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Solutions of the 2D quasi-geostrophic equation in Hölder spaces

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Abstract

The 2D quasi-geostrophic equation

$$\partial_t \theta + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, \quad u = \mathscr{R}^\perp(\theta)$$

is a two-dimensional model of the 3D hydrodynamics equations. When $\alpha \leqslant \frac{1}{2}$, the issue of existence and uniqueness concerning this equation becomes difficult. It is shown here that this equation with either $\kappa = 0$ or $\kappa > 0$ and $0 \leqslant \alpha \leqslant \frac{1}{2}$ has a unique local in time solution corresponding to any initial datum in the space $C^r \cap L^q$ for $r > 1$ and $q > 1$.

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Keywords: 2D quasi-geostrophic equation; Hölder space; Local existence

1. Introduction

The paper concerns itself with the 2D quasi-geostrophic (QG) equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = (u_1, u_2) = \nabla^\perp \psi, \quad (-\Delta)^{1/2}\psi = \theta, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^2$, $t \geqslant 0$, $\kappa \geqslant 0$ is the diffusion coefficient, $\alpha \in [0, 1]$ is a parameter, $\theta = \theta(x, t)$ is a scalar representing the temperature, u is the velocity field and ψ is the usual stream

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function. In addition to its applications in geophysics [6,11], the 2D QG equation (1.1) serves as a two-dimensional model of the 3D hydrodynamics equations. Solutions of (1.1) behave strikingly analogously to those of the 3D hydrodynamics equations and the study of (1.1) may provide clues to the millennium prize problems on the 3D Navier–Stokes equations.

The goal of this work is to establish existence and uniqueness results for (1.1) when the initial datum

$$\theta(x, 0) = \theta_0(x) \tag{1.2}$$

is given in a Hölder space. We distinguish between the inviscid QG equation, namely (1.1) with $\kappa = 0$ and the dissipative QG equation with $\kappa > 0$. For the inviscid QG equation, the fundamental issue of global existence for classical solutions remains open. Several local existence results represent the current status of art. The pioneering work of Constantin, Majda and Tabak [6] provided the first local existence result for θ_0 in the Sobolev space H^s with $s \geqslant 3$. Chae in [2] studied solutions corresponding to θ_0 in the Triebel–Lizorkin space $F^s_{p,q}$ with $s > 1 + 2/p$ and obtained local existence and blow-up criterion. In a very recent work [9], Córdoba and Córdoba managed to prove a local result for $\theta_0 \in H^s$ with $s > 2$ by making use of the duality of BMO with the Hardy space $\mathscr{H}$. Other progress on the issue of finite time blowup includes the geometric approach of Córdoba [8] and Córdoba and Fefferman [10].

For the dissipative QG equation, current research on the existence of solutions indicates that $\alpha = \frac{1}{2}$ is a critical index. In the sub-critical case, namely $\alpha > \frac{1}{2}$, solutions at several regularity levels, including solutions in the classical sense, have been shown to be global in time [7,12,15]. The theory of global existence and regularity for this case is thus in a satisfactory state. In the critical case $\alpha = \frac{1}{2}$, classical solutions are known to be global if their initial L^∞-norm is comparable to κ [5]. For initial data of arbitrary size, the global existence of classical solutions has not been established. It is hoped that the resolution of this problem will shed light on the millennium prize problem for the 3D Navier–Stokes equations. The super-critical case $\alpha < \frac{1}{2}$ is even harder to deal with and work on this case is more recent. For $\alpha \leqslant \frac{1}{2}$, Chae and Lee [3] established a global existence result under the assumption that θ_0 is small in the Besov space $B^{2-2\alpha}_{2,1}$. Córdoba and Córdoba [9] obtained a local existence result for $\theta_0 \in H^s$ with $s + \alpha > 2$ and a global result for small data in H^s with $s > 2$ or in $H^{3/2}$ in the case of $\alpha = \frac{1}{2}$. It is worth mentioning that other topics involving the 2D dissipative QG equation such as vanishing viscosity limit and large-time behavior have also been investigated [1,13,14].

In this paper, we are interested in solutions in the Hölder class C^r, and part of our goal has been to reduce regularity assumptions on the initial data to the minimum required for uniqueness. We shall show that the QG equation (1.1) with either $\kappa = 0$ or $\kappa > 0$ and $\alpha \in [0, \frac{1}{2}]$ possesses a unique local in time solution for any initial datum $\theta_0 \in C^r \cap L^q$ with $r > 1$ and $q > 1$. The functional setting $C^r \cap L^q$ allows us to control the velocity field u in terms of θ. According to the second equation in (1.1), u is related to θ through thetwo-dimensional Riesz transforms,

$$u = \nabla^\perp \Lambda^{-1} \theta = \mathscr{R}^\perp(\theta), \tag{1.3}$$

where $\Lambda = (-\Delta)^{1/2}$ and $\mathscr{R}^{\perp} = (-\mathscr{R}_2, \mathscr{R}_1)$ with $\mathscr{R}_1$ and $\mathscr{R}_2$ being the two-dimensional Riesz transforms. Riesz transforms do not necessarily map C^r to C^r, but they are bounded on $C^r \cap L^q$.

The rest of this paper is organized as follows. In Section 2, we review the characterization of Hölder spaces and gather several important estimates. In particular, the boundedness of Riesz transforms on $C^r \cap L^q$ is demonstrated here. Section 3 presents two key commutator estimates. Section 4 proves the existence result for the inviscid QG equation and Section 5 is devoted to the dissipative QG equation.

2. Hölder spaces

This is a preparatory section in which we review the characterization of the Hölder class functions and gather several estimates to be utilized in subsequent sections. A portion of the materials presented in this section can be found in [4].

We start with a dyadic decomposition of $\mathbb{R}^d$, where $d > 0$ is an integer. It is a classical result that there exist two radial functions $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$ satisfying

$$\mathit{supp}\,\chi \subset \{\xi : |\xi| \leqslant 4/3\}, \quad \mathit{supp}\,\phi \subset \{\xi : 3/4 < |\xi| < 8/3\},$$

$$\chi(\xi) + \sum_{j \geqslant 0} \phi(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^d.$$

For the purpose of isolating different Fourier frequencies, define the operators Δ_i for $i \in \mathbb{Z}$ as follows:

$$\Delta_i u = \begin{cases} 0 & \text{if } i \leqslant -2; \\ \chi(D)u = \int h(y)u(x-y)\,\mathrm{d}y & \text{if } i = -1; \\ \phi(2^{-i}D)u = 2^{id}\int g(2^i y)u(x-y)\,\mathrm{d}y & \text{if } i \geqslant 0, \end{cases} \tag{2.1}$$

where $h = \chi^{\vee}$ and $g = \phi^{\vee}$ are the inverse Fourier transforms of χ and ϕ, respectively.

