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Local Well-Posedness for the Hall-MHD Equations with Fractional Magnetic Diffusion

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Abstract. The Hall-magnetohydrodynamics (Hall-MHD) equations, rigorously derived from kinetic models, are useful in describing many physical phenomena in geophysics and astrophysics. This paper studies the local well-posedness of classical solutions to the Hall-MHD equations with the magnetic diffusion given by a fractional Laplacian operator, $(-\Delta)^\alpha$. Due to the presence of the Hall term in the Hall-MHD equations, standard energy estimates appear to indicate that we need $\alpha \geq 1$ in order to obtain the local well-posedness. This paper breaks the barrier and shows that the fractional Hall-MHD equations are locally well-posed for any $\alpha > \frac{1}{2}$. The approach here fully exploits the smoothing effects of the dissipation and establishes the local bounds for the Sobolev norms through the Besov space techniques. The method presented here may be applicable to similar situations involving other partial differential equations.

1. Introduction

This paper focuses on the Hall-magnetohydrodynamics (Hall-MHD) equations with fractional magnetic diffusion,

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = B \cdot \nabla B, \\
\partial_t B + u \cdot \nabla B + \nabla \times ((\nabla \times B) \times B) + (-\Delta)^\alpha B = B \cdot \nabla u, \\
\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\
u(x,0) = u_0(x), \quad B(x,0) = B_0(x),
\end{cases}
\tag{1.1}
$$

where $x \in \mathbb{R}^d$ with $d \geq 2$, $u = u(x,t)$ and $B = B(x,t)$ are vector fields representing the velocity and the magnetic field, respectively, $p = p(x,t)$ denotes the pressure, $\alpha > 0$ is a parameter and the fractional Laplacian $(-\Delta)^\alpha$ is defined through the Fourier transform,

$$
\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \, \widehat{f}(\xi).
$$

For notational convenience, we also use Λ for $(-\Delta)^{\frac{1}{2}}$. The Hall-MHD equations with the usual Lapalcian dissipation were derived in [1] from kinetic models. The Hall-MHD equations differ from the standard incompressible MHD equations in the Hall term $\nabla \times ((\nabla \times B) \times B)$, which is important in the study of magnetic reconnection (see, e.g., [8,13]). The Hall-MHD equations have been mathematically investigated in several works ([1,4–7]). Global weak solutions of (1.1) with both Δu and ΔB and local classical solutions of (1.1) with ΔB (with or without Δu) were obtained in [4]. In addition, a blowup criterion and the global existence of small classical solutions were also established in [4]. These results were later sharpened by [5].

We examine the issue of whether or not (1.1) is locally well-posedness when the fractional power $\alpha < 1$. Previously local solutions of (1.1) were obtained for $\alpha = 1$ ([4,5]). Standard energy estimates appear to indicate that $\alpha \geq 1$ is necessary in order to obtain local bounds for the solutions in Sobolev spaces. This

requirement comes from the estimates of the regularity-demanding Hall term $\nabla \times ((\nabla \times B) \times B)$. To understand more precisely the issue at hand, we perform a short energy estimate on the essential part of the equation for B,

$$\partial_t B + \nabla \times ((\nabla \times B) \times B) + (-\Delta)^\alpha B = 0.$$

The global L^2-bound

$$\|B(t)\|_{L^2}^2 + 2 \int_0^t \|\Lambda^\alpha B(\tau)\|_{L^2}^2 d\tau = \|B_0\|_{L^2}^2 \tag{1.2}$$

follows from the simple fact

$$\int \nabla \times ((\nabla \times B) \times B) \cdot B = \int ((\nabla \times B) \times B) \cdot (\nabla \times B) = 0.$$

To obtain the H^1-bound, we invoke the equation for $\|\nabla B\|_{L^2}^2$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2 + \|\Lambda^\alpha \nabla B\|_{L^2}^2 = -\sum_{i=1}^d \int \partial_i \nabla \times ((\nabla \times B) \times B) \cdot \partial_i B.$$

We can indeed shift one-derivative, namely

$$\int \partial_i \nabla \times ((\nabla \times B) \times B) \cdot \partial_i B = \int ((\nabla \times B) \times \partial_i B) \cdot \partial_i \nabla \times B.$$

Hölder's inequality allows us to conclude that

$$\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2 + \|\Lambda^\alpha \nabla B\|_{L^2}^2 \leq \|\nabla B\|_{L^2} \|\nabla B\|_{L^\infty} \|\nabla \nabla \times B\|_{L^2}.$$

