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THE 2D INCOMPRESSIBLE MAGNETOHYDRODYNAMICS EQUATIONS WITH ONLY MAGNETIC DIFFUSION*

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Abstract. This paper examines the global (in time) regularity of classical solutions to the two-dimensional (2D) incompressible magnetohydrodynamics (MHD) equations with only magnetic diffusion. Here the magnetic diffusion is given by the fractional Laplacian operator $(-\Delta)^\beta$. We establish the global regularity for the case when $\beta > 1$. This result significantly improves previous work which requires $\beta > \frac{3}{2}$ and brings us closer to the resolution of the well-known global regularity problem on the 2D MHD equations with standard Laplacian magnetic diffusion, namely, the case when $\beta = 1$.

Key words. MHD equations, partial dissipation, classical solutions, global regularity

AMS subject classifications. 35Q35, 35B35, 35B65, 76D03

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1. Introduction. This paper focuses on the initial-value problem for the two-dimensional (2D) incompressible magnetohydrodynamics (MHD) equations with fractional Laplacian magnetic diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\, t>0,\\ \partial_t b + u\cdot\nabla b + (-\Delta)^\beta b = b\cdot\nabla u, \quad x\in\mathbb{R}^2,\, t>0,\\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \quad x\in\mathbb{R}^2,\, t>0,\\ u(x,0)=u_0(x), \quad b(x,0)=b_0(x), \quad x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where the fractional Laplacian operator $(-\Delta)^\beta$ is defined through the Fourier transform

$$\widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta}\,\widehat{f}(\xi)$$

with $\widehat{f}$ being the Fourier transform of f, namely,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx.$$

When $\beta = 1$, (1.1) reduces to the MHD equations with Laplacian magnetic diffusion, which models many significant phenomena such as the magnetic reconnection in astrophysics and geomagnetic dynamo in geophysics (see, e.g., [20]).

What we care about here is the global regularity problem, namely, whether the solutions of (1.1) emanating from sufficiently smooth data (u_0, b_0) remain regular for

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all time. The main result of this paper states that (1.1) with any $\beta > 1$ always possesses a unique global solution. More precisely, we have the following theorem.

THEOREM 1.1. *Consider* (1.1) *with* $\beta > 1$. *Assume that* $(u_0, b_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$, $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$, *and* $j_0 = \nabla \times b_0$ *satisfying*

$$\|\nabla j_0\|_{L^\infty} < \infty.$$

Then (1.1) *has a unique global solution* (u, b) *satisfying, for any* $T > 0$,

$$(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)), \quad \nabla j \in L^1([0, T]; L^\infty(\mathbb{R}^2)),$$

where $j = \nabla \times b$.

We now review some recent work to put our result in a proper content. Due to their prominent roles in modeling many phenomena in astrophysics, geophysics, and plasma physics, the MHD equations have been studied extensively mathematically. Duvaut and Lions constructed a global Leray–Hopf weak solution and a local strong solution of the three-dimensional incompressible MHD equations [9]. Sermange and Temam further examined the properties of these solutions [22]. More recent work on the MHD equations develops regularity criteria in terms of the velocity field and deals with the MHD equations with dissipation and magnetic diffusion given by general Fourier multiplier operators such as the fractional Laplacian operators (see, e.g., [6, 7, 10, 11, 13, 14, 23, 25, 26, 27, 28, 30, 31, 32]).

Another direction that has generated considerable interest recently is the global regularity problem on the MHD equations with partial dissipation, especially in the 2D case (see, e.g., [3, 4, 12, 15, 16, 17, 29, 33]). Since the global regularity of the 2D MHD equations with both Laplacian dissipation and magnetic diffusion is easy to obtain while the regularity of the completely inviscid MHD equations appears to be out of reach, it is natural to examine the MHD equations with partial dissipation. One particular partial dissipation case is (1.1). When $\beta \geq 1$, any solution of (1.1) with $(u_0, b_0) \in H^1$ generates a global weak solution (u, b) that remains bound in H^1 for all time (see, e.g., [3, 15]). However, it is not clear if such weak solutions are unique or their regularity can be improved to be in H^2 if the initial data (u_0, b_0) is in H^2. The result of this paper obtains the uniqueness and global regularity for the case when $\beta > 1$. Previous global regularity results require that $\beta > \frac{3}{2}$ (see, e.g., [13, 23, 30, 31]). Our approach here does not appear to be able to extend to the borderline case when $\beta = 1$, which is currently being studied by a different method [19]. Progress has also been made on several other partial dissipation cases of the MHD equations. Lin, Xu, and Zhang recently studied the MHD equations with the Laplacian dissipation in the velocity equation but without magnetic diffusion and were able to establish the global existence of small solutions after translating the magnetic field by a constant vector [16, 17, 29]. Cao and Wu examined the anisotrophic 2D MHD equations with horizontal dissipation and vertical magnetic diffusion (or vertical dissipation and horizontal magnetic diffusion) and obtained the global regularity for this partial dissipation case [3]. The anisotrophic MHD equations with horizontal dissipation and horizontal magnetic diffusion were also investigated very recently and progress was also made (see [4]).