For $i \in \mathbb{Z}$, S_i is the sum of Δ_j with $j \leqslant i-1$, i.e.,

$$S_i u = \Delta_{-1}u + \Delta_0 u + \Delta_1 u + \cdots + \Delta_{i-1}u = \int_{\mathbb{R}^d} h(2^i y)u(x-y)\,\mathrm{d}y.$$

It can be shown for any tempered distribution f that $S_i f \to f$ in the distributional sense, as $i \to \infty$.

For any $r \in \mathbb{R}$ and $p, q \in [1, \infty]$, the Besov space $B^r_{p,q}$ consists of all tempered distributions f such that the sequence $\{2^{jr}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}}$ belongs to $l^q(\mathbb{Z})$. When both p and q are equal to ∞, the Besov space $B^r_{p,q}$ reduces to the Hölder space C^r, i.e., $B^r_{\infty,\infty} = C^r$. More explicitly, C^r with $r \in \mathbb{R}$ contains any function f satisfying

$$\|f\|_{C^r} \equiv \sup_{j\in\mathbb{Z}} 2^{jr}\|\Delta_j f\|_{L^\infty} < \infty. \tag{2.2}$$

It is easy to check that C^r endowed with the norm defined in (2.2) is a Banach space.

For $r \geqslant 0$, C^r is closely related to the classical Hölder space $\tilde{C}^r$ equipped with the norm

$$\|f\|_{\tilde{C}^r} \equiv \sum_{|\beta| \leqslant [r]} \|\partial^\beta f\|_{L^\infty} + \sup_{x \neq y} \frac{|\partial^{[r]} f(x) - \partial^{[r]} f(y)|}{|x-y|^{r-[r]}}. \tag{2.3}$$

In fact, if r is not an integer, then the norms in (2.2) and (2.3) are equivalent and $C^r = \tilde{C}^r$. The proof for this equivalence is classical and can be found in [4]. When r is an integer, say $r = k$, $\tilde{C}^r$ is the space of bounded functions with bounded j-th derivatives for any $j \leqslant k$. In particular, $\tilde{C}^1$ contains the usual Lipschitz functions and is sometimes denoted by *Lip*. As a consequence of Bernstein's lemma (stated below), $\tilde{C}^r$ is a subspace of C^r. Explicit examples can be constructed to show that such an inclusion is genuine. In addition, according to Proposition 2.2, $\tilde{C}^r$ includes $C^{r+\varepsilon}$ for any $\varepsilon > 0$. In summary, for any integer $k \geqslant 0$ and $\varepsilon > 0$,

$$C^{k+\varepsilon} \subset \tilde{C}^k \subset C^k.$$

Proposition 2.1 (*Bernstein's Lemma*). *Let $d > 0$ be an integer and $R_2 > R_1 > 0$ be two real numbers.*

(i) *If $1 \leqslant p \leqslant q \leqslant \infty$ and supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leqslant R_1 2^j\}$, then*

$$\max_{|\alpha|=k} \|\partial^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C 2^{jk+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)},$$

where $C > 0$ is a constant depending on k and R_1 only.

(ii) *If $p \in [1, \infty]$ and supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : R_1 2^j \leqslant |\xi| \leqslant R_2 2^j\}$, then*

$$C^{-1} 2^{jk} \|f\|_{L^p(\mathbb{R}^d)} \leqslant \max_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d)} \leqslant C 2^{jk} \|f\|_{L^p(\mathbb{R}^d)},$$

where $C > 0$ is a constant depending on k, R_1 and R_2 only.

Proposition 2.2. *There exists a constant C such that for any $\varepsilon > 0$ and $f \in C^\varepsilon$,*

$$\|f\|_{L^\infty} \leqslant \frac{C}{\varepsilon} \|f\|_{C^0} \log_2 \left(e + \frac{\|f\|_{C^\varepsilon}}{\|f\|_{C^0}} \right) \leqslant \frac{C}{\varepsilon} \|f\|_{C^\varepsilon}. \tag{2.4}$$

In the 2D QG equation (1.1), the velocity field u is determined by θ through the 2D Riesz transforms, namely (1.3). Riesz transforms do not necessarily map a Hölder space C^r to itself, but their action on C^r is indeed bounded in $C^r \cap L^p$ for any $p \in (1, \infty)$. The precise statement is presented in Proposition 2.3, followed by a proof. We first recall a general result concerning the boundedness of Fourier multiplier operators on Hölder spaces.

Proposition 2.3. *Let $d > 0$ be an integer and F be a infinitely differentiable function on $\mathbb{R}^d$. Assume that for some $R > 0$ and $m \in \mathbb{R}$,*

$$F(\lambda\xi) = \lambda^m F(\xi)$$

holds for any $\xi \in \mathbb{R}^d$ with $|\xi| > R$ and $\lambda \geqslant 1$. Then the Fourier multiplier operator $F(D)$ maps continuously from C^r to C^{r-m} for any $r \in \mathbb{R}$.

Notice that the Fourier transforms of the 2D Riesz transforms $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by

$$\widehat{\mathcal{R}_k f}(\xi) = -i\frac{\xi_k}{|\xi|}\widehat{f}(\xi), \quad \xi \in \mathbb{R}^2, \; k = 1, 2.$$

Proposition 2.4. *Let $r \in \mathbb{R}$ and $p \in (1, \infty)$. Then, there exists a constant C depending on r and p alone such that*

$$\|\mathcal{R}_k f\|_{C^r} \leqslant C\|f\|_{C^r \cap L^p},$$

where $k = 1$ or 2.

