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Stability and large-time behavior for the 2D Boussineq system with horizontal dissipation and vertical thermal diffusion

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Abstract. This paper solves the stability and large-time behavior problem on perturbations near the hydrostatic equilibrium of the two-dimensional Boussinesq system with horizontal dissipation and vertical thermal diffusion. The spatial domain Ω is $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being the 1D periodic box and $\mathbb{R}$ being the whole line. The results presented in this paper establish the observed stabilizing phenomenon and stratifying patterns of the buoyancy-driven fluids as mathematically rigorous facts. The stability and large-time behavior problem concerned here is difficult due to the lack of the vertical dissipation and horizontal thermal diffusion. To make up for the missing regularization, we exploit the smoothing and stabilizing effect due to the coupling and interaction between the temperature and the fluids. By constructing suitable energy functional and introducing the orthogonal decomposition of the velocity and the temperature into their horizontal averages and oscillation parts, we are able to establish the stability in the Sobolev space H^2 and obtain algebraic decay rates for the oscillation parts in the H^1-norm.

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1. Introduction

The goal of this paper is three-fold: first, to reveal and rigorously confirm the stabilizing phenomenon of the temperature on buoyancy-driven fluids; second, to assess the effect of the domain on the stability and large-time behavior of perturbations near the hydrostatic equilibrium; and third, to develop an

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effective approach for the stability problem and decay properties on solutions to the partially dissipated systems of partial differential equations.

The hydrostatic equilibrium is a significant steady-state of many geophysical fluids [9,11]. Our atmosphere is mostly in hydrostatic equilibrium, with the upward pressure gradient force balanced out by the downward gravitational force. Understanding the stability and large-time behavior of perturbations near this special equilibrium may help explain and predict some of the severe weather phenomena [3,7].

Several studies have recently been conducted to rigorously understand the influence of the temperature on the stability of perturbations near the hydrostatic equilibrium. Since the Boussinesq models are the most relevant models for buoyancy-driven fluids, these studies are based on several incompressible Bousinesq systems. The work of Doering, Wu, Zhao and Zheng [6] investigated the stability of the hydrostatic equilibrium to the 2D Boussinesq system with only kinematic dissipation (without thermal diffusion) and rigorously proved the global asymptotic stability of any perturbation near the hydrostatic equilibrium [6]. In addition, extensive numerical simulations are performed in [6] to corroborate the analytical results and predict some phenomena that are not proven. The work of Tao, Wu, Zhao and Zheng [12] resolves several important issues left open in [6]. In particular, [12] provides a precise description of the final buoyancy distribution in case of general initial conditions and the explicit decay rate of the velocity field or the total mechanical energy. The paper of Castro, Córdoba and Lear successfully established the stability and large time behavior on the 2D Boussinesq equations with velocity damping instead of dissipation [4]. We shall refrain from describing more results at this moment but defer them until the later part of this introduction.

This paper focuses on the following 2D anisotropic Boussinesq system

$$\begin{cases} \partial_t U + U\cdot\nabla U = -\nabla P + \nu\,\partial_{11}U + \Theta\mathbf{e}_2, \quad x\in\Omega,\ t>0,\\ \partial_t\Theta + U\cdot\nabla\Theta = \eta\,\partial_{22}\Theta,\\ \nabla\cdot U = 0, \end{cases} \tag{1.1}$$

where U denotes the fluid velocity, P the pressure, Θ the temperature, and $\nu > 0$ the kinematic viscosity and η the thermal diffusivity, respectively. Here $\mathbf{e}_2$ is the unit vector in the vertical direction and the spatial domain Ω is given by

$$\Omega = \mathbb{T}\times\mathbb{R},$$

with $\mathbb{T} = [0,1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line. (1.1) models anisotropic buoyancy-driven fluids in the circumstance when the vertical dissipation and the horizontal thermal diffusion are negligible [11].

We are mainly concerned with the stability and the precise large-time behavior of perturbations near the hydrostatic equilibrium (U_{he}, Θ_{he}) with

$$U_{he} = 0, \quad \Theta_{he} = x_2.$$

For the static velocity U_{he}, the momentum equation is satisfied when the pressure gradient is balanced by the buoyancy force, namely

$$-\nabla P_{he} + \Theta_{he}\,\mathbf{e}_2 = 0 \quad \text{or} \quad P_{he} = \frac{1}{2}x_2^2.$$

To understand the desired stability, we write the equation of the perturbation denoted by (u, p, θ), where

$$u = U - U_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \Theta - \Theta_{he}.$$

It follows easily from (1.1) that the perturbation (u, p, θ) satisfies the following anisotropic Boussinesq equations with horizontal dissipation and vertical thermal diffusion

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\,\partial_{11}u + \theta\mathbf{e}_2, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\,\partial_{22}\theta, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2}$$

The system (1.2) obeyed by the perturbations differs from the original system (1.1) by a single term, u_2 in the equation of θ. Without this term, the L^2-norm of the velocity u to (1.1) can grow in time due to the buoyancy forcing term $\theta\mathbf{e}_2$. As pointed out in [2], solutions of the 3D Boussinesq equations with even full dissipation and thermal diffusion can actually grow in time. This term in (1.2) helps balance out $\theta\mathbf{e}_2$ in the energy estimates. Therefore, the buoyancy forcing no longer plays a destabilizing role in (1.2).

However, the lack of full kinematic dissipation in the momentum equation becomes the main obstacle in the stability problem concerned here. Even when the temperature is identically zero, $\theta \equiv 0$, the fluid itself may not even be stable. In fact, when $\theta \equiv 0$, the fluid is governed by the 2D anisotropic incompressible Navier-Stokes equastions

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla P + \nu\partial_{11}u, \\ \nabla\cdot u = 0 \end{cases} \tag{1.3}$$

or, in terms of the vorticity $\omega = \nabla\times u$,

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu\partial_{11}\omega, \\ u = \nabla^\perp\Delta^{-1}\omega := (-\partial_2, \partial_1)\Delta^{-1}\omega. \end{cases} \tag{1.4}$$

The stability problem on (1.3) in the Sobolev setting H^2 remains an open problem in the whole space case $\mathbb{R}^2$. The global well-posedness of (1.3) follows from the classical Yudovich approach due to the boundedness of the vorticity itself from (1.4). But the vorticity gradient $\nabla\omega$ or more generally the second-order derivatives of u in any Lesbesgue space L^p with $2 \le p \le \infty$ can potentially grow rather rapidly in time. In fact, it appears that the best upper bound we have for the whole space cases are double exponential in time,

$$\|\nabla\omega(t)\|_{L^q(\mathbb{R}^2)} \le \big(\|\nabla\omega(0)\|_{L^q(\mathbb{R}^2)}\big)^{e^{C\|\omega(0)\|_{L^\infty(\mathbb{R}^2)}t}}.$$

Indeed in the case of the 2D Euler equation in a unit disk, Kiselev and Sverak were able to construct an explicit vorticity solution whose gradient grows double exponentially [8]. It is then clear that the stability problem would not be possible if the temperature θ does not stabilize the fluid.

In a previous work by Ben Said, Pandey and Wu [1], we explored the influence of the temperature on the fluid when we study the stability of the Boussinesq system in the whole space $\mathbb{R}^2$,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\,\partial_{22} u + \theta \mathbf{e}_2, \quad x\in\mathbb{R}^2,\, t>0,\\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\,\partial_{11}\theta,\\ \nabla\cdot u = 0,\\ u(x,0)=u_0(x),\quad \theta(x,0)=\theta_0(x). \end{cases} \tag{1.5}$$

We remark that the Boussinesq systems are asymmetrical and the exchange of the vertical dissipation with the horizontal one leads to different regularity properties. As a consequence, (1.2) can not be dealt with via the same approach as the one for (1.5). We slightly elaborate on how we exploited the stabilizing effect of the temperature in the system (1.5) and explain why the same mechanism would not work for (1.2). To unearth the influence of the temperature on the fluid, we make use of the coupling in (1.5) to seek special structure in the system. To do so, we first apply the Leray projection operator

$$\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$$

to the velocity equation in (1.5) to eliminate the pressure

$$\partial_t u = \nu\partial_{22}u + \mathbb{P}(\theta\mathbf{e}_2) - \mathbb{P}(u\cdot\nabla u). \tag{1.6}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(\theta\mathbf{e}_2) = \theta\mathbf{e}_2 - \nabla\Delta^{-1}\nabla\cdot(\theta\mathbf{e}_2) = \begin{bmatrix} -\partial_1\partial_2\Delta^{-1}\theta \\ \theta - \partial_2^2\Delta^{-1}\theta \end{bmatrix}. \tag{1.7}$$

Inserting (1.7) in (1.6) and writing (1.6) in terms of its component equations, we obtain

$$\begin{cases} \partial_t u_1 = \nu\,\partial_{22}u_1 - \partial_1\partial_2\Delta^{-1}\theta + N_1,\\ \partial_t u_2 = \nu\,\partial_{22}u_2 + \partial_1\partial_1\Delta^{-1}\theta + N_2, \end{cases} \tag{1.8}$$

where N_1 and N_2 are the nonlinear terms,

$$N_1 = -(u\cdot\nabla u_1 - \partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u)),\quad N_2 = -(u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)).$$

By differentiating the equations of (1.8) as well as the equation of θ in t and making several substitutions, we find that (u,θ) actually satisfies the wave equations

$$\begin{cases} \partial_{tt}u - (\eta\partial_{11} + \nu\partial_{22})\partial_t u + \nu\eta\partial_{11}\partial_{22}u + \partial_{11}\Delta^{-1}u = N_3,\\ \partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_4, \end{cases} \tag{1.9}$$

where N_3 and N_4 represent the nonlinear terms. In comparison with (1.5), the wave structure in (1.9) exhibits more smoothing and stabilizing properties. In particular, the extra smoothing given by the wave term $\partial_{11}\Delta^{-1}u$ is in the horizontal direction. (1.5) originally has vertical dissipation and this extra horizontal smoothing makes up for what is lacking in (1.5). This is one of the main reasons that [1] was able to solve the stability problem on (1.5).