Therefore, it appears that we need $\alpha \geq 1$ in order to bound the term $\|\nabla \nabla \times B\|_{L^2}$ on the right-hand side. More generally, the energy inequality involving the H^σ-norm

$$\frac{1}{2}\frac{d}{dt}\|B\|_{H^\sigma}^2 + \|\Lambda^\alpha B\|_{H^\sigma}^2 \leq C \|B\|_{H^\sigma} \|\nabla B\|_{L^\infty} \|\nabla B\|_{H^\sigma}.$$

also appears to demand that $\alpha \geq 1$ in order to bound $\|\nabla B\|_{H^\sigma}$.

This paper obtains the local existence and uniqueness of solutions to (1.1) with any $\alpha > \frac{1}{2}$. More precisely, we prove the following theorem.

Theorem 1.1. *Consider* (1.1) *with* $\alpha > \frac{1}{2}$. *Assume* $(u_0, B_0) \in H^\sigma(\mathbb{R}^d)$ *with* $\sigma > 1 + \frac{d}{2}$, *and* $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. *Then there exist* $T_0 = T_0(\|(u_0, B_0)\|_{H^\sigma}) > 0$ *and a unique solution* (u, B) *of* (1.1) *on* $[0, T_0]$ *such that*

$$(u, B) \in L^\infty([0, T_0]; H^\sigma(\mathbb{R}^d)).$$

In addition, for any $\sigma' < \sigma$,

$$(u, B) \in C([0, T_0]; H^{\sigma'}(\mathbb{R}^d))$$

and $\|(u(t), B(t))\|_{H^\sigma}$ *is continuous from the right on* $[0, T_0)$.

The essential idea of proving Theorem 1.1 is to fully exploit the dissipation in the equation for B and estimate the Sobolev norm $\|(u, B)\|_{H^\sigma}$ via Besov space techniques. We identify H^σ with the Besov space $B_{2,2}^\sigma$ and suitably shift the derivatives in the nonlinear term. The definition of Besov spaces and related facts used in this paper are provided in the appendix. The rest of this paper is divided into two sections followed by an appendix. Section 2 states and proves the result for the local a priori bound. Section 3 presents the complete proof of Theorem 1.1. The appendix supplies the definitions of the Littlewood–Paley decomposition and Besov spaces.

2. Local A Priori Bound

This section establishes a local a priori bound for smooth solutions of (1.1), which is the key component in the proof of Theorem 1.1. The result for the local a priori bound can be stated as follows.

Proposition 2.1. *Consider* (1.1) *with* $\alpha > \frac{1}{2}$. *Assume the initial data* $(u_0, B_0) \in H^\sigma(\mathbb{R}^d)$ *with* $\sigma > 1 + \frac{d}{2}$. *Let* (u, B) *be the corresponding solution. Then, there exists* $T_0 = T_0(\|(u_0, B_0)\|_{H^\sigma}) > 0$ *such that, for* $t \in [0, T_0]$,

$$\|(u(t), B(t))\|_{H^\sigma} \le C(\alpha, T_0, \|(u_0, B_0)\|_{H^\sigma})$$

and

$$\int_0^{T_0} \|\Lambda^\alpha B(s)\|_{H^\sigma}^2 \, ds \le C(\alpha, T_0, \|(u_0, B_0)\|_{H^\sigma}).$$

Proof of Proposition 2.1. The proof identifies the Sobolev space H^σ with the Besov space $B_{2,2}^\sigma$ and resorts to Besov space techniques.

Let $l \in \mathbb{Z}$ be an integer and let Δ_l denote the homogeneous frequency localized operator. Applying Δ_l to (1.1) yields

$$\partial_t \Delta_l u + \Delta_l(u \cdot \nabla u) + \nabla \Delta_l p = \Delta_l(B \cdot \nabla B),$$
$$\partial_t \Delta_l B + \Delta_l(u \cdot \nabla B) + \Delta_l \nabla \times ((\nabla \times B) \times B) + (-\Delta)^\alpha \Delta_l B = \Delta_l(B \cdot \nabla u).$$

