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THE TWO-DIMENSIONAL INCOMPRESSIBLE BOUSSINESQ EQUATIONS WITH GENERAL CRITICAL DISSIPATION*

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Abstract. The two-dimensional incompressible Boussinesq equations with partial or fractional dissipation have recently attracted considerable attention and the global regularity issue has been extensively investigated. This paper aims at the global regularity in the case when the dissipation is critical. The critical dissipation refers to $\alpha + \beta = 1$ when $\Lambda^\alpha \equiv (-\Delta)^{\frac{\alpha}{2}}$ and Λ^β represent the fractional Laplacian dissipation in the velocity and the temperature equations, respectively. When $\alpha = 1$ and $\beta = 0$ or when $\alpha = 0$ and $\beta = 1$, the global regularity was obtained in [T. Hmidi, S. Keraani, and F. Rousset, *J. Differential Equations*, 249 (2010), pp. 2147–2174; T. Hmidi, S. Keraani, and F. Rousset, *Comm. Partial Differential Equations*, 36 (2011), pp. 420–445]. However, the approaches there do not apply to the situation when $\alpha + \beta = 1$ with both $\alpha > 0$ and $\beta > 0$. The novelty here is to reduce the critical Boussinesq system to a critical active scalar equation or, more precisely, the generalized critical surface quasi-geostrophic equation. When α is restricted to a suitable range, the global regularity of the critical Boussinesq system can be obtained by exploiting the global regularity of this scalar equation and the global bound for a combined quantity of the vorticity and the temperature.

Key words. Boussinesq equations, critical dissipation, global regularity

AMS subject classifications. 35Q35, 35B35, 35B65, 76D03

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1. Introduction.

This paper studies the global (in time) regularity of solutions to the two-dimensional (2D) incompressible Boussinesq equations with a general critical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu\, \Lambda^\alpha u = -\nabla p + \theta \mathbf{e}_2, \qquad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0, \qquad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa\, \Lambda^\beta \theta = 0, \qquad x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x),\ \theta(x,0) = \theta_0(x), \qquad x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the 2D velocity, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu > 0$, $\kappa > 0$, $0 < \alpha < 1$, and $0 < \beta < 1$ are real parameters. Here $\Lambda = \sqrt{-\Delta}$ represents the Zygmund operator

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with Λ^α being defined through the Fourier transform, namely,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where the Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

Equation (1.1) generalizes the standard 2D Boussinesq equations in which the dissipation is given by the Laplacian operator. The Boussinesq equations with the standard Laplacian model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Raleigh–Bernard convection (see, e.g., [9, 17, 25, 29, 33, 34]). The generalization to include the fractional Laplacian is mainly mathematical. Mathematically, the 2D Boussinesq equations serve as a lower-dimensional model of the three-dimensional (3D) hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier–Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [26], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows.

This paper emphasizes the mathematics of (1.1) and is devoted to the global well-posedness problem. We remark that this problem has recently attracted considerable attention (see, e.g., [3, 5, 6, 12, 15, 16, 18, 19, 20, 21, 22, 24, 27]). This paper aims at the global regularity of (1.1) with a general critical dissipation

$$0 < \alpha < 1, \quad 0 < \beta < 1, \quad \alpha + \beta = 1. \tag{1.2}$$

This is an extremely difficult problem. When α and β are in the range in (1.2), standard energy estimates do not yield the global bounds in any Sobolev spaces. It appears that one needs at least $\alpha \geq 2$ or $\beta \geq 2$ in order to bound the derivatives of the solution [5, 21]. Therefore novel ideas are necessary to deal with the 2D Boussinesq equations with such a low dissipation as specified in (1.2). In fact, even the subcritical case $\alpha + \beta > 1$ is not trivial and only certain ranges of α and β in this case are resolved. Miao and Xue in [27] obtained the global regularity for (1.1) with α and β in the intervals

$$\alpha \in \left(\frac{6-\sqrt{6}}{4}, 1 \right), \quad \beta \in \left(1-\alpha, \min\left\{ \frac{7+2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2-2\alpha \right\} \right).$$

Another subcritical case was solved by Constantin and Vicol [12], who verified the global regularity of (1.1) with

$$\nu > 0, \quad \kappa > 0, \quad \alpha \in (0,2), \quad \beta \in (0,2), \quad \beta > \frac{2}{2+\alpha}.$$

The papers [19, 20] are the first to deal with the critical case $\alpha + \beta = 1$. [19] proved the global regularity for the special case $\alpha = 1$ and $\beta = 0$ while [20] for the case $\alpha = 0$ and $\beta = 1$. Unfortunately, the approaches in [19, 20] do not apply to the general critical case (1.2). When the critical fractional dissipation is split between the velocity equation and the temperature equation in (1.1), the situation becomes very complicated. Magically we are able to reduce (1.1) with the general critical case (1.2) into a critical active scalar equation or, more precisely, the generalized critical surface quasi-geostrophic equation (specified below). When α is restricted to a suitable range, the global regularity of the critical Boussinesq system (1.1) can be obtained by exploiting the global regularity of the critical scalar equation. More precisely we are able to prove the following theorem.

THEOREM 1.1. *Let $\alpha_0 < \alpha < 1$ and $\alpha + \beta = 1$, where*

$$\alpha_0 = \frac{23 - \sqrt{145}}{12} \approx 0.9132. \tag{1.3}$$

Assume that $u_0 \in B^\sigma_{2,1}(\mathbb{R}^2)$ with $\sigma \geq \frac{5}{2}$ and $\theta_0 \in B^2_{2,1}(\mathbb{R}^2)$. Then (1.1) *has a unique global solution (u, θ) satisfying, for any $0 < T < \infty$,*

$$\begin{aligned} &u \in C([0,T]; B^\sigma_{2,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^{\sigma+\alpha}_{2,1}(\mathbb{R}^2)),\\ &\theta \in C([0,T]; B^2_{2,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^{2+\beta}_{2,1}(\mathbb{R}^2)). \end{aligned} \tag{1.4}$$

Here $B^s_{q,r}$ with $s \in \mathbb{R}$ and $q, r \in [1, \infty]$ denotes an inhomogeneous Besov space and its precise definition is given in section 4. The key component in the proof of Theorem 1.1 is to establish the global a priori bounds in the class defined in (1.4). This does not appear to be trivial and the energy methods are not sufficient for this purpose. Although the global bounds for u in $L^\infty([0,T]; L^2)$ and θ in $L^\infty([0,T]; L^q)$ with $q \in [2, \infty]$ can be easily obtained, the global bounds for the derivatives are not evident. To avoid the pressure term, we resort to the vorticity formulation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_1 \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega \qquad \text{or} \quad u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \tag{1.5}$$

However, the "vortex stretching" term $\partial_1 \theta$ appears to prevent us from proving any global bound for ω. A natural idea would be to eliminate $\partial_1 \theta$ from the vorticity equation. For notational convenience, we set $\nu = \kappa = 1$ in (1.1) throughout the rest of this paper. Realizing that $\Lambda^\alpha \omega - \partial_1 \theta = \Lambda^\alpha(\omega - \Lambda^{-\alpha}\partial_1 \theta)$, we can hide $\partial_1 \theta$ by considering the new quantity

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha}\partial_1,$$

which satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla]\theta + \Lambda^{\beta - \alpha}\partial_1 \theta. \tag{1.6}$$

Here we have used the standard commutator notation

$$[\mathcal{R}_\alpha, u \cdot \nabla]\theta = \mathcal{R}_\alpha(u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.$$

Equation (1.6) can be obtained by taking the difference of the vorticity equation and the resulting equation after applying $\mathcal{R}_\alpha$ to the temperature equation. Although (1.6) appears to be more complicated than the vorticity equation, the commutator term $[\mathcal{R}_\alpha, u \cdot \nabla]\theta$ is less singular than $\partial_1 \theta$ in the vorticity equation. By obtaining a suitable bound for $[\mathcal{R}_\alpha, u \cdot \nabla]\theta$, we are able to obtain a global bound for $\|G\|_{L^2}$ when $\alpha > \frac{4}{5}$. In addition, by fully exploiting the dissipation with $\alpha > \frac{4}{5}$, a global bound is also established for $\|G\|_{L^q}$ when q is in the range

$$2 < q < q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}. \tag{1.7}$$

This global bound for $\|G\|_{L^q}$ enables us to gain further regularity for G. In fact, we establish that, for $\alpha_0 < \alpha$ and $s \leq 3\alpha - 2$, the Besov norm $\|G\|_{B^s_{q,\infty}}$ obeys, for any

$T > 0$ and $t \le T$,

$$\|G(t)\|_{B^s_{q,\infty}} \le C, \tag{1.8}$$

where C is a constant depending on T and the norms of the initial data.

In contrast to the critical case with $\alpha = 1$ and $\kappa = 0$ dealt with in [19], the general critical case appears to be more difficult. The regularity of G here does not translate to the regularity on the vorticity ω since the corresponding regularity of $\mathcal{R}_\alpha\theta$ is not known for $\alpha < 1$. This paper offers a different approach by gaining further regularity through the temperature equation

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda^\beta\theta = 0. \tag{1.9}$$

Since u is determined by ω through the Biot–Savart law, or $u = \nabla^\perp\Delta^{-1}\omega$ and $\omega = G + \mathcal{R}_\alpha\theta$, we can decompose u into two parts,

$$u = \nabla^\perp\Delta^{-1}\omega = \nabla^\perp\Delta^{-1}G + \nabla^\perp\Delta^{-1}\mathcal{R}_\alpha\theta \equiv \widetilde{u} + v. \tag{1.10}$$

For $\alpha_0 < \alpha$ and as a consequence of (1.8), $\widetilde{u}$ is regular in the sense that

$$\|\nabla\widetilde{u}\|_{L^\infty} = \|\nabla\nabla^\perp\Delta^{-1}G\|_{L^\infty} \le C\,\|G\|_{B^s_{q,\infty}} \le C.$$

In addition, when $\alpha + \beta = 1$, v in terms of θ can be written as

$$v = \nabla^\perp\Delta^{-1}\Lambda^{-(1-\beta)}\,\partial_1\,\theta.$$

Therefore, (1.9) is almost a generalized critical surface quasi-geostrophic (SQG)-type equation first studied in [10] except that u here contains an extra regular velocity $\widetilde{u}$. We remark that there is a large literature on the SQG equation and interested readers may consult [4, 7, 8, 10, 11, 12, 13, 14, 23] and the references therein. Since energy estimates do not appear to yield the desired global a priori bounds, we employ the approach of Constantin and Vicol [12] to establish the global regularity of (1.9) and (1.10). Different from the Schwartz class setting in [12], the initial data θ_0 here is in H^1 with $\|\nabla\theta_0\|_{L^\infty} < \infty$. The precise global existence and uniqueness of (1.9) and (1.10) obtained here can be stated as follows.

THEOREM 1.2. *Let $\beta \in (0,1)$ and $0 < T < \infty$. Let $\widetilde{u}$ be a 2D vector field satisfying $\nabla\cdot\widetilde{u} = 0$ and*

$$M \equiv \max\left\{\|\widetilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}, \|\nabla\widetilde{u}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}\right\} < \infty.$$

Consider the generalized critical SQG-type equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda^\beta\theta = 0, \quad x\in\mathbb{R}^2,\ t>0,\\ u = \widetilde{u} + v, \quad v = -\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta, \quad x\in\mathbb{R}^2,\ t>0,\\ \theta(x,0) = \theta_0(x), \quad x\in\mathbb{R}^2. \end{cases} \tag{1.11}$$

Assume that $\theta_0 \in H^1(\mathbb{R}^2)$ with

$$\|\nabla\theta_0\|_{L^\infty} < \infty.$$

Then (1.11) has a unique solution $\theta \in C([0,T]; H^1(\mathbb{R}^2))$ satisfying

$$\|\nabla\theta\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T).$$

In order to prove this theorem, we need to convert the operator relating v and $\partial_1\theta$, namely, $\nabla^\perp\Lambda^{-3+\beta}$ into an integral form. Since $\beta \in (0,1)$, the standard Riesz potential formula does not appear to apply here (see, e.g., [31]). Nevertheless, $\nabla^\perp \Lambda^{-3+\beta}$ can be represented through an integral kernel by making use of the inverse Fourier transform of functions of the form $\frac{P_k(\xi)}{|\xi|^{k+2-\beta}}$, where P_k is a harmonic polynomial of degree k ([31, p. 73]). As a special consequence, the symmetric part of ∇v can be represented as

$$S(\nabla v) \equiv \frac{1}{2}(\nabla v + (\nabla v)^t) = C \int_{\mathbb{R}^2} \frac{\sigma(x-y)}{|x-y|^{1+\beta}} \left(\partial_1 \theta(x) - \partial_1 \theta(y)\right) dy$$

with

$$\sigma(z) = \frac{1}{|z|^2} \begin{bmatrix} -2z_1 z_2 & z_1^2 - z_2^2 \\ z_1^2 - z_2^2 & 2z_1 z_2 \end{bmatrix}.$$

More details can be found in section 2.

