$. This section investigates the inviscid limit of the 2D Boussinesq equations with Yudovich-type initial data

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta\, e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{5.1}$$

where $\nu > 0$, $\alpha \in (0, 1]$ and $\beta \in (1, 2]$. When we set $\nu = 0$, (5.1) becomes the Euler–Boussinesq equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{5.2}$$

The initial data (u_0, θ_0) will be in the Yudovich class (as specified below). The main goal is to rigorously establish (5.2) as the inviscid limit of (5.1) with an explicit convergence rate. As a preparation, we first prove the global existence and uniqueness of (5.1) for either $\nu > 0$ or $\nu = 0$.

Theorem 5.1. *Consider (5.1) with either $\nu > 0$ or $\nu = 0$, $\alpha \in (0, 1]$ and $\beta \in (1, 2]$. Let $q > \frac{2}{\beta - 1}$. Assume u_0 satisfying $\nabla \cdot u_0 = 0$ and $u_0 \in L^2(\mathbb{R}^2)$, $\omega_0 \equiv \nabla \times u_0 \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Assume $\theta_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$. Then, there exists a unique solution (u, θ) to (5.1) such that, for some $r \in (2, q)$,*

$$u \in C_{loc}([0, \infty); L^2 \cap L^\infty), \quad \omega \in L^\infty_{loc}(0, \infty; L^q \cap L^\infty),$$

$$\theta \in C([0, \infty); L^2 \cap L^q) \cap L^2(0, \infty; H^{\frac{\beta}{2}}) \cap L^1_{loc}(0, \infty; B^{1+\frac{2}{r}}_{r,1}). \tag{5.3}$$

Furthermore, the bounds for (u, θ, ω) in the class (5.3) are independent of ν even in the case when $\nu > 0$.

Theorem 5.1 can be slightly improved if we are willing to assume that θ_0 is in the Besov space setting $\theta_0 \in L^2(\mathbb{R}^2) \cap \dot{B}^{1-\beta}_{\infty,1}(\mathbb{R}^2)$. The approach is similar to that for theorem 5.1 and we state it as a corollary (see also [16]).

Corollary 5.2. *Consider (5.1) with either $\nu > 0$ or $\nu = 0$, $\alpha \in (0, 1]$ and $\beta \in (1, 2]$. Assume u_0 satisfying $\nabla \cdot u_0 = 0$ and $u_0 \in L^2(\mathbb{R}^2)$, $\omega_0 \equiv \nabla \times u_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Assume $\theta_0 \in L^2(\mathbb{R}^2) \cap \dot{B}^{1-\beta}_{\infty,1}(\mathbb{R}^2)$. Then, there exists a unique solution (u, θ) to (5.1) such that,*

$$u \in C_{loc}([0, \infty); L^2 \cap L^\infty), \quad \omega \in L^\infty_{loc}(0, \infty; L^2 \cap L^\infty),$$

$$\theta \in C_{loc}([0, \infty); L^2 \cap \dot{B}^{1-\beta}_{\infty,1}) \cap L^1_{loc}(0, \infty; \dot{B}^1_{\infty,1}).$$

With this global existence and uniqueness result at our disposal, we can prove the following inviscid limit result.

Theorem 5.3. *Assume the initial data (u_0, θ_0) is the Yudovich-type data, as specified in theorem 5.1. Let $(u^{(\nu)}, \theta^{(\nu)})$ be the corresponding solution of (5.1) with $\nu > 0$, $\alpha \in (0, 1]$ and $\beta \in (1, 2]$ while (u, θ) be the corresponding solution of (5.2) with $\beta \in (1, 2]$. Let $T > 0$. Then the difference $(u^{(\nu)} - u, \theta^{(\nu)} - \theta)$ satisfies, for any $0 < t \leqslant T$,*

$$\|(u^{(\nu)} - u, \theta^{(\nu)} - \theta)\|_{L^2} \leqslant L^{1 - e^{-b(t)}} (\nu t \|\Lambda^\alpha u\|_{L^\infty_t L^2})^{e^{-b(t)}}, \tag{5.4}$$

where L and b are given by

$$L = 2\|u\|_{L^\infty([0,T];L^\infty)}, \quad b(t) = 2e \|\omega\|_{L^\infty([0,T];L^\infty)} t + \int_0^t \|\theta(\tau)\|_{B^{1+\frac{1}{r}}_{r,1}} \, d\tau$$

with $\omega = \nabla \times u$.