Taking the inner product with $(\Delta_l u, \Delta_l B)$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta_l u\|_{L^2}^2 + \|\Delta_l B\|_{L^2}^2\right) + C_0 2^{2\alpha l}\|\Delta_l B\|_{L^2}^2 \le K_1 + K_2 + K_3 + K_4 + K_5, \tag{2.1}$$

where

$$K_1 = -\int [\Delta_l, u \cdot \nabla]u \cdot \Delta_l u, \qquad K_2 = -\int [\Delta_l, u \cdot \nabla]B \cdot \Delta_l B,$$

$$K_3 = \int [\Delta_l, B \cdot \nabla]B \cdot \Delta_l u, \qquad K_4 = \int [\Delta_l, B \cdot \nabla]u \cdot \Delta_l B,$$

$$K_5 = -\int \Delta_l \nabla \times ((\nabla \times B) \times B) \cdot \Delta_l B.$$

Note that we have used the standard commutator notation,

$$[\Delta_l, u \cdot \nabla]u = \Delta_l(u \cdot \nabla u) - u \cdot \nabla(\Delta_l u)$$

and applied the lower bound, for a constant $C_0 > 0$,

$$\int \Delta_l B \cdot (-\Delta)^\alpha \Delta_l B \ge C_0 2^{2\alpha l}\|\Delta_l B\|_{L^2}^2.$$

Using the notion of paraproducts, we write

$$K_1 = K_{11} + K_{12} + K_{13},$$

where

$$K_{11} = \sum_{|k-l|\le 2} \int \left(\Delta_l(S_{k-1}u \cdot \nabla \Delta_k u) - S_{k-1}u \cdot \nabla \Delta_l \Delta_k u\right) \cdot \Delta_l u,$$

$$K_{12} = \sum_{|k-l|\le 2} \int \left(\Delta_l(\Delta_k u \cdot \nabla S_{k-1}u) - \Delta_k u \cdot \nabla \Delta_l S_{k-1}u\right) \cdot \Delta_l u,$$

$$K_{13} = \sum_{k\ge l-1} \int \left(\Delta_l(\Delta_k u \cdot \nabla \widetilde{\Delta}_k u) - \Delta_k u \cdot \nabla \Delta_l \widetilde{\Delta}_k u\right) \cdot \Delta_l u$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. By Hölder's inequality and a standard commutator estimate,

$$
\begin{aligned}
|K_{11}| &\leq C \, \|\nabla S_{l-1} u\|_{L^\infty} \, \|\Delta_l u\|_{L^2} \sum_{|k-l| \leq 2} \|\Delta_k u\|_{L^2} \\
&\leq C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l u\|_{L^2} \sum_{|k-l| \leq 2} \|\Delta_k u\|_{L^2}.
\end{aligned}
$$

Since the summation over k for fixed l above consists of only a finite number of terms and, as we shall later in the proof, the norm generated by each term is a multiple of that generated by the typical term, it suffices to keep the typical term with $k = l$ and ignore the summation. This would help keep our presentation concise. We will invoke this practice throughout the rest of the paper. By Hölder's inequality, K_{12} is bounded by

$$
|K_{12}| \leq C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l u\|_{L^2}^2.
$$

By Hölder's inequality and Bernstein's inequality,

$$
|K_{13}| \leq C \, \|\Delta_l u\|_{L^2} \, \|\nabla u\|_{L^\infty} \sum_{k \geq l-1} 2^{l-k} \, \|\Delta_k u\|_{L^2}.
$$

Therefore,

$$
|K_1| \leq C \, \|\Delta_l u\|_{L^2} \, \|\nabla u\|_{L^\infty} \left(\|\Delta_l u\|_{L^2} + \sum_{k \geq l-1} 2^{l-k} \, \|\Delta_k u\|_{L^2} \right).
$$

Similarly, K_2, K_3 and K_4 are bounded by

$$
\begin{aligned}
|K_2| \leq{}& C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l B\|_{L^2}^2 + C \, \|\nabla B\|_{L^\infty} \, \|\Delta_l u\|_{L^2} \|\Delta_l B\|_{L^2} \\
&+ C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l B\|_{L^2} \sum_{k \geq l-1} 2^{l-k} \, \|\Delta_k B\|_{L^2}, \\
|K_3| \leq{}& C \, \|\nabla B\|_{L^\infty} \, \|\Delta_l u\|_{L^2} \left(\|\Delta_l B\|_{L^2} + \sum_{k \geq l-1} 2^{l-k} \, \|\Delta_k B\|_{L^2} \right), \\
|K_4| \leq{}& C \, \|\nabla B\|_{L^\infty} \, \|\Delta_l u\|_{L^2} \|\Delta_l B\|_{L^2} + C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l B\|_{L^2}^2 \\
&+ C \, \|\nabla u\|_{L^\infty} \, \|\Delta_l B\|_{L^2} \sum_{k \geq l-1} 2^{l-k} \, \|\Delta_k B\|_{L^2}.
\end{aligned}
$$