The gained regularity in θ via Theorem 1.2 allows us to assert the desired regularity in the velocity and the vorticity. Especially, $\|\omega\|_{L^\infty}$ is bounded on any time interval $[0, T]$. Further regularity leading to (1.4) is established through energy estimates in Besov space settings. With these global bounds at our disposal, the global existence part of Theorem 1.1 follows from a local existence through a standard procedure such as the successive approximation and an extension of the local solution into a global one with the aid of the global a priori bounds. The uniqueness of solutions in the class (1.4) is clear.

The rest of this paper is organized as follows. Section 2 represents the relation $v = \nabla^\perp \Delta^{-1} \Lambda^{-(1-\beta)} \partial_1 \theta$ as an integral. Integral formulas for ∇v and its symmetric part are also derived in this section. Section 3 proves Theorem 1.2. Section 4 provides the definitions of functional spaces such as the Besov spaces and related facts. In addition, a commutator estimate in the Besov space setting is also proven in this section. This commutator estimate will be used extensively in the sections that follow. Sections 5 and 6 establish global a priori bounds for $\|G\|_{L^2}$ and for $\|G\|_{L^q}$, where q satisfies (1.7). Section 7 proves Theorem 1.1. To do so, we first obtain a global bound of G in the Besov space $B^s_{q,\infty}$ with any $s \le 3\alpha - 2$. This global bound and Theorem 1.2 yield the proof of Theorem 1.1.

2. Representing $\boldsymbol{v = \nabla^\perp \Lambda^{-3-\beta} \partial_1 \theta}$ and $\boldsymbol{\nabla v}$ as integrals. In this section we represent $v = \nabla^\perp \Lambda^{-3-\beta} \partial_1 \theta$, its gradient ∇v, and the symmetric part of ∇v as integrals. These integral representations will be used in the next section. More precisely, we prove the following lemma.

LEMMA 2.1. *Let θ be a smooth function of $\mathbb{R}^2$ which is sufficiently rapidly decreasing at ∞. Let $\beta \in (0,1)$ and v be given by*

$$v = \nabla^\perp \Lambda^{-3-\beta} \partial_1 \theta. \tag{2.1}$$

Then v and ∇v can be written as

$$\begin{aligned} v(x) &= C(\beta) \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{1+\beta}} \partial_1 \theta(y)\, dy, \\ \nabla v(x) &= C(\beta) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \int_{\mathbb{R}^2} \frac{1}{|x-y|^{1+\beta}} \partial_1 \theta(y)\, dy \\ &\quad - (1+\beta) C(\beta) \int_{\mathbb{R}^2} \frac{(x-y)^\perp \otimes (x-y)}{|x-y|^{3+\beta}} \partial_1 \theta(y)\, dy, \end{aligned} \tag{2.2}$$

where $C(\beta)$ is a constant depending on β only, and $a \otimes b$ denotes the tensor product of two vectors a and b, namely, $a \otimes b = (a_i b_j)$. Especially the symmetric part of ∇v, *denoted by* $S(\nabla v)$, *is given by*

$$S(\nabla v) \equiv \frac{1}{2}(\nabla v + (\nabla v)^t) = C \int_{\mathbb{R}^2} \frac{\sigma(x-y)}{|x-y|^{1+\beta}} \left(\partial_1\theta(x) - \partial_1\theta(y)\right) dy, \tag{2.3}$$

where C *is a constant depending on* β *only and*

$$\sigma(z) = \frac{1}{|z|^2} \begin{bmatrix} -2z_1z_2 & z_1^2 - z_2^2 \\ z_1^2 - z_2^2 & 2z_1z_2 \end{bmatrix}.$$

To derive the formulas in Lemma 2.1, we need a fact stated in the following lemma, which can be found in Stein's book ([31, p. 73]). We note that the Fourier transform $\widehat{f}(\xi)$ in Stein's book ([31, p. 46]) is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i\, x\cdot\xi} f(x)\, dx,$$

which is normally the definition of the inverse Fourier transform. This explains why the Fourier transform $\widehat{f}$ in Stein's book is here changed to the inverse Fourier transform $f^\vee$.

Lemma 2.2. *Let* $P_k(x)$ *with* $x \in \mathbb{R}^d$ *be a homogeneous harmonic polynomial of degree* k, *where* $k \geq 1$ *is an integer. Let* $0 < \alpha < d$. *Then, for a constant* $C(d,k,\alpha)$ *depending on* $d, k,$ *and* α *only*

$$\left(\frac{P_k(\xi)}{|\xi|^{d+k-\alpha}}\right)^\vee = C(d,k,\alpha)\, \frac{P_k(x)}{|x|^{k+\alpha}}$$

in the sense that

$$\int_{\mathbb{R}^d} \frac{P_k(\xi)}{|\xi|^{d+k-\alpha}}\, \phi(\xi)\, d\xi = C(d,k,\alpha) \int_{\mathbb{R}^d} \frac{P_k(x)}{|x|^{k+\alpha}}\, \phi^\vee(x)\, dx$$

for every ϕ *which is sufficiently rapidly decreasing at* ∞, *and whose inverse Fourier transform has the same property.*

With this lemma at our disposal, we can now prove Lemma 2.1.

Proof of Lemma 2.1. By (2.1),

$$\widehat{v}(\xi) = i\, \xi^\perp |\xi|^{-3+\beta}\, \widehat{\partial_1\theta}(\xi).$$

According to Lemma 2.2,

$$v(x) = i\, \left(\xi^\perp|\xi|^{-3+\beta}\right)^\vee * \partial_1\theta = C(\beta)\frac{x^\perp}{|x|^{1+\beta}} * \partial_1\theta = C(\beta) \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{1+\beta}}\, \partial_1\theta(y)\, dy.$$

Since $\beta \in (0,1)$, the kernel in the representation of v is not singular and we have

$$\begin{aligned} \nabla v &= C(\beta) \int_{\mathbb{R}^2} \nabla_x \left[\frac{(x-y)^\perp}{|x-y|^{1+\beta}}\right] \partial_1\theta(y)\, dy \\ &= C(\beta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \int_{\mathbb{R}^2} \frac{1}{|x-y|^{1+\beta}}\, \partial_1\theta(y)\, dy \\ &\quad - (1+\beta)C(\beta) \int_{\mathbb{R}^2} \frac{(x-y)^\perp \otimes (x-y)}{|x-y|^{3+\beta}}\, \partial_1\theta(y)\, dy. \end{aligned}$$

Equation (2.3) is obtained by taking the symmetric part of (2.2) and inserting $\partial_1\theta(x)$ in the resulting integral. The inserted term does not contribute to the integral thanks to the fact that, for any $r > 0$,

$$\int_{|z|=r} \sigma(z)\, dz = 0.$$

This completes the proof of Lemma 2.1. ◻

3. Global regularity for an active scalar with critical dissipation. The goal of this section is to prove Theorem 1.2, which states the global regularity of a generalized SQG-type equation with critical dissipation. The proof is obtained by modifying the approach of Constantin and Vicol [12]. Different from [12], the functional setting here is weaker.

To prove Theorem 1.2, we first establish the global existence and uniqueness of (1.11) when $\widetilde{u}$ is smooth and θ_0 is smooth and decays sufficiently fast at infinity.

THEOREM 3.1. *Let $\beta \in (0,1]$ and $0 < T < \infty$. Let $\theta_0 \in C^\infty(\mathbb{R}^2)$ and decay sufficiently fast at ∞. Let $\widetilde{u}$ be a smooth $2D$ vector field satisfying $\nabla \cdot \widetilde{u} = 0$ and*

$$M \equiv \max\left\{\|\widetilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}, \|\nabla\widetilde{u}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}\right\} < \infty.$$

Then (1.11) has a unique global smooth solution θ on $[0,T]$. In addition,

$$\|\nabla\theta\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \le C(M, \|\theta_0\|_{L^\infty}, \|\nabla\theta_0\|_{L^\infty}, T), \tag{3.1}$$

where the constant C in the inequality above depends only on the quantities inside the parenthesis.

We will be more specific about the decay requirement on θ_0 in the proof of Theorem 3.1. The rest of this section proves Theorem 3.1 and then Theorem 1.2. The proof of Theorem 3.1 employs the method of Constantin and Vicol [12]. We recall a basic concept. For a given $\delta > 0$, a function f is said to have only small shocks with a parameter δ or simply $f \in OSS_\delta$ if there is $L > 0$ such that

$$|f(x) - f(y)| \le \delta \quad \text{whenever } |x - y| < L. \tag{3.2}$$

The proof of Theorem 3.1 consists of two main parts. The first part shows that $\theta \in OSS_\delta$ for a suitable $\delta = \delta(\|\theta_0\|_{L^\infty})$ implies (3.1). L in (3.2) is not required to be big in order for this part of the result to hold. What we really need here is a smoothness property on θ and it suffices for $\theta \in OSS_\delta$ with a small L, say $L < 1$. The second part proves that $\theta_0 \in OSS_{\frac{\delta}{4}}$ for some $L > 0$ implies that $\theta \in OSS_\delta$ for the same L. For the sake of clarity , we present each part as a proposition.

PROPOSITION 3.2. *Let $\beta \in (0,1]$ and $0 < T < \infty$. Let $\widetilde{u}$ be a vector field satisfying $\nabla \cdot \widetilde{u} = 0$ and*

$$M \equiv \max\left\{\|\widetilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}, \|\nabla\widetilde{u}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}\right\} < \infty.$$

If θ is in OSS_δ uniformly on $[0,T]$ with δ given by

$$\delta = \frac{C}{\|\theta_0\|_{L^\infty}^{\frac{2\beta}{2-\beta}}} \tag{3.3}$$

for a suitable pure constant $C > 0$ (independent of θ_0), then $\nabla\theta$ is bounded as in (3.1).

PROPOSITION 3.3. *Let $\beta \in (0, 1]$ and $0 < T < \infty$. Let $\widetilde{u}$ be a vector field satisfying $\nabla \cdot \widetilde{u} = 0$ and*

$$M \equiv \max\left\{\|\widetilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}, \|\nabla\widetilde{u}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}\right\} < \infty.$$

Assume that $\theta_0 \in OSS_{\delta/4}$ for some $\delta > 0$. Then $\theta \in OSS_\delta$ uniformly on $[0, T]$.

With the two propositions above in our disposal, we can now prove Theorem 3.1.