Proof. Using the operator Δ_{-1} defined in (2.1), we divide $\mathcal{R}_k f$ into two parts,

$$\mathcal{R}_k f = \Delta_{-1}\mathcal{R}_k f + (1 - \Delta_{-1})\mathcal{R}_k f. \tag{2.5}$$

Since *supp* $\chi(\xi) \cap$ *supp* $\phi(2^{-j}\xi) = \emptyset$ for $j \geqslant 1$, the operator $\Delta_j \Delta_{-1} = 0$ when $j \geqslant 1$. Thus, according to (2.2),

$$\begin{aligned}\|\Delta_{-1}\mathcal{R}_k f\|_{C^r} &= \sup_{j\in\mathbb{Z}} 2^{jr}\|\Delta_j\Delta_{-1}\mathcal{R}_k f\|_{L^\infty}\\ &= \max\left\{2^{-r}\|\Delta_{-1}\Delta_{-1}\mathcal{R}_k f\|_{L^\infty}, \|\Delta_0\Delta_{-1}\mathcal{R}_k f\|_{L^\infty}\right\}\\ &\leqslant \max\{1, 2^{-r}\}\|\Delta_{-1}\mathcal{R}_k f\|_{L^\infty}.\end{aligned}$$

Let q be the conjugate of p, namely $1/p + 1/q = 1$. It then follows from the basic fact that Riesz transforms are bounded on L^p for any $p \in (1, \infty)$ that

$$\begin{aligned}\|\Delta_{-1}\mathcal{R}_k f\|_{C^r} &\leqslant \max\{1, 2^{-r}\}\|h * (\mathcal{R}_k f)\|_{L^\infty}\\ &\leqslant \max\{1, 2^{-r}\}\|h\|_{L^q}\|\mathcal{R}_k f\|_{L^p} = C\|f\|_{L^p},\end{aligned}$$

where C is a constant depending on r and p alone. To estimate the second part in (2.5), we apply Proposition 2.3 with $F(\xi) = (1 - \chi(\xi))(-\mathrm{i}\xi_k)/|\xi|$ and $m = 0$ and conclude that it maps C^r to C^r. This concludes the proof of Proposition 2.4. $\square$

For notational convenience, we write $Y_{r,p}$ for $C^r \cap L^p$ from now on. For $r \in \mathbb{R}$ and $p \geqslant 1$, $Y_{r,p}$ is a Banach space if endowed with the norm $\| \;\|_{r,p}$, where

$$\|f\|_{r,p} = \|f\|_{C^r} + \|f\|_{L^p}.$$

Finally we introduce the notion of paraproduct [4]. The usual product uv of two functions u and v can be decomposed into three parts. More precisely, using the notion of paraproduct, we can write

$$uv = T_u v + T_v u + R(u, v), \tag{2.6}$$

where

$$T_u v = \sum_j S_{j-1}u \cdot \Delta_j v, \quad R(u, v) = \sum_{|i-j|\leqslant 1} \Delta_i u \cdot \Delta_j v.$$

We remark that the decomposition in (2.6) allows one to distinguish different types of terms in the product of uv. The Fourier frequencies of u and v in $T_u v$ and $T_v u$ are separated from each other while those of the terms in $R(u, v)$ are close to each other. Using the decomposition in (2.6), one can show that for $s > 0$

$$\|uv\|_{C^s} \leqslant C(\|u\|_{C^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{C^s}). \tag{2.7}$$

3. Two commutator estimates

Two major commutator estimates are stated and proved in this section. For future references, these estimates are presented in the context of the Besov space $B^r_{p,\infty}$. When $p = \infty$, $B^r_{p,\infty}$ becomes the Hölder space C^r and these bounds become the desired ones.

Lemma 3.1. *Let $j \geqslant -1$ be an integer, $r \in \mathbb{R}$ and $p \in [1, \infty]$. Then,*

$$\|[u \cdot \nabla, \Delta_j]\theta\|_{L^p} \leqslant C2^{-jr}(\|\nabla\theta\|_{L^\infty}\|u\|_{B^r_{p,\infty}} + \|\nabla u\|_{L^\infty}\|\theta\|_{B^r_{p,\infty}}), \tag{3.1}$$

where C is a pure constant and the brackets [,] *represents the commutator, namely*

$$[u \cdot \nabla, \Delta_j]\theta = u \cdot \nabla(\Delta_j\theta) - \Delta_j(u \cdot \nabla\theta).$$

In particular, if $p = \infty$, (3.1) *becomes*

$$\|[u \cdot \nabla, \Delta_j]\theta\|_{L^\infty} \leqslant C2^{-jr}(\|\nabla\theta\|_{L^\infty}\|u\|_{C^r} + \|\nabla u\|_{L^\infty}\|\theta\|_{C^r}). \tag{3.2}$$

Eq. (3.1) is suitable for situations when u and θ are equally regular. If $\nabla\theta$ is not known to be bounded in L^∞, then (3.1) fails. The lemma that follows provides a new estimate which needs no information about $\nabla\theta$. As a trade-off, u is required to be in $B^{r+1}_{p,\infty}$. The importance of these lemmas will be seen in the proofs of Theorem 4.1 and Theorem 5.1.

Lemma 3.2. *Let $j \geqslant -1$, $r \in \mathbb{R}$ and $p \in [1, \infty]$. Then, for some pure constant C,*

$$\|[u \cdot \nabla, \Delta_j]\theta\|_{L^p} \leqslant C2^{-jr}(\|\nabla u\|_{L^\infty}\|\theta\|_{B^r_{p,\infty}} + \|\theta\|_{L^\infty}\|u\|_{B^{r+1}_{p,\infty}}). \tag{3.3}$$

In the special case of $p = \infty$, (3.3) *becomes*

$$2^{jr}\|[u \cdot \nabla, \Delta_j]\theta\|_{L^\infty} \leqslant C(\|\nabla u\|_{L^\infty}\|\theta\|_{C^r} + \|\theta\|_{L^\infty}\|u\|_{C^{r+1}}). \tag{3.4}$$

Proof of Lemma 3.1. Utilizing the paraproduct notations T and R, we decompose $[u \cdot \nabla, \Delta_j]\theta$ into five parts,

$$[u \cdot \nabla, \Delta_j]\theta = [u_i \cdot \partial_i, \Delta_j]\theta = K_1 + K_2 + K_3 + K_4 + K_5,$$

where

$$\begin{aligned}
K_1 &= [T_{u_i}\partial_i, \Delta_j]\theta,\\
K_2 &= -\Delta_j T_{\partial_i\theta}u_i,\\
K_3 &= T_{\partial_i\Delta_j\theta}u_i,\\
K_4 &= -\Delta_j R(u_i, \partial_i\theta),\\
K_5 &= R(u_i, \partial_i\Delta_j\theta).
\end{aligned} \tag{3.5}$$

Returning to the definition of T, we can write

$$K_1 = \sum_{j'\in\mathbb{Z}} [S_{j'-1}(u_i), \Delta_j]\partial_i \Delta_{j'}\theta$$
$$= \sum_{j'\in\mathbb{Z}} (S_{j'-1}(u_i)\Delta_j(\partial_i\Delta_{j'}\theta) - \Delta_j S_{j'-1}(u_i)(\partial_i\Delta_{j'}\theta)). \tag{3.6}$$