We start with the proof of theorem 5.1.

Proof. The local existence can be established through a standard procedure. We provide the global *a priori* bounds needed for the global existence. Obviously,

$$\|\theta(t)\|_{L^2} \leqslant \|\theta_0\|_{L^2}, \qquad \|\theta(t)\|_{L^q} \leqslant \|\theta_0\|_{L^q}, \qquad \|u(t)\|_{L^2} \leqslant \|u_0\|_{L^2} + t \|\theta_0\|_{L^2}.$$

Since ω satisfies

$$\partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_{x_1} \theta,$$

it is clear (see, e.g., [4, 16]) that, for any $\nu \geqslant 0$,

$$\|\omega(t)\|_{L^q \cap L^\infty} \leqslant \|\omega_0\|_{L^q \cap L^\infty} + \int_0^t \|\partial_{x_1} \theta(\tau)\|_{L^q \cap L^\infty} \, d\tau. \tag{5.5}$$

To obtain a global bound for $\|\omega(t)\|_{L^q \cap L^\infty}$, we make use of the smoothing effect in the θ-equation. By Lemma 5.4 below, for any $2 < r < q$ and $0 < \epsilon < \beta$,

$$\sup_{j \geqslant -1} 2^{(\beta-\epsilon)j} \|\Delta_j \theta\|_{L^1_t L^\infty} \leqslant C \|\theta_0\|_{L^r} (1 + \|\omega\|_{L^1_t (L^\infty \cap L^q)}).$$

Choosing $2 < r < q$ and $0 < \epsilon < \beta$ such that $1 + \frac{2}{r} < \beta - \epsilon$, we have, by Bernstein's inequality,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leqslant \sup_{j \geqslant -1} 2^{(\beta-\epsilon)j} \|\Delta_j \theta\|_{L^1_t L^\infty},$$

$$\|\nabla\theta(\tau)\|_{L^q \cap L^\infty} \leqslant \sum_{j \geqslant -1} 2^j \|\Delta_j \theta\|_{L^q \cap L^\infty} \leqslant C \sum_{j \geqslant -1} 2^{(1+\frac{2}{r})j} \|\Delta_j \theta\|_{L^r} = C \|\theta\|_{B^{1+\frac{2}{r}}_{r,1}}. \tag{5.6}$$

Combining these estimates, we obtain

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leqslant C \|\theta_0\|_{L^r} \left(1 + t\|\omega_0\|_{L^\infty \cap L^q} + \int_0^t \|\theta\|_{L^1_\tau B^{1+\frac{2}{r}}_{r,1}} \, d\tau\right).$$

By Gronwall's inequality, for C depending on $\|\theta_0\|_{L^2 \cap L^q}$ and $\|\omega_0\|_{L^\infty \cap L^q}$ only,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leqslant C \, e^{Ct}.$$

Consequently, by (5.5), $\|\omega\|_{L^\infty_t (L^\infty \cap L^q)} \leqslant C \, e^{Ct}$. In addition, by Gagliardo–Nirenberg inequality,

$$\|u\|_{L^\infty} \leqslant C \|u\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^\infty}^{\frac{1}{2}},$$

which yields the global bound for $\|u\|_{L^\infty}$. This completes the proof for the global bounds. We remark that these global bounds are independent of ν even in the case when $\nu > 0$. Next we show that any two solutions satisfying (5.3) must coincide. This approach is now well-understood (see, e.g., [12, 43]). Assume that (u_1, θ_1) and (u_2, θ_2) are two solutions of (5.1) and let p_1 and p_2 be the associated pressures, respectively. Consider the differences

$$u = u_2 - u_1, \quad \theta = \theta_2 - \theta_1, \quad p = p_2 - p_1,$$

which satisfy

$$\partial_t u + u \cdot \nabla u_2 + u_1 \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2,$$
$$\partial_t \theta + u \cdot \nabla \theta_2 + u_1 \cdot \nabla \theta + \Lambda^\beta \theta = 0.$$