The system (1.2) concerned here can also be converted into a system of wave equations

$$\begin{cases} \partial_{tt} u - (\eta\partial_{22} + \nu\partial_{11})\partial_t u + \nu\eta\partial_{11}\partial_{22} u + \partial_{11}\Delta^{-1} u = N_5, \\ \partial_{tt}\theta - (\eta\partial_{22} + \nu\partial_{11})\partial_t \theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_6, \end{cases} \tag{1.10}$$

where N_5 and N_6 contain the nonlinear terms. (1.10) reveals more regularizing properties than (1.2), but the extra smoothing given by the wave term $\partial_{11}\Delta^{-1}u$ is in the horizontal direction. The system (1.2) itself has horizontal kinematic dissipation and what we really need is the vertical regularization. Therefore the extra horizontal smoothing in the wave equation (1.10) does not appear to help. This explains the difference between (1.2) and (1.5) as well as why the extra smoothing from the coupling with the temperature does not help us with the stability problem concerned here.

The spatial domain here is taken to be $\Omega = \mathbb{T} \times \mathbb{R}$. We explain how we can take advantage of this domain to help with our stability problem. The horizontal variable is in a periodic domain and the Fourier transform in the horizontal variable is represented by a sequence of Fourier modes. Our idea is to separate the zeroth horizontal Fourier mode from the non-zero modes. To be more precise, we introduce several notation. Let $f = f(x_1, x_2)$ be a function defined on $\mathbb{T} \times \mathbb{R}$ that is integrable in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$. We define its horizontal average $\overline{f}$ by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{1.11}$$

Clearly, $\overline{f}$ represents the zeroth Fourier mode of f. We decompose f into $\overline{f}$ and the corresponding oscillation portion $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{1.12}$$

$\widetilde{f}$ contains all non-zero Fourier modes. The decomposition in (1.12) has some special properties. First of all, this decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any non-negative integer k. In fact,

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} = 0,$$

where $(g, h)_{\dot{H}^k(\Omega)}$ denotes the inner product in the homogeneous Sobolev space $\dot{H}^k$. Furthermore, $\widetilde{f}$ admits strong versions of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\, \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}, \quad \|\widetilde{f}\|_{L^\infty(\Omega)} \le C\, \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

By invoking the decompositions

$$u = \overline{u} + \widetilde{u}, \quad \theta = \overline{\theta} + \widetilde{\theta}$$

in the estimates of the H^2-norm of (u, θ) and applying the aforementioned properties, we are able to deal with the nonlinear terms suitably, even when there is only horizontal dissipation.

We are ready to present our main results. The first result establishes the H^2-stability while the second result provides the decay rates of the oscillation portion $(\widetilde{u}, \widetilde{\theta})$.

Theorem 1.1. *Let $\mathbb{T} = [0,1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (1.2) has a unique global solution (u, θ) that remains uniformly bounded for all time, for any $t \geq 0$,

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{H^2}^2 d\tau + \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \leq C\varepsilon^2$$

for some constant $C > 0$.

Theorem 1.1 asserts that the solution of (1.2) emanating from any small initial perturbation (in the H^2-sense) is always global (in time) and remains comparable to the initial size. This result takes advantage of the domain $\Omega = \mathbb{T} \times \mathbb{R}$ to decompose both u and θ into their horizontal averages and oscillation parts in order to handle the nonlinear terms. When the spatial domain is the whole space $\mathbb{R}^2$, no such decomposition is possible and the stability problem on (1.2) in $\mathbb{R}^2$ remains an open problem.

Theorem 1.1 implies that $\|\partial_1 \theta(\tau)\|_{L^2}^2$ is also time integrable. The temperature equation has no horizontal dissipation and this horizontal regularization reflects the extra smoothing and stabilizing resulting from the coupling and interaction of the temperature and the fluid.

The next theorem rigorously establishes what we have observed in numerical simulations of buoyancy-driven stratified fluids (see, e.g., [6]). Perturbations governed by the Boussinesq systems near the hydrostatic equilibrium are observed to stratify and eventually approach their horizontal averages while the oscillation parts of both u and θ are observed to decay to zero. The following theorem verifies that indeed the oscillation parts $\widetilde{u}$ and $\widetilde{\theta}$ decays to zero at algebraic rates.

Theorem 1.2. *Let $u_0, \theta_0 \in H^2(\Omega)$ with $\nabla \cdot u_0 = 0$. Assume that (u_0, θ_0) satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon\,,$$

for sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution of (1.2). Then the oscillation part $(\widetilde{u}, \widetilde{\theta})$ satisfies the following algebraic decay in time,

$$\|\widetilde{u}\|_{H^1} + \|\widetilde{\theta}\|_{H^1} \leq c(1+t)^{-\frac{1}{2}},$$

for some constant $c > 0$ and for all $t \geq 0$. In addition, $(\widetilde{u}, \widetilde{\theta})$ has the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

As a consequence of Theorem 1.2, the solution (u, θ) of (1.2) approaches the horizontal average $(\overline{u}, \overline{\theta})$ asymptotically, and the Boussinesq system (1.2) evolves to the following 1D system eventually

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \eta \partial_2^2 \overline{\theta}. \end{cases}$$

How do the results differ from some of the closely related work? We have previously described several related work [4,6,12]. None of the previous work has investigated the stability problem on the Boussinesq system when the velocity equation involves only the horizontal dissipation. As aforementioned, the Boussinesq systems are asymmetrical, and exchanging the x_1 and x_2 variables results in systems with quite different properties. A previous work of Ben Said, Pandey and Wu [1] examined the case when the velocity equation has the vertical dissipation and when the spatial domain is $\mathbb{R}^2$. The approach of [1] can not be extended to the case when the velocity dissipation is horizontal. As we explained before, the two cases are different and the vertical dissipation cases is more favorable in the sense that the extra horizontal dissipation from the wave structure complements the vertical dissipation.

We now explain the main lines in the proofs for Theorem 1.1 and Theorem 1.2. Since the local well-posedness of (1.2) follows from a standard procedure (see, e.g., [10]), the proof of Theorem 1.1 reduces to obtaining the global H^2-bound for the solution (u, θ) of (1.2). We use the bootstrapping argument (see, e.g., [13]). To initiate the argument, we first construct a suitable energy functional $E(t)$. In order for the estimates involving E to be self-contained, we need to include two pieces $E_1(t)$ and $E_2(t)$,

$$E(t) = E_1(t) + E_2(t).$$

The first piece $E_1(t)$ includes the H^2-norm of (u, θ) and the corresponding time integral part due to the partial dissipation, namely

$$E_1(t) := \max_{0 \le \tau \le t} (\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau.$$

The second piece $E_2(t)$ comes from the extra smoothing reflected in the wave equation (1.10). As aforementioned, the wave term $\partial_{11}\Delta^{-1}\theta$ in (1.10) provides a weak horizontal smoothing for θ, which complements the vertical thermal diffusion. Therefore,

$$E_2(t) := \int_0^t \|\partial_1 \theta\|_{L^2}^2 \, d\tau.$$

Our main efforts are then devoted to proving the *a priori* energy inequality, for $t > 0$,

$$E(t) \le c_1 E(0) + c_2 E(t)^{\frac{3}{2}}. \tag{1.13}$$

The proof of (3.1) is naturally divided into two parts. The first part focuses on the estimate of E_1 and we obtain

$$E_1 \leq E_1(0) + c_3\, E_1(t)^{\frac{3}{2}} + c_4\, E_2(t)^{\frac{3}{2}}. \tag{1.14}$$