Proof of Theorem 3.1. Since $\theta_0 \in C^\infty(\mathbb{R}^2)$ and decays sufficiently fast at ∞, it is easy to check $\theta_0 \in OSS_{\delta/4}$ for δ given by (3.3). There are two different ways to achieve this. The first is to use the simple inequality

$$|\theta_0(x) - \theta_0(y)| \le \|\nabla\theta_0\|_{L^\infty}|x - y|$$

and take $L = \delta/(4\|\nabla\theta_0\|_{L^\infty})$. Alternatively, we first take $R > 0$ such that $|\theta_0(x)| < \delta/8$ for any $|x| \ge R$. Then, by the uniform continuity of θ_0 in the disk $|x| \le 2R$, there is $L_1 > 0$ such that

$$|\theta_0(x) - \theta_0(y)| \le \delta/4$$

when $|x - y| \le L_1$. Taking $L = \min\{R, L_1\}$, we obtain $\theta_0 \in OSS_{\delta/4}$. By Proposition 3.3, $\theta \in OSS_\delta$ uniformly on $[0, T]$. By Proposition 3.2, $\nabla\theta$ satisfies (3.1). The global existence and uniqueness follow from a local well-posedness and an extension to a global solution through the global bound for $\|\nabla\theta\|_{L^\infty}$ in (3.1). We omit further details. This completes the proof of Theorem 3.1. $\square$

Proof of Theorem 1.2. We first regularize $\widetilde{u}$ and the initial data. For $\epsilon > 0$, we define ρ_ϵ to be the standard mollifier, namely,

$$\rho(x) = \rho(|x|) \in C_0^\infty(\mathbb{R}^2), \quad \rho \ge 0, \quad \int_{\mathbb{R}^2} \rho(x)\, dx = 1, \quad \rho_\epsilon(x) = \epsilon^{-2}\rho\left(\frac{x}{\epsilon}\right).$$

In addition, let χ_ϵ be the standard smooth cutoff, namely, $\chi_\epsilon(x) = \chi(\epsilon x), \chi \in C_0^\infty(\mathbb{R}^2)$, and $\chi(x) = 1$ in $|x| \le 2$. Now we define

$$\widetilde{u}^\epsilon = \rho_\epsilon * \widetilde{u}, \qquad \theta_0^\epsilon = \rho_\epsilon * (\chi_\epsilon\theta_0),$$

and consider the following regularized initial-value problem

$$\begin{cases} \partial_t\theta^\epsilon + u^\epsilon \cdot \nabla\theta^\epsilon + \Lambda^\beta\theta^\epsilon = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ u^\epsilon = \widetilde{u}^\epsilon + v^\epsilon, \quad v^\epsilon = -\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta^\epsilon, \quad x \in \mathbb{R}^2,\ t > 0, \\ \theta^\epsilon(x, 0) = \theta_0^\epsilon(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{3.4}$$

Then $\theta_0^\epsilon \in C^\infty(\mathbb{R}^2)$ and has the decay properties required in Theorem 3.1. Therefore, by Theorem 3.1, (3.4) has a unique global smooth solution θ^ϵ satisfying

$$\|\nabla\theta^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \le C(M, \|\theta_0^\epsilon\|_{L^\infty}, \|\nabla\theta_0^\epsilon\|_{L^\infty}, T).$$

Since

$$\|\theta_0^\epsilon\|_{L^\infty} \le \|\theta_0\|_{L^\infty} \le \sqrt[3]{6}\,\|\nabla\theta_0\|_{L^\infty}^{\frac{1}{3}}\,\|\theta_0\|_{L^2}^{\frac{1}{3}}\,\|\nabla\theta_0\|_{L^2}^{\frac{1}{3}}$$

and $\|\nabla\theta_0^\epsilon\|_{L^\infty} \le \|\nabla\theta_0\|_{L^\infty}$, $\nabla\theta^\epsilon$ admits a global bound that is uniform with respect to ϵ,

$$\|\nabla\theta^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T).$$

In addition, a simple energy estimate shows that

$$\|\theta^\epsilon\|_{H^1} \le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T). \tag{3.5}$$

In fact, (3.5) follows from the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta^\epsilon\|_{L^2}^2 + \|\Lambda^{1+\frac{\beta}{2}}\theta^\epsilon\|_{L^2}^2 \le (\|\nabla\widetilde{u}^\epsilon\|_{L^\infty} + \|\nabla v^\epsilon\|_{L^\infty})\|\nabla\theta^\epsilon\|_{L^2}^2,$$

and the global bounds $\|\nabla\widetilde{u}^\epsilon\|_{L^\infty} \le \|\nabla\widetilde{u}\|_{L^\infty}$ and

$$\begin{aligned} \|\nabla v^\epsilon\|_{L^\infty} &= \|\nabla\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta^\epsilon\|_{L^\infty} \\ &\le C(\|\theta^\epsilon\|_{L^2} + \|\nabla\theta^\epsilon\|_{L^\infty}) \\ &\le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T). \end{aligned} \tag{3.6}$$

The interpolation inequality in (3.6) can be proven through the Littlewood–Paley decomposition and the details are deferred to the end of section 4. Then θ^ϵ has a weak limit $\theta \in H^1$, namely,

$$\theta^\epsilon \rightharpoonup \theta \quad \text{weakly in } H^1 \text{ as } \epsilon \to 0. \tag{3.7}$$

In addition, it can be shown that

$$\theta^\epsilon \to \theta \quad \text{strongly in } L^2 \text{ as } \epsilon \to 0. \tag{3.8}$$

This can be achieved by estimating the difference $\theta^{\epsilon_1} - \theta^{\epsilon_2}$ through energy estimates. Let u^{ϵ_1} and u^{ϵ_2} be the corresponding velocities. Then $\overline{\theta} = \theta^{\epsilon_1} - \theta^{\epsilon_2}$ and $\overline{u} = u^{\epsilon_1} - u^{\epsilon_2}$ satisfy

$$\partial_t\overline{\theta} + \overline{u}\cdot\nabla\theta^{\epsilon_1} + u^{\epsilon_2}\cdot\nabla\overline{\theta} + \Lambda^\beta\overline{\theta} = 0.$$

Therefore,

$$\frac{d}{dt}\|\overline{\theta}\|_{L^2}^2 + 2\|\Lambda^{\beta/2}\overline{\theta}\|_{L^2}^2 = -2\int_{\mathbb{R}^2}\overline{u}\cdot\nabla\theta^{\epsilon_1}\overline{\theta}\,dx.$$

Noticing that

$$\overline{u} = u^{\epsilon_1} - u^{\epsilon_2} = \widetilde{u}^{\epsilon_1} - \widetilde{u}^{\epsilon_2} + v^{\epsilon_1} - v^{\epsilon_2},$$

we obtain, by Hölder's inequality,

$$\begin{aligned} \left|\int_{\mathbb{R}^2}\overline{u}\cdot\nabla\theta^{\epsilon_1}\overline{\theta}\,dx\right| &\le \|\widetilde{u}^{\epsilon_1} - \widetilde{u}^{\epsilon_2}\|_{L^\infty}\|\nabla\theta^{\epsilon_1}\|_{L^2}\|\overline{\theta}\|_{L^2} \\ &\quad + \|v^{\epsilon_1} - v^{\epsilon_2}\|_{L^{\frac{2}{\beta}}}\|\nabla\theta^{\epsilon_1}\|_{L^{\frac{2}{1-\beta}}}\|\overline{\theta}\|_{L^2}. \end{aligned}$$

Due to $\|\widetilde{u}^{\epsilon_1} - \widetilde{u}^{\epsilon_2}\|_{L^\infty} \le C|\epsilon_1 - \epsilon_2|M$ and recalling (3.5), we have

$$\|\widetilde{u}^{\epsilon_1} - \widetilde{u}^{\epsilon_2}\|_{L^\infty}\|\nabla\theta^{\epsilon_1}\|_{L^2}\|\overline{\theta}\|_{L^2} \le C\,|\epsilon_1 - \epsilon_2|\,\|\overline{\theta}\|_{L^2},$$

where $C = C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T)$. Since

$$v^{\epsilon_1} - v^{\epsilon_2} = -\nabla^\perp \Delta^{-1} \Lambda^{-(1-\beta)} \partial_1 \overline{\theta},$$

and by the Hardy–Littlewood–Sobolev inequality,

$$\|v^{\epsilon_1} - v^{\epsilon_2}\|_{L^{\frac{2}{\beta}}} \le C \, \|\Lambda^{-(1-\beta)}\overline{\theta}\|_{L^{\frac{2}{\beta}}} \le C \, \|\overline{\theta}\|_{L^2}.$$

By a simple interpolation inequality,

$$\|\nabla\theta^{\epsilon_1}\|_{L^{\frac{2}{1-\beta}}} \le \|\nabla\theta^{\epsilon_1}\|_{L^2}^{1-\beta} \|\nabla\theta^{\epsilon_1}\|_{L^\infty}^{\beta} \le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T).$$

Therefore,

$$\frac{d}{dt}\|\overline{\theta}\|_{L^2}^2 + 2\|\Lambda^{\beta/2}\overline{\theta}\|_{L^2}^2 \le C \, |\epsilon_1 - \epsilon_2| \, \|\overline{\theta}\|_{L^2} + C \, \|\overline{\theta}\|_{L^2}^2.$$

Consequently,

$$\begin{aligned} \|\overline{\theta}(t)\|_{L^2} &\equiv \|\theta^{\epsilon_1}(t) - \theta^{\epsilon_2}(t)\|_{L^2} \\ &\le C \, (|\epsilon_1 - \epsilon_2| \, + \, \|\theta_0^{\epsilon_1} - \theta_0^{\epsilon_2}\|_{L^2}) e^{Ct} \\ &\le C \, (|\epsilon_1 - \epsilon_2| \, + \|\theta_0^{\epsilon_1} - \theta_0\|_{L^2} + \|\theta_0 - \theta_0^{\epsilon_2}\|_{L^2}) \, e^{Ct} \\ &\le C \, \max\{\epsilon_1, \epsilon_2\} \, e^{Ct}, \end{aligned}$$

where $C = C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T)$. This proves (3.8). By the interpolation inequality, for $0 < s' < 1$,

$$\|\theta^\epsilon - \theta\|_{H^{s'}} \le C \, \|\theta^\epsilon - \theta\|_{L^2}^{1-s'/s} \, \|\theta^\epsilon - \theta\|_{H^s}^{s'/s},$$

(3.7) and (3.8) imply that, for $0 < s' < 1$,

$$\theta^\epsilon \to \theta \quad \text{strongly in } H^{s'} \text{ as } \epsilon \to 0.$$

This strong convergence allows us to show that θ is the corresponding global solution of (1.11) satisfying $\theta \in L^\infty([0,T]; H^1)$ and $\|\nabla\theta\|_{L^\infty} \le C(M, \|\theta_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}, T)$. The continuity of θ in time, namely, $\theta \in C([0,T]; H^1)$, follows from a standard approach (see, e.g., [26, p. 111] or [1, p. 138]) and we omit the details. Finally, it is not hard to verify that such solutions are unique. This completes the proof of Theorem 1.2. ∎

We now turn to the proofs of Propositions 3.2 and 3.3. To prove Proposition 3.2, we will make use of the following lower bound obtained in [12]. This lower bound improves a pointwise inequality of Córdoba and Córdoba [13].