Taking the inner product with (u, θ) and integrating by parts lead to

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \nu \|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}} \theta\|_{L^2}^2 \leqslant \|u\|_{L^2} \|\theta\|_{L^2} + J_1 + J_2, \tag{5.7}$$

where

$$J_1 = -\int u \cdot \nabla u_2 \cdot u \, dx, \qquad J_2 = -\int u \cdot \nabla \theta_2 \cdot \theta \, dx.$$

For notational convenience, we let $\delta > 0$ and write

$$Y_\delta^2(t) = \delta^2 + \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2.$$

For any $\rho \in [2, \infty)$, we have

$$|J_1| \leqslant \|u\|_{L^\infty}^{\frac{2}{\rho}} \|\nabla u_2\|_{L^\rho} \|u\|_{L^2}^{2-\frac{2}{\rho}}.$$

Furthermore,

$$\|u\|_{L^\infty} \leqslant \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} = L(t), \quad \|\nabla u_2\|_{L^\rho} \leqslant \rho \sup_{\rho \geqslant 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho} \leqslant \rho M(t)$$

where we have used the fact that $\sup_{\rho \geqslant 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho}$ is bounded due to $\omega_2 \in L^q \cap L^\infty$. Therefore, by optimizing the bound over ρ, we have

$$|J_1| \leqslant M(t) \rho \left(\frac{L(t)}{Y_\delta} \right)^{\frac{2}{\rho}} Y_\delta^2 \leqslant 2e\, M(t) \left[\log L(t) - \log Y_\delta(t) \right] Y_\delta^2(t).$$

To bound J_2, we recall (5.6) to obtain

$$|J_2| \leqslant \|\nabla \theta_2\|_{L^\infty} \|u\|_{L^2} \|\theta\|_{L^2} \leqslant \|\theta_2\|_{B_{r,1}^{1+\frac{1}{r}}} Y_\delta^2(t).$$

Inserting the bounds above in (5.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_\delta(t) \leqslant 2e\, M(t) \left[\log L(t) - \log Y_\delta(t) \right] Y_\delta(t) + (1 + \|\theta_2\|_{B_{r,1}^{1+\frac{1}{r}}}) Y_\delta(t).$$

It then follows from applying the Osgood inequality that

$$Y_\delta(t) \leqslant L(t)^{1-B(t)} Y_\delta(0)^{B(t)},$$

where

$$B(t) \equiv \exp\left(-\int_0^t (2e\, M(\tau) + \|\theta_2(\tau)\|_{B_{r,1}^{1+\frac{1}{r}}}) \, \mathrm{d}\tau \right).$$

Since $Y_\delta(0) = \delta$, we obtain by letting $\delta \to 0$ that

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \equiv 0$$

for any $t > 0$. This proves the uniqueness and thus theorem 5.1. $\qquad\square$

We now prove theorem 5.3.

Proof of theorem 5.3. Since $(u^{(\nu)}, \theta^{(\nu)})$ and (u, θ) solve (5.1) and (5.2), respectively, the differences

$$\widetilde{u} = u^{(\nu)} - u, \qquad \widetilde{\theta} = \theta^{(\nu)} - \theta, \qquad \widetilde{p} = p^{(\nu)} - p,$$

satisfy

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla u + u^{(\nu)} \cdot \nabla \widetilde{u} + \nu \Lambda^\alpha \widetilde{u} + \nu \Lambda^\alpha u = -\nabla \widetilde{p} + \widetilde{\theta} e_2,$$
$$\partial_t \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta + u^{(\nu)} \cdot \nabla \widetilde{\theta} + \Lambda^\beta \widetilde{\theta} = 0.$$