The second part proves

$$E_2 \leq c_5 E_1(0) + c_6\, E_1(t) + c_7\, E_1(t)^{\frac{3}{2}} + c_8\, E_2(t)^{\frac{3}{2}}, \tag{1.15}$$

where c_1 through c_8 are all constants. Adding (1.14) to a suitable multiple of (1.15) yields (1.13). An application of the bootstrapping argument to (1.13) then implies the desired global bound. The proofs of (1.14) and (1.15) involves various techniques such as the aforementioned orthogonal decomposition, Poincaré type inequalities and various anisotropic inequalities. We slightly elaborate on the estimate of (1.15). Due to the lack of the horizontal thermal diffusion, the time integral of $\|\partial_1\theta\|_{L^2}^2$ can not be bounded via the equation of θ. The strategy here is to make use of the vorticity equation,

$$\partial_1\theta = \partial_t\omega + u\cdot\nabla\omega - \nu\partial_{11}\omega.$$

We shift the time integrability of $\|\partial_1\theta\|_{L^2}^2$ to other terms involving the velocity and the vorticity

$$\begin{aligned}\int_0^t \|\partial_1\theta\|_{L^2}^2\, d\tau &= \int_0^t\int \partial_1\theta\,\partial_t\omega\, dxd\tau \\ &\quad - \nu\int_0^t\int \partial_1\theta\,\partial_{11}\omega\, dxd\tau + \int_0^t\int \partial_1\theta\,(u\cdot\nabla\omega)dx\, d\tau.\end{aligned}$$

We further transfer the time derivative in the first term on the right from $\partial_t\omega$ and invoke the equation of θ,

$$\int_0^t\int \partial_1\theta\,\partial_t\omega\, dxd\tau = \int \partial_1\,\theta(t)\omega(t)dx - \int \partial_1\,\theta_0\omega_0 dx - \int_0^t\int \omega\,\partial_1\partial_t\theta\, dx\, d\tau.$$

This process generates many terms, but fortunately we are able to prove (1.15) after a lengthy estimates of all the terms.

To prove the algebraic decay rates on the H^1-norm of the oscillation part stated in Theorem 1.2, we write the system governing $(\widetilde{u}, \widetilde{\theta})$ by first taking the horizontal average of (1.2) and then taking the difference of (1.2) and the horizontal average,

$$\begin{cases}\partial_t\widetilde{u} + \widetilde{u\cdot\nabla\widetilde{u}} + \widetilde{u_2}\partial_2\overline{u} - \nu\partial_1^2\widetilde{u} + \nabla\widetilde{p} = \widetilde{\theta}e_2\,,\\ \partial_t\widetilde{\theta} + \widetilde{u\cdot\nabla\widetilde{\theta}} + \widetilde{u_2}\partial_2\overline{\theta} - \eta\partial_2^2\widetilde{\theta} + \widetilde{u_2} = 0.\end{cases} \tag{1.16}$$

The estimate of the H^1-norm is naturally divided into the computations of $\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}$ and $\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}$. One main difficulty is that the equation of $\widetilde{\theta}$ has only vertical diffusion, but the aforementioned Poincaré inequality can only bound a function in terms of its horizontal derivatives. As a consequence of this disparity, some of the nonlinear parts related to $\widetilde{\theta}$ can not be bounded appropriately and require the upper bounds involving $\|\widetilde{\theta}\|_{L^2}$. To deal with these terms, we seek extra smoothing and stabilizing effect on $\widetilde{\theta}$ by exploiting the coupling in (1.16). More precisely, we include the following extra term along with the H^1-norm to form a Lyapunov functional,

$$-\delta(\widetilde{u}_2, \widetilde{\theta}),$$

where $\delta > 0$ is a small constant and $(\widetilde{u}_2, \widetilde{\theta})$ denotes the L^2-inner product. The time derivative of this inner product generates $\delta\|\widetilde{\theta}\|_{L^2}^2$, which help balance $\|\widetilde{\theta}\|_{L^2}^2$ from the nonlinearity. After a long process of estimating the nonlinear terms, we are able to establish the following energy inequality

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta})\Big) + \nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 \leq 0.$$

Especially, this inequality implies that the H^1-norm satisfies, for any $0 \leq s < t$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2). \tag{1.17}$$

In addition, we obtain the time integrability of $\|\widetilde{\theta}\|_{L^2}^2$,

$$\int_0^\infty \|\widetilde{\theta}(t)\|_{L^2}^2\, dt < \infty,$$

which, together with the time integrability bounds from Theorem 1.1, implies the time integrability of $\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2$,

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\, dt < \infty. \tag{1.18}$$

A elementary lemma applied to (1.17) and (1.18) leads to the desired algebraic decay and the asymptotic behavior as $t \to \infty$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq C\,(1+t)^{-\frac{1}{2}}, \quad t(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

The rest of this paper is divided into three main sections. Section 2 serves as a preparation for the proofs of Theorems 1.1 and 1.2. It lists several crucial properties on the orthogonal decomposition such as the Poincaré type inequality for the oscillation part $\widetilde{f}$. In addition, it also provides anisotropic inequalities involving triple products defined on the domain Ω. Section 3 presents the proof of Theorem 1.1. This section is further divided into three subsections. Two of the subsections are devoted to the proofs (1.14) and (1.15). Section 4 proves Theorem 1.2.

2. Preliminaries

This section makes several preparations. The first few lemmas are related to the decomposition (1.12). Lemma 2.1 provides basic properties of the decomposition (1.12) while Lemma 2.2 compares the 1D Sobolev inequalities on the whole line $\mathbb{R}$ and on bounded domains. Lemma 2.3 presents anisotropic upper bounds for triple products as well as for the L^∞-norm on the domain Ω. Lemmas 2.4 and 2.5 contain strong versions of the Poincaré type inequalities for the oscillation part and anisotropic upper bounds when only the oscillation part is involved.

The first lemma provides several properties of $\overline{f}$ and $\widetilde{f}$ to be used in the proof of our main results.

Lemma 2.1. *Let $\Omega = \mathbb{T}\times\mathbb{R}$. Assume that f defined on Ω is sufficiently regular, say $f \in H^2(\Omega)$. Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.11) and (1.12). Then*

(a) *The average operator $\overline{f}$ and the oscillation operator $\widetilde{f}$ commute with partial derivatives,*

$$\overline{\partial_1 f} = \partial_1\overline{f} = 0, \quad \overline{\partial_2 f} = \partial_2\overline{f}, \quad \widetilde{\partial_1 f} = \partial_1\widetilde{f}, \quad \widetilde{\partial_2 f} = \partial_2\widetilde{f}, \quad \overline{\widetilde{f}} = 0.$$

(b) *If f is a divergence-free vector field, namely $\nabla\cdot f = 0$, then $\overline{f}$ and $\widetilde{f}$ are also divergence-free,*

$$\nabla\cdot\overline{f} = 0 \quad \textit{and} \quad \nabla\cdot\widetilde{f} = 0.$$

(c) *$\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k$ for any integer $k \geq 0$, namely*

$$(\overline{f}, \widetilde{f})_{\dot{H}^k(\Omega)} := \int_\Omega \overline{D^k f}\cdot\widetilde{D^k f}\,dx = 0, \quad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}.$$

In particular,

$$\|\overline{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)} \quad \textit{and} \quad \|\widetilde{f}\|_{\dot{H}^k(\Omega)} \leq \|f\|_{\dot{H}^k(\Omega)}.$$

The orthogonality is actually more general and holds for any integrable functions,

$$\int_\Omega \overline{f}\cdot\widetilde{g}\,dx = 0.$$

The properties given in Lemma 2.1 can be easily verified via (1.11) and (1.12).

The second lemma makes a comparison between the elementary 1D inequality on the whole line $\mathbb{R}$ and its bounded domain version.

Lemma 2.2. *For any 1D function $f \in H^1(\mathbb{R})$,*

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\,\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\|f'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}.$$

For any bounded domain such as $\mathbb{T} = [0,1]$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \leq \sqrt{2}\,\|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})}\,\|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})},$$

in particular, if the function f has mean zero such as the oscillation part $\tilde{f}$,

$$\|f\|_{L^\infty(\mathbb{T})} \leq C\,\|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})}\,\|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})}.$$

The next lemma provides anisotropic upper bounds for triple products and for the L^∞-norm of a 2D function. Anisotropic Sobolev inequalities have become a necessary tool in the study of anisotropic equations. The whole space version of these type of inequalities has previously been used in [5] in the 2D cases and in [14] in the 3D case.

Lemma 2.3. *Let $\Omega = \mathbb{T}\times\mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega fgh\,dx\Big| \le C\|f\|_{L^2}^{\frac12}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}\,. \tag{2.1}$$

For any $f \in H^2(\Omega)$, we have

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_{L^2}^{\frac14}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14} \times(\|\partial_2 f\|_{L^2}+\|\partial_1\partial_2 f\|_{L^2})^{\frac14}.$$

When f in Lemma 2.3 is replaced by the oscillation part $\tilde{f}$, the lower-order part in (2.1) can be removed.