LEMMA 3.4. *Let $\beta \in (0, 2)$ and $q \in [1, \infty]$. Let $f \in C^1(\mathbb{R}^d)$ decay sufficiently fast at ∞. Then the pointwise lower bound holds,*

$$\nabla f \cdot \Lambda^\beta(\nabla f) \ge \frac{1}{2}\Lambda^\beta \left(|\nabla f|^2\right) + \frac{1}{2} D(\nabla f) + \frac{|\nabla f|^{2+\frac{\beta q}{q+d}}}{C_0 \, \|f\|_{L^q}^{\frac{\beta q}{q+d}}},$$

where $C_0 = C_0(d, \beta, q)$ is a constant and D is given by the principal value (P.V.) integral

$$D(g) = C(d, \beta) \, P.V. \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^2}{|x-y|^{d+\beta}} \, dy \ge 0.$$

Proof of Proposition 3.2. We show that, if

$$\theta \in OSS_\delta \quad \text{with} \quad \delta = \frac{C}{\|\theta_0\|_{L^\infty}^{\frac{2\beta}{2-\beta}}}$$

for a suitable C, then $\nabla\theta$ satisfies (3.1). Clearly, $\nabla\theta$ satisfies

$$\partial_t(\nabla\theta) + u\cdot\nabla(\nabla\theta) + \Lambda^\beta(\nabla\theta) = -\nabla u(\nabla\theta).$$

Dotting with $\nabla\theta$ and applying Lemma 3.4 yield

$$\begin{aligned}\frac{1}{2}(\partial_t + u\cdot\nabla + \Lambda^\beta)|\nabla\theta|^2 + \frac{1}{2}D(\nabla\theta) + \frac{|\nabla\theta|^{2+\beta}}{C_0\,\|\theta_0\|_{L^\infty}^\beta}\\ \le |\nabla\theta\cdot S(\nabla u)\cdot\nabla\theta| \le \|S(\nabla u)\|_{L^\infty}\,|\nabla\theta(x)|^2,\end{aligned} \tag{3.9}$$

where $S(\nabla u)$ denotes the symmetric part of ∇u, or $S(\nabla u) = \frac{1}{2}((\nabla u)+(\nabla u)^t)$. Clearly

$$\|S(\nabla u)\|_{L^\infty} \le \|\nabla\widetilde{u}\|_{L^\infty} + \|S(\nabla v)\|_{L^\infty} \le M + \|S(\nabla v)\|_{L^\infty}.$$

Now we bound $\|S(\nabla v)\|_{L^\infty}$. According to Lemma 2.1,

$$|S(\nabla v(x))| \le C\left|\int_{\mathbb{R}^2}\frac{\sigma(x-y)}{|x-y|^{1+\beta}}\,(\partial_1\theta(x)-\partial_1\theta(y))\,dy\right|.$$

To further the estimate, we split the integral over $\mathbb{R}^2$ into three parts:

$$|x-y|\le\rho,\quad \rho<|x-y|\le L,\quad |x-y|>L$$

for $0<\rho<L$ to be specified later. By Hölder's inequality, the integral over $|x-y|\le\rho$ is bounded by

$$\left|\int_{|x-y|\le\rho}\frac{1}{|x-y|^{1+\beta}}\,|\partial_1\theta(x)-\partial_1\theta(y)|\,dy\right| \le \sqrt{D(\nabla\theta)}\,\rho^{1-\frac{\beta}{2}}.$$

Through integration by parts, the integral over $\rho<|x-y|\le L$ is bounded by

$$\begin{aligned}\left|\int_{\rho<|x-y|\le L}\frac{1}{|x-y|^{2+\beta}}\,|\theta(x)-\theta(y)|\,dy\right| &\le \delta\left|\int_{\rho<|x-y|\le L}\frac{1}{|x-y|^{2+\beta}}\;dy\right|\\ &= C\,\delta\,\rho^{-\beta}.\end{aligned}$$

We remark that a rigorous justification of the estimate above can be carried out through smooth cutoff functions. Again through integration by parts, the integral over $|x-y|>L$ is bounded by

$$\left|\int_{|x-y|>L}\frac{1}{|x-y|^{2+\beta}}\,|\theta(x)-\theta(y)|\,dy\right| \le C\,\|\theta_0\|_{L^\infty}L^{-\beta}.$$

Therefore,

$$\begin{aligned}\|S(\nabla u)\|_{L^\infty}\,|\nabla\theta(x)|^2 &\le C\,\sqrt{D(\nabla\theta)}\,\rho^{1-\frac{\beta}{2}}\,|\nabla\theta(x)|^2 + C\,\delta\,\rho^{-\beta}\,|\nabla\theta(x)|^2\\ &\quad + (M + C\,\|\theta_0\|_{L^\infty}L^{-\beta})\,|\nabla\theta(x)|^2\\ &\le \frac{1}{2}D(\nabla\theta) + C\,\rho^{2-\beta}\,|\nabla\theta(x)|^4 + C\,\delta\,\rho^{-\beta}\,|\nabla\theta(x)|^2\\ &\quad + (M + C\,\|\theta_0\|_{L^\infty}L^{-\beta})\,|\nabla\theta(x)|^2,\end{aligned}$$

where C's are pure constants depending on β only. If we choose ρ and δ as

$$\rho = \frac{1}{(4C\,C_0\|\theta_0\|_{L^\infty}^{\beta})^{\frac{1}{2-\beta}}\,|\nabla\theta(x)|} \quad \text{and} \quad \delta = \frac{1}{C^{1+\beta}\,(4C_0\|\theta_0\|_{L^\infty}^{\beta})^{\frac{2}{2-\beta}}},$$

then

$$\|S(\nabla u)\|_{L^\infty}\,|\nabla\theta(x)|^2 \le \frac{1}{2}D(\nabla\theta) + \frac{|\nabla\theta|^{2+\beta}}{2C_0\,\|\theta_0\|_{L^\infty}^{\beta}} + (M + C\,\|\theta_0\|_{L^\infty}L^{-\beta})\,|\nabla\theta(x)|^2.$$

Inserting this bound into (3.9) yields

$$\frac{1}{2}(\partial_t + u\cdot\nabla + \Lambda^\beta)|\nabla\theta|^2 + \frac{|\nabla\theta(x)|^{2+\beta}}{2C_0\,\|\theta_0\|_{L^\infty}^{\beta}} \le (M + C\,\|\theta_0\|_{L^\infty}L^{-\beta})\,|\nabla\theta(x)|^2. \tag{3.10}$$

This differential inequality allows us to conclude that

$$|\nabla\theta(x,t)| \le \max\{\|\nabla\theta_0\|_{L^\infty}, \widetilde{C}(M, \|\theta_0\|_{L^\infty})\}, \tag{3.11}$$

where $\widetilde{C}$ is given by

$$\widetilde{C} = \left(2C_0(M + C\,\|\theta_0\|_{L^\infty}L^{-\beta}\right)^{\frac{1}{\beta}}\,\|\theta_0\|_{L^\infty}. \tag{3.12}$$

To show (3.11), we define $\Gamma : [0,\infty) \to [0,\infty)$ to be a nondecreasing C^2 convex function satisfying $\Gamma(\rho) = 0$ for $0 \le \rho \le \max\{\|\nabla\theta_0\|_{L^\infty}, \widetilde{C}\}$ and $\Gamma(\rho) > 0$ for $\rho > \max\{\|\nabla\theta_0\|_{L^\infty}, \widetilde{C}\}$. Multiplying (3.10) by $\Gamma'(|\nabla\theta|^2)$ and applying the lower bound

$$\Lambda^\beta\,(\Gamma(f)) \le \Gamma'(f)\,\Lambda^\beta f, \tag{3.13}$$

where is valid for any $\beta \in (0,2)$ and any convex function Γ (see [8, 13]), we obtain

$$\begin{aligned}&\frac{1}{2}(\partial_t + u\cdot\nabla + \Lambda^\beta)\,\Gamma(|\nabla\theta|^2)\\ &\qquad \le (M + C\,\|\theta_0\|_{L^\infty}L^{-\beta})\,|\nabla\theta(x)|^2\left[1 - \frac{|\nabla\theta(x,t)|^\beta}{\widetilde{C}^\beta}\right]\Gamma'(|\nabla\theta|^2).\end{aligned}$$

It is easy to verify that the right-hand side of the inequality above is always less than or equal to zero due to the definition of Γ. Therefore,

$$(\partial_t + u\cdot\nabla + \Lambda^\beta)\,\Gamma(|\nabla\theta|^2) \le 0.$$

Thanks to $\nabla\cdot u = 0$ and the positivity of the integral

$$\int_{\mathbb{R}^2} f(x)|f(x)|^{2p-2}\;\Lambda^\beta f(x)\,dx \ge 0,$$

we obtain $\|\Gamma(|\nabla\theta|^2)\|_{L^{2p}} \le \|\Gamma(|\nabla\theta_0|^2)\|_{L^{2p}}$ for any $1 \le p < \infty$. Letting $p \to \infty$, we have

$$\|\Gamma(|\nabla\theta(t)|^2)\|_{L^\infty} \le \|\Gamma(|\nabla\theta_0|^2)\|_{L^\infty} \le 0,$$

which implies (3.11). This completes the proof of Proposition 3.2. ◻

To prove Proposition 3.3, we need the lower bound in the following lemma. This lemma can be proven by following the lines of that for Lemma 3.4 (see [12]).

LEMMA 3.5. *Let $\beta \in (0,2)$. Let $h \in \mathbb{R}^2$ and $\delta_h\theta(x,t) \equiv \theta(x+h,t) - \theta(x,t)$. Then the following pointwise lower bound holds,*

$$\delta_h\theta\, \Lambda^\beta \delta_h\theta \geq \frac{1}{2}\Lambda^\beta(\delta_h\theta)^2 + D_h(\delta_h\theta) + C\, \frac{|\delta_h\theta(x,t)|^{2+\beta}}{\|\theta\|^\beta_{L^\infty}|h|^\beta},$$

where C is a constant depending on β only, and

$$D_h(\delta_h\theta) = c\, P.V. \int_{\mathbb{R}^2} \frac{(\delta_h\theta(x,t) - \delta_h\theta(y,t))^2}{|x-y|^{2+\beta}}\, dy.$$

Proof of Proposition 3.3. Let $h \in \mathbb{R}^2$ and consider the evolution of

$$g(x,t;h) = (\delta_h\theta(x,t))^2\, \Phi(h), \tag{3.14}$$

where $\Phi(h) = e^{-\Psi(h)}$ with Ψ satisfying

$$\Psi(h) = \Psi(|h|) \ \text{ is nondecreasing}, \quad \Psi(0) = 0, \quad \Psi(h) \to +\infty \quad \text{as} \quad |h| \to \infty. \tag{3.15}$$

An explicit form of Ψ will be specified later. First of all, $\delta_h\theta$ satisfies

$$(\partial_t + u\cdot\nabla + \delta_h u \cdot \nabla_h + \Lambda^\beta)\delta_h\theta = 0, \tag{3.16}$$

where $\delta_h u(x,t) = u(x+h,t) - u(x,t)$. Multiplying (3.16) by $\delta_h\theta(x,t)\,\Phi(h)$ and applying Lemma 3.5, we obtain

$$\begin{aligned}&(\partial_t + u\cdot\nabla + \delta_h u\cdot\nabla_h + \Lambda^\beta)g + 2D_h(\delta_h\theta)\,\Phi(h) + C\,\Phi(h)\,\frac{|\delta_h\theta(x,t)|^{2+\beta}}{\|\theta\|^\beta_{L^\infty}|h|^\beta}\\ &\qquad \leq (\delta_h\theta(x,t))^2\, \delta_h u\cdot\nabla_h\Phi(h).\end{aligned}$$

Noticing that $\nabla_h\Phi(h) = -\Phi(h)\nabla_h\Psi(h)$, we have

$$(\delta_h\theta(x,t))^2\,\delta_h u\cdot\nabla_h\Phi(h) = -(\delta_h\theta(x,t))^2\,\Phi(h)\,\delta_h u\cdot\nabla_h\Psi(h).$$

Therefore, if we write $\mathcal{L} \equiv \partial_t + u\cdot\nabla + \delta_h u\cdot\nabla_h + \Lambda^\beta$, we obtain

$$\mathcal{L}g + 2D_h(\delta_h\theta)\,\Phi(h) + C\,\frac{g^{1+\frac{\beta}{2}}}{\Phi^{\frac{\beta}{2}}(h)\|\theta\|^\beta_{L^\infty}|h|^\beta} \leq g\,|\delta_h u|\,\Psi'(|h|). \tag{3.17}$$

Thanks to the assumptions on $\widetilde{u}$,

$$|\delta_h\widetilde{u}| \leq \widetilde{M} \equiv \min\{M|h|, 2\|u\|_{L^\infty(0,T;L^\infty)}\} < \infty.$$

Therefore, $u = \widetilde{u} + v$ satisfies

$$|\delta_h u| \leq \widetilde{M} + |\delta_h v|. \tag{3.18}$$

According to Lemma 2.1,

$$v(x,t) = C\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{1+\beta}}\,\partial_1\theta(y)\,dy.$$

Since $\beta \in (0,1)$ and θ is bounded and decays sufficiently fast at ∞, we obtain after integration by parts,

$$v_1(x,t) = (1+\beta)C \int_{\mathbb{R}^2} \frac{(x_1-y_1)(x_2-y_2)}{|x-y|^{3+\beta}}\,\theta(y)\,dy,$$
$$v_2(x,y) = C \int_{\mathbb{R}^2} \left[\frac{1}{|x-y|^{1+\beta}} + \frac{(1+\beta)(x_1-y_1)^2}{|x-y|^{3+\beta}}\right]\theta(y)\,dy.$$