We now explain our proof of Theorem 1.1. Since the local (in time) well-posedness can be established following a standard approach, our efforts are devoted to proving global a priori bounds for (u, b) in the initial functional setting H^s with $s > 2$. (u, b) indeed admits a global H^1-bound, but direct energy estimates do not appear to easily yield a global H^2-bound. The difficulty comes from the nonlinear term in the velocity equation due to the lack of dissipation. To bypass this difficulty, we first make use of the magnetic diffusion $(-\Delta)^\beta b$ with $\beta > 1$ to show the global bound for $\|\omega\|_{L^q}$ and $\|j\|_{L^q}$ for $2 \le q \le \frac{2}{2-\beta}$. (The range is modified to $2 \le q < \infty$ when $\beta = 2$.) The magnetic diffusion is further exploited to obtain a global bound for j in the space-time Besov space $L^1_t B^s_{q,1}$, namely, for any $T > 0$ and $t \le T$

$$\|j\|_{L^1_t B^s_{q,1}} \le C(T, u_0, b_0) < \infty \quad \text{with} \quad 2 \le q \le \frac{2}{2-\beta}, \quad \frac{2}{q} < s < 2\beta - 1. \tag{1.2}$$

Roughly speaking, this global bound provides the time integrability of the L^q-norm of j up to $(2\beta - 1)$-derivative. This global bound is proved through Besov space techniques. Special consequences of this global bound include the time integrability of $\|j\|_{L^\infty}$ and of $\|\nabla j\|_{L^r}$ for $r > q$. To gain higher regularity, we go through an iterative process. The bound $\|\nabla j\|_{L^1_t L^r} < \infty$ allows us to further prove a global bound for $\|\omega\|_{L^r}$ and $\|j\|_{L^r}$ with $r > q$, which can be employed to prove (1.2) with q replaced by r. Repeating this process leads to the global bound in (1.2) for any $q \in [2, \infty)$, which especially implies, for any $t > 0$,

$$\int_0^t \|\nabla j\|_{L^\infty} d\tau < \infty. \tag{1.3}$$

Equation (1.3) can then be further used to obtain a global bound for $\|\omega\|_{L^\infty}$. The time integrability of $\|j\|_{L^\infty}$ and the boundedness of $\|\omega\|_{L^\infty_{t,x}}$ are enough to prove a global bound for (u, b) in H^s.

The rest of this paper is divided into three sections. The second section provides the definition of inhomogeneous Besov spaces and related facts such as Bernstein's inequality. The third section proves the global L^q-bound for (ω, j), while the fourth section establishes the global bound in (1.2). The last section gains higher regularity through an iterative process and proves Theorem 1.1.

2. Functional spaces. This section provides the definition of Besov spaces and related facts used in the subsequent sections. Materials presented here can be found in several books and many papers (see, e.g., [1, 2, 18, 21, 24]).

We start with notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. It is a simple fact in analysis that there exist two radially symmetric functions $\Psi, \Phi \in \mathcal{S}$ such that

$$\text{supp}\widehat{\Psi} \subset B\left(0, \frac{4}{3}\right), \qquad \text{supp}\widehat{\Phi} \subset \mathcal{A}\left(0, \frac{3}{4}, \frac{8}{3}\right),$$

$$\widehat{\Psi}(\xi) + \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^d,$$

where $B(0, r)$ denotes the ball centered at the origin with radius r, $\mathcal{A}(0, r_1, r_2)$ denotes the annulus centered at the origin with the inner radius r_1 and the outer r_2, $\Phi_0 = \Phi$, and $\Phi_j(x) = 2^{jd}\Phi_0(2^j x)$ or $\widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi)$.

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0 & \text{if } j \le -2, \\ \Psi * f & \text{if } j = -1, \\ \Phi_j * f & \text{if } j = 0, 1, 2, \dots. \end{cases} \tag{2.2}$$

DEFINITION 2.1. *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$, and $s \in \mathbb{R}$ consists of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (2.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein-type inequalities for fractional derivatives.

PROPOSITION 2.2. *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

(1) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \le K 2^j\}$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p, and q only.

3. Global L^q-bound for (ω, j). This section proves a global a priori bound for ω and j in the Lebesgue space $L^q(\mathbb{R}^2)$ for q in a suitable range. More precisely, we prove the following proposition.

PROPOSITION 3.1. *Assume (u_0, b_0) satisfies the conditions stated in Theorem 1.1. Let (u, b) be the corresponding solution of (1.1) with $\beta > 1$. Then, for any q satisfying*

$$2 \le q \le \frac{2}{2 - \beta} \tag{3.1}$$

(the range of q is modified to $2 \le q < \infty$ when $\beta = 2$), and for any $T > 0$ and $t \le T$, there exists a constant $C = C(T, u_0, b_0)$ such that

$$\|\omega(t)\|_{L^q} \le C, \qquad \|j(t)\|_{L^q} \le C. \tag{3.2}$$

To prove Proposition 3.1, we first provide two simple bounds. A simple energy estimate yields the global L^2-bound of (1.1) with $\beta \ge 0$.

LEMMA 3.2. *Assume* (u_0, b_0) *satisfies the conditions stated in Theorem* 1.1. *Let* (u, b) *be the corresponding solution of* (1.1) *with* $\beta \geq 0$. *Then, for any* $t \geq 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta b(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2,$$

where $\Lambda = \sqrt{-\Delta}$.

In addition, by resorting to the equations of ω and j,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j + (-\Delta)^\beta j = b \cdot \nabla \omega + 2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2), \\ \omega(x, 0) = \omega_0(x), \quad \omega(x, 0) = j_0(x), \end{cases} \tag{3.3}$$

the global H^1-bound on (u, b) can be obtained in a similar fashion as in [3].