Lemma 2.4. *Let $\Omega = \mathbb{T}\times\mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega \widetilde{f}gh\,dx\Big| \le C\|\widetilde{f}\|_{L^2}^{\frac12}\|\partial_1\widetilde{f}\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}. \tag{2.2}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C\|\widetilde{f}\|_{L^2}^{\frac14}\|\partial_1\widetilde{f}\|_{L^2}^{\frac14}\|\partial_2\widetilde{f}\|_{L^2}^{\frac14}\|\partial_1\partial_2\widetilde{f}\|_{L^2}^{\frac14}\,.$$

Lemma 2.5. *Let $\overline{f}$ and $\widetilde{f}$ be defined as in (1.11) and (1.12). If $\|\partial_1\widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\|\partial_1\widetilde{f}\|_{L^2(\Omega)},$$

where C is a pure constant. In addition, if $\|\partial_1\widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C\|\partial_1\widetilde{f}\|_{H^1(\Omega)}.$$

As a direct consequence of Lemma 2.5 and the inequality (2.2), one has

$$\Big|\int_\Omega \widetilde{f}gh\,dx\Big| \le C\|\partial_1\widetilde{f}\|_{L^2}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}. \tag{2.3}$$

The last lemma provides an explicit decay rate in (2.5) for functions that are integrable and are decreasing in a general sense, namely (2.4).

Lemma 2.6. *Let $f = f(t)$ be a nonnegative function satisfying , for two constants $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty f(\tau)d\tau < C_0 \quad \text{and} \quad f(t) \le C_1 f(s) \quad \text{for any} \quad 0 \le s < t. \tag{2.4}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$ and for any $t > 0$,

$$f(t) \le C_2(1+t)^{-1}. \tag{2.5}$$

Furthermore, $f(t)$ has the following large-time asymptotic behavior,

$$\lim_{t\to\infty} t\,f(t) = 0.$$

3. The H^2 nonlinear stability

This section is devoted to the proof of Theorem 1.1, which asserts the global existence, uniqueness and stability of solutions to (1.2).

Proof of Theorem 1.1. Since the local well-posedness of (1.2) follows from a standard procedure (see, e.g., [10]), our attention is focused on the global H^2-bound of the solution (u, θ). We use the bootstrapping argument. To set up the argument, we first define a suitable energy functional $E(t)$. In order for the estimate involving E to be self-contained, we need to include two pieces, $E_1(t)$ and $E_2(t)$

$$E(t) = E_1(t) + E_2(t).$$

The first piece $E_1(t)$ includes the H^2-norm of (u, θ) and the corresponding time integral part due to the partial dissipation, namely

$$E_1(t) := \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\eta\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau.$$

The second piece $E_2(t)$ comes from the extra smoothing reflected in the wave equation (1.10). As aforementioned, the wave term $\partial_{11}\Delta^{-1}\theta$ in (1.10) provides a weak horizontal smoothing for θ, which complements the vertical thermal diffusion. Therefore,

$$E_2(t) := \int_0^t \|\partial_1\theta\|_{L^2}^2\, d\tau.$$

Our main efforts are then devoted to proving the *a priori* energy inequality, for $t > 0$,

$$E(t) \le c_1 E(0) + c_2 E(t)^{\frac{3}{2}}. \tag{3.1}$$

The proof of (3.1) consists of two main parts. The first part focuses on the estimate of E_1 and we obtain

$$E_1 \le E_1(0) + c_3\, E_1(t)^{\frac{3}{2}} + c_4\, E_2(t)^{\frac{3}{2}}. \tag{3.2}$$

The second part proves

$$E_2 \le c_5 E_1(0) + c_6\, E_1(t) + c_7\, E_1(t)^{\frac{3}{2}} + c_8\, E_2(t)^{\frac{3}{2}}, \tag{3.3}$$

where c_1 through c_8 are all constants. Adding (3.2) with $1/(2c_6)$ of (3.3) yields the desired inequality in (3.1). The bootstrapping argument applied to (3.1) then yields the desired global H^2-bound on (u, θ). We provide more details. We set the initial data (u_0, θ_0) to be sufficiently small, say

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon := \frac{1}{4\sqrt{c_1}c_2}.$$

If we make the ansatz that

$$E(t) \le \frac{1}{4c_2^2}, \tag{3.4}$$

then (3.1) implies

$$E(t) \le c_1 E(0) + \frac{1}{2}E(t), \qquad \frac{1}{2}E(t) \le c_1 E(0)$$

or

$$E(t) \le 2c_1 E(0) \le 2c_1 \|(u_0,\theta_0)\|_{H^2}^2 \le \frac{2c_1}{16c_1c_2^2} = \frac{1}{8c_2^2}. \tag{3.5}$$

Since the bound in (3.5) is smaller than in the ansatz (3.4), the bootstrapping argument then implies that, for any $t>0$,

$$E(t) \le \frac{1}{8c_2^2} \quad \text{or} \quad \|(u(t),\theta(t))\|_{H^2} \le \sqrt{2c_1}\,\varepsilon,$$

which yields the desired global bound and stability.

For the sake of clarity, the rest of this section is divided into three subsections. The first two subsections prove (3.2) and (3.3), respectively, while the third subsection shows the uniqueness of the solutions. □

3.1. Proof of (3.2)

This subsection proves (3.2). We start with the global L^2-bound

$$\|(u(t),\theta(t))\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1 u\|_{L^2}^2 d\tau + 2\eta\int_0^t \|\partial_2\theta\|_{L^2}^2 d\tau = \|(u_0,\theta_0)\|_{L^2}^2. \tag{3.6}$$

Next we estimate the H^1-norm via the temperature equation and the vorticity equation

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu\partial_{11}\omega + \partial_1\theta, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\partial_{22}\theta. \end{cases} \tag{3.7}$$

Taking the inner product of $(\omega,\nabla\theta)$ with (3.7), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|\partial_1\omega\|_{L^2}^2 + \eta\|\partial_2\nabla\theta\|_{L^2}^2 \\ &\quad = -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,dx + \int(\partial_1\theta\cdot\omega - \nabla u_2\cdot\nabla\theta)\,dx \\ &\quad := I_1 + I_2. \end{aligned} \tag{3.8}$$

Due to $\nabla\cdot u = 0$, there exists a stream function ψ so that $u = \nabla^\perp\psi = (-\partial_2\psi,\partial_1\psi)$ and $\Delta\psi = \omega$. Hence

$$\begin{aligned} I_2 :&= \int(\partial_1\theta\cdot\omega - \nabla u_2\cdot\nabla\theta)dx = \int(\partial_1\theta\Delta\psi - \nabla\partial_1\psi\cdot\nabla\theta)dx \\ &= \int(-\theta\Delta\partial_1\psi + \Delta\partial_1\psi\theta)dx \\ &= 0. \end{aligned} \tag{3.9}$$

To make use of the anisotropic dissipation, we further split I_1 into four terms,

$$\begin{aligned} I_1 :&= -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,dx \\ &= -\int \partial_1 u_1(\partial_1\theta)^2 dx - \int \partial_1 u_2\partial_1\theta\partial_2\theta dx \\ &\quad - \int \partial_2 u_1\partial_1\theta\partial_2\theta dx - \int \partial_2 u_2(\partial_2\theta)^2 dx \\ &:= I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \tag{3.10}$$

The goal here is to obtain upper bounds that are time-integrable. By $\nabla\cdot u = 0$, integration by parts, Lemma 2.4 and Young's inequaliy

$$
\begin{aligned}
I_{11} :&= -\int \partial_1 u_1 (\partial_1\theta)^2 dx = -2\int u_2 \partial_1\theta\partial_1\partial_2\theta dx\\
&\le c\,\|\partial_1\partial_1\partial_2\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\partial_1\theta\|_{L^2}^{\frac12}\|u_2\|_{L^2}\\
&\le c\,\|\partial_2\theta\|_{H^2}^{\frac32}\|\partial_1\theta\|_{L^2}^{\frac12}\|u\|_{H^2}\\
&\le c\,\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big).
\end{aligned}\tag{3.11}
$$

By Lemmas 2.1 and 2.5,

$$
\begin{aligned}
I_{12} :&= -\int \partial_1 u_2 \partial_1\theta\partial_2\theta dx = -\int \partial_1\widetilde{u_2}\partial_1\widetilde{\theta}\partial_2\theta dx\\
&\le c\,\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}\\
&\le c\,\|\partial_1 u\|_{H^2}\|\partial_2\theta\|_{H^2}\|\theta\|_{H^2}\\
&\le c\,\|\theta\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned}\tag{3.12}
$$

I_{13} contains two terms with "bad" derivatives $\partial_2 u_1$ and $\partial_1\theta$, so we need to invoke the decompositions $u = \tilde{u} + \bar{u}$ and $\theta = \widetilde{\theta} + \overline{\theta}$,

$$
\begin{aligned}
I_{13} :&= -\int \partial_2 u_1 \partial_1\theta\partial_2\theta dx\\
&= -\int \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta} dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta} dx\\
&\quad - \int \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta} dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta} dx\\
&:= I_{131} + I_{132} + I_{133} + I_{134}.
\end{aligned}\tag{3.13}
$$