Because $\Delta_j\Delta_{j'} = 0$ for $|j - j'| \geqslant 2$ and

$$supp\, \widehat{S_{j'-1}(u_i)(\partial_i}\Delta_{j'}\theta) \subset \{\xi : 2^{j'-3} \leqslant |\xi| \leqslant 2^{j'+1}\},$$

the sum in (3.6) only involves those terms with j' satisfying $|j' - j| \leqslant 4$. Thus,

$$K_1 = \sum_{|j'-j|\leqslant 4} 2^{jd} \int h(2^j(x-y))(S_{j'-1}(u_i)(x) - S_{j'-1}(u_i)(y))(\partial_i\Delta_{j'}\theta)(y)\,\mathrm{d}y$$
$$= \sum_{|j'-j|\leqslant 4} \int h(y)(S_{j'-1}(u_i)(x) - S_{j'-1}(u_i)(x - 2^{-j}y))(\partial_i\Delta_{j'}\theta)(x - 2^{-j}y)\,\mathrm{d}y.$$

For $r \in \mathbb{R}$, $p \in (1, \infty]$ and a pure constant C,

$$\|K_1\|_{L^p} \leqslant C2^{-j}\|\nabla u_i\|_{L^\infty}\|\Delta_j\partial_i\theta\|_{L^p} \leqslant C2^{-jr}2^{j(r-1)}\|\Delta_j\partial_i\theta\|_{L^p}\|\nabla u_i\|_{L^\infty}$$
$$\leqslant C2^{-jr}2^{jr}\|\Delta_j\theta\|_{L^p}\|\nabla u_i\|_{L^\infty} \leqslant C2^{-jr}\|\theta\|_{B^r_{p,\infty}}\|\nabla u\|_{L^\infty}, \tag{3.7}$$

where we have used Proposition 2.1 in the third inequality.

To estimate K_2 and K_3, we first write them as

$$K_2 = -\sum_{j'} \Delta_j(S_{j'-1}(\partial_i\theta)\Delta_{j'}(u_i)), \quad K_3 = \sum_{j'} S_{j'-1}(\partial_i\Delta_j\theta)\Delta_{j'}(u_i).$$

Similarly, only terms with j' satisfying $|j - j'| \leqslant 4$ survive in the sums above. Thus, we have for $r \in \mathbb{R}$ and $p \in (1, \infty]$

$$\|K_2\|_{L^p} \leqslant C2^{-jr}\|S_{j-1}(\partial_i\theta)\Delta_j(u_i)\|_{B^r_{p,\infty}} \leqslant C2^{-jr}\|\nabla\theta\|_{L^\infty}\|u\|_{B^r_{p,\infty}},$$
$$\|K_3\|_{L^p} \leqslant C\|S_{j-1}(\partial_i\theta)\|_{L^\infty}\|\Delta_j u_i\|_{L^p} \leqslant C2^{-jr}\|\nabla\theta\|_{L^\infty}\|u\|_{B^r_{p,\infty}}, \tag{3.8}$$

where C's in the above inequalities are pure constants. By the definition of R,

$$K_4 = -\sum_{|j'-j''|\leqslant 1} \Delta_j(\Delta_{j'}(u_i)\Delta_{j''}(\partial_i\theta)), \quad K_5 = \sum_{|j'-j''|\leqslant 1} \Delta_{j'}(u_i)\Delta_{j''}(\Delta_j\partial_i\theta).$$

Obviously, only a finite number of terms in the sums above are nonzero. So,

$$\|K_4\|_{L^p} \leqslant C2^{-jr}\|\Delta_j(u_i)\Delta_j(\partial_i\theta)\|_{B^r_{p,\infty}} \leqslant C2^{-jr}\|\nabla\theta\|_{L^\infty}\|u\|_{B^r_{p,\infty}}, \tag{3.9}$$

$$\|K_5\|_{L^p} \leqslant C\|\Delta_j\partial_i\theta\|_{L^\infty}\|\Delta_j(u_i)\|_{L^p} \leqslant C2^{-jr}\|\nabla\theta\|_{L^\infty}\|u\|_{B^r_{p,\infty}}. \tag{3.10}$$

Gathering the estimates in (3.7)–(3.10), we establish the desired inequality in (3.1). When $p = \infty$, the Besov space $B^r_{p,\infty}$ reduces to the Hölder space C^r and (3.1) to (3.2). $\square$

Proof of Lemma 3.2. As in the proof of Lemma 3.1, we decompose $[u\cdot\nabla, \Delta_j]\theta$ as the sum of K_1, K_2, K_3, K_4 and K_5. The estimate for K_1 remains effective, but different bounds are needed for K_2, K_3, K_4 and K_5. Recall that

$$K_2 = -\sum_{|j'-j|\leqslant 4} \Delta_j(S_{j'-1}(\partial_i\theta)\Delta_{j'}(u_i)).$$

Let $j\geqslant 0$ since the case $j=-1$ can be handled similarly. Applying the definition of Δ_j in (2.1) and integrating by parts, we obtain

$$\begin{aligned}K_2 &= -\sum_{|j'-j|\leqslant 4} 2^{jd}\int g(2^j(x-y))\partial_i(S_{j'-1}\theta\Delta_{j'}u_i)(y)\,\mathrm{d}y\\ &= -\sum_{|j'-j|\leqslant 4} 2^j 2^{jd}\int (\partial_i g)(2^j(x-y))(S_{j'-1}\theta\Delta_{j'}u_i)(y)\,\mathrm{d}y.\end{aligned}$$

Therefore, for a pure constant C,

$$\begin{aligned}\|K_2\|_{L^p} &\leqslant 2^j\|\nabla g\|_{L^1}\|S_{j-1}\theta\Delta_j u\|_{L^p}\\ &\leqslant C2^j\|\theta\|_{L^\infty}\|\Delta_j u\|_{L^p}\leqslant C2^{-jr}\|\theta\|_{L^\infty}\|u\|_{B^{r+1}_{p,\infty}}.\end{aligned}\tag{3.11}$$

The estimate for K_3 is direct. In fact, by Proposition 2.1,

$$\begin{aligned}\|K_3\|_{L^p} &\leqslant C\|\partial_i\Delta_j\theta\|_{L^\infty}\|\Delta_j u_i\|_{L^p}\\ &\leqslant C2^j\|\Delta_j\theta\|_{L^\infty}\|\Delta_j u\|_{L^p}\leqslant C2^{-jr}\|\theta\|_{L^\infty}\|u\|_{B^{r+1}_{p,\infty}}.\end{aligned}\tag{3.12}$$

K_4 can be similarly estimated as K_2.

$$\|K_4\|_{L^p}\leqslant C2^j\|\nabla g\|_{L^1}\|\Delta_j\theta\|_{L^\infty}\|\Delta_j u\|_{L^p}\leqslant C2^{-jr}\|\theta\|_{L^\infty}\|u\|_{B^{r+1}_{p,\infty}}.\tag{3.13}$$

Finally, we have

$$\|K_5\|_{L^p}\leqslant C\|\Delta_j u_i\|_{L^p}\|\Delta_j\partial_i\theta\|_{L^\infty}\leqslant C2^{-jr}\|\theta\|_{L^\infty}\|u\|_{B^{r+1}_{p,\infty}}.\tag{3.14}$$

Combining (3.7)–(3.14) yields (3.3). □

4. The inviscid QG equation

This section is devoted to the inviscid QG equation. We prove that it has a unique local solution for any initial datum $\theta_0\in C^r\cap L^q$ with $r>1$ and $q>1$. More precisely, we have the following theorem.