LEMMA 3.3. *Assume* (u_0, b_0) *satisfies the conditions stated in Theorem* 1.1. *Let* (u, b) *be the corresponding solution of* (1.1) *with* $\beta \geq 1$. *Then, for any* $T > 0$ *and* $t \leq T$, *there exists a constant* $C = C(T, \|(u_0, b_0)\|_{H^1})$ *such that*

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta j(\tau)\|_{L^2}^2 \, d\tau \leq C. \tag{3.4}$$

Proof of Proposition 3.1. Multiplying the first equation in (3.3) by $\omega|\omega|^{q-2}$, integrating in space and applying Hölder's inequality, we have

$$\frac{1}{q}\frac{d}{dt}\|\omega\|_{L^q}^q = \int b \cdot \nabla j \, \omega|\omega|^{q-2} \leq \|\omega\|_{L^q}^{q-1} \|b\|_{L^\infty} \|\nabla j\|_{L^q}.$$

Recall the simple Sobolev inequalities, for $\beta > 1$,

$$\|b\|_{L^\infty} \leq C\|b\|_{L^2}^{1-\frac{1}{1+\beta}} \|\Lambda^\beta j\|_{L^2}^{\frac{1}{1+\beta}}, \qquad \|\nabla j\|_{L^q} \leq C\, \|j\|_{L^2}^{1-\frac{2(q-1)}{\beta q}} \|\Lambda^\beta j\|_{L^2}^{\frac{2(q-1)}{\beta q}}, \tag{3.5}$$

and note that (3.1) ensures that $\frac{2(q-1)}{\beta q} \leq 1$. By Young's inequality,

$$\|b\|_{L^\infty} \|\nabla j\|_{L^q} \leq C\,(\|b\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2).$$

Therefore,

$$\|\omega(t)\|_{L^q} \leq \|\omega_0\|_{L^q} + C\int_0^t (\|b\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\, dt$$

and the bounds in Lemmas 3.2 and 3.3 yield the global bound for $\|\omega(t)\|_{L^q}$. To get a global bound for $\|j\|_{L^q}$, we first obtain from the equation of j in (3.3)

$$\frac{1}{q}\frac{d}{dt}\|j\|_{L^q}^q + \int j|j|^{q-2}\,(-\Delta)^\beta j = K_1 + K_2 + K_3 + K_4 + K_5, \tag{3.6}$$

where $K_1, \dots, K_5$ are given by

$$K_1 = \int b \cdot \nabla \omega \, j|j|^{q-2},$$

$$K_2 = 2\int \partial_1 b_1 \, \partial_2 u_1 \, j|j|^{q-2},$$

$$K_3 = 2\int \partial_1 b_1\, \partial_1 u_2\, j|j|^{q-2},$$
$$K_4 = 2\int \partial_1 u_1\, \partial_2 b_1\, j|j|^{q-2},$$
$$K_5 = 2\int \partial_1 u_1\, \partial_1 b_2\, j|j|^{q-2}.$$

According to [8], we have the lower bound, for $C_0 = C_0(\beta, q)$,

$$\int j|j|^{q-2}\,(-\Delta)^\beta j \ge C_0 \int |\Lambda^\beta(|j|^{\frac{q}{2}})|^2. \tag{3.7}$$

By integration by parts and Hölder's inequality,

$$\begin{aligned}|K_1| &= \frac{2(q-1)}{q}\left|\int \omega\, b\cdot\nabla(|j|^{\frac{q}{2}})\,|j|^{\frac{q}{2}-1}\right|\\ &\le C\,\|b\|_{L^\infty}\,\|\omega\|_{L^q}\,\|j\|_{L^q}^{\frac{q}{2}-1}\,\|\nabla(|j|^{\frac{q}{2}})\|_{L^2}.\end{aligned}$$

By the trivial embedding inequality, for $\beta \ge 1$,

$$\|\nabla(|j|^{\frac{q}{2}})\|_{L^2} \le \||j|^{\frac{q}{2}}\|_{H^\beta} \le \||j|^{\frac{q}{2}}\|_{L^2} + \|\Lambda^\beta(|j|^{\frac{q}{2}})\|_{L^2} = \|j\|_{L^q}^{\frac{q}{2}} + \|\Lambda^\beta(|j|^{\frac{q}{2}})\|_{L^2},$$

we obtain

$$\begin{aligned}|K_1| &\le \frac{C_0}{2}\int |\Lambda^\beta(|j|^{\frac{q}{2}})|^2 + C\,\|b\|_{L^\infty}\,\|\omega\|_{L^q}\,\|j\|_{L^q}^{q-1} + C\,\|b\|_{L^\infty}^2\,\|\omega\|_{L^q}^2\,\|j\|_{L^q}^{q-2}\\ &\le \frac{C_0}{2}\int |\Lambda^\beta(|j|^{\frac{q}{2}})|^2 + C\,(1+\|b\|_{L^\infty})\|b\|_{L^\infty}(\|\omega\|_{L^q}^q + \|j\|_{L^q}^q).\end{aligned}$$

K_2 can be easily bounded. In fact, by the Sobolev inequality with $\beta > 1$,

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\,\|f\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{\beta}}\,\|\Lambda^\beta f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{\beta}},$$

we have

$$\begin{aligned}|K_2| &\le 2\|\partial_1 b_1\|_{L^\infty}\,\|\omega\|_{L^q}\,\|j\|_{L^q}^{q-1}\\ &\le C\,\|\partial_1 b_1\|_{L^2}^{1-\frac{1}{\beta}}\,\|\Lambda^\beta\partial_1 b_1\|_{L^2}^{\frac{1}{\beta}}\,(\|\omega\|_{L^q}^q + \|j\|_{L^q}^q)\\ &\le C\,\|j\|_{L^2}^{1-\frac{1}{\beta}}\,\|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}}\,(\|\omega\|_{L^q}^q + \|j\|_{L^q}^q),\end{aligned} \tag{3.8}$$

where the simple inequality $\|\partial_1 b_1\|_{L^2} \le \|\nabla b\|_{L^2} = \|\nabla\times b\|_{L^2} = \|j\|_{L^2}$. Clearly, K_3, K_4, and K_5 admit the same bound as in (3.8). Collecting the estimates, we have