Due to Lemma 2.1,

$$
I_{131} := -\int_\Omega \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta} dx = \int_{\mathbb{R}} \partial_2\overline{u_1}\partial_2\overline{\theta}\int_{\mathbb{T}} \partial_1\widetilde{\theta}\, dx_1 dx_2 = \int_{\mathbb{R}} \partial_2\overline{u_1}\partial_2\overline{\theta}\partial_1\overline{\widetilde{\theta}} dx_2 = 0. \tag{3.14}
$$

According to Lemma 2.4 and Young's inequaliy

$$
\begin{aligned}
I_{132} :&= -\int_\Omega \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta} dx\\
&\le c\,\|\partial_2\overline{\theta}\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le c\,\|\partial_2\theta\|_{H^2}\|u\|_{H^2}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac12}\\
&\le c\,\|\partial_2\theta\|_{H^2}^{\frac32}\|\partial_1 u\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\\
&\le c\,\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big).
\end{aligned}\tag{3.15}
$$

Similarly,

$$\begin{aligned}
I_{133} &= -\int_\Omega \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx\\
&\le c\|\partial_2\overline{u_1}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac32}\\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned}\tag{3.16}$$

I_{134} can be similarly bounded as I_{133}. In fact

$$\begin{aligned}
I_{134} &:= -\int_\Omega \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx\\
&\le c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_1}\|_{L^2}\\
&\le c\|u\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac32}\\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned}\tag{3.17}$$

Inserting (3.14), (3.15), (3.16) and (3.17) in (3.13), we get

$$I_{13}\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big).\tag{3.18}$$

By $\nabla\cdot u=0$, and Lemmas 2.1 and 2.4,

$$\begin{aligned}
I_{14} : &= -\int \partial_2 u_2(\partial_2\theta)^2dx = \int \partial_1\widetilde{u_1}(\partial_2\theta)^2dx\\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}\\
&\le \|\theta\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_2\theta\|_{H^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned}\tag{3.19}$$

Inserting the bounds in (3.11), (3.12), (3.18), (3.19) in (3.10) leads to

$$I_1\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big).\tag{3.20}$$

By (3.20), (3.9) and (3.8) and the fact $\|\omega\|_{L^2}=\|\nabla u\|_{L^2}$,

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\eta\|\partial_2\nabla\theta\|_{L^2}^2+\nu\|\partial_1\omega\|_{L^2}^2\\
&\qquad\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big).
\end{aligned}\tag{3.21}$$

Integrating (3.21) in time over $[0,t]$ yields,

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+2\eta\int_0^t\|\partial_2\nabla\theta\|_{L^2}^2d\tau+2\nu\int_0^t\|\partial_1 w\|_{L^2}^2d\tau\\
&\qquad\le\|\nabla u_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2+c\,E_1(t)^{\frac32}+c\,E_2(t)^{\frac32}.
\end{aligned}\tag{3.22}$$

Applying ∇ to the first equation of (3.7) and dotting with $\nabla\omega$ and applying Δ to the second equation of (3.7) and dotting with $\Delta\theta$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)+\eta\|\partial_2\nabla\theta\|_{L^2}^2+\nu\|\partial_1\nabla\omega\|_{L^2}^2\\
&\quad=-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx-\int\Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx+\int(\nabla\partial_1\theta\cdot\nabla\omega-\Delta u_2\cdot\Delta\theta)dx\\
&\quad:=J_1+J_2+J_3.
\end{aligned}\tag{3.23}$$

Since $\nabla\cdot u=0$, there exists a stream function ψ such that we can write $u=\nabla^\perp\psi=(-\partial_2\psi,\partial_1\psi)$ and $\Delta\psi=\omega$. Hence,

$$\begin{aligned}
J_3:&=\int(\nabla\partial_1\theta\cdot\nabla\omega-\Delta u_2\Delta\theta)\,dx=\int(\nabla\partial_1\theta\cdot\nabla\omega-\Delta\partial_1\psi\Delta\theta)\,dx\\
&=\int(\nabla\partial_1\theta\cdot\nabla\omega-\partial_1\omega\,\Delta\theta)\,dx=\int(\nabla\partial_1\theta\cdot\nabla\omega+\partial_1\nabla\omega\ \cdot\nabla\theta)\,dx\\
&=\int\partial_1(\nabla\theta\cdot\nabla\omega)\,dx=0.
\end{aligned}$$

Integrating by parts, one can write J_2 as follows

$$\begin{aligned}
J_2:&=-\int\Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx\\
&=-\int\Delta\theta\Delta u_1\partial_1\theta dx-\int\Delta\theta\Delta u_2\partial_2\theta dx\\
&\quad-2\int\Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx-2\int\Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx\\
&:=J_{21}+J_{22}+J_{23}+J_{24}.
\end{aligned}\tag{3.24}$$

To deal with J_{21}, we invoke the decompositions $u=\overline{u}+\widetilde{u}$ and $\theta=\overline{\theta}+\widetilde{\theta}$ to write

$$\begin{aligned}
J_{21}:&=-\int\Delta\theta\Delta u_1\partial_1\theta dx=-\int\Delta\theta\Delta u_1\partial_1\widetilde{\theta}dx\\
&=-\int\Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx-\int\Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx\\
&\quad-\int\Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx-\int\Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx\\
&:=J_{211}+J_{212}+J_{213}+J_{214}.
\end{aligned}\tag{3.25}$$

According to Lemma 2.1,

$$J_{211}:=-\int\Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx=\int_{\mathbb{R}}\Delta\overline{u_1}\Delta\overline{\theta}\int_{\mathbb{T}}\partial_1\widetilde{\theta}\,dx_1dx_2=\int_{\mathbb{R}}\Delta\overline{u_1}\Delta\overline{\theta}\partial_1\overline{\widetilde{\theta}}dx_2=0.\tag{3.26}$$

Furthermore, J_{212} can be written explicitly as

$$\begin{aligned} J_{212} &:= -\int \Delta\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\ &= -\int \partial_{11}\overline{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx - \int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{11}\widetilde{\theta}dx - \int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{22}\widetilde{\theta}dx \\ &:= J_{2121} + J_{2122} + J_{2123}. \end{aligned} \tag{3.27}$$

Due to Lemma 2.1,

$$J_{2121} := -\int \underbrace{\partial_{11}\overline{u_1}}_{=0}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx = 0. \tag{3.28}$$

Integrating by parts and using Lemma 2.1 yield

$$\begin{aligned} J_{2122} &:= -\int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{11}\widetilde{\theta}dx = -\frac{1}{2}\int \partial_{22}\overline{u_1}\partial_1(\partial_1\widetilde{\theta})^2dx \\ &= \frac{1}{2}\int \partial_{22}\underbrace{\partial_1\overline{u_1}}_{=0}(\partial_1\widetilde{\theta})^2dx = 0. \end{aligned} \tag{3.29}$$

It follows from Lemma 2.4 and Young's inequality that

$$\begin{aligned} J_{2123} &:= -\int \partial_{22}\overline{u_1}\partial_1\widetilde{\theta}\partial_{22}\widetilde{\theta}dx \\ &\le c\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_{22}\overline{u_1}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \end{aligned} \tag{3.30}$$

Inserting the bounds in (3.28), (3.29) and (3.30) in (3.27) yields

$$J_{212} \le c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \tag{3.31}$$

By Lemmas 2.1 and 2.4,

$$\begin{aligned} J_{213} &:= -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx = -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\partial_{22}\overline{\theta}dx \\ &\le c\|\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.32}$$

Applying Lemma 2.4, we have

$$
\begin{aligned}
J_{214} &:= -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\
&\le c\|\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big).
\end{aligned} \tag{3.33}
$$

Collecting (3.26), (3.31), (3.32), (3.33) and inserting them in (3.25), we get

$$
J_{21} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big). \tag{3.34}
$$

Using the decompositions of u and θ, we can split J_{22} into four terms,

$$
\begin{aligned}
J_{22} &:= -\int \Delta\theta\Delta u_2\partial_2\theta dx \\
&= -\int \Delta\overline{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&\quad -\int \Delta\widetilde{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\widetilde{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&:= J_{221}+J_{222}+J_{223}+J_{224}.
\end{aligned} \tag{3.35}
$$

We start with J_{221}. By the divergence free condition of u, and Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{221} &:= -\int \Delta\overline{\theta}\Delta\overline{u_2}\partial_2\theta dx = -\int \partial_{22}\overline{\theta}\partial_{22}\overline{u_2}\partial_2\theta dx \\
&= \int \partial_{22}\overline{\theta}\partial_2\underbrace{\partial_1\overline{u_1}}_{=0}\partial_2\theta dx = 0.
\end{aligned} \tag{3.36}
$$

Similarly,

$$
J_{223} := -\int \Delta\widetilde{\theta}\Delta\overline{u_2}\partial_2\theta dx = 0. \tag{3.37}
$$

According to Lemmas 2.1 and 2.4 and Young's inequality,

$$
\begin{aligned}
J_{222} &:= -\int \Delta\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&= -\int \partial_2\partial_2\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&\le c\|\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.38}
$$

By Lemma 2.4 and Young's inequality,

$$
\begin{aligned}
J_{224} &:= -\int \Delta\widetilde{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&\le c\|\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.39)
\end{aligned}
$$