Using the cancelation property,

$$
\int B \times \Delta_l (\nabla \times B) \cdot \Delta_l (\nabla \times B) = 0,
$$

we can rewrite K_5 as

$$
\begin{aligned}
K_5 &= \int \left(\Delta_l (B \times (\nabla \times B)) - B \times (\Delta_l \nabla \times B) \right) \cdot \Delta_l \nabla \times B \\
&= \int [\Delta_l, B \times](\nabla \times B) \cdot \Delta_l (\nabla \times B) = K_{51} + K_{52} + K_{53},
\end{aligned}
$$

where

$$K_{51} = \sum_{|k-l|\leq 2} \int \left(\Delta_l(S_{k-1}B \times (\nabla \times \Delta_k B)) - S_{k-1}B \times (\nabla \times \Delta_l \Delta_k B)\right) \cdot \Delta_l \nabla \times B,$$

$$K_{52} = \sum_{|k-l|\leq 2} \int \left(\Delta_l(\Delta_k B \times (\nabla \times S_{k-1}B)) - \Delta_k B \times (\nabla \times \Delta_l S_{k-1}B)\right) \cdot \Delta_l \nabla \times B,$$

$$K_{53} = \sum_{k\geq l-1} \int \left(\Delta_l(\Delta_k B \times (\nabla \times \widetilde{\Delta}_k B)) - \Delta_k B \times (\nabla \times \Delta_l \widetilde{\Delta}_k B)\right) \cdot \Delta_l \nabla \times B.$$

By Hölder's inequality and a standard commutator estimate,

$$|K_{51}| \leq C \sum_{|k-l|\leq 2} \|\nabla S_{k-1}B\|_{L^\infty} \|\Delta_k B\|_{L^2} \|\Delta_l \nabla \times B\|_{L^2}$$
$$\leq C\, 2^l \, \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2}^2.$$

By Hölder's inequality,

$$|K_{52}| \leq C\, 2^l \, \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2}^2.$$

By Hölder's inequality and Bernstein's inequality,

$$|K_{53}| \leq C\, 2^l \, \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2} \sum_{k\geq l-1} \|\Delta_k B\|_{L^2}.$$

Therefore,

$$|K_5| \leq C\, 2^l \, \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2} \left(\|\Delta_l B\|_{L^2} + \sum_{k\geq l-1} \|\Delta_k B\|_{L^2}\right).$$

Inserting the estimates above in (2.1), we obtain

$$\frac{d}{dt}\left(\|\Delta_l u\|_{L^2}^2 + \|\Delta_l B\|_{L^2}^2\right) + C_0 2^{2\alpha l}\|\Delta_l B\|_{L^2}^2$$
$$\leq C\,\|(\nabla u, \nabla B)\|_{L^\infty}\left(\|\Delta_l u\|_{L^2}^2 + \|\Delta_l B\|_{L^2}^2\right)$$
$$+ C\,\|(\nabla u, \nabla B)\|_{L^\infty}\left[\left(\sum_{k\geq l-1} 2^{l-k}\|\Delta_k u\|_{L^2}\right)^2 + \left(\sum_{k\geq l-1} 2^{l-k}\|\Delta_k B\|_{L^2}\right)^2\right]$$
$$+ C\, 2^l\|\nabla B\|_{L^\infty}\|\Delta_l B\|_{L^2}^2 + C\, 2^l\|\nabla B\|_{L^\infty}\|\Delta_l B\|_{L^2}\sum_{k\geq l-1}\|\Delta_k B\|_{L^2}.$$