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\|j\|_{L^q}^q + \frac{C_0}{2}\int |\Lambda^\beta(|j|^{\frac{q}{2}})|^2\\ &\quad\le C\,(\|b\|_{L^\infty} + \|b\|_{L^\infty}^2 + \|j\|_{L^2}^{1-\frac{1}{\beta}}\,\|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}})(\|\omega\|_{L^q}^q + \|j\|_{L^q}^q).\end{aligned}$$

Thanks to the time integrability of $\|b\|_{L^\infty}^2$ (see (3.5) for a bound) and $\|j\|_{L^2}^{1-\frac{1}{\beta}}\,\|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}}$ on any finite time interval and the global bound for $\|\omega\|_{L^q}$, this differential inequality yields the global bound for $\|j\|_{L^q}$. This completes the proof of Proposition 3.1. ∎

4. Global $L^1_t B^s_{q,1}$-bound for j. This section establishes a global bound for j in the space-time Besov space $L^1_t B^s_{q,1}$, where q satisfies (3.1) and $\frac{2}{q} < s < 2\beta - 1$. For $\beta > 1$, this global bound provides a better integrability than the one given in (3.4) and will be exploited to gain higher regularity in the next section.

PROPOSITION 4.1. *Assume (u_0, b_0) satisfies the conditions stated in Theorem 1.1. Let (u, b) be the corresponding solution of (1.1) with $\beta > 1$. Then, for any q and s satisfying*

$$2 \le q \le \frac{2}{2-\beta}, \qquad \frac{2}{q} < s < 2\beta - 1 \tag{4.1}$$

(the range of q is modified to $2 \le q < \infty$ when $\beta = 2$), and for any $T > 0$ and $t \le T$, there exists a constant $C = C(q, s, T, u_0, b_0)$ such that

$$\|j\|_{L^1_t B^s_{q,1}} \le C. \tag{4.2}$$

We remark that the proof of this theorem makes use of the global L^q-bound for $\|\omega\|_{L^q}$ and $\|j\|_{L^q}$ and this explains why we need to restrict q to the range in (4.1). It can be seen from the proof that this theorem remains valid for any $q \ge 2$ as long as $\|\omega\|_{L^q}$ and $\|j\|_{L^q}$ are bounded. We now prove Proposition 4.1.

Proof of Proposition 4.1. Let $k \ge 0$ be an integer. Applying Δ_k to the equation of j in (3.3), multiplying by $\Delta_k j\, |\Delta_k j|^{q-2}$, and integrating in space, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\Delta_k j\|^q_{L^q} + \int \Delta_k j\, |\Delta_k j|^{k-2} (-\Delta)^\beta \Delta_k j = L_1 + \cdots + L_6, \tag{4.3}$$

where $L_1, \dots, L_6$ are given by

$$\begin{aligned}
L_1 &= -\int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(u \cdot \nabla j),\\
L_2 &= \int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(b \cdot \nabla \omega),\\
L_3 &= 2\int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(\partial_1 b_1 \partial_2 u_1),\\
L_4 &= 2\int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(\partial_1 b_1 \partial_1 u_2),\\
L_5 &= -2\int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(\partial_1 u_1 \partial_2 b_1),\\
L_6 &= -2\int \Delta_k j\, |\Delta_k j|^{q-2}\, \Delta_k(\partial_1 u_1 \partial_1 b_2).
\end{aligned}$$

The term associated the magnetic diffusion admits the lower bound (see [5])

$$\int \Delta_k j\, |\Delta_k j|^{q-2} (-\Delta)^\beta \Delta_k j \ge C_1 2^{2\beta k} \|\Delta_k j\|^q_{L^q}$$

for $C_1 = C_1(\beta, q)$. By Bony's paraproducts decomposition, we write

$$L_1 = L_{11} + L_{12} + L_{13} + L_{14} + L_{15},$$

where

$$L_{11} = -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} [\Delta_k, S_{l-1}u\cdot\nabla]\Delta_l j,$$

$$L_{12} = -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} (S_{l-1}u - S_k u)\cdot\nabla\Delta_k\Delta_l j,$$

$$L_{13} = -\int \Delta_k j\, |\Delta_k j|^{q-2}\, S_k u\cdot\nabla\Delta_k j,$$

$$L_{14} = -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} \Delta_k(\Delta_l u\cdot\nabla S_{l-1}j),$$

$$L_{15} = -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{l\ge k-1} \Delta_k(\Delta_l u\cdot\nabla\widetilde{\Delta}_l j)$$

with $\widetilde{\Delta}_l = \Delta_{l-1} + \Delta_l + \Delta_{l+1}$. Thanks to $\nabla\cdot S_k u = 0$, we have $L_{13} = 0$. By Hölder's inequality, a standard commutator estimate, and Bernstein's inequality,

$$\begin{aligned} |L_{11}| &\le \|\Delta_k j\|_{L^q}^{q-1} \sum_{|k-l|\le 2} \|[\Delta_k, S_{l-1}u\cdot\nabla]\Delta_l j\|_{L^q} \\ &\le C\,\|\Delta_k j\|_{L^q}^{q-1}\, \|\nabla S_{k-1}u\|_{L^q}\, \|\Delta_k j\|_{L^\infty} \\ &\le C\,\|\Delta_k j\|_{L^q}^{q-1}\, \|\omega\|_{L^q}\, 2^{k\frac{2}{q}} \|\Delta_k j\|_{L^q}. \end{aligned} \tag{4.4}$$