Inserting (3.36), (3.37), (3.38) and (3.39) in (3.35), we obtain

$$
J_{22} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.40)
$$

To bound J_{23}, we start by writing it into a summation of four integrals,

$$
\begin{aligned}
J_{23} &:= -2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx \\
&= -2\int \Delta\theta\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\theta\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\overline{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\
&\quad - 2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx - 2\int \Delta\overline{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&:= J_{231}+J_{232}+J_{233}+J_{234}. \qquad (3.41)
\end{aligned}
$$

Using Lemma 2.1, we can write J_{231} as

$$
\begin{aligned}
J_{231} &:= -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx = -2\int \Delta\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&= -2\int \partial_1\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&:= J_{2311}+J_{2312}. \qquad (3.42)
\end{aligned}
$$

By $\nabla\cdot u=0$, integration by parts and Lemma 2.4,

$$
\begin{aligned}
J_{2311} &:= -2\int \partial_1\widetilde{u_1}(\partial_1\partial_1\widetilde{\theta})^2 dx = 2\int \partial_2\widetilde{u_2}(\partial_1\partial_1\widetilde{\theta})^2 dx \\
&= -4\int \widetilde{u_2}\partial_1\partial_1\widetilde{\theta}\partial_2\partial_1\partial_1\widetilde{\theta} dx \\
&\le c\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.43)
\end{aligned}
$$

By Lemma 2.4 and Young's inequality,

$$\begin{aligned} J_{2312} &:= -2\int \partial_2\partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta}dx \\ &\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.44}$$

Collecting (3.43) and (3.44) and inserting them into (3.42), we get

$$J_{231} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \tag{3.45}$$

To bound J_{232}, we use Lemmas 2.1 and 2.4 to obtain

$$\begin{aligned} J_{232} &:= -2\int \Delta\overline{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\ &= -2\int \partial_2\partial_2\overline{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta}dx \\ &\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.46}$$

By the decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$,

$$\begin{aligned} J_{233} &:= -2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\ &= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \Delta\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\ &= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\ &= J_{2331} + J_{2332} + J_{2333}. \end{aligned} \tag{3.47}$$

According to Lemma 2.4 and Young's inequality,

$$\begin{aligned} J_{2331} &:= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx \\ &\le c\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\ &\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.48}$$

Using integration by parts, Lemma 2.1, Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
J_{2332} &:= -2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= 2\int \partial_1\widetilde{\theta}\underbrace{\partial_1\partial_2\overline{u_1}}_{=0}\partial_1\partial_2\widetilde{\theta}dx + 2\int \partial_1\widetilde{\theta}(\partial_2\overline{u_1}\partial_1\partial_1\partial_2\widetilde{\theta})dx \\
&= 2\int \partial_1\widetilde{\theta}(\partial_2\overline{u_1}\partial_1\partial_1\partial_2\widetilde{\theta})dx \\
&= 2\int_{\mathbb{R}} \partial_2\overline{u_1}\Big(\int_{\mathbb{T}} \partial_1\widetilde{\theta}(\partial_1\partial_1\partial_2\widetilde{\theta})dx_1\Big)dx_2 \\
&\le 2\int_{\mathbb{R}} |\partial_2\overline{u_1}|\|\partial_1\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}dx_2 \\
&\le 2\|\partial_2\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} \\
&\le c\|\partial_2\overline{u_1}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|^2_{L^2} + \|\partial_2\theta\|^2_{H^2}\Big). \qquad (3.49)
\end{aligned}
$$

Due to Lemma 2.4,

$$
\begin{aligned}
J_{2333} &:= -2\int \partial_2\partial_2\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\overline{u_1}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_2\theta\|^2_{H^2}. \qquad (3.50)
\end{aligned}
$$

Combining (3.48), (3.49), (3.50) and inserting them in (3.47), we get

$$
J_{23} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|^2_{H^2} + \|\partial_2\theta\|^2_{H^2} + \|\partial_1\theta\|^2_{L^2}\Big). \qquad (3.51)
$$

By Lemmas 2.1 and 2.4,

$$
\begin{aligned}
J_{234} &:= -2\int \Delta\overline{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \partial_2\partial_2\overline{\theta}\partial_2 u_1\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\overline{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\partial_2\overline{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2 u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_2\theta\|^2_{H^2}. \qquad (3.52)
\end{aligned}
$$

Inserting (3.45), (3.46), (3.51) and (3.52) in (3.41), we obtain

$$
J_{23} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|^2_{H^2} + \|\partial_1\theta\|^2_{L^2} + \|\partial_2\theta\|^2_{H^2}\Big). \qquad (3.53)
$$

By $u=\overline{u}+\widetilde{u}$ and $\theta=\overline{\theta}+\widetilde{\theta}$ and Lemma 2.1,

$$\begin{aligned}
J_{24} &:= -2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx\\
&= -2\int(\partial_1 u_2\partial_1\partial_2\theta\Delta\theta+\partial_2 u_2\partial_2\partial_2\theta\Delta\theta)dx\\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta-2\int\partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx\\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta dx-2\int\partial_2 \overline{u_2}\partial_2\partial_2\theta\Delta\theta dx-2\int\partial_2 \widetilde{u_2}\partial_2\partial_2\theta\Delta\theta dx\\
&:= J_{241}+J_{242}+J_{243}. \qquad (3.54)
\end{aligned}$$

We start with J_{241}. By Lemma 2.4 and Young's inequality we have

$$\begin{aligned}
J_{241} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta dx\\
&\le c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\Delta\theta\|_{L^2}^{\frac12}\|\partial_2\Delta\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}\\
&\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac32}\\
&\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.55)
\end{aligned}$$

Next, using the divergence free condition of u and Lemma 2.1,

$$J_{242} := -2\int \partial_2\overline{u_2}\partial_2\partial_2\theta\Delta\theta dx = 2\int\partial_1\overline{u_1}\partial_2\partial_2\theta\Delta\theta dx=0. \qquad (3.56)$$

According to Lemma 2.4 and Young's inequality,

$$\begin{aligned}
J_{244} &:= -2\int \partial_2\widetilde{u_2}\partial_2\partial_2\theta\Delta\theta dx\\
&\le c\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\Delta\theta\|_{L^2}^{\frac12}\|\partial_2\Delta\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\theta\|_{L^2}\\
&\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac32}\\
&\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.57)
\end{aligned}$$

Collecting (3.55), (3.56), and (3.57) and inserting them in (3.54), we obtain

$$J_{24}\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.58)$$

Thus, by (3.34), (3.40), (3.53), (3.58), and (3.24),

$$J_2\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (3.59)$$

It remains to bound J_1. To do so, we split it into four integrals

$$\begin{aligned} J_1 &:= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx \\ &= -\int \partial_1 u_1(\partial_1\omega)^2\,dx - \int \partial_1 u_2\partial_1\omega\partial_2\omega\,dx \\ &\quad -\int \partial_2 u_1\partial_1\omega\partial_2\omega\,dx - \int \partial_2 u_2(\partial_2\omega)^2\,dx \\ &:= J_{11}+J_{12}+J_{13}+J_{14}. \end{aligned} \tag{3.60}$$

Due to Lemmas 2.1 and 2.4,

$$\begin{aligned} J_{11} &:= -\int \partial_1 u_1(\partial_1\omega)^2\,dx \\ &= -\int \partial_1\widetilde{u_1}(\partial_1\widetilde{\omega})(\partial_1\widetilde{\omega})\,dx \\ &\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\omega}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.61}$$

According to Lemmas 2.1 and 2.4,

$$\begin{aligned} J_{12} &:= -\int \partial_1 u_2\partial_1\omega\partial_2\omega\,dx \\ &= -\int \partial_1\widetilde{u_2}\partial_1\widetilde{\omega}\partial_2\omega\,dx \\ &\le c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.62}$$

Making use of the orthogonal decomposition of u_1 and ω and Lemma 2.1, we can write J_{13} as

$$\begin{aligned} J_{13} &:= -\int \partial_2 u_1\partial_1\omega\partial_2\omega\,dx = -\int \partial_2 u_1\partial_1\widetilde{\omega}\partial_2\omega\,dx \\ &= -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\overline{\omega}\,dx - \int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\widetilde{\omega}\,dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx \\ &= J_{131}+J_{132}+J_{133}. \end{aligned} \tag{3.63}$$

According to Lemma 2.1,

$$J_{131} := -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\overline{\omega}\,dx = 0. \tag{3.64}$$

To bound J_{132} we use Lemma 2.4

$$\begin{aligned} J_{132} &:= -\int \partial_2\overline{u_1}\partial_1\widetilde{\omega}\partial_2\widetilde{\omega}\,dx \\ &\le c\|\partial_2\overline{u_1}\|_{L^2}\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12} \\ &\le c\|u\|_{H^2}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac32}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.65}$$

Similarly,

$$\begin{aligned} J_{133} &:= -\int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx \\ &\le c\|\partial_2\omega\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\nabla\omega\|_{L^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.66}$$