Therefore,

$$\begin{aligned} |\delta_h v| &\le C \int_{\mathbb{R}^2} \frac{1}{|x-y|^{1+\beta}} |\delta_h \theta(y,t)|\,dy \\ &\le C \int_{|x-y|\le 1} \frac{1}{|x-y|^{1+\beta}} |\delta_h \theta(y,t)|\,dy + C \int_{|x-y|>1} \frac{1}{|x-y|^{1+\beta}} |\delta_h \theta(y,t)|\,dy \\ &\le C\,\|\theta_0\|_{L^\infty\cap L^2}. \qquad (3.19) \end{aligned}$$

Inserting the estimates of (3.18) and (3.19) in (3.17) and using $\Phi(h) \le 1$, we obtain

$$\mathcal{L}g + C_1 \frac{g^{1+\frac{\beta}{2}}}{\|\theta_0\|^{\beta}_{L^\infty}|h|^\beta} \le C_2\, g\,(\widetilde{M} + \|\theta_0\|_{L^2\cap L^\infty})\,\Psi'(|h|), \qquad (3.20)$$

where C_1 and C_2 are constants depending on β only. Now we choose

$$\Psi(|h|) = \frac{C_1\,(\delta_0)^\beta}{C_2(1-\beta)\|\theta_0\|^\beta_{L^\infty}(\widetilde{M} + \|\theta_0\|_{L^2\cap L^\infty})}|h|^{1-\beta}.$$

Certainly this choice satisfies (3.15). Then (3.20) becomes

$$\mathcal{L}g \le \frac{C_1\,g}{|h|^\beta\|\theta_0\|^\beta_{L^\infty}}\left(\delta_0^\beta - g^{\frac{\beta}{2}}\right). \qquad (3.21)$$

We can then conclude from (3.21) that

$$g(x,t;h) \le \delta_0^2 \quad \text{for } |h| \le L.$$

This can be proven by following a similar argument as in the proof of Proposition 3.2. More precisely, we multiply (3.21) by $\widetilde{\Gamma}'(g)$ with $\widetilde{\Gamma}$ being a nondecreasing smooth convex function on $[0,\infty)$ satisfying $\widetilde{\Gamma}(\rho) = 0$ for $0 \le \rho \le \delta_0^2$ and positive for $\rho \ge \delta_0^2$. Thanks to (3.13),

$$\mathcal{L}\left(\widetilde{\Gamma}(g)\right) \le \frac{C_1\,g}{|h|^\beta\|\theta_0\|^\beta_{L^\infty}}(\delta_0^\beta - g^{\frac{\beta}{2}})\,\widetilde{\Gamma}'(g) \le 0.$$

By first estimating the L^q-norm of $\widetilde{\Gamma}(g)$ and then sending $q \to \infty$, we obtain that the L^∞-norm of $\widetilde{\Gamma}(g)$ is bounded by its initial L^∞-norm. Since initially $g \le \delta_0^2$ and thus the initial L^∞-norm of $\widetilde{\Gamma}(g)$ is zero, we have the L^∞-norm of $\widetilde{\Gamma}(g)$ is zero for all time. Therefore $g \le \delta_0^2$ and

$$|\delta_h\theta(x,t)| \le \delta_0\,(\Phi(h))^{-\frac{1}{2}} \le \delta_0\,e^{\frac{1}{2}\Psi(L)} \le 4\delta_0.$$

This completes the proof of Proposition 3.3. $\square$

4. Besov spaces and a commutator estimate. This section provides the definitions of some of the functional spaces and related facts to be used in the subsequent sections . More details can be found in several books and many papers (see, e.g., [1, 2, 28, 30, 32]). In addition, we prove a commutator estimate to be used extensively in the sections that follow.

To introduce the Besov spaces, we start with some notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \, dx = 0, \, |\gamma| = 0, 1, 2, \dots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ denotes the space of multinomials. For each $j \in \mathbb{Z}$, we write

$$A_j = \left\{ \xi \in \mathbb{R}^d : \; 2^{j-1} \le |\xi| < 2^{j+1} \right\}. \tag{4.1}$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{4.2}$$

DEFINITION 4.1. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}'_0$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{4.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0 & \text{if } j \le -2, \\ \Psi * f & \text{if } j = -1, \\ \Phi_j * f & \text{if } j = 0, 1, 2, \dots. \end{cases} \tag{4.4}$$

DEFINITION 4.2. *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way.

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (4.4). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequality is a useful tool on Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein-type inequalities for fractional derivatives.

PROPOSITION 4.3. *Let* $\alpha \geq 0$. *Let* $1 \leq p \leq q \leq \infty$.
(1) *If* f *satisfies*

$$\mathit{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K2^j\}$$

for some integer j *and a constant* $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If* f *satisfies*

$$\mathit{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j *and constants* $0 < K_1 \leq K_2$, *then*

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 *and* C_2 *are constants depending on* $\alpha, p,$ *and* q *only.*

The rest of this section is devoted to the proof of a commutator estimate. We need the following lemma.

LEMMA 4.4. *Let* $\delta \in (0,1)$. *Let* $q,\, q_1,\, q_2, r_1, r_2 \in [1,\infty]$ *satisfying*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \qquad \frac{1}{r_1} + \frac{1}{r_2} = 1.$$

If $f \in \mathring{B}^\delta_{q_1,r_1}(\mathbb{R}^d)$, $|x|^{\delta+\frac{d}{r_1}}\phi \in L^{r_2}(\mathbb{R}^d)$, *and* $g \in L^{q_2}(\mathbb{R}^d)$, *then*

$$\|\phi * (fg) - f\phi * g\|_{L^q} \leq C\, \||x|^{\delta+\frac{d}{r_1}}\phi\|_{L^{r_2}}\, \|f\|_{\mathring{B}^\delta_{q_1,r_1}}\, \|g\|_{L^{q_2}}, \tag{4.5}$$

where C *is a constant independent of* $f, g,$ *and* h. *In the borderline case when* $\delta = 1$, (4.5) *still holds if* $\|f\|_{\mathring{B}^\delta_{q_1,r_1}}$ *is replaced by* $\|\nabla f\|_{L^{q_1}}$ *and* $r_2 = 1$.

The case when $\delta = 1$, $q_1 = q$, and $q_2 = \infty$ was obtained in [19, p. 2153]. The fractional case with $\delta \in (0,1)$, $q_1 = q$, and $q_2 = \infty$ was obtained by Chae and Wu in [6]. Here we need to make use of the more general case when $q_1 > q$.

Proof of Lemma 4.4. By Minkowski's inequality followed by Hölder's inequality, for any $q \in [1,\infty]$,

$$\begin{aligned}
\|\phi * (fg) - f(\phi * g)\|_{L^q} &= \left[\int \left|\int \phi(z)\,(f(x) - f(x-z))g(x-z)\,dz\right|^q dx\right]^{1/q} \\
&\leq \int |\phi(z)|\, \|f(\cdot) - f(\cdot - z))\|_{L^{q_1}}\, \|g\|_{L^{q_2}} dz \\
&\leq \|g\|_{L^{q_2}} \left\| \frac{\|f(\cdot) - f(\cdot - z))\|_{L^{q_1}}}{|z|^{\delta+\frac{d}{r_1}}} \right\|_{L^{r_1}} \left\| |z|^{\delta+\frac{d}{r_1}} |\phi(z)| \right\|_{L^{r_2}}.
\end{aligned}$$

Equation (4.5) then follows from the definition of $\mathring{B}^\delta_{q_1,r_1}$. $\square$

PROPOSITION 4.5. *Let* $\alpha \in (0,1)$. *Let* $s \in (0,1)$ *and* $\delta \in (0,1)$ *satisfy* $s + 1 - \alpha - \delta < 0$. *Let* $q \in [2,\infty)$, $r \in [1,\infty]$, $q_1, q_2 \in [2,\infty]$ *satisfy* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then, for a constant* C,

$$\|[\mathcal{R}_\alpha, f]g\|_{B^s_{q,r}} \leq C\, \|f\|_{B^\delta_{q_1,\infty}}\, \|g\|_{B^{s+1-\alpha-\delta}_{q_2,r}}. \tag{4.6}$$

When $\delta = 1$, (4.6) *is still valid if* $\|f\|_{B^\delta_{q_1,\infty}}$ *is replaced by* $\|\nabla f\|_{L^{p_1}}$.

Proof. Let $k \geq -1$ be an integer. By the notion of paraproducts, we write

$$\Delta_k[\mathcal{R}_\alpha, f]g = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \sum_{|j-k|\leq 2} \Delta_k \left(\mathcal{R}_\alpha(S_{j-1}f\, \Delta_j g) - S_{j-1}f\, \mathcal{R}_\alpha \Delta_j g\right), \\ J_2 &= \sum_{|j-k|\leq 2} \Delta_k \left(\mathcal{R}_\alpha(\Delta_j f\, S_{j-1}g) - \Delta_j f\, \mathcal{R}_\alpha S_{j-1} g\right), \\ J_3 &= \sum_{j\geq k-1} \Delta_k \left(\mathcal{R}_\alpha(\Delta_j f\, \widetilde{\Delta}_j g) - \Delta_j f\, \mathcal{R}_\alpha \widetilde{\Delta}_j g\right) \end{aligned}$$

with $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$. We first note that, if the Fourier transform of F is supported in the annulus around radius 2^j, then $\mathcal{R}_\alpha F$ can be represented as a convolution,

$$\mathcal{R}_\alpha F = h_j * F, \qquad h_j(x) = 2^{(d+1-\alpha)j}\, h_0(2^j x) \tag{4.7}$$

for a function h_0 in the Schwartz class $\mathcal{S}$. This can be obtained by simply examining the Fourier transform of $\mathcal{R}_\alpha F$. By the definition of $B^s_{q,r}$,

$$\begin{aligned} \|[\mathcal{R}_\alpha, f]g\|_{B^s_{q,r}} &= \left\|2^{sk}\|\Delta_k[\mathcal{R}_\alpha, f]g\|_{L^q}\right\|_{l^r} \\ &\leq \left\|2^{sk}\|J_1\|_{L^q}\right\|_{l^r} + \left\|2^{sk}\|J_2\|_{L^q}\right\|_{l^r} + \left\|2^{sk}\|J_3\|_{L^q}\right\|_{l^r}. \end{aligned} \tag{4.8}$$

Applying Lemma 4.4, we find

$$\begin{aligned} \|J_1\|_{L^q} &\leq C\, 2^{(1-\alpha)k}\, \||x|^\delta 2^{dk} h_0(2^k x)\|_{L^1}\, \|S_{j-1}f\|_{\mathring{B}^\delta_{q_1,\infty}}\, \|\Delta_j g\|_{L^{q_2}} \\ &\leq C\, 2^{(1-\alpha-\delta)k}\, \|f\|_{\mathring{B}^\delta_{q_1,\infty}}\, \|\Delta_j g\|_{L^{q_2}}. \end{aligned}$$

Thus,

$$\left\|2^{sk}\|J_1\|_{L^q}\right\|_{l^r} \leq C\, \|f\|_{\mathring{B}^\delta_{q_1,\infty}}\, \|g\|_{B^{s+1-\alpha-\delta}_{q_2,r}}. \tag{4.9}$$

By Bernstein's inequality, we have

$$\begin{aligned} \|J_2\|_{L^q} &\leq C\, 2^{(1-\alpha-\delta)j} \|f\|_{B^\delta_{q_1,\infty}}\, \|S_{j-1}g\|_{L^{q_2}} \\ &\leq C\, 2^{(1-\alpha-\delta)j}\, \|f\|_{B^\delta_{q_1,\infty}} \sum_{m\leq k-2} \|\Delta_m g\|_{L^{q_2}} \\ &\leq C\, \|f\|_{B^\delta_{q_1,\infty}} \sum_{m\leq k-2} 2^{(1-\alpha-\delta)(k-m)}\, 2^{(1-\alpha-\delta)m} \|\Delta_m g\|_{L^{q_2}}. \end{aligned}$$