Here we have also applied the simple fact that for fixed k, the summation is for a finite number of l satisfying $|k-l|\le 2$ and the estimate for the term with the index l is only a constant multiple of the bound for the term with the index k. In addition, the simple bound

$$\|\nabla S_{k-1}u\|_{L^q} \le \|\nabla u\|_{L^q} \le C\|\omega\|_{L^q}$$

is also used here. It is easily seen that L_{12} obeys the same bound as in (4.4). By Hölder's inequality and Bernstein's inequality (both lower bound and upper bound parts),

$$\begin{aligned} |L_{14}| &\le C\,\|\Delta_k j\|_{L^q}^{q-1}\, \|\Delta_k u\|_{L^q}\, \|\nabla S_{k-1}j\|_{L^\infty} \\ &\le C\,\|\Delta_k j\|_{L^q}^{q-1}\, 2^{-k}\, \|\nabla\Delta_k u\|_{L^q} \sum_{m\le k-1} 2^{(1+\frac{2}{q})m} \|\Delta_m j\|_{L^q} \\ &\le C\,\|\omega\|_{L^q}\, \|\Delta_k j\|_{L^q}^{q-1}\, 2^{-k} \sum_{m\le k-1} 2^{(1+\frac{2}{q})m} \|\Delta_m j\|_{L^q}. \end{aligned}$$

By the divergence-free condition, $\nabla\cdot\Delta_l u = 0$,

$$\begin{aligned} |L_{15}| &\le C\,\|\Delta_k j\|_{L^q}^{q-1} \sum_{l\ge k-1} 2^k \|\Delta_l u\|_{L^q}\, \|\widetilde{\Delta}_l j\|_{L^\infty} \\ &\le C\,\|\Delta_k j\|_{L^q}^{q-1} \sum_{l\ge k-1} 2^{k-l} \|\nabla\Delta_l u\|_{L^q}\, 2^{l\frac{2}{q}}\, \|\widetilde{\Delta}_l j\|_{L^q} \\ &\le C\,\|\omega\|_{L^q}\, \|\Delta_k j\|_{L^q}^{q-1}\, 2^{k\frac{2}{q}} \sum_{l\ge k-1} 2^{(k-l)(1-\frac{2}{q})}\, \|\Delta_l j\|_{L^q}. \end{aligned}$$

We have thus completed the estimate for L_1. To bound L_2, we also decompose it into five terms,

$$L_2 = L_{21} + L_{22} + L_{23} + L_{24} + L_{25},$$

where

$$\begin{aligned}
L_{21} &= -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} [\Delta_k, S_{l-1} b\cdot\nabla]\Delta_l \omega,\\
L_{22} &= -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} (S_{l-1} b - S_k b)\cdot\nabla\Delta_k\Delta_l \omega,\\
L_{23} &= -\int \Delta_k j\, |\Delta_k j|^{q-2}\, S_k b\cdot\nabla\Delta_k \omega,\\
L_{24} &= -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} \Delta_k(\Delta_l b\cdot\nabla S_{l-1}\omega),\\
L_{25} &= -\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{l\ge k-1} \Delta_k(\Delta_l b\cdot\nabla\widetilde{\Delta}_l\omega).
\end{aligned}$$

The estimates of these terms are similar to those in L_1, but there are some differences, as can be seen from the bounds below.

$$\begin{aligned}
|L_2| &\le C\, 2^{k\frac{2}{q}}\, \|\Delta_k j\|_{L^q}^{q-1}\, \|\nabla b\|_{L^q}\, \|\omega\|_{L^q} + C\, 2^k\, \|\Delta_k j\|_{L^q}^{q-1}\, \|b\|_{L^\infty}\, \|\omega\|_{L^q}\\
&\quad + C\, 2^{k\frac{2}{q}}\, \|\Delta_k j\|_{L^q}^{q-1} \sum_{l\ge k-1} 2^{(k-l)(1-\frac{2}{q})} \|\nabla\Delta_l b\|_{L^q}\, \|\Delta_l\omega\|_{L^q}\\
&\le C\, 2^{k\frac{2}{q}}\, \|\Delta_k j\|_{L^q}^{q-1}\, \|\nabla b\|_{L^q}\, \|\omega\|_{L^q} + C\, 2^k\, \|\Delta_k j\|_{L^q}^{q-1}\, \|b\|_{L^\infty}\, \|\omega\|_{L^q}.
\end{aligned}$$

To bound L_3, we decompose it into three terms as

$$L_3 = L_{31} + L_{32} + L_{33},$$

where

$$\begin{aligned}
L_{31} &= 2\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} \Delta_k(S_{l-1}\partial_1 b_1 \Delta_l \partial_2 u_1),\\
L_{32} &= 2\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{|k-l|\le 2} \Delta_k(\Delta_l\partial_1 b_1 S_{l-1} \partial_2 u_1),\\
L_{33} &= 2\int \Delta_k j\, |\Delta_k j|^{q-2} \sum_{l\ge k-1} \Delta_k(\Delta_l\partial_1 b_1 \widetilde{\Delta}_l \partial_2 u_1).
\end{aligned}$$