Thus, by (3.64), (3.65), (3.66), and (3.63),

$$J_{13} \le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \tag{3.67}$$

Due to $\nabla\cdot u = 0$, Lemma 2.1, and the inequality (2.3)

$$\begin{aligned} J_{14} &:= -\int \partial_2 u_2(\partial_2\omega)^2\,dx \\ &= -\int \partial_1\widetilde{u_1}(\partial_2\overline{\omega}+\partial_2\widetilde{\omega})^2\,dx \\ &= -2\int \partial_1\widetilde{u_1}\partial_2\overline{\omega}\partial_2\widetilde{\omega}\,dx - 2\int \partial_1\widetilde{u_1}(\partial_2\widetilde{\omega})^2\,dx \\ &\le c\Big(\|\partial_2\overline{\omega}\|_{L^2}+\|\partial_2\widetilde{\omega}\|_{L^2}\Big)\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.68}$$

Collecting the results obtained in (3.61), (3.62), (3.67), (3.68) and inserting them in (3.60), we obtain

$$J_1 \le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2\,. \tag{3.69}$$

Combining the upper bounds in (3.59), (3.69) and inserting them in (3.23), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)+\eta\|\partial_2\nabla\theta\|_{L^2}^2+\nu\|\partial_1\nabla\omega\|_{L^2}^2 \\ &\quad\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.70}$$

Integrating (3.70) in time over [0,t], we get

$$\begin{aligned} &\|\nabla\omega\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2+2\eta\int_0^t\|\partial_2\nabla\theta\|_{L^2}^2\,d\tau+2\nu\int_0^t\|\partial_1\nabla\omega\|_{L^2}^2\,d\tau \\ &\quad\le c\int_0^t\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big)\,d\tau+\|\Delta u_0\|_{L^2}^2+\|\Delta\theta_0\|_{L^2}^2 \\ &\quad\le \|\Delta u_0\|_{L^2}^2+\|\Delta\theta_0\|_{L^2}^2+c\,E_1(t)^{\frac{3}{2}}+c\,E_2(t)^{\frac{3}{2}}. \end{aligned} \tag{3.71}$$

(3.2) then follows from (3.6), (3.22) and (3.71).

3.2. Proof of (3.3)

This subsection proves (3.3). We estimate the time integral of $\|\partial_1\theta\|_{L^2}^2$. Since the equation of θ has no horizontal dissipation, we need to use the coupling in the vorticity equation with the equation of θ,

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu\partial_{11}\omega + \partial_1\theta, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\partial_{22}\theta. \end{cases} \tag{3.72}$$

Dotting the first equation of (3.72) by $\partial_1\theta$ and then integrating in space, we get

$$\begin{aligned} \|\partial_1\theta\|_{L^2}^2 &= \int \partial_1\theta(\partial_t\omega - \nu\partial_{11}\omega + u\cdot\nabla\omega)dx \\ &= \frac{d}{dt}\int \partial_1\theta\omega dx - \int \omega\partial_1\partial_t\theta dx - \nu\int \partial_1\theta\partial_{11}\omega dx + \int \partial_1\theta(u\cdot\nabla\omega)dx \\ &:= A + B + C + D. \end{aligned}$$

Due to Hölder inequality and Cauchy's inequality, we have

$$\begin{aligned} \int_0^t A d\tau &:= \int_0^t \frac{d}{dt}\int \partial_1\theta\omega dx\, d\tau \\ &= \int \partial_1\theta(t)\omega(t)dx - \int \partial_1\theta_0\omega_0 dx \\ &\le \|\partial_1\theta\|_{L^2}\|\omega\|_{L^2} + \|\partial_1\theta_0\|_{L^2}\|\omega_0\|_{L^2} \\ &\le \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big). \end{aligned} \tag{3.73}$$

Integrating by parts and using the second equation in (3.72), we write B as

$$\begin{aligned} B &:= -\int \omega\,\partial_1\partial_t\theta\,dx = \int \partial_1\omega\,\partial_t\theta\,dx \\ &= \int \partial_1\omega\Big(\eta\partial_{22}\theta - u\cdot\nabla\theta - u_2\Big)dx \\ &= \eta\int \partial_1\omega\,\partial_2\partial_2\theta dx - \int \partial_1\omega\,u_2 dx - \int \partial_1\omega\,u\cdot\nabla\theta\,dx \\ &:= B_1 + B_2 + B_3. \end{aligned} \tag{3.74}$$

By Hölder's inequality,

$$B_1 := \eta\int \partial_1\omega\,\partial_2\partial_2\theta dx \le \eta\|\partial_1\omega\|_{L^2}\|\partial_2\partial_2\theta\|_{L^2} \le \|\partial_1 u\|_{H^2}^2 + \frac{\eta^2}{4}\|\partial_2\theta\|_{H^2}^2. \tag{3.75}$$

Integrating by parts and using Lemma 2.1 and 2.5, we have

$$\begin{aligned} B_2 &:= -\int \partial_1\omega u_2 dx = -\int \partial_1\widetilde{\omega}u_2 dx = \int \widetilde{\omega}\partial_1 u_2 dx \\ &\le \|\widetilde{\omega}\|_{L^2}\|\partial_1 u_2\|_{L^2} \le \|\partial_1\widetilde{\omega}\|_{L^2}\|\partial_1 u_2\|_{L^2} \le \|\partial_1 u\|_{H^2}^2. \end{aligned} \tag{3.76}$$

By Lemma 2.1, one can decompose B_3 as

$$\begin{aligned} B_3 &:= -\int \partial_1\omega\, u\cdot\nabla\theta dx \\ &= -\int \partial_1\widetilde{\omega} u_1\partial_1\widetilde{\theta} dx - \int \partial_1\widetilde{\omega} u_2\partial_2\theta dx \\ &:= B_{31} + B_{32}. \end{aligned} \tag{3.77}$$

Due to Lemma 2.4,

$$\begin{aligned} B_{31} &:= -\int \partial_1\widetilde{\omega} u_1\partial_1\widetilde{\theta} dx \\ &\le c\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|u_1\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac12} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_2\theta\|_{H^2}^{\frac12} \\ &\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.78}$$

Similarly,

$$\begin{aligned} B_{32} &:= -\int \partial_1\widetilde{\omega} u_2\partial_2\theta dx \\ &\le c\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\theta\|_{L^2}^{\frac12}\|\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|u_2\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_2\theta\|_{H^2} \\ &\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.79}$$

In view of (3.78), (3.79) and (3.77), we have

$$B_3 \le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{3.80}$$

Combining (3.75), (3.76), (3.80) and inserting them in (3.74) yield

$$B \le 2\|\partial_1 u\|_{H^2}^2 + \frac{\eta^2}{4}\|\partial_2\theta\|_{H^2}^2 + c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big).$$

Hence,

$$\begin{aligned} \int_0^t B d\tau &\le 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\eta^2}{4}\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\ &\quad + c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big)d\tau. \end{aligned} \tag{3.81}$$

To bound the integral C, we use both Hölder's inequality and Young's inequality

$$C := -\nu\int \partial_1\theta\partial_{11} w dx \le \nu\|\partial_1\theta\|_{L^2}\|\partial_{11} w\|_{L^2} \le \frac14\|\partial_1\theta\|_{L^2}^2 + \nu^2\|\partial_1 u\|_{H^2}^2.$$

Hence,

$$\int_0^t C d\tau \le \frac14\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + \nu^2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau. \tag{3.82}$$

Due to Lemma 2.1, D can be written as

$$
\begin{aligned}
D &:= \int \partial_1\theta(u\cdot\nabla\omega)dx = \int \partial_1\widetilde{\theta}(u\cdot\nabla\omega)dx \\
&= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1u_2dx - \int \partial_1\widetilde{\theta}u_1\partial_1\partial_2u_1dx \\
&\quad + \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1u_2dx - \int \partial_1\widetilde{\theta}u_2\partial_2\partial_2u_1dx \\
&:= D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{3.83}
$$

The integrals D_1 up to D_3 can be bounded as follows. By using Lemmas 2.1 and 2.4,

$$
\begin{aligned}
D_1 &:= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1u_2dx \\
&= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1\widetilde{u_2}dx \\
&\le c\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big),
\end{aligned} \tag{3.84}
$$

$$
\begin{aligned}
D_2 &:= -\int \partial_1\widetilde{\theta}u_1\partial_1\partial_2u_1dx \\
&= -\int \partial_1\widetilde{\theta}u_1\partial_1\partial_2\widetilde{u_1}dx \\
&\le c\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big),
\end{aligned} \tag{3.85}
$$

$$
\begin{aligned}
D_3 &:= \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1u_2dx \\
&= \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1\widetilde{u_2}dx \\
&\le c\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_2\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.86}
$$

Using the fact that $\overline{u_2} = 0$ and the inequality (2.3), D_4 can be bounded by

$$\begin{aligned} D_4 &:= -\int \partial_1\widetilde{\theta} u_2 \partial_2\partial_2 u_1 dx = -\int \partial_1\widetilde{\theta}\widetilde{u_2}\partial_2\partial_2 u_1 dx \\ &\le c\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_1\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.87}$$

In view of (3.83), collecting the bounds in (3.84), (3.85), (3.86) and (3.87), we get

$$D \le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)$$

Hence,

$$\begin{aligned} \int_0^t D\, d\tau &\le c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau \\ &\le cE_1(t)^{\frac{3}{2}} + cE_2(t)^{\frac{3}{2}}. \end{aligned} \tag{3.88}$$

Therefore, combining the estimates (3.73), (3.81), (3.82) and (3.88), we obtain

$$\begin{aligned} \int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau &\le \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big) \\ &\quad + 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\eta^2}{4}\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\ &\quad + \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + \nu^2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau \\ &\quad + c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau \\ &\le \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + cE_1(0) + cE_1(t) + cE_1(t)^{\frac{3}{2}} + cE_2(t)^{\frac{3}{2}}, \end{aligned}$$

which yields the desired inequality (3.3).