Since $s+1-\alpha-\delta < 0$, we obtain, by applying Young's inequality for series,

$$\left\|2^{sk}\|J_2\|_{L^q}\right\|_{l^r} \leq C\, \|f\|_{B^\delta_{q_1,\infty}}\, \|g\|_{B^{s+1-\alpha-\delta}_{q_2,r}}. \tag{4.10}$$

Similarly, we have

$$\|J_3\|_{L^q} \leq C \sum_{j\geq k-1} 2^{(1-\alpha-\delta)j}\, \|f\|_{B^\delta_{q_1,\infty}}\, \|\Delta_j g\|_{L^{q_2}}.$$

Therefore, for $s > 0$, by Young's inequality for series,

$$\begin{aligned}\left\|2^{sk}\|J_3\|_{L^q}\right\|_{l^r} &\le C \left\| 2^{sk} \sum_{j\ge k-1} 2^{(1-\alpha-\delta)j}\, \|f\|_{B^\delta_{q_1,\infty}}\, \|\Delta_j g\|_{L^{q_2}} \right\|_{l^r} \\ &\le C\, \|f\|_{B^\delta_{q_1,\infty}}\, \|g\|_{B^{s+1-\alpha-\delta}_{q_2,r}}. \end{aligned} \tag{4.11}$$

Combining (4.8), (4.9), (4.10), and (4.11), we obtain the desired bound in (4.6). This completes the proof of Proposition 4.5. □

We have used a simple inequality, namely, (3.6) in the proof of Theorem 1.2 and we now prove it. Its proof is deferred to this section since it involves the Littlewood–Paley decomposition and Bernstein's inequality described above. By Bernstein's inequality,

$$\begin{aligned}\|\nabla\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta^\epsilon\|_{L^\infty} &\le \|\Delta_{-1}\nabla\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta^\epsilon\|_{L^\infty} + \sum_{j=0}^{\infty}\|\Delta_j\nabla\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta^\epsilon\|_{L^\infty} \\ &\le C\,\|\Delta_{-1}\theta^\epsilon\|_{L^2} + \sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\theta^\epsilon\|_{L^\infty} \\ &\le C\,\|\theta^\epsilon\|_{L^2} + \sum_{j=0}^{\infty}2^{(\beta-1)j}\|\nabla\Delta_j\theta^\epsilon\|_{L^\infty},\end{aligned}$$

which reduces to (3.6) by recalling that $\beta \in (0,1)$.

5. Global L^2-bound for G. This section establishes a global a priori bound for $\|G\|_{L^2}$. Recall that

$$G = \omega - \mathcal{R}_\alpha\theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha}\partial_1 \tag{5.1}$$

and G satisfies

$$\partial_t G + u\cdot\nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u\cdot\nabla]\theta + \Lambda^{1-2\alpha}\partial_1\theta. \tag{5.2}$$

This global bound is valid for any $\frac{4}{5} < \alpha < 1$.

Theorem 5.1. *Assume (u_0,θ_0) satisfies the assumptions stated in Theorem* 1.1. *Let (u,θ) be the corresponding solution of* (1.1). *If $\frac{4}{5} < \alpha < 1$, then G defined in* (5.1) *has a global L^2-bound, namely, for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}G(\tau)\|_{L^2}^2\, d\tau \le C(T,u_0,\theta_0),$$

where $C(T,u_0,\theta_0)$ is a constant depending on T and $\|u_0\|_{L^2}$ and $\|\theta_0\|_{L^2\cap L^\infty}$ only.

To prove Theorem 5.1, the following elementary global a priori bounds will be used.

Lemma 5.2. *Assume that u_0 and θ_0 satisfy the conditions stated in Theorem* 1.1. *Then the corresponding solution (u,θ) of* (1.1) *obeys the global bounds, for any $t > 0$,*

$$\|\theta(t)\|_{L^r} \le \|\theta_0\|_{L^r} \qquad \textit{for any } 2\le r\le\infty,$$
$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}u(\tau)\|_{L^2}^2\, d\tau \le \left(\|u_0\|_{L^2}^2 + t\|\theta_0\|_{L^2}^2\right)^2.$$

Proof of Theorem 5.1. Taking the inner product of (5.2) with G, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 = J_1 + J_2, \tag{5.3}$$

where

$$J_1 = \int G\,[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,dx = \int G\,\nabla\cdot[\mathcal{R}_\alpha, u]\theta\,dx,$$
$$J_2 = \int G\,\Lambda^{1-2\alpha}\partial_1\theta\,dx.$$

Let $q_1, q_2 \in [2,\infty]$ satisfying $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{2}$. By Hölder's inequality and Proposition 4.5,

$$\begin{aligned}|J_1| &\le \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}\,\|\Lambda^{1-\frac{\alpha}{2}}\Lambda^{-1}\nabla\cdot[\mathcal{R}_\alpha, u]\theta\|_{L^2}\\ &\le C\,\|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}\|u\|_{B^\alpha_{q_1,\infty}}\,\|\theta\|_{B^{2-\frac{5}{2}\alpha}_{q_2,2}}.\end{aligned}$$

Setting $q_1=\frac{2}{\alpha}$ and noticing that

$$u = \nabla^\perp(-\Delta)^{-1}\omega = \nabla^\perp(-\Delta)^{-1}(G+\Lambda^{-\alpha}\partial_1\theta),$$

we obtain, by the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}\|u\|_{\mathring{B}^\alpha_{q_1,\infty}} &\le C\,\|\Lambda^\alpha u\|_{L^{q_1}} \le C\,\|\Lambda^{\alpha-1}\omega\|_{L^{q_1}}\\ &\le C\,\|\Lambda^{\alpha-1}G\|_{L^{q_1}} + C\,\|\theta\|_{L^{q_1}}\\ &\le C\,\|G\|_{L^2} + C\,\|\theta_0\|_{L^{q_1}}.\end{aligned}\tag{5.4}$$

So, we get

$$\|u\|_{B^\alpha_{q_1,\infty}} \le \|u\|_{\mathring{B}^\alpha_{q_1,\infty}} + C\|u\|_{L^2} \le C\|G\|_{L^2} + C\,\|\theta_0\|_{L^{q_1}} + C\|u\|_{L^2}.$$

Thanks to the embedding $L^{q_2}\hookrightarrow B^{2-\frac{5}{2}\alpha}_{q_2,2}$ when $\alpha>\frac{4}{5}$, we have

$$\|\theta\|_{B^{2-\frac{5}{2}\alpha}_{q_2,2}} \le C\,\|\theta_0\|_{L^{q_2}}. \tag{5.5}$$

Therefore, by Young's inequality,

$$|J_1| \le \frac{1}{4}\,\|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + C\,\|\theta_0\|_{L^{q_2}}^2\|G\|_{L^2}^2 + C\,\|\theta_0\|_{L^{q_2}}^2\left(\|\theta_0\|_{L^{q_1}}^2 + \|u_0\|_{L^2}^2 + t^2\|\theta_0\|_{L^2}^2\right),$$

where the C's are pure constants. Since $\frac{4}{5}<\alpha<1$,

$$\begin{aligned}|J_2| &\le \|\theta_0\|_{L^2}\,\|\Lambda^{2-2\alpha}G\|_{L^2}\\ &\le \|\theta_0\|_{L^2}\,\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{4(\frac{1}{\alpha}-1)}\,\|G\|_{L^2}^{5-\frac{4}{\alpha}}\\ &\le \frac{1}{4}\,\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C\,\|\theta_0\|_{L^2}^{\frac{2\alpha}{5\alpha-4}}\|G\|_{L^2}^2.\end{aligned}$$

Inserting the bounds for J_1 and J_2 in (5.3) and applying Gronwall's inequality yield the desired bound. This completes the proof of Theorem 5.1. $\square$

6. Global L^q-bound for G with $2 < q < q_0$. This section proves a global bound for $\|G\|_{L^q}$ with $2 < q < q_0$, where q_0 is specified in (6.1). This global bound is valid for any $\frac{4}{5} < \alpha < 1$.

THEOREM 6.1. *Let $\frac{4}{5} < \alpha < 1$ and $\alpha + \beta = 1$. Consider* (1.1) *with $u_0 \in H^2$ and $\theta_0 \in B^2_{2,1}$. Let (u, θ) be the corresponding smooth solution. Assume that $2 < q < q_0$ with q_0 given by*

$$q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}. \tag{6.1}$$

Then, for any $T > 0$ and $t \leq T$,

$$\|G(t)\|_{L^q} \leq C(T, u_0, \theta_0),$$

where $C(T, u_0, \theta_0)$ is a constant depending on T, u_0, and θ_0 only.

Proof. Taking the inner product of (5.2) with $G|G|^{q-2}$, we have

$$\frac{1}{q}\frac{d}{dt}\|G\|^q_{L^q} + \int G|G|^{q-2}\Lambda^\alpha G\, dx = K_1 + K_2, \tag{6.2}$$

where

$$K_1 = \int G|G|^{q-2}\, [\mathcal{R}_\alpha, u \cdot \nabla]\theta\, dx,$$
$$K_2 = \int G|G|^{q-2}\, \Lambda^{1-2\alpha}\partial_1\theta\, dx.$$

By a pointwise inequality for fractional Laplacians (see [13]) and a Sobolev embedding inequality,

$$\int G|G|^{q-2}\Lambda^\alpha G\, dx \geq C \int |\Lambda^{\frac{\alpha}{2}}|G|^{\frac{q}{2}}|^2\, dx \geq C_0\, \|G\|^q_{L^{\frac{2q}{2-\alpha}}},$$

where $C, C_0 > 0$ are constants. To bound K_1, we have for any $s \in (0, 1)$,

$$\begin{aligned} |K_1| &\leq \left| \int \Lambda^s(G|G|^{q-2})\, \Lambda^{1-s}(\Lambda^{-1}\nabla \cdot [\mathcal{R}_\alpha, u]\theta)\, dx \right| \\ &\leq \|\Lambda^s(G|G|^{q-2})\|_{L^2}\, \|\Lambda^{1-s}([\mathcal{R}_\alpha, u]\theta)\|_{L^2}. \end{aligned}$$

By Proposition 4.5, (5.4), and (5.5), we obtain

$$\begin{aligned} \|\Lambda^{1-s}[\mathcal{R}_\alpha, u]\theta\|_{L^2} &= \|[\mathcal{R}_\alpha, u]\theta\|_{\mathring{H}^{1-s}} \\ &\leq C\, \|u\|_{B^\alpha_{q_1,\infty}}\, \|\theta\|_{B^{2-s-2\alpha}_{q_2,2}} + C\, \|u\|_{L^2}\, \|\theta_0\|_{L^2} \\ &\leq C\, (\|G\|_{L^2} + \|\theta_0\|_{L^{q_1}} + \|u\|_{L^2})\|\theta_0\|_{L^{q_2}} + C\, \|u\|_{L^2}\, \|\theta_0\|_{L^2} \equiv B(t), \end{aligned}$$

where $2 - 2\alpha < s$, $q_1 = \frac{2}{\alpha}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$, and $B(t)$ is a smooth function of t depending on $\|u_0\|_{L^2}$ and $\|\theta_0\|_{L^2 \cap L^{q_1} \cap L^{q_2}}$ only. Choosing s satisfying

$$2 + s - \alpha - \frac{2(2-\alpha)}{q} = \frac{\alpha}{2}$$

and applying Lemma 6.2 below, we obtain

$$\|\Lambda^s(G|G|^{q-2})\|_{L^2} \leq C\, \|G\|^{q-2}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|_{\mathring{H}^{2+s-\alpha-\frac{2(2-\alpha)}{q}}} = C\, \|G\|^{q-2}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|_{\mathring{H}^{\frac{\alpha}{2}}}.$$

Combining the estimates, we obtain, for any $t \le T$,

$$|K_1| \le B(t)\, \|G\|^{q-2}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|_{\mathring{H}^{\frac{\alpha}{2}}} \le \frac{C_0}{2}\|G\|^q_{L^{\frac{2q}{2-\alpha}}} + C\,\|G\|^{\frac{q}{2}}_{\mathring{H}^{\frac{\alpha}{2}}},$$

where C is a constant depending on T and the norms of the initial data. Since $2-2\alpha < s$,

$$\begin{aligned}|K_2| &\le \|(1+\Lambda)^s(G|G|^{q-2})\|_{L^2}\|(1+\Lambda)^{2-2\alpha-s}\theta\|_{L^2}\\ &\le C(\|\Lambda^s(G|G|^{q-2})\|_{L^2} + \|G\|^{q-1}_{L^2})\,\|\theta_0\|_{L^2}\\ &\le \frac{C_0}{2}\|G\|^q_{L^{\frac{2q}{2-\alpha}}} + C\,\|G\|^{\frac{q}{2}}_{\mathring{H}^{\frac{\alpha}{2}}} + C\|G\|^{q-1}_{L^2}\|\theta_0\|_{L^2}.\end{aligned}$$

Inserting the estimates for K_1 and K_2 in (6.2) and using the fact in Theorem 5.1 that, for $\frac{q}{2} \le 2$ and any $T>0$,

$$\int_0^T \|G\|^{\frac{q}{2}}_{\mathring{H}^{\frac{\alpha}{2}}}\, dt < \infty,$$

we then obtain the desired global bound. The conditions on the indices

$$q \ge 2, \qquad 2-s-2\alpha < 0, \quad 2+s-\alpha-\frac{2(2-\alpha)}{q} = \frac{\alpha}{2}$$

are fulfilled if

$$\alpha > \frac{4}{5}, \qquad 2 \le q < q_0 \equiv \frac{8-4\alpha}{8-7\alpha}.$$

This completes the proof of Theorem 6.1. ◻

We have used the following lemma in the proof of Theorem 6.1. This lemma generalizes a previous inequality of [19, p. 2170].