These terms can be bounded by

$$\begin{aligned}
|L_{31}| &\le C\, \|\Delta_k j\|_{L^q}^{q-1}\, \|S_{k-1}\partial_1 b_1\|_{L^\infty}\, \|\Delta_k \partial_2 u_1\|_{L^q}\\
&\le C\, \|\Delta_k j\|_{L^q}^{q-1}\, 2^{k\frac{2}{q}}\, \|j\|_{L^q}\, \|\omega\|_{L^q},
\end{aligned}$$

where we have used the bound $\|\nabla u\|_{L^q} \le C\, \|\omega\|_{L^q}$ and $\|\nabla b\|_{L^q} \le C\, \|j\|_{L^q}$. Clearly, L_{32} admits the same bound. L_{33} is bounded by

$$\begin{aligned}
|L_{33}| &\le C\, \|\Delta_k j\|_{L^q}^{q-1} \sum_{l\ge k-1} 2^{l\frac{2}{q}}\, \|\Delta_l \partial_1 b_1\|_{L^q}\, \|\Delta_l \partial_2 u_1\|_{L^q}\\
&\le C\, \|\omega\|_{L^q}\, \|\Delta_k j\|_{L^q}^{q-1} \sum_{l\ge k-1} 2^{l\frac{2}{q}}\, \|\Delta_l j\|_{L^q}.
\end{aligned}$$

Collecting all the estimates above, we obtain

$$\frac{d}{dt}\|\Delta_k j\|_{L^q} + C_1\, 2^{2\beta k}\|\Delta_k j\|_{L^q} \le RHS(t),$$

where $RHS(t)$ denotes the bound

$$\begin{aligned} RHS(t) \equiv{}& C\,\|\omega\|_{L^q}\, 2^{k\frac{2}{q}}\,\|\Delta_k j\|_{L^q} \\ &+ C\|\omega\|_{L^q}\, 2^{-k} \sum_{m\le k-1} 2^{(1+\frac{2}{q})m}\|\Delta_m j\|_{L^q} \\ &+ C2^{k\frac{2}{q}}\,\|\omega\|_{L^q} \sum_{l\ge k-1} 2^{(k-l)(1-\frac{2}{q})}\,\|\Delta_l j\|_{L^q} \\ &+ C2^{k\frac{2}{q}}\,\|j\|_{L^q}\,\|\omega\|_{L^q} + C\,2^k\,\|b\|_{L^\infty}\,\|\omega\|_{L^q} \\ &+ C\|\omega\|_{L^q} \sum_{l\ge k-1} 2^{l\frac{2}{q}}\,\|\Delta_l j\|_{L^q}. \end{aligned} \tag{4.5}$$

Integrating in time, we have

$$\|\Delta_k j(t)\|_{L^q} \le e^{-C_1\,2^{2\beta k}t}\,\|\Delta_k j(0)\|_{L^q} + \int_0^t e^{-C_1\,2^{2\beta k}(t-\tau)}\,RHS(\tau)\,d\tau.$$

For any fixed $t>0$, we take the L^1-norm on $[0,t]$ and apply Young's inequality to obtain

$$\int_0^t \|\Delta_k j(\tau)\|_{L^q}\,d\tau \le C\,2^{-2\beta k}\|\Delta_k j(0)\|_{L^q} + C\,2^{-2\beta k}\int_0^t RHS(\tau)\,d\tau.$$

Multiplying by 2^{ks} and summing over $k=0,1,2,\dots$, we obtain

$$\begin{aligned} \|j\|_{L^1_t B^s_{q,1}} &= \int_0^t \|\Delta_{-1} j(\tau)\|_{L^q}\,d\tau + \sum_{k=0}^{\infty} 2^{ks}\int_0^t \|\Delta_k j(\tau)\|_{L^q}\,d\tau \\ &\le C(t) + C\,\|j_0\|_{B^{s-2\beta}_{q,1}} + \sum_{k=0}^{\infty} 2^{k(s-2\beta)}\int_0^t RHS(\tau)\,d\tau, \end{aligned} \tag{4.6}$$

where we have applied the global bound $\|\Delta_{-1} j(\tau)\|_{L^q} \le \|j\|_{L^q}$. For clarity of presentation, we write

$$\sum_{k=0}^{\infty} 2^{k(s-2\beta)}\int_0^t RHS(\tau)\,d\tau \equiv M_1+M_2+M_3+M_4+M_5+M_6,$$

where, according to (4.5),

$$\begin{aligned} M_1 &= C\,\|\omega\|_{L^\infty_t L^q}\sum_{k=0}^{\infty} 2^{k(s+\frac{2}{q}-2\beta)}\int_0^t \|\Delta_k j(\tau)\|_{L^q}\,d\tau, \\ M_2 &= C\,\|\omega\|_{L^\infty_t L^q}\sum_{k=0}^{\infty}\sum_{m\le k-1} 2^{(m-k)(1-s+2\beta)}\,2^{(s+\frac{2}{q}-2\beta)m}\int_0^t \|\Delta_m j\|_{L^q}\,d\tau, \\ M_3 &= C\,\|\omega\|_{L^\infty_t L^q}\sum_{k=0}^{\infty} 2^{k(1+s-2\beta)}\sum_{l\ge k-1} 2^{(-1+\frac{2}{q})l}\int_0^t \|\Delta_l j(\tau)\|_{L^q}\,d\tau, \end{aligned}$$

$$
\begin{aligned}
M_4 &= C\, t\, \|\omega\|_{L_t^\infty L^q}\, \|j\|_{L_t^\infty L^q} \sum_{k=0}^{\infty} 2^{k(s+\frac{2}{q}-2\beta)},\\
M_5 &= C\, \|\omega\|_{L_t^\infty L^q}\, \|b\|_{L_t^1 L^\infty} \sum_{k=0}^{\infty} 2^{k(s+1-2\beta)},\\
M_6 &= C\, \|\omega\|_{L_t^\infty L^q} \sum_{k=0}^{\infty} 2^{k(s-2\beta)} \sum_{l\ge k-1} 2^{l\frac{2}{q}} \int_0^t \|\Delta_l j(\tau)\|_{L^q}\, d\tau.
\end{aligned}
$$