Multiplying the inequality above by $2^{2\sigma l}$ and summing over $l \in \mathbb{Z}$, invoking the global bound for the L^2-norm of (u, B) and the equivalence of the norms

$$\|f\|_{H^\sigma}^2 \sim \|f\|_{L^2}^2 + \|f\|_{\dot{H}^\sigma}^2,$$

we have

$$\|u(t)\|_{H^\sigma}^2 + \|B(t)\|_{H^\sigma}^2 + C_0 \int_0^t \|\Lambda^\alpha B(\tau)\|_{H^\sigma}^2 \, d\tau$$

$$\leq \|u_0\|_{H^\sigma}^2 + \|B_0\|_{H^\sigma}^2 + C \int_0^t \|(\nabla u, \nabla B)\|_{L^\infty} \left(\|u(\tau)\|_{\dot{H}^\sigma}^2 + \|B(\tau)\|_{\dot{H}^\sigma}^2 \right) d\tau$$

$$+ C \sum_{l \in \mathbb{Z}} 2^{(2\sigma+1)l} \int_0^t \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2}^2 \, d\tau$$

$$+ C \sum_{l \in \mathbb{Z}} 2^{(2\sigma+1)l} \int_0^t \|\nabla B\|_{L^\infty} \left(\sum_{k \geq l-1} \|\Delta_k B\|_{L^2} \right)^2 d\tau. \qquad (2.2)$$

To derive the inequality above, we have used Young's inequality for series convolution

$$\sum_{l \in \mathbb{Z}} 2^{2\sigma l} \left(\sum_{k \geq l-1} 2^{l-k} \|\Delta_k u\|_{L^2} \right)^2 = \sum_{l \in \mathbb{Z}} \left(\sum_{k \geq l-1} 2^{(\sigma+1)(l-k)} 2^{\sigma k} \|\Delta_k u\|_{L^2} \right)^2$$

$$\leq C \sum_{l \in \mathbb{Z}} 2^{2\sigma l} \|\Delta_l u\|_{L^2}^2 \leq C \|u\|_{\dot{H}^\sigma}^2.$$

We further bound the last two terms in (2.2),

$$L_1 \equiv C \sum_{l \in \mathbb{Z}} 2^{(2\sigma+1)l} \int_0^t \|\nabla B\|_{L^\infty} \|\Delta_l B\|_{L^2}^2 \, d\tau,$$

$$L_2 \equiv C \sum_{l \in \mathbb{Z}} 2^{(2\sigma+1)l} \int_0^t \|\nabla B\|_{L^\infty} \left(\sum_{k \geq l-1} \|\Delta_k B\|_{L^2} \right)^2 d\tau.$$

Set $\theta = 1 - \frac{1}{2\alpha}$. For $\alpha > \frac{1}{2}$, $\theta \in (0,1)$. By Hölder's inequality,

$$L_1 = C \int_0^t \|\nabla B\|_{L^\infty} \sum_{l \in \mathbb{Z}} \left(2^{2\sigma l} \|\Delta_l B\|_{L^2}^2 \right)^\theta \left(2^{2(\sigma+\alpha)l} \|\Delta_l B\|_{L^2}^2 \right)^{(1-\theta)} d\tau$$

$$\leq C \int_0^t \|\nabla B\|_{L^\infty} \left(\sum_{l \in \mathbb{Z}} 2^{2\sigma l} \|\Delta_l B\|_{L^2}^2 \right)^\theta \left(\sum_{l \in \mathbb{Z}} 2^{2(\sigma+\alpha)l} \|\Delta_l B\|_{L^2}^2 \right)^{(1-\theta)} d\tau$$

$$\leq C \int_0^t \|\nabla B\|_{L^\infty}^{\frac{1}{\theta}} \|B\|_{\dot{H}^\sigma}^2 \, d\tau + \frac{C_0}{4} \int_0^t \|B(\tau)\|_{\dot{H}^{\sigma+\alpha}}^2 \, d\tau.$$

By Young's inequality for series convolution and an interpolation inequality,

$$L_2 = C \int_0^t \|\nabla B\|_{L^\infty} \sum_{l \in \mathbb{Z}} \left(\sum_{k \geq l-1} 2^{(l-k)(\sigma+\frac{1}{2})} 2^{(\sigma+\frac{1}{2})k} \|\Delta_k B\|_{L^2} \right)^2 d\tau$$

$$\leq C \int_0^t \|\nabla B\|_{L^\infty} \|B\|_{\dot{H}^{\sigma+\frac{1}{2}}}^2 \, d\tau \leq C \int_0^t \|\nabla B\|_{L^\infty} \|B\|_{\dot{H}^\sigma}^{2\theta} \|B\|_{\dot{H}^{\sigma+\alpha}}^{2(1-\theta)} \, d\tau$$

$$\leq C \int_0^t \|\nabla B\|_{L^\infty}^{\frac{1}{\theta}} \|B\|_{\dot{H}^\sigma}^2 \, d\tau + \frac{C_0}{4} \int_0^t \|B(\tau)\|_{\dot{H}^{\sigma+\alpha}}^2 \, d\tau.$$