Lemma 6.2. *Let $s \in (0,1)$, $\alpha \in (0,1)$, and $q \in [2,\infty)$. Then,*

$$\|\Lambda^s(G|G|^{q-2})\|_{L^2} \le C\|G\|^{q-2}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|_{\mathring{B}^s_{\frac{2q}{2(2-\alpha)-q(1-\alpha)},2}}.$$

Especially,

$$\|\Lambda^s(G|G|^{q-2})\|_{L^2} \le C\|G\|^{q-2}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|_{\mathring{H}^{2+s-\alpha-\frac{2(2-\alpha)}{q}}}. \tag{6.3}$$

Proof. Identifying $\mathring{H}^s$ with the Besov space $\mathring{B}^s_{2,2}$, we have

$$\|\Lambda^s(G|G|^{q-2})\|^2_{L^2} = \int \frac{\|G|G|^{q-2}(\cdot - z) - G|G|^{q-2}(\cdot)\|^2_{L^2}}{|z|^{2+2s}}\, dz.$$

Using the simple inequality, for $q \ge 2$,

$$|a|a|^{q-2} - b|b|^{q-2}| \le C\,|a-b|(|a|^{q-2} + |b|^{q-2})$$

and Hölder's inequality, we have

$$\|G|G|^{q-2}(\cdot - z) - G|G|^{q-2}(\cdot)\|^2_{L^2} \le C\,\|G\|^{2(q-2)}_{L^{\frac{2q}{2-\alpha}}}\, \|G(\cdot - z) - G(\cdot)\|^2_{L^{\frac{2q}{2(2-\alpha)-q(1-\alpha)}}}.$$

Therefore,

$$\|\Lambda^s(G|G|^{q-2})\|^2_{L^2} \le C\,\|G\|^{2(q-2)}_{L^{\frac{2q}{2-\alpha}}}\, \|G\|^2_{\mathring{B}^s_{\frac{2q}{2(2-\alpha)-q(1-\alpha)},2}}.$$

Equation (6.3) holds due to the embedding $\mathring{B}^{2+s-\alpha-\frac{2(2-\alpha)}{q}}_{2,2} \hookrightarrow \mathring{B}^s_{\frac{2q}{2(2-\alpha)-q(1-\alpha)},2}$. This completes the proof of Lemma 6.2. ◻

7. Proof of Theorem 1.1. This section proves Theorem 1.1. To do so, we need more regularity for G. By fully exploiting the dissipation, we are able to show in Proposition 7.1 below that G in $B^s_{q,\infty}$ with any $s \le 3\alpha - 2$ and $2 \le q < q_0$ is bounded for all time.

PROPOSITION 7.1. *Consider the IVP* (1.1) *with* $\alpha_0 < \alpha < 1$. *Assume the initial data* (u_0, θ_0) *satisfy the conditions stated in Theorem* 1.1 *and let* (u, θ) *be the corresponding solution. If the indices* s *and* q *satisfy*

$$0 < s \le 3\alpha - 2, \quad \frac{2}{2\alpha - 1} < q < q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}, \tag{7.1}$$

then G *obeys the global a priori bound, for any* $T > 0$,

$$\|G\|_{L^\infty(0,T;B^s_{q,\infty})} \le C,$$

where C *is a constant depending on* T *and the initial norms only.*

This proposition will be proven at the end of this section. With this proposition at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The key part of the proof is to establish the global a priori bounds of the solution in the functional setting (1.4). According to Proposition 7.1, for any $T > 0$, and s and q satisfying (7.1), we have

$$G \equiv \omega - \Lambda^{-\alpha}\partial_1\theta \in L^\infty(0, T; B^s_{q,\infty}). \tag{7.2}$$

Since $u = \nabla^\perp\Delta^{-1}\omega$, we have

$$u = \nabla^\perp\Delta^{-1}\, G + \nabla^\perp\Delta^{-1}\,\Lambda^{-\alpha}\partial_1\theta \equiv \widetilde{u} + v.$$

Since $\alpha_0 < \alpha < 1$, we can choose q satisfying (7.1) such that

$$\frac{2}{q} - (3\alpha - 2) < 0.$$

In fact, thanks to $\alpha_0 < \alpha$, we have

$$\frac{2}{3\alpha - 2} < \frac{8 - 4\alpha}{8 - 7\alpha}$$

and any q satisfying

$$\frac{2}{3\alpha - 2} < q < q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}$$

would work. By the embedding $B^0_{\infty,1} \hookrightarrow L^\infty$, Bernstein's inequality, and the boundedness of Riesz transforms on L^q,

$$\begin{aligned}
\|\nabla\widetilde{u}\|_{L^\infty} &\le \sum_{j=-1}^{\infty} \|\nabla\nabla^\perp\Delta^{-1}\Delta_j G\|_{L^\infty} \\
&\le C \sum_{j=-1}^{\infty} 2^{j\frac{2}{q}} \|\Delta_j G\|_{L^q} \\
&\le \|G\|_{B^s_{q,\infty}} \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q} - s)}.
\end{aligned}$$

Taking $s \le 3\alpha - 2$ but close to $3\alpha - 2$, we have $\frac{2}{q} - s < 0$ and thus

$$M \equiv \|\nabla \widetilde{u}\|_{L^\infty(0,T;L^\infty)} < \infty.$$

Applying Theorem 1.2 to the equation for θ,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda^{1-\alpha}\theta = 0, \\ u = \widetilde{u} + v, \quad v \equiv -\nabla^\perp \Delta^{-1} \Lambda^{-\alpha} \partial_1 \theta, \end{cases}$$

we obtain

$$\theta \in C([0,T];H^1), \quad \|\nabla\theta\|_{L^\infty(0,T;L^\infty)} \le C(M, \|\theta_0\|_{L^\infty}, \|\nabla\theta_0\|_{L^\infty}, T) < \infty. \tag{7.3}$$

According to the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega + \Lambda^\alpha \omega = \partial_1 \theta, \tag{7.4}$$

we have, thanks to $\omega_0 \in B^{\sigma-1}_{2,1}(\mathbb{R}^2)$ with $\sigma \ge \frac{5}{2}$ and $B^1_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$,

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^T \|\partial_1\theta(\tau)\|_{L^\infty}\, d\tau. \tag{7.5}$$

(7.3) and (7.5) will lead us to the global a priori bounds in (1.4). To this goal, we employ the following estimate, for any integer $j \ge -1$ and any $t \in (0,T]$,

$$\begin{aligned} \frac{2^{\alpha j}}{j+2} \int_0^t \|\Delta_j \omega\|_{L^\infty}\, d\tau \le C &\left(\|\omega_0\|_{L^\infty} + \int_0^t \|\partial_1\theta(\tau)\|_{L^\infty}\, d\tau \right) \\ &\times \left(1 + t + \|u\|_{L^2} + \|\omega\|_{L^1_t L^\infty} \right) \equiv h(t), \end{aligned} \tag{7.6}$$

where C is a constant independent of j. (7.6) can be established by applying Besov-space-type estimates to (7.4) and more details can be found in [28, p. 134]. As a consequence, we have, for any $0 < \alpha' < \alpha$,

$$\begin{aligned} \int_0^t \sum_{j=-1}^\infty 2^{\alpha' j} \|\Delta_j \omega\|_{L^\infty}\, d\tau &= \sum_{j=-1}^\infty 2^{\alpha' j} \int_0^t \|\Delta_j\omega\|_{L^\infty}\, d\tau \\ &\le h(t) \sum_{j=-1}^\infty 2^{(\alpha'-\alpha)j}/(j+1) \le C\, h(t). \end{aligned} \tag{7.7}$$

Therefore, by Bernstein's inequality and $\|\Delta_j \nabla u\|_{L^\infty} \le C\, \|\Delta_j \omega\|_{L^\infty}$ for $j \ge 0$,

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{j=0}^\infty \|\Delta_j \nabla u\|_{L^\infty} \le C\|u\|_{L^2} + C \sum_{j=0}^\infty \|\Delta_j \omega\|_{L^\infty}. \tag{7.8}$$

Combining (7.7) and (7.8) yields, for any $0 < t \le T$,

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau < \infty. \tag{7.9}$$

Through standard Besov-space-type estimates, we can show that

$$Y(t) \equiv \|u(t)\|_{B^\sigma_{2,1}} + \|\theta(t)\|_{B^2_{2,1}}$$

obeys the differential inequality

$$\frac{d}{dt}Y(t) + C\,(\|u\|_{B^{\sigma+\alpha}_{2,1}} + \|\theta\|_{B^{2+\beta}_{2,1}}) \le C\,(1 + \|\nabla u(t)\|_{L^\infty} + \|\nabla\theta(t)\|_{L^\infty})Y(t).$$

Gronwall's inequality together with (7.3) and (7.9) yields the desired property in (1.4). The continuity of u and θ in time, namely, $u \in C([0,T]; B^{\sigma}_{2,1})$ and $\theta \in C([0,T]; B^{2}_{2,1})$, follows from a standard approach (see, e.g., [26, p. 111] or [1, p. 138]). We thus have obtained all the global a priori bounds in (1.4). The global existence part can then be obtained by first establishing the local existence through a standard process such as the successive approximation and then extending to all time via the global a priori bounds. The uniqueness of solutions in the functional setting (1.4) is clear and we thus omit the details. This completes the proof of Theorem 1.1. ∎

We finally prove Proposition 7.1.