Since $\frac{2}{q}-2\beta<0$, we can choose an integer $k_0>0$ such that

$$
C\, \|\omega\|_{L_t^\infty L^q}\, 2^{k_0(\frac{2}{q}-2\beta)} \le \frac{1}{16}.
$$

We can split the sum in M_1 into two parts,

$$
\begin{aligned}
M_1 &= C\, \|\omega\|_{L_t^\infty L^q} \sum_{k=0}^{k_0} 2^{k(s+\frac{2}{q}-2\beta)} \int_0^t \|\Delta_k j(\tau)\|_{L^q}\, d\tau\\
&\quad + C\, \|\omega\|_{L_t^\infty L^q} \sum_{k=k_0+1}^{\infty} 2^{k(s+\frac{2}{q}-2\beta)} \int_0^t \|\Delta_k j(\tau)\|_{L^q}\, d\tau\\
&= C(t,u_0,b_0) + \frac{1}{16}\|j\|_{L_t^1 B^s_{q,1}},
\end{aligned}
$$

where we have used the bound

$$
\int_0^t \|\Delta_k j(\tau)\|_{L^q}\, d\tau \le \int_0^t \|j(\tau)\|_{L^q}\, d\tau \le C.
$$

To deal with M_2, we first realize that $1-s+2\beta>0$ and $(m-k)(1-s+2\beta)<0$, and we apply Young's inequality for series convolution to obtain

$$
M_2 \le C\, \|\omega\|_{L_t^\infty L^q} \sum_{k=0}^{\infty} 2^{k(s+\frac{2}{q}-2\beta)} \int_0^t \|\Delta_k j(\tau)\|_{L^q}\, d\tau,
$$

which obeys the same bound as M_1, namely,

$$
M_2 \le C(t,u_0,b_0) + \frac{1}{16}\|j\|_{L_t^1 B^s_{q,1}}.
$$

To bound M_3, we exchange the order of two sums to get

$$
M_3 = C\, \|\omega\|_{L_t^\infty L^q} \sum_{l=-1}^{\infty} 2^{(-1+\frac{2}{q})l} \int_0^t \|\Delta_l j(\tau)\|_{L^q}\, d\tau \sum_{k=0}^{l+1} 2^{k(1+s-2\beta)}.
$$

Since $1+s-2\beta<0$, we have

$$
\sum_{k=0}^{l+1} 2^{k(1+s-2\beta)} \le C
$$

and thus

$$
M_3 = C\, \|\omega\|_{L_t^\infty L^q} \sum_{l=-1}^{\infty} 2^{(-1+\frac{2}{q}-s)l}\, 2^{ls} \int_0^t \|\Delta_l j(\tau)\|_{L^q}\, d\tau.
$$

Noticing that $-1+\frac{2}{q}-s<0$, we take a positive integer l_0 such that

$$C\,\|\omega\|_{L^\infty_t L^q}\, 2^{(-1+\frac{2}{q}-s)l_0} < \frac{1}{16}.$$

Then, M_3 is bounded by

$$M_3 \le C(t,u_0,b_0) + \frac{1}{16}\|j\|_{L^1_t B^s_{q,1}}.$$

Since $s+1-2\beta<0$, we clearly have

$$M_4+M_5 \le C\,t\,\|\omega\|_{L^\infty_t L^q}\,\|j\|_{L^\infty_t L^q} + C\,\|\omega\|_{L^\infty_t L^q}\,\|b\|_{L^1_t L^\infty}.$$

M_6 can be bounded in a similar fashion as M_3 and we have, for $\frac{2}{q}<s$,

$$M_6 \le C(t,u_0,b_0) + \frac{1}{16}\|j\|_{L^1_t B^s_{q,1}}.$$

Inserting the estimates above in (4.6), we have

$$\|j\|_{L^1_t B^s_{q,1}} \le C(t,u_0,b_0) + C\,\|j_0\|_{B^{s-2\beta}_{q,1}} + \frac{1}{4}\,\|j\|_{L^1_t B^s_{q,1}}.$$

This completes the proof of Proposition 4.1. □

5. Higher regularity through an iterative process and proof of Theorem 1.1. This section explores some consequences of Proposition 4.1. In particular, we obtain global bounds for ∇j in $L^1_t L^\infty_x$ and ω in $L^\infty_{t,x}$, which are sufficient for the proof of Theorem 1.1. We now state the proposition for high regularity.

PROPOSITION 5.1. *Assume (u_0,b_0) satisfies the conditions stated in Theorem* 1.1. *Let (u,b) be the corresponding solution of* (1.1) *with $\beta>1$. Then, for any $T>0$ and $t\le T$, there exists a constant $C=C(T,u_0,b_0)$ such that*

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau \le C, \qquad \|\omega(t)\|_{L^\infty}\le C.$$

The first step is the following integrability result, as a special consequence of Proposition 4.1.