Theorem 4.1. *Consider solutions of the 2D inviscid QG equation*

$$\partial_t\theta + u\cdot\nabla\theta = 0,\qquad u=\mathscr{R}^\perp(\theta)=\nabla^\perp\Lambda^{-1}\theta,\tag{4.1}$$

corresponding to initial data $\theta_0 \in Y_{r,q} \equiv C^r \cap L^q$. *If* $r > 1$ *and* $q > 1$, *then* (4.1) *has a unique solution* θ *on the time interval* $[0, T]$, *where* $T > 0$ *depends on* $\|\theta_0\|_{r,q}$ *only. The solution* θ *is in the space*

$$L^\infty([0, T]; Y_{r,q}) \cap Lip([0, T]; Y_{r-1,q}) \cap C([0, T]; Y_{s,q})$$

with $s \in [r - 1, r)$.

The rest of this section is devoted to the proof of Theorem 4.1. For the sake of a clear presentation, we divide it into two subsections. An a priori bound is proven in the first subsection. The second subsection proves Theorem 4.1, with the aid of the a priori estimate.

4.1. An a priori estimate

Proposition 4.2. *Let* $r > 1$ *and* $q > 1$. *Let* $T > 0$ *and* $\theta_0 \in Y_{r,q}$. *If* θ *solves the 2D inviscid QG equation* (4.1) *with the initial datum* θ_0 *on the time interval* $[0, T]$ *and* $\theta(\cdot, t) \in Y_{r,q}$ *for* $t \in [0, T]$, *then*

$$\|\theta(\cdot, t)\|_{r,q} \leqslant \|\theta_0\|_{r,q} \exp\left(C \int_0^t \|\theta(\cdot, \tau)\|_{r,q}\, \mathrm{d}\tau\right)$$

for $t \in [0, T]$, *where C is a constant depending on r and q only.*

Proof. According to Proposition 2.4, $\theta \in Y_{r,q}$ implies that $u \in Y_{r,q}$. Thus, for $t \in [0, T]$, we can define the flow map $X(\cdot, t)$ satisfying

$$\begin{cases} \partial_t X(x, t) = u(X(x, t), t), \\ X(x, 0) = x. \end{cases} \tag{4.2}$$

Let $j \geqslant -1$ and apply the operator Δ_j to both sides of the inviscid QG equation to yield

$$\partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta = [u \cdot \nabla, \Delta_j]\theta,$$

where $[u \cdot \nabla, \Delta_j]\theta = u \cdot \nabla(\Delta_j \theta) - \Delta_j(u \cdot \nabla\theta)$. This equation can be rewritten in the form

$$\Delta_j \theta(x, t) = \Delta_j \theta_0(X^{-1}(x, t)) + \int_0^t [u \cdot \nabla, \Delta_j](X(X^{-1}(x, t), s), s)\, \mathrm{d}s.$$

If we take the L^∞-norm, then

$$\|\Delta_j \theta(\cdot, t)\|_{L^\infty} \leqslant \|\Delta_j \theta_0\|_{L^\infty} + \int_0^t \|[u \cdot \nabla, \Delta_j]\theta(\cdot, s)\|_{L^\infty}\, \mathrm{d}s. \tag{4.3}$$

Applying Lemma 3.1 with $p = \infty$, we obtain

$$\|\theta(\cdot, t)\|_{C^r} \leqslant \|\theta_0\|_{C^r} + \int_0^t (\|\nabla\theta(\cdot, s)\|_{L^\infty} \|u(\cdot, s)\|_{C^r} + \|\nabla u(\cdot, s)\|_{L^\infty} \|\theta(\cdot, s)\|_{C^r})\, \mathrm{d}s.$$

According to (2.4), for $r > 1$ and a constant C depending on r only,

$$\|\nabla\theta\|_{L^\infty} \leqslant C\|\theta\|_{C^1} \log\left(e + \frac{\|\theta\|_{C^r}}{\|\theta\|_{C^1}}\right) \leqslant C\|\theta\|_{C^r}.$$

Similarly, $\|\nabla u\|_{L^\infty} \leqslant C\|u\|_{C^r}$. Therefore, for C depending on r only,

$$\|\theta(\cdot,t)\|_{C^r} \leqslant \|\theta_0\|_{C^r} + C\int_0^t \|u(\cdot,s)\|_{C^r}\|\theta(\cdot,s)\|_{C^r}\,\mathrm{d}s.$$

By Gronwall's inequality and Proposition 2.4,

$$\|\theta(\cdot,t)\|_{C^r} \leqslant \|\theta_0\|_{C^r} \exp\left(C\int_0^t \|\theta(\cdot,s)\|_{r,q}\,\mathrm{d}s\right). \tag{4.4}$$

Adding the usual estimate $\|\theta(\cdot,t)\|_{L^q} \leqslant \|\theta_0\|_{L^q}$ to (4.4) completes the proof of Proposition 4.2. □

4.2. Proof of Theorem 4.1

Proof of Theorem 4.1. The proof starts with the construction of a successive approximation sequence $\{\theta^{(n)}\}$ satisfying

$$\begin{cases} \theta^{(1)} = S_2(\theta_0), \\ \partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} = 0, \\ u^{(n)} = \nabla^\perp\Lambda^{-1}\theta^{(n)}, \\ \theta^{(n+1)}(x,0) = S_{n+2}\theta_0. \end{cases} \tag{4.5}$$

The rest of the proof can be divided into two major steps. The first step establishes the existence of $T_1 > 0$ such that $\{\theta^{(n)}(\cdot,t)\}$ is bounded uniformly in $Y_{r,p}$ for any $t\in[0,T_1]$. The second step verifies for some $T_2\in[0,T_1]$ that $\{\theta^{(n)}\}$ is a Cauchy sequence in $C([0,T_2];Y_{r-1,q})$.