Inserting the estimates above in (2.2) and invoking the embedding inequalities

$$\|\nabla B\|_{L^\infty} \leq C \|B\|_{H^\sigma} \qquad \text{for } \sigma > 1 + \frac{d}{2},$$

we have

$$\|u(t)\|_{H^\sigma}^2 + \|B(t)\|_{H^\sigma}^2 + C_0 \int_0^t \|B(\tau)\|_{\dot{H}^{\sigma+\alpha}}^2 \, d\tau$$

$$\leq \|u_0\|_{H^\sigma}^2 + \|B_0\|_{H^\sigma}^2 + C \int_0^t \left(\|u(t)\|_{H^\sigma}^2 + \|B(t)\|_{H^\sigma}^2 \right)^\gamma \, d\tau, \qquad (2.3)$$

for a constant $\gamma > 1$. This inequality implies a local bound for $\|u(t)\|_{H^\sigma}^2 + \|B(t)\|_{H^\sigma}^2$, namely for some $T_0 = T_0(\|(u_0, B_0)\|_{H^\sigma}) > 0$ such that, for $t \in [0, T_0]$,

$$\|u(t)\|_{H^\sigma} + \|B(t)\|_{H^\sigma} \leq C(u_0, B_0, \alpha, T_0)$$

and

$$\int_0^{T_0} \|B(\tau)\|_{H^{\sigma+\alpha}}^2 \, d\tau < \infty. \qquad (2.4)$$

This completes the proof of Proposition 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. Local Existence and Uniqueness

This section proves Theorem 1.1.

Proof of Theorem 1.1. The local existence and uniqueness can be obtained through an approximation procedure. Here we use the Friedrichs method, a smoothing approach through filtering the high frequencies. For each positive integer n, we define

$$\widehat{\mathcal{J}_n f}(\xi) = \chi_{B_n}(\xi) \, \widehat{f}(\xi),$$

where B_n denotes the closed ball of radius n centered at 0 and χ_{B_n} denotes the characteristic functions on B_n. Denote

$$H_n^\sigma \equiv \left\{ f \in H^\sigma(\mathbb{R}^d), \; \mathrm{supp}\widehat{f} \subset B_n \right\}.$$

We seek a solution $(u, B) \in H_n^\sigma$ satisfying

$$\begin{cases} \partial_t u + \mathcal{J}_n \mathcal{P}(\mathcal{J}_n \mathcal{P} u \cdot \nabla \mathcal{J}_n \mathcal{P} u) = \mathcal{J}_n \mathcal{P}(\mathcal{J}_n \mathcal{P} B \cdot \nabla \mathcal{J}_n \mathcal{P} B), \\ \partial_t B + \mathcal{J}_n \mathcal{P}(\mathcal{J}_n \mathcal{P} u \cdot \nabla \mathcal{J}_n \mathcal{P} B) + \mathcal{J}_n \mathcal{P}(\nabla \times ((\nabla \times \mathcal{J}_n \mathcal{P} B) \times \mathcal{J}_n \mathcal{P} B)) \\ \quad + (-\Delta)^\alpha \mathcal{J}_n \mathcal{P} B = \mathcal{J}_n \mathcal{P}(\mathcal{J}_n \mathcal{P} B \cdot \nabla \mathcal{J}_n \mathcal{P} u), \\ u(x, 0) = (\mathcal{J}_n u_0)(x), \quad B(x, 0) = (\mathcal{J}_n B_0)(x), \end{cases} \qquad (3.1)$$

where $\mathcal{P}$ denotes the projection onto divergence-free vector fields.

For each fixed $n \geq 1$, it is not very hard, although tedious, to verify that the right-hand side of (3.1) satisfies the Lipschitz condition in H_n^σ and, by Picard's theorem, (3.1) has a unique global (in time) solution. The uniqueness implies that

$$\mathcal{J}_n \mathcal{P} u = u, \quad \mathcal{J}_n \mathcal{P} B = B$$

and ensures the divergence-free conditions $\nabla \cdot u = 0$ and $\nabla \cdot B = 0$. Then, (3.1) is simplified to

$$\begin{cases} \partial_t u + \mathcal{J}_n \mathcal{P}(u \cdot \nabla u) = \mathcal{J}_n \mathcal{P}(B \cdot \nabla B), \\ \partial_t B + \mathcal{J}_n \mathcal{P}(u \cdot \nabla B) + \mathcal{J}_n \mathcal{P}(\nabla \times ((\nabla \times B) \times B)) + (-\Delta)^\alpha B = \mathcal{J}_n \mathcal{P}(B \cdot \nabla u). \end{cases}$$