PROPOSITION 5.2. *Assume (u_0,b_0) satisfies the conditions stated in Theorem* 1.1. *Let (u,b) be the corresponding solution of* (1.1) *with $\beta>1$. Then, for any r satisfying*

$$2\le r\le\infty \quad \text{if } \beta>\tfrac{4}{3}, \quad \text{and} \quad 2\le r<\frac{2}{4-3\beta} \quad \text{if } \beta\le\tfrac{4}{3}, \tag{5.1}$$

and, for any $T>0$ and $t\le T$, there exists a constant $C=C(T,u_0,b_0)$ such that

$$\int_0^t \|\nabla j(\tau)\|_{L^r}\,d\tau \le C.$$

We remark that the range for r, namely, (5.1), is bigger than the one for q in (3.1). An immediate consequence is the global bound for $\|\omega\|_{L^r}$ and $\|j\|_{L^r}$, as explained in the proof of Proposition 5.1.

Proof of Proposition 5.2. By Bernstein's inequality,

$$\begin{aligned}\|\nabla j\|_{L^r} &\le \sum_{k=-1}^{\infty} \|\Delta_k \nabla j\|_{L^r}\\ &\le \sum_{k=-1}^{\infty} 2^{k(1+\frac{2}{q}-\frac{2}{r}-s)} 2^{ks} \|\Delta_k j\|_{L^q}.\end{aligned}$$

In the case when $\beta > \frac{4}{3}$, we can choose q and s satisfying (4.1), say, $q = 3$ and $s = \frac{5}{3}$, such that

$$1 + \frac{2}{q} - s \le 0,$$

and consequently, for any $r \in [2, \infty]$,

$$\|\nabla j\|_{L^r} \le \sum_{k=-1}^{\infty} 2^{ks} \|\Delta_k j\|_{L^q} \equiv \|j\|_{B^s_{q,1}}.$$

In the case when $\beta \le \frac{4}{3}$, we can choose q and s satisfying (4.1) and r satisfying (5.1) such that

$$1 + \frac{2}{q} - \frac{2}{r} - s \le 0$$

and again

$$\|\nabla j\|_{L^r} \le \|j\|_{B^s_{q,1}}.$$

Proposition 5.2 then follows from Proposition 4.1. ∎

We now prove Proposition 5.1.

Proof of Proposition 5.1. Proposition 5.2 allows us to obtain a global bound for $\|\omega\|_{L^r}$. In fact, it follows from the vorticity equation that

$$\frac{1}{r}\frac{d}{dt}\|\omega\|^r_{L^r} \le \|b\|_{L^\infty} \|\nabla j\|_{L^r} \, \|\omega\|^{r-1}_{L^r}.$$

Due to the Sobolev inequality, for $q > 2$,

$$\|b\|_{L^\infty} \le C \|b\|^{\frac{q-2}{2(q-1)}}_{L^2} \, \|j\|^{\frac{q}{2q-2}}_{L^q}$$

and the time integrability of $\|\nabla j\|_{L^r}$ from Proposition 5.2, we obtain the global bound

$$\|\omega(t)\|_{L^r} \le \|\omega_0\|_{L^r} + \|b\|_{L^\infty_t L^\infty_x} \int_0^t \|\nabla j\|_{L^r} \, d\tau < \infty.$$

By going through the proof of Proposition 3.1 with q replaced by r, we can show that

$$\|j(t)\|_{L^r} \le C.$$

As a consequence of the global bounds for $\|\omega(t)\|_{L^r}$ and $\|j(t)\|_{L^r}$, we can prove Proposition 4.1 again with q replaced by r. This iterative process allows us to establish the global bound

$$\|j\|_{L^1_t B^s_{r,1}} < \infty$$

for any $r \in [2, \infty)$ and $\frac{2}{r} < s < 2\beta - 1$. As a special consequence, we have, for any $t > 0$,

$$\int_0^t \|\nabla j\|_{L^\infty} \, d\tau < \infty. \tag{5.2}$$

In fact,

$$\begin{aligned} \|\nabla j\|_{L^\infty} &\le \sum_{k=-1}^{\infty} \|\Delta_k \nabla j\|_{L^\infty} \le \sum_{k=-1}^{\infty} 2^{k(1+\frac{2}{r})} \, \|\Delta_k j\|_{L^r} \\ &= \sum_{k=-1}^{\infty} 2^{k(1+\frac{2}{r}-s)} \, 2^{ks} \|\Delta_k j\|_{L^r} = \sum_{k=-1}^{\infty} 2^{ks} \|\Delta_k j\|_{L^r} \equiv \|j\|_{B^s_{r,1}}, \end{aligned}$$

where we have choose r large and $s < 2\beta - 1$ such that

$$1 + \frac{2}{r} - s \le 0.$$

The global bound in (5.2) further allows us to show that

$$\|\omega\|_{L^\infty} < \infty, \tag{5.3}$$

which follows from the inequality

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \|b\|_{L^\infty_t L^\infty} \int_0^t \|\nabla j\|_{L^\infty} \, d\tau.$$

This completes the proof of Proposition 5.1. ◻

Finally we prove Theorem 1.1.

Proof of Theorem 1.1. The proof of Theorem 1.1 is divided into two main steps. The first step is to construct a local (in time) solution, while the second step extends the local solution into a global one by making use of the global a priori bounds obtained in Proposition 5.1. The construction of a local solution is quite standard and is thus omitted here. The global bounds in Proposition 5.1 are sufficient in proving the global bound

$$\|(u, b)\|_{H^s} < \infty.$$

This completes the proof of Theorem 1.1. ◻

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