Proof of Proposition 7.1. Recall that G satisfies

$$\partial_t G + u\cdot\nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u\cdot\nabla]\theta + \Lambda^{1-2\alpha}\partial_1\theta. \tag{7.10}$$

Let $j \ge -1$. Applying Δ_j to (7.10) and taking the inner product with $\Delta_j G|\Delta_j G|^{q-2}$, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|^q_{L^q} + \int \Delta_j G|\Delta_j G|^{q-2}\Lambda^\alpha\Delta_j G\,dx = K_1^{(j)} + K_2^{(j)} + K_3^{(j)}, \tag{7.11}$$

where

$$\begin{aligned} K_1^{(j)} &= \int \Delta_j G|\Delta_j G|^{q-2}\,\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,dx,\\ K_2^{(j)} &= -\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)\,dx,\\ K_3^{(j)} &= \int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(\Lambda^{1-2\alpha}\partial_1\theta)\,dx. \end{aligned} \tag{7.12}$$

For $j \ge 0$, the Fourier transform of $\Delta_j G$ is supported away from the origin and the dissipative part admits a lower bound,

$$\int \Delta_j G|\Delta_j G|^{q-2}\Lambda^\alpha\Delta_j G\,dx \ge C\,2^{\alpha j}\|\Delta_j G\|^q_{L^q} \tag{7.13}$$

for a constant C that depends on q and α only (see, e.g., [7, 35]). For $j = -1$, the dissipative part is still nonnegative and can be neglected. By Hölder's inequality,

$$|K_1^{(j)}| \le \|\Delta_j G\|^{q-1}_{L^q}\,\|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q}.$$

Furthermore, $\|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q}$ can be estimated in a similar fashion as in the proof of Proposition 4.5 and is bounded by

$$\|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q} \le C\,2^{(2-2\alpha)j}\,\|u\|_{\mathring{B}^{\alpha}_{q,\infty}}\,\|\theta\|_{L^\infty} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}.$$

In addition, as in (5.4),

$$\begin{aligned} \|u\|_{\mathring{B}^{\alpha}_{q,\infty}} &\le C\,\|\Lambda^\alpha u\|_{L^q} \le C\,\|\Lambda^{\alpha-1}G\|_{L^q} + C\,\|\theta\|_{L^q}\\ &\le C\,\|G\|_{L^{\tilde q}} + C\,\|\theta_0\|_{L^q}, \end{aligned} \tag{7.14}$$

where $\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\alpha}{2} + \frac{1}{2}$. For $\alpha \in (\alpha_0, 1)$, we have $2 < \widetilde{q} < q$ and thus $\|G\|_{L^{\widetilde{q}}} < C$. Therefore,

$$|K_1^{(j)}| \le B(t)\, 2^{(2-2\alpha)j}\, \|\Delta_j G\|_{L^q}^{q-1}, \tag{7.15}$$

where $B_1(t)$ is a smooth function of t that depends on the initial norms only. To estimate $K_2^{(j)}$, we apply the notion of paraproducts to write

$$\Delta_j(u\cdot\nabla G) = J_1 + J_2 + J_3 + J_4 + J_5, \tag{7.16}$$

where

$$\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} \left(\Delta_j(S_{k-1}u\cdot\nabla\Delta_k G) - S_{k-1}u\cdot\nabla\Delta_j\Delta_k G\right),\\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_3 &= S_j u\cdot\nabla\Delta_j G,\\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G)
\end{aligned}$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Inserting (7.16) into (7.12) naturally splits the integral in $K_2^{(j)}$ into five parts $K_2^{(j)} = K_{21}^{(j)} + K_{22}^{(j)} + K_{23}^{(j)} + K_{24}^{(j)} + K_{25}^{(j)}$. By Hölder's inequality,

$$|K_{21}^{(j)}| \le \|\Delta_j G\|_{L^q}^{q-1} \sum_{|j-k|\le 2} \|(\Delta_j(S_{k-1}u\cdot\nabla\Delta_k G) - S_{k-1}u\cdot\nabla\Delta_j\Delta_k G)\|_{L^q}.$$

Since the summation above is for k satisfying $|j-k|\le 2$ and can be replaced by a constant multiple of the representative term with $k = j$, we obtain, by a standard commutator estimate and Bernstein's inequality,

$$\begin{aligned}
|K_{21}^{(j)}| &\le C\,\|\Delta_j G\|_q^{q-1}\, 2^{(1-\alpha)j}\, \|\Lambda^\alpha S_{j-1}u\|_{L^q}\, \|\Delta_j G\|_{L^\infty}\\
&\le C\,\|\Delta_j G\|_q^{q-1}\, 2^{(1-\alpha+\frac{2}{q})j}\, \|\Lambda^\alpha u\|_{L^q}\, \|\Delta_j G\|_{L^q}\\
&\le C\, 2^{(1-\alpha+\frac{2}{q})j}\, \|\Lambda^\alpha u\|_{L^q}\, \|\Delta_j G\|_{L^q}^q.
\end{aligned}$$

The second term $K_{22}^{(j)}$ can be bounded by

$$\begin{aligned}
|K_{22}^{(j)}| &\le C\,\|\Delta_j G\|_{L^q}^{q-1}\, \|\Delta_j u\|_{L^q}\, 2^j \|\Delta_j G\|_{L^\infty}\\
&\le C\, 2^{(1-\alpha+\frac{2}{q})j}\, \|\Lambda^\alpha\Delta_j u\|_{L^q}\, \|\Delta_j G\|_{L^q}^q,
\end{aligned}$$

where we have used the lower bound part of the Bernstein inequality

$$2^{\alpha j}\,\|\Delta_j u\|_{L^q} \le C\,\|\Lambda^\alpha\Delta_j u\|_{L^q}.$$

This inequality is valid for $j \ge 0$. In the case when $j = -1$, this inequality is not needed and it suffices to apply the upper bound part of the Bernstein inequality

$$\|\Delta_{-1}u\|_{L^q} \le C\,\|\Delta_{-1}u\|_{L^2}.$$

Due to the divergence-free condition $\nabla \cdot u = 0$, we have $K_{23}^{(j)} = 0$. By Hölder's inequality and Bernstein's inequality,

$$
\begin{aligned}
|K_{24}^{(j)}| &\le C\, \|\Delta_j G\|_{L^q}^{q-1} \|\Delta_j u \cdot \nabla S_{j-1} G\|_{L^q} \\
&\le C\, 2^{-\alpha j} \|\Delta_j G\|_{L^q}^{q-1} \|\Lambda^\alpha \Delta_j u\|_{L^q} \sum_{m \le j-1} 2^{(1+\frac{2}{q})m} \|\Delta_m G\|_{L^q} \\
&\le C\, 2^{(1-\alpha+\frac{2}{q})j} \|\Delta_j G\|_{L^q}^{q-1} \|\Lambda^\alpha \Delta_j u\|_{L^q} \sum_{m \le j-1} 2^{(1+\frac{2}{q})(m-j)} \|\Delta_m G\|_{L^q}, \\
|K_{25}^{(j)}| &\le C\, \|\Delta_j G\|_{L^q}^{q-1} \sum_{k \ge j-1} 2^j \|\Delta_j (\Delta_k u \widetilde{\Delta}_k G)\|_{L^q} \\
&\le C\, 2^{(1-\alpha+\frac{2}{q})j} \|\Delta_j G\|_{L^q}^{q-1} \sum_{k \ge j-1} 2^{(j-k)(\alpha-\frac{2}{q})} \|\Lambda^\alpha \Delta_k u\|_{L^q} \|\widetilde{\Delta}_k G\|_{L^q}.
\end{aligned}
$$

Collecting the estimates for $K_2^{(j)}$ and bounding $\|\Lambda^\alpha u\|_{L^q}$ as in (7.14), we have

$$
\begin{aligned}
|K_2^{(j)}| \le C\, 2^{(1-\alpha+\frac{2}{q})j} \|\Delta_j G\|_{L^q}^{q-1} \Bigg[&\|\Delta_j G\|_{L^q} + \sum_{m \le j-1} 2^{(1+\frac{2}{q})(m-j)} \|\Delta_m G\|_{L^q} \\
&+ \sum_{k \ge j-1} 2^{(j-k)(\alpha-\frac{2}{q})} \|\Delta_k G\|_{L^q} \Bigg].
\end{aligned} \tag{7.17}
$$

By Hölder's inequality and Bernstein's inequality,

$$
|K_3^{(j)}| \le 2^{2(1-\alpha)j} \|\Delta_j G\|_{L^q}^{q-1} \|\Delta_j \theta\|_{L^q} \le 2^{2(1-\alpha)j} \|\Delta_j G\|_{L^q}^{q-1} \|\theta_0\|_{L^q}. \tag{7.18}
$$

Combining (7.11), (7.13), (7.15), (7.17), and (7.18), we obtain

$$
\frac{d}{dt} \|\Delta_j G\|_{L^q} + C\, 2^{\alpha j} \|\Delta_j G\|_{L^q} \le C\, B(t)\, 2^{(2-2\alpha)j} + C\, 2^{(1-\alpha+\frac{2}{q})j} L(t),
$$

where $B(t)$ is a smooth function of t depending on the initial norms only and

$$
L = \|\Delta_j G\|_{L^q} + \sum_{m \le j-1} 2^{(1+\frac{2}{q})(m-j)} \|\Delta_m G\|_{L^q} + \sum_{k \ge j-1} 2^{(j-k)(\alpha-\frac{2}{q})} \|\Delta_k G\|_{L^q}.
$$

Integrating in time leads to

$$
\begin{aligned}
\|\Delta_j G(t)\|_{L^q} \le e^{-C 2^{\alpha j} t} \|\Delta_j G(0)\|_{L^q} &+ C\, \widetilde{B}(t)\, 2^{(2-3\alpha)j} \\
&+ C\, 2^{(1-\alpha+\frac{2}{q})j} \int_0^t e^{-C 2^{\alpha j}(t-\tau)} L(\tau)\, d\tau,
\end{aligned}
$$

where $\widetilde{B}(t)$ is a smooth function of t depending on the initial norms only. Multiplying each side by 2^{sj} with $0 < s \le 3\alpha - 2$ and taking sup with respect to j, we find that

$$
\|G(t)\|_{B^s_{q,\infty}} \le \|G(0)\|_{B^s_{q,\infty}} + C\, \widetilde{B}(t) + L_1 + L_2 + L_3, \tag{7.19}
$$

where

$$
\begin{aligned}
L_1 &= \sup_{j \ge -1} C\, 2^{(1-\alpha+\frac{2}{q})j} \int_0^t e^{-C 2^{\alpha j}(t-\tau)} 2^{sj} \|\Delta_j G(\tau)\|_{L^q}\, d\tau, \\
L_2 &= \sup_{j \ge -1} C\, 2^{(1-\alpha+\frac{2}{q})j} \int_0^t e^{-C 2^{\alpha j}(t-\tau)} 2^{sj} \sum_{m \le j-1} 2^{(1+\frac{2}{q})(m-j)} \|\Delta_m G\|_{L^q}\, d\tau, \\
L_3 &= \sup_{j \ge -1} C\, 2^{(1-\alpha+\frac{2}{q})j} \int_0^t e^{-C 2^{\alpha j}(t-\tau)} 2^{sj} \sum_{k \ge j-1} 2^{(j-k)(\alpha-\frac{2}{q})} \|\Delta_k G\|_{L^q}\, d\tau.
\end{aligned}
$$

Since $1-2\alpha+\frac{2}{q}<0$, we choose an integer j_0 such that, for $j\ge j_0$,

$$C\,2^{(1-2\alpha+\frac{2}{q})j}\le \frac{1}{8}.$$

Then

$$\begin{aligned} L_1 &\le \sup_{-1\le j\le j_0} C\,2^{(1-\alpha+\frac{2}{q})j}\int_0^t e^{-C2^{\alpha j}(t-\tau)}2^{sj}\|\Delta_j G(\tau)\|_{L^q}\,d\tau \\ &\quad+\sup_{j_0+1\le j<\infty} C\,2^{(1-\alpha+\frac{2}{q})j}\int_0^t e^{-C2^{\alpha j}(t-\tau)}2^{sj}\|\Delta_j G(\tau)\|_{L^q}\,d\tau \\ &\le C\,\|G\|_{L^\infty(0,T;L^q)}\,2^{(1-2\alpha+\frac{2}{q}+s)j_0}+\frac{1}{8}\|G(t)\|_{L^\infty(0,T;B^s_{q,\infty})}. \end{aligned}$$

L_2 and L_3 obey the same bound. Inserting these bounds into (7.19), we find

$$\|G\|_{L^\infty(0,T;B^s_{q,\infty})}\le C\,(1+\|G(0)\|_{B^s_{q,\infty}})+C\,\|G\|_{L^\infty(0,T;L^q)},$$

where we have written C for $\max_{0\le t\le T}\widetilde{B}(t)$. Since $u_0\in B^\sigma_{2,1}$ with $\sigma\ge\frac{5}{2}$ and $\theta_0\in B^2_{2,1}$,

$$\omega_0\in B^{\sigma-1}_{2,1}(\mathbb{R}^2)\hookrightarrow B^s_{q,\infty}(\mathbb{R}^2),\quad G(0)=\omega_0-\Lambda^{-\alpha}\partial_1\theta_0\in B^s_{q,\infty}.$$

This finishes the proof of Proposition 7.1. $\square$

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