We denote this solution by (u^n, B^n). As in the proof of Proposition 2.1, we can show that (u^n, B^n) satisfies

$$\|(u^n, B^n)\|_{H^\sigma}^2 \leq \|(u_0^n, B_0^n)\|_{H^\sigma}^2 + C \int_0^t \|(u^n(s), B^n(s))\|_{H^\sigma}^{2\gamma} \, ds \qquad (3.2)$$

for some $\gamma > 1$. Due to $\|(u_0^n, B_0^n)\|_{H^\sigma} \leq \|(u_0, B_0)\|_{H^\sigma}$, this inequality is uniform in n. This allows us to obtain a uniform local bound

$$\sup_{t \in [0,T_0]} \|(u^n(t), B^n(t))\|_{H^\sigma} \leq M(\alpha, T_0, \|(u_0, B_0)\|_{H^\sigma}). \tag{3.3}$$

As in (2.4), we also have the uniform local bound for the time integral

$$\int_0^{T_0} \|(\Lambda^\alpha B^n)(s)\|_{H^\sigma}^2 \, ds \leq M(\alpha, T_0, \|(u_0, B_0)\|_{H^\sigma}).$$

Furthermore, these uniform bounds allow us to show that

$$\|(u^n, B^n) - (u^m, B^m)\|_{L^2} \to 0 \qquad \text{as} \quad n, m \to \infty. \tag{3.4}$$

This is shown through standard energy estimates for $\|(u^n, B^n) - (u^m, B^m)\|_{L^2}$. The process involves many terms, but most of them can be handled in a standard fashion (see, e.g., [9, p. 107]). We provide the detailed energy estimate for the term that is special here, namely the Hall term $\nabla \times ((\nabla \times B) \times B)$. In the process of the energy estimates, we need to bound the term

$$\int \left(\nabla \times ((\nabla \times B^n) \times B^n) - \nabla \times ((\nabla \times B^m) \times B^m) \right) \cdot (B^n - B^m) \, dx$$

$$= \int \left(\nabla \times ((\nabla \times (B^n - B^m)) \times B^n) \right) \cdot (B^n - B^m) \, dx$$

$$+ \int \left(\nabla \times ((\nabla \times B^m) \times (B^n - B^m)) \right) \cdot (B^n - B^m) \, dx. \tag{3.5}$$

The first term on the right-hand side of (3.5) is zero,

$$\int \left(\nabla \times ((\nabla \times (B^n - B^m)) \times B^n) \right) \cdot (B^n - B^m) \, dx$$

$$= \int \left((\nabla \times (B^n - B^m)) \times B^n \right) \cdot (\nabla \times (B^n - B^m)) \, dx = 0.$$

For the second term on the right of (3.5), by the simple vector identity

$$\nabla \times ((\nabla \times B^m) \times (B^n - B^m))$$
$$= (B^n - B^m) \cdot \nabla (\nabla \times B^m) - (\nabla \times B^m) \cdot \nabla (B^n - B^m),$$

we have

$$\left| \int \left(\nabla \times ((\nabla \times B^m) \times (B^n - B^m)) \right) \cdot (B^n - B^m) \, dx \right|$$

$$\leq \|\nabla(\nabla \times B^m)\|_{L^{\frac{d}{\alpha}}} \|B^n - B^m\|_{L^{\frac{2d}{d-2\alpha}}} \|B^n - B^m\|_{L^2}$$

$$\leq C\|\Lambda^\alpha B^m\|_{H^\sigma}^2 \|B^n - B^m\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha(B^n - B^m)\|_{L^2}^2,$$

where we have used

$$\|\nabla^2 f\|_{L^{\frac{d}{\alpha}}} \leq C\|\Lambda^\alpha f\|_{H^\sigma}, \quad \|f\|_{L^{\frac{2d}{d-2\alpha}}} \leq C\|\Lambda^\alpha f\|_{L^2}.$$