Step 1: A similar argument as in the proof of Proposition 4.2 yields the following bound for $\{\theta^{(n+1)}\}$,

$$\|\theta^{(n+1)}(\cdot,t)\|_{r,q} \leqslant \|S_{n+2}\theta_0\|_{r,q} \exp\left(C_0\int_0^t \|\theta^{(n)}(\cdot,s)\|_{r,q}\,\mathrm{d}s\right),$$

where C_0 is a constant depending on r and q only. Choose T_1 and M satisfying

$$M = 2\|\theta_0\|_{r,q} \quad \text{and} \quad \exp(C_0MT_1)\leqslant 2 \quad \text{or} \quad T_1 = \frac{\ln(2)}{2C_0\|\theta_0\|_{r,q}}.$$

Then $\|\theta^{(n)}(\cdot,t)\|_{r,q}\leqslant M$ for all n and $t\in[0,T_1]$. In fact,

$$\|\theta^{(1)}\|_{r,q} = \|S_2(\theta_0)\|_{r,q} \leqslant \|\theta_0\|_{r,q} < M$$

and $\|\theta^{(k)}(\cdot,t)\|_{r,q}\leqslant M$ leads to

$$\|\theta^{(n+1)}(\cdot,t)\|_{r,q} \leqslant \|\theta_0\|_{r,q} \exp(C_0\,MT_1) \leqslant M.$$

Furthermore, for $r>1$,

$$\begin{aligned} \|\partial_t\theta^{(n)}\|_{C^{r-1}} &= \|u^{(n)}\cdot\nabla\theta^{(n+1)}\|_{C^{r-1}} \\ &\leqslant C(\|u^{(n)}\|_{L^\infty}\|\nabla\theta^{(n+1)}\|_{C^{r-1}} + \|u^{(n)}\|_{C^{r-1}}\|\nabla\theta^{(n+1)}\|_{L^\infty}) \\ &\leqslant C\|u^{(n)}\|_{C^{r-1}}\|\theta^{(n+1)}\|_{C^r} \leqslant CM^2, \end{aligned}$$

where C is constant depending on r only. Thus,

$$\partial_t \theta^{(n)} \in L^\infty([0,T_1];Y_{r-1,q}) \quad \text{or} \quad \theta^{(n)} \in Lip([0,T_1];Y_{r-1,q}) \tag{4.6}$$

with uniform bounds.

Step 2: To show that $\{\theta^{(n)}\}$ is a Cauchy sequence in $Y_{r-1,q}$, we consider the difference $\eta^{(n)} = \theta^{(n)} - \theta^{(n-1)}$. Rigorously speaking, we should consider the more general difference $\eta^{(m,n)} = \theta^{(m)} - \theta^{(n)}$, but the analysis for $\eta^{(m,n)}$ is parallel to what we shall present for $\eta^{(n)}$ and we thus consider $\eta^{(n)}$ for the sake of a concise presentation. It follows from (4.5) that $\{\eta^{(n)}\}$ satisfies

$$\begin{cases} \eta^{(1)} = S_2(\theta_0) - \theta_0, \\ \partial_t \eta^{(n+1)} + u^{(n)} \cdot \nabla \eta^{(n+1)} = w^{(n)} \cdot \nabla \theta^{(n)}, \\ w^{(n)} = \mathscr{R}^\perp(\theta^{(n)}) = \nabla^\perp \Lambda^{-1} \eta^{(n)}, \\ \eta^{(n+1)}(x,0) = \eta_0^{(n+1)}(x) = \Delta_{n+1}\theta_0. \end{cases}$$

Proceeding as in the proof of Proposition 4.2, we obtain for any integer $j \geqslant -1$,

$$\begin{aligned} \|\Delta_j \eta^{(n+1)}(\cdot,t)\|_{L^\infty} \leqslant \|\Delta_j \eta_0^{(n+1)}\|_{L^\infty} &+ \int_0^t \|[u^{(n)} \cdot \nabla, \Delta_j]\eta^{(n+1)}(\cdot,s)\|_{L^\infty}\, \mathrm{d}s \\ &+ \int_0^t \|\Delta_j (w^{(n)} \cdot \nabla\theta^{(n)})(\cdot,s)\|_{L^\infty}\, \mathrm{d}s. \end{aligned}$$

Bounding the last two terms in the above inequality by (3.3) and (2.7), respectively, we have

$$\begin{aligned} \|\eta^{(n+1)}(\cdot,t)\|_{C^{r-1}} &\leqslant \|\eta_0^{(n+1)}\|_{C^{r-1}} \\ &+ C\int_0^t (\|\nabla u^{(n)}(\cdot,s)\|_{L^\infty}\|\eta^{(n+1)}(\cdot,s)\|_{C^{r-1}} + \|\eta^{(n+1)}(\cdot,s)\|_{L^\infty}\|u^{(n)}(\cdot,s)\|_{C^r})\,\mathrm{d}s \\ &+ C\int_0^t (\|w^{(n)}(\cdot,s)\|_{L^\infty}\|\nabla\theta^{(n)}(\cdot,s)\|_{C^{r-1}} + \|w^{(n)}(\cdot,s)\|_{C^{r-1}}\|\nabla\theta^{(n)}(\cdot,s)\|_{L^\infty})\,\mathrm{d}s, \end{aligned}$$

where C's are constants depending on r only. Since $r > 1$, Proposition 2.2 implies,

$$\begin{aligned} &\|\nabla u^{(n)}\|_{L^\infty} \leqslant C\|u^{(n)}\|_{C^r}, \quad \|\eta^{(n+1)}\|_{L^\infty} \leqslant C\|\eta^{(n+1)}\|_{C^{r-1}}, \\ &\|\nabla\theta^{(n)}\|_{L^\infty} \leqslant C\|\theta^{(n)}\|_{C^r}, \quad \|w^{(n)}\|_{L^\infty} \leqslant C\|w^{(n)}\|_{C^{r-1}}. \end{aligned}$$

Therefore, for constants C depending on r only,

$$\begin{aligned} \|\eta^{(n+1)}(\cdot,t)\|_{C^{r-1}} \leqslant \|\eta_0^{(n+1)}\|_{C^{r-1}} &+ C\int_0^t \|\eta^{(n+1)}(\cdot,s)\|_{C^{r-1}}\|u^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s \\ &+ C\int_0^t \|w^{(n)}(\cdot,s)\|_{C^{r-1}}\|\theta^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s \end{aligned}$$