Taking the inner product with $(\widetilde{u}, \widetilde{\theta})$ and integrating by parts lead to

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2 \right) + \nu \|\Lambda^{\frac{\alpha}{2}} \widetilde{u}\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}} \widetilde{\theta}\|_{L^2}^2$$
$$\leqslant \|\widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} + K_1 + K_2 + K_3, \tag{5.8}$$

where

$$K_1 = -\int \widetilde{u} \cdot \nabla u \cdot \widetilde{u} \, \mathrm{d}x, \quad K_2 = -\int \widetilde{u} \cdot \nabla \theta \widetilde{\theta} \, \mathrm{d}x, \quad K_3 = -\nu \int \widetilde{u} \cdot \Lambda^\alpha u \, \mathrm{d}x.$$

Let $\delta > 0$ and set $Z_{\nu,\delta}^2(t) = \delta^2 + \|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2$. The estimate for K_3 is easy,

$$|K_3| \leqslant \nu \|\widetilde{u}\|_{L^2} \|\Lambda^\alpha u\|_{L^2} \leqslant \nu Z_{\nu,\delta}(t) \|\Lambda^\alpha u\|_{L^2}.$$

K_1 can be bounded in a similar fashion as J_1,

$$|K_1| \leqslant 2e\,\overline{M} \left[\log \overline{L} - \log Z_{\nu,\delta}(t) \right] Z_{\nu,\delta}^2(t),$$

where L and M are given by

$$\overline{L} = \|u^{(\nu)}\|_{L^\infty([0,T];L^\infty)} + \|u\|_{L^\infty([0,T];L^\infty)}, \quad \overline{M} = \sup_{t\in[0,T]} \sup_{\rho \geqslant 2} \frac{\|\nabla u\|_{L^\rho}}{\rho}.$$

We note that $\overline{L}$ is independent of ν since the bound for $\|u^{(\nu)}\|_{L^\infty([0,T];L^\infty)}$ is independent of ν. Finally, K_2 admits the following bound

$$|K_2| \leqslant \|\theta\|_{B_{r,1}^{1+\frac{1}{r}}} Z_{\nu,\delta}^2(t).$$

Inserting the estimates for K_1, K_2 and K_3 in (5.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_{\nu,\delta}(t) \leqslant \nu \|\Lambda^\alpha u\|_{L^2} + (1 + \|\theta\|_{B_{r,1}^{1+\frac{1}{r}}}) Z_{\nu,\delta}(t)$$
$$+ 2e\,\overline{M} \left[\log \overline{L} - \log Z_{\nu,\delta}(t) \right] Z_{\nu,\delta}(t).$$

By the Osgood inequality, we have, for any $t \leqslant T$,

$$Z_{\nu,\delta}(t) \leqslant L^{1-\mathrm{e}^{-b(t)}} \left(Z_{\nu,\delta}(0) + \nu t \, \|\Lambda^\alpha u\|_{L_t^\infty L^2} \right)^{\mathrm{e}^{-b(t)}},$$

where

$$b(t) = 2e\overline{M}t + \int_0^t \|\theta(\tau)\|_{B_{r,1}^{1+\frac{1}{r}}} \,\mathrm{d}\tau.$$

Letting $\delta \to 0$, we obtain (5.4). This completes the proof of theorem 5.3. $\qquad\square$

Finally we prove an estimate that has been used in the proof of theorem 5.1.