Putting together the estimates for all the terms, we obtain

$$\frac{d}{dt}\|B^n - B^m\|_{L^2}^2 \leq C\|\Lambda^\alpha B^m\|_{H^\sigma}^2 \|B^n - B^m\|_{L^2}^2 + C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Noticing that $\|\Lambda^\alpha B^m\|_{H^\sigma}^2$ is time integrable, Gronwall's inequality yields the desired convergence (3.4). Let (u, B) be the limit. Due to the uniform bound (3.3), $(u, B) \in H^\sigma$ for $t \in [0, T_0]$. By the interpolation inequality, for any $0 < \sigma' < \sigma$,

$$\|f\|_{H^{\sigma'}} \leq C_\sigma \|f\|_{L^2}^{1 - \frac{\sigma'}{\sigma}} \|f\|_{H^\sigma}^{\frac{\sigma'}{\sigma}},$$

we further obtain the strong convergence

$$\|(u^n, B^n) - (u, B)\|_{H^{\sigma'}} \to 0 \qquad \text{as} \quad n \to \infty$$

and consequently, $(u, B) \in C([0, T_0]; H^{\sigma'})$. This strong convergence makes it easy to check that (u, B) satisfies the Hall-MHD equation in (1.1). In addition, the time continuity in $(u, B) \in C([0, T_0]; H^{\sigma'})$ allows to show the weak time continuity

$$(u, B) \in C_W([0, T_0]; H^{\sigma}) \quad \text{or} \quad t \mapsto \int (u(x, t), B(x, t)) \cdot \phi(x) \, dx \quad \text{is continuous}$$

for any $\phi \in H^{-\sigma}$. To show the right (in time) continuity of $\|(u(t), B(t))\|_{H^{\sigma}}$, we make use of the energy inequality, for any $t > \widetilde{t}$,

$$\|(u(t), B(t))\|_{H^{\sigma}}^2 \leq \|(u(\widetilde{t}), B(\widetilde{t}))\|_{H^{\sigma}}^2 + C \int_{\widetilde{t}}^{t} \|(u(s), B(s))\|_{H^{2\sigma}}^{\gamma} \, ds,$$

This inequality can be obtained in a similar fashion as (3.2). Then,

$$\lim_{t \to \widetilde{t}+} \|(u(t), B(t))\|_{H^{\sigma}} \leq \|(u(\widetilde{t}), B(\widetilde{t}))\|_{H^{\sigma}}.$$

By the weak continuity in time,

$$\|(u(\widetilde{t}), B(\widetilde{t}))\|_{H^{\sigma}} \leq \lim_{t \to \widetilde{t}+} \|(u(t), B(t))\|_{H^{\sigma}}.$$

The desired right (in time) continuity of $\|(u(t), B(t))\|_{H^{\sigma}}$ then follows. This completes the proof of Theorem 1.1. $\qquad \square$

Appendix A. Besov Spaces

This appendix provides the definitions of some of the functional spaces and related facts used in the previous sections. Materials presented in this appendix can be found in several books and many papers (see, e.g., [2,3,10–12]).

We start with several notations. $\mathcal{S}$ denotes the usual Schwartz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x) \, x^{\gamma} \, dx = 0, \ |\gamma| = 0, 1, 2, \dots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^{\perp} = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d : \ 2^{j-1} \leq |\xi| < 2^{j+1} \right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j \in \mathbb{Z}} \subset \mathcal{S}$ such that

$$\text{supp}\,\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \backslash \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \backslash \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{A.1}$$

Definition A.1. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}_{p,q}^s$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\dot{B}_{p,q}^s} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \dots. \end{cases} \tag{A.2}$$

Definition A.2. The inhomogeneous Besov space $B_{p,q}^s$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B_{p,q}^s} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Besov spaces $\dot{B}_{p,q}^s$ and $B_{p,q}^s$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\dot{B}_{p,q}^s} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.3. *For any $s \in \mathbb{R}$,*

$$\dot{H}^s \sim \dot{B}^s_{2,2}, \quad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$\dot{B}^s_{q,\min\{q,2\}} \hookrightarrow \dot{W}^s_q \hookrightarrow \dot{B}^s_{q,\max\{q,2\}}.$$

In particular, $\dot{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \dot{B}^0_{q,\max\{q,2\}}$.

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. There will be no confusion if we keep in mind that Δ_j's associated with the homogeneous Besov spaces is defined in (A.1) while those associated with the inhomogeneous Besov spaces are defined in (A.2). Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (A.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition A.4. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

(1) *If f satisfies*

$$supp \, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1 \, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*

$$supp \, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1 \, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2 \, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

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