It follows from a basic energy estimate that

$$\|\eta^{(n+1)}(\cdot,t)\|_{L^q} \leqslant \|\eta_0^{(n+1)}\|_{L^q} + \int_0^t \|w^{(n)}(\cdot,s)\|_{L^q}\|\nabla\theta^{(n)}(\cdot,s)\|_{L^\infty}\,\mathrm{d}s.$$

Adding the last two inequalities yields

$$\|\eta^{(n+1)}(\cdot,t)\|_{r-1,q} \leqslant \|\eta_0^{(n+1)}\|_{r-1,q} + C\int_0^t \|\eta^{(n+1)}(\cdot,s)\|_{C^{r-1}}\|u^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s + C\int_0^t \|w^{(n)}(\cdot,s)\|_{r-1,q}\|\theta^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s.$$

The components of $w^{(n)}$ are the Riesz transforms of $\eta^{(n)}$ and thus, according to Proposition 2.4,

$$\|w^{(n)}\|_{r-1,q} \leqslant C\|\eta^{(n)}\|_{r-1,q}.$$

We thus have reached an iterative relationship between $\|\eta^{(n)}\|_{r-1,q}$ and $\|\eta^{(n+1)}\|_{r-1,q}$,

$$\|\eta^{(n+1)}(\cdot,t)\|_{r-1,q} \leqslant \|\eta_0^{(n+1)}\|_{r-1,q} + C_1\int_0^t \|\eta^{(n+1)}(\cdot,s)\|_{C^{r-1}}\|u^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s + C_1\int_0^t \|\eta^{(n)}(\cdot,s)\|_{r-1,q}\|\theta^{(n)}(\cdot,s)\|_{C^r}\,\mathrm{d}s, \tag{4.7}$$

where the constants are labeled as C_1 for the purpose of defining T_2. It has been shown in Step 1 that for $t \leqslant T_1$,

$$\|u^{(n)}(\cdot,t)\|_{r,q} \leqslant M, \quad \|\theta^{(n)}(\cdot,t)\|_{r,q} \leqslant M.$$

Now, choose $T_2 > 0$ satisfying

$$T_2 \leqslant T_1, \quad C_1 M T_2 \leqslant \tfrac{1}{4}$$

and we shall show that $\{\eta^{(n)}(\cdot,t)\}$ is a Cauchy sequence in $Y_{r-1,q}$ for $t \leqslant T_2$. For any given $\varepsilon > 0$, choose a large N such that for any $n \geqslant N$,

$$\|\eta_0^{(n)}\|_{r-1,q} = \|\Delta_n\theta_0\|_{r-1,q} \leqslant \frac{\varepsilon}{2}.$$

If $\|\eta^{(n)}(\cdot,t)\|_{r-1,q} \leqslant \varepsilon$ for $t \leqslant T_2$, then (4.7) implies

$$\|\eta^{(n+1)}(\cdot,t)\|_{r-1,q} \leqslant \frac{\varepsilon}{2} + C_1\varepsilon M T_2 + C_1 M\int_0^t \|\eta^{(n+1)}(\cdot,s)\|_{r-1,q}\,\mathrm{d}s$$

valid for any $t \leqslant T_2$. It then follows from Gronwall's inequality that

$$\|\eta^{(n+1)}(\cdot,t)\|_{r-1,q} \leqslant \varepsilon$$

for any $t \leqslant T_2$. This completes Step 2.

We conclude from Steps 1 and 2 above that there exists a θ satisfying

$$\theta \in L^\infty([0,T_1]; Y_{r,q}) \cap Lip([0,T_1]; Y_{r-1,q})$$

such that $\theta^{(n)}$ converges to θ in $C([0,T_2]; Y_{r-1,q})$. By an interpolation inequality, $\theta^{(n)}$ also converges to θ in $C([0,T_2]; Y_{s,q})$ with $s \in [r-1, r)$. Thus, we have

$$\theta \in C([0,T_2]; Y_{s,q}).$$

The proof of uniqueness follows the same procedure as in Step 2, so we omit the details. This completes the proof of Theorem 4.1. □

5. The dissipative QG equation

Attention of this section will be focused on the 2D dissipative QG equation

$$\partial_t \theta + u \cdot \nabla\theta + \kappa\Lambda^{2\alpha}\theta = 0, \quad u = \mathscr{R}^{\perp}(\theta). \tag{5.1}$$

We show that the (5.1) always has a local in time solution corresponding to any initial datum in $C^r \cap L^q$ with $r > 1$ and $q > 1$.

Theorem 5.1. *Consider the 2D dissipative QG equation* (5.1) *with* $\kappa > 0$ *and* $0 \leqslant \alpha \leqslant \frac{1}{2}$. *Assume that the initial datum* θ_0 *is in* $Y_{r,q} \equiv C^r \cap L^q$, *where* $r > 1$ *and* $q > 1$. *Then there exists a* $T > 0$ *depending on* $\|\theta_0\|_{r,q}$ *only such that* (5.1) *has a unique solution* $\theta(x, t)$ *for* $t \in [0, T]$. *Furthermore,* θ *satisfies*

$$\theta \in L^{\infty}([0, T]; Y_{r,q}) \cap C([0, T]; Y_{r-1,q}) \cap Lip([0, T]; C^{r-1}).$$

We first recall the positivity lemma.

Lemma 5.2. *Let* $\alpha \in [0, 1]$ *and* $p \in [2, \infty)$. *If* $\Lambda^{2\alpha}\theta \in L^p$, *then*

$$\int_{\mathbb{R}^2} |\theta|^{p-2}\theta\Lambda^{2\alpha}\theta \,\mathrm{d}x \geqslant 0.$$

This lemma was first proved in [12]. Very recently, Córdoba and Córdoba skillfully proved a point-wise inequality involving the operator $\Lambda^{2\alpha}$ with $\alpha \in [0, 1]$ and deduced as a special consequence this inequality [9].

Proposition 5.3. *Let* $\alpha \in [0, 1]$ *and* $\theta \in \mathbb{S}$, *the Schwartz class. Then,*

$$2\theta\Lambda^{2\alpha}\theta(x) \geqslant \Lambda^{2\alpha}\theta^2(x)$$

for any $x \in \mathbb{R}^2$.

It is easily seen that the positivity lemma allows us to show the maximum principle for solutions of (5.1) with any $\alpha \in [0, 1]$,

$$\|\theta\|_{L^p} \leqslant \|\theta_0\|_{L^p}, \tag{5.2}$$

where $p \in [1, \infty]$. The point-wise inequality in (5.3) actually leads to a L^p-decay estimate, as stated in the following proposition.