Lemma 5.4. *Assume that $\nabla \cdot u = 0$ and θ solves*

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \theta(x,\,0) = \theta_0(x). \end{cases} \tag{5.9}$$

Let $r \in [2, \infty)$. Then θ obeys the a priori estimate: for any integer $j \geqslant 0$,

$$2^{\beta j} \|\Delta_j \theta\|_{L_t^1 L^r} \leqslant C \|\Delta_j \theta_0\|_{L^r}$$

$$+ C \left[\|\Delta_{-1} \nabla u\|_{L_t^1 L^\infty} + \sum_{m=0}^{j-2} \|\Delta_m \omega\|_{L_t^1 L^\infty} \right] \|\Delta_j \theta\|_{L_t^\infty L^r}$$

$$+ C \|\omega\|_{L_t^1 L^\infty} \left[\|\Delta_j \theta\|_{L^r} + \sum_{m \leqslant j-2} 2^{(m-j)} \|\Delta_m \theta\|_{L^r} + \sum_{k \geqslant j-1} 2^{(j-k)} \|\Delta_k \theta\|_{L^r} \right]. \tag{5.10}$$

In particular,

$$2^{\beta j} \|\Delta_j \theta\|_{L_t^1 L^r} \leqslant C \|\theta_0\|_{L^r} \left(1 + \|\omega\|_{L_t^1 L^q} + (j+2)\|\omega\|_{L_t^1 L^\infty} \right). \tag{5.11}$$

Proof of lemma 5.4. Applying Δ_j to (5.9) and taking inner product with $\Delta_j \theta |\Delta_j \theta|^{r-2}$, we obtain

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \theta\|_{L^r}^r + \int \Delta_j \theta \, |\Delta_j \theta|^{r-2} \Lambda^\beta \theta \, dx = - \int \Delta_j \theta |\Delta_j \theta|^{r-2} \Delta_j (u \cdot \nabla \theta) \, \mathrm{d}x. \tag{5.12}$$

The dissipative term obeys the lower bound (see [11, 41])

$$\int \Delta_j \theta \, |\Delta_j \theta|^{r-2} \Lambda^\beta \theta \, \mathrm{d}x \geqslant C \, 2^{\beta j} \|\Delta_j \theta\|_{L^r}^r.$$

Using the notion of para-products, we write

$$\Delta_j(u \cdot \nabla\theta) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$J_{11} = \sum_{|j-k|\leqslant 2} [\Delta_j, S_{k-1}u \cdot \nabla]\Delta_k\theta,$$

$$J_{12} = \sum_{|j-k|\leqslant 2} (S_{k-1}u - S_j u) \cdot \nabla\Delta_j\Delta_k\theta,$$

$$J_{13} = S_j u \cdot \nabla\Delta_j\theta,$$

$$J_{14} = \sum_{|j-k|\leqslant 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1}\theta),$$

$$J_{15} = \sum_{k\geqslant j-1} \Delta_j(\Delta_k u \cdot \nabla\widetilde{\Delta}_k\theta).$$

Since $\nabla \cdot u = 0$, we have

$$\int J_{13}|\Delta_j\theta|^{r-2}\Delta_j\theta \, \mathrm{d}x = 0.$$

By Hölder's inequality,

$$\left| \int J_{11}|\Delta_j\theta|^{r-2}\Delta_j\theta \, \mathrm{d}x \right| \leqslant \|J_{11}\|_{L^r}\|\Delta_j\theta\|_{L^r}^{r-1}.$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla\Delta_k\theta(y)\,\mathrm{d}y,$$

where Φ_j is the kernel of the operator Δ_j (see the appendix). By Young's inequality for convolution,

$$\begin{aligned}
\|J_{11}\|_{L^r} &\leqslant \||x|\Psi_j(x)\|_{L^1}\,\|\nabla S_{j-1}u\|_{L^\infty}\,\|\nabla\Delta_j\theta\|_{L^r} \\
&\leqslant \||x|\Psi_0(x)\|_{L^1}\,\|\nabla S_{j-1}u\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r} \\
&= C\,\|\nabla S_{j-1}u\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r},
\end{aligned}$$

where we have used the definition of Φ_j and Bernstein's inequality (see the appendix) in the second inequality above. By Bernstein's inequality,

$$\begin{aligned}
\|J_{12}\|_{L^r} &\leqslant C\|\Delta_j u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^r} \\
&\leqslant C\,\|\nabla\Delta_j u\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r} \\
&\leqslant C\,\|\omega\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r};
\end{aligned}$$