Proposition 5.4. *Let* $p = 2^k$ *for an integer* $k \geqslant 1$. *If* θ *solves* (5.1) *with an initial data* $\theta_0 \in L^p$, *then the* L^p*-norm of* θ *decays algebraically in time. More precisely,*

$$\|\theta(\cdot, t)\|_{L^p} \leqslant \frac{\|\theta_0\|_{L^p}}{(1 + \kappa C_p \gamma t \|\theta_0\|_{L^2}^{-\gamma p} \|\theta_0\|_{L^p}^{\gamma p})^{1/\gamma p}},$$

where $\gamma = \alpha/(p-2)$ *and* C_p *is a constant depending on p and* α *only.*

Part of the proof of Theorem 5.1 is to obtain a uniform bound for a successive approximation sequence. The a priori estimate in the following proposition and its derivation will be useful for this purpose.

Proposition 5.5. *Let κ, α, r and q be as in Theorem* 5.1. *Assume that $\theta(\cdot, t) \in Y_{r,q}$ is a solution of the* 2*D dissipative QG equation with $\theta_0 \in Y_{r,q}$ for $t \in [0, T]$. Then, for some constant C depending on r and q only,*

$$\|\theta(\cdot, t)\|_{r,q} \leqslant \|\theta_0\|_{r,q} \exp\left(C \int_0^t \|\theta(\cdot, \tau)\|_{r,q} \, \mathrm{d}\tau\right)$$

is valid for any $t \leqslant T$.

Proof. Because of (5.2), it suffices to bound the C^r-norm. To proceed, we estimate $\|\Delta_j \theta\|_{L^\infty}$ for any integer $j \geqslant -1$ ($\Delta_j = 0$ for $j \leqslant -2$ according to (2.1)). Since $\Lambda^{2\alpha}$ and Δ_j commutes,

$$\partial_t \Delta_j \theta + u \cdot \nabla(\Delta_j \theta) + \kappa \Lambda^{2\alpha} \Delta_j \theta = [u \cdot \nabla, \Delta_j]\theta. \tag{5.3}$$

We first bound the L^p-norm of $\Delta_j \theta$ and then let $p \to \infty$. For $p \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\Delta_j \theta|^p \, \mathrm{d}x + \mathrm{I} = \mathrm{II} + \mathrm{III},$$

where I, II and III correspond to the terms in (5.3), namely

$$\mathrm{I} = \kappa p \int |\Delta_j \theta|^{p-2} \Delta_j \theta \Lambda^{2\alpha}(\Delta_j \theta) \, \mathrm{d}x,$$

$$\mathrm{II} = -p \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (u \cdot \nabla \Delta_j \theta) \, \mathrm{d}x,$$

$$\mathrm{III} = p \int |\Delta_j \theta|^{p-2} \Delta_j \theta [u \cdot \nabla, \Delta_j]\theta \, \mathrm{d}x.$$

The second term II is equal to zero after integration by parts,

$$\mathrm{II} = -\int u \cdot \nabla(|\Delta_j \theta|^p) \, \mathrm{d}x = 0.$$

Lemma 5.2 implies that $\mathrm{I} \geqslant 0$. Applying Hölder's inequality to III,

$$\mathrm{III} \leqslant p \|\Delta_j \theta\|_{L^p}^{p-1} \|[u \cdot \nabla, \Delta_j]\theta\|_{L^p}$$

and then combining these estimates yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \theta\|_{L^p} \leqslant \|[u \cdot \nabla, \Delta_j]\theta\|_{L^p}.$$

Integrating with respect to t and letting $p \to \infty$, we obtain

$$\|\Delta_j \theta(\cdot, t)\|_{L^\infty} \leqslant \|\Delta_j \theta_0\|_{L^\infty} + \int_0^t \|[u \cdot \nabla, \Delta_j]\theta(\cdot, s)\|_{L^\infty} \, \mathrm{d}s.$$

This is identical to inequality (4.3) in the proof of Proposition 4.2. We thus omit further details. □

Proof of Theorem 5.1. . The same strategy as in the proof of Theorem 4.1 also works here, so we shall just present the major lines. Consider a successive approximation sequence $\{\theta^{(n)}\}$ satisfying

$$\begin{cases} \theta^{(1)} = S_2(\theta_0), \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \kappa \Lambda^{2\alpha} \theta^{(n+1)} = 0, \\ u^{(n)} = \nabla^{\perp} \Lambda^{-1} \theta^{(n)}, \\ \theta^{(n+1)}(x,0) = \theta_0^{(n+1)}(x) = S_{n+2}\theta_0. \end{cases}$$

Proposition 5.5 allows us to show that there exists a $T_1 > 0$ depending on $\|\theta_0\|_{r,q}$ only such that $\{\theta^{(n)}(\cdot, t)\}$ is bounded uniformly in $Y_{r,q}$ for any integer $n > 0$ and $t \in [0, T_1]$. In addition, for $\alpha \leqslant \frac{1}{2}$ and $r > 1$,

$$\begin{aligned} \|\partial_t \theta^{(n+1)}\|_{C^{r-1}} &\leqslant \|u^{(n)} \cdot \nabla \theta^{(n+1)}\|_{C^{r-1}} + \kappa \|\Lambda^{2\alpha} \theta^{(n+1)}\|_{C^{r-1}} \\ &\leqslant C(\|u^{(n)}\|_{L^\infty} \|\nabla \theta^{(n+1)}\|_{C^{r-1}} + \|u^{(n)}\|_{C^{r-1}} \|\nabla \theta^{(n+1)}\|_{L^\infty}) \\ &\quad + \kappa \|\theta^{(n+1)}\|_{C^r} \\ &\leqslant (C\|u^{(n)}\|_{C^{r-1}} + \kappa) \|\theta^{(n+1)}\|_{C^r} \leqslant CM^2, \end{aligned}$$

where C is a constant depending on κ. This uniform bound allows us to conclude that $\theta \in Lip([0, T_1]; C^{r-1})$. The sequence $\{\theta^{(n)}\}$ is then shown to be Cauchy in $C([0, T_2]; Y_{r-1,q})$ for some $T_2 \in [0, T_1]$ by considering the difference

$$\eta^{(n+1)} = \theta^{(n+1)} - \theta^{(n)}.$$

Clearly, $\eta^{(n+1)}$ satisfies

$$\partial_t \eta^{(n+1)} + u^{(n)} \cdot \nabla \eta^{(n+1)} + \kappa \Lambda^{2\alpha} \eta^{(n+1)} = w^{(n)} \cdot \nabla \theta^{(n)}, \quad w^{(n)} = \nabla^{\perp} \Lambda^{-1} \eta^{(n)}.$$

The rest is then similar to Step 2 in the proof of Theorem 4.1 and we omit the details. □

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