$$\begin{aligned}
\|J_{14}\|_{L^r} &\leqslant C\|\Delta_j u\|_{L^\infty}\|\nabla S_{j-1}\theta\|_{L^r} \\
&\leqslant C\,\|\nabla\Delta_j u\|_{L^\infty} \sum_{m\leqslant j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r} \\
&\leqslant C\,\|\omega\|_{L^\infty} \sum_{m\leqslant j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r};
\end{aligned}$$

$$\begin{aligned}
\|J_{15}\|_{L^r} &\leqslant C \sum_{k\geqslant j-1} 2^{(j-k)}\|\nabla\Delta_k u\|_{L^\infty}\|\Delta_k\theta\|_{L^r} \\
&\leqslant C\,\|\omega\|_{L^\infty} \sum_{k\geqslant j-1} 2^{(j-k)}\|\Delta_k\theta\|_{L^r}.
\end{aligned}$$

Inserting the estimates above in (5.12), we have

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^r} + C\,2^{\beta j}\|\Delta_j\theta\|_{L^r} \leqslant C\,\|\nabla S_{j-1}u\|_{L^\infty}\|\Delta_j\theta\|_{L^r}$$

$$+\, C\,\|\omega\|_{L^\infty}\left[\|\Delta_j\theta\|_{L^r} + \sum_{m\leqslant j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r} + \sum_{k\geqslant j-1} 2^{(j-k)}\|\Delta_k\theta\|_{L^r}\right].$$

Integrating in time, taking L_t^1 and multiplying by $2^{\beta j}$, we obtain (5.10). This completes the proof of lemma 5.4. $\qquad\square$

Appendix A. Functional spaces and Osgood inequality

This appendix provides the definitions of some of the functional spaces and related facts used in the previous sections. In addition, the Osgood inequality used in the proofs of theorems 5.1 and 5.3 is also provided here for the convenience of readers. Materials presented in this appendix can be found in several books and many papers (see, e.g., [4, 5, 34, 38, 40]).

We start with several notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{\xi \in \mathbb{R}^d : \ 2^{j-1} \leqslant |\xi| < 2^{j+1}\right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & , \quad \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & , \quad \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A.1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leqslant -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.2}$$

In addition to the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k.$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (A.1) that $S_j \to Id$ as $j \to \infty$ in the distributional sense.

Definition A.1. *The inhomogeneous Besov space $B_{p,q}^s$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B_{p,q}^s} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty,$$

where $\Delta_j f$ is as defined in (A.2).

We have also used the space–time space defined below.

Definition A.2. *For $t > 0$, $s \in \mathbb{R}$ and $1 \leqslant p, q, r \leqslant \infty$, the space-time space $\widetilde{L}_t^r B_{p,q}^s$ is defined through the norm*

$$\|f\|_{\widetilde{L}_t^r B_{p,q}^s} \equiv \|2^{js}\|\Delta_j f\|_{L_t^r L^p}\|_{l^q}.$$

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition A.3. *Let $\alpha \geqslant 0$. Let $1 \leqslant p \leqslant q \leqslant \infty$.*

(1) If f satisfies

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leqslant K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_1 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) If f satisfies

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leqslant |\xi| \leqslant K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leqslant K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leqslant \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_2 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Finally we recall the Osgood inequality.

Proposition A.4. *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geqslant 0$ satisfies*

$$\int_0^\infty \frac{1}{\omega(r)} \, \mathrm{d}r = \infty.$$

Suppose that $\rho(t) > 0$ satisfies

$$\rho(t) \leqslant a + \int_{t_0}^t \alpha(s)\omega(\rho(s)) \, \mathrm{d}s$$

for some constant $a \geqslant 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then

$$-\Omega(\rho(t)) + \Omega(a) \leqslant \int_{t_0}^t \alpha(\tau) \, \mathrm{d}\tau,$$

where

$$\Omega(x) = \int_x^1 \frac{\mathrm{d}r}{\omega(r)}.$$

Acknowledgments

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