3.3. Uniqueness

For the sake of completeness, we provide the proof for the uniqueness part of Theorem 1.1, although the proof is not difficult. We show that two solutions $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ of (1.2) with one of them in the H^2-regularity class say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T, H^2(\Omega))$ must coincide. Their difference (u^*, p^*, θ^*) with $u^* = u^{(1)} - u^{(2)}$, $p^* = p^{(1)} - p^{(2)}$, $\theta^* = \theta^{(1)} - \theta^{(2)}$ satisfies, according to (1.2)

$$\begin{cases} \partial_t u^* + u^{(2)}\cdot\nabla u^* + u^*\cdot\nabla u^{(1)} + \nabla p^* = \nu\,\partial_{11}u^* + \theta^*\mathbf{e}_2, \\ \partial_t\theta^* + u^{(2)}\cdot\nabla\theta^* + u^*\cdot\nabla\theta^{(1)} + u_2^* = \eta\,\partial_{22}\theta^*, \\ \nabla\cdot u^* = 0, \\ u^*(x,0) = 0, \quad \theta^*(x,0) = 0. \end{cases} \tag{3.89}$$

We estimate the difference (u^*, p^*, θ^*) in $L^2(\Omega)$. Dotting (3.89) by (u^*, θ^*) and applying the divergence free condition of u^*, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2 &+ \nu\|\partial_1 u^*\|_{L^2}^2 + \eta\|\partial_2\theta^*\|_{L^2}^2\\ &= -\int u^*\cdot\nabla u^{(1)}\cdot u^*\,dx - \int u^*\cdot\nabla\theta^{(1)}\cdot\theta^*\,dx\\ &:= I_1 + I_2.\end{aligned}\tag{3.90}$$

By Lemma 2.3 and the uniformly global bound for $\|u^{(1)}\|_{H^2}$,

$$\begin{aligned}I_1 &:= -\int u^*\cdot\nabla u^{(1)}\cdot u^*\,dx\\ &\le c\|u^*\|_{L^2}^{\frac12}\Big(\|u^*\|_{L^2}+\|\partial_1 u^*\|_{L^2}\Big)^{\frac12}\underbrace{\|\nabla u^{(1)}\|_{L^2}^{\frac12}\|\partial_2\nabla u^{(1)}\|_{L^2}^{\frac12}}_{\le c}\|u^*\|_{L^2}\\ &\le c\|u^*\|_{L^2}\Big(\|u^*\|_{L^2}+\|\partial_1 u^*\|_{L^2}\Big)+\|u^*\|_{L^2}^2\\ &\le c\|u^*\|_{L^2}^2+\frac{\nu}{2}\|\partial_1 u^*\|_{L^2}^2.\end{aligned}\tag{3.91}$$

By Lemma 2.3 and the uniformly global bound for $\|\theta^{(1)}\|_{H^2}$,

$$\begin{aligned}I_2 &:= -\int u^*\cdot\nabla\theta^{(1)}\cdot\theta^*\,dx\\ &\le c\underbrace{\|\nabla\theta^{(1)}\|_{L^2}^{\frac12}\Big(\|\nabla\theta^{(1)}\|_{L^2}+\|\partial_1\nabla\theta^{(1)}\|_{L^2}\Big)^{\frac12}}_{\le c}\|\theta^*\|_{L^2}^{\frac12}\|\partial_2\theta^*\|_{L^2}^{\frac12}\|u^*\|_{L^2}\\ &\le c\|\theta^*\|_{L^2}^{\frac12}\|\partial_2\theta^*\|_{L^2}^{\frac12}\|u^*\|_{L^2}\\ &\le c\|u^*\|_{L^2}\Big(\|\theta^*\|_{L^2}+\|\partial_2\theta^*\|_{L^2}\Big)\\ &\le c\|u^*\|_{L^2}^2+c\|\theta^*\|_{L^2}^2+\frac{\eta}{2}\|\partial_2\theta^*\|_{L^2}^2.\end{aligned}\tag{3.92}$$

Inserting the estimates (3.91) and (3.92) in (3.90) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2 &+ \nu\|\partial_1 u^*\|_{L^2}^2 + \eta\|\partial_2\theta^*\|_{L^2}^2\\ &\le c\Big(\|u^*\|_{L^2}^2+\|\theta^*\|_{L^2}^2\Big)+\frac{\nu}{2}\|\partial_1 u^*\|_{L^2}^2+\frac{\eta}{2}\|\partial_2\theta^*\|_{L^2}^2\end{aligned}$$

or

$$\frac{d}{dt}\|(u^*,\theta^*)\|_{L^2}^2+\nu\|\partial_1 u^*\|_{L^2}^2+\eta\|\partial_2\theta^*\|_{L^2}^2\le c\|(u^*,\theta^*)\|_{L^2}^2.\tag{3.93}$$

Gronwall's inequality applied to (3.93) implies that $\|u^*\|_{L^2}^2 = \|\theta^*\|_{L^2}^2 = 0$. This completes the proof of Theorem 1.1. □

4. Decay rates result

This section proves Theorem 1.2, which asserts the algebraic decay rates for the H^1-norm of the oscillation part $(\widetilde{u}, \widetilde{\theta})$ of the solution to (1.2).

Proof of Theorem 1.2. We start the proof of Theorem 1.2 by writing the system governing the horizontal average $(\overline{u}, \overline{\theta})$, namely,

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \eta \partial_2^2 \overline{\theta}. \end{cases} \tag{4.1}$$

Taking the difference of (1.2) and (4.1), we get

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u_2} \partial_2 \overline{u} - \nu \partial_1^2 \widetilde{u} + \nabla \widetilde{p} = \widetilde{\theta} e_2, \\ \partial_t \widetilde{\theta} + \widetilde{u \cdot \nabla \widetilde{\theta}} + \widetilde{u_2} \partial_2 \overline{\theta} - \eta \partial_2^2 \widetilde{\theta} + \widetilde{u_2} = 0. \end{cases} \tag{4.2}$$

Dotting the system (4.2) by $(\widetilde{u}, \widetilde{\theta})$ yields,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) + \nu \|\partial_1 \widetilde{u}\|_{L^2}^2 + \eta \|\partial_2 \widetilde{\theta}\|_{L^2}^2 \\ &= - \int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx - \int \widetilde{u_2} \partial_2 \overline{u} \cdot \widetilde{u} dx - \int \widetilde{u \cdot \nabla \widetilde{\theta}} \cdot \widetilde{\theta} dx - \int \widetilde{u_2} \partial_2 \overline{\theta} \cdot \widetilde{\theta} dx \\ &:= A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{4.3}$$

By the divergence-free condition of u and Lemma 2.1,

$$A_1 := - \int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = \underbrace{- \int u \cdot \nabla \widetilde{u} \cdot \widetilde{u} dx}_{=0} + \underbrace{\int \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx}_{=0} = 0. \tag{4.4}$$

Similarly,

$$A_3 := \int \widetilde{u \cdot \nabla \widetilde{\theta}} \cdot \widetilde{\theta} dx = 0. \tag{4.5}$$

By Lemma 2.4, the divergence free condition of u, and Lemmas 2.1 and 2.5,

$$\begin{aligned} A_2 &:= - \int \widetilde{u_2} \partial_2 \overline{u} \cdot \widetilde{u} dx \\ &\le c \|\partial_2 \overline{u}\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le c \|\partial_2 \overline{u}\|_{L^2} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{L^2}^2. \end{aligned} \tag{4.6}$$

By Hölder's inequality, and Lemmas 2.2 and 2.5,

$$\begin{aligned} A_4 &:= - \int \widetilde{u_2} \partial_2 \overline{\theta} \cdot \widetilde{\theta} dx \\ &\le c \|\partial_2 \overline{\theta}\|_{L^\infty_{x_2}} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\le c \|\partial_2 \overline{\theta}\|_{H^1} \|\partial_1 \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\ &\le c \|\theta\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2} \\ &\le c \|\theta\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \end{aligned} \tag{4.7}$$

Collecting the estimates (4.4), (4.5), (4.6), (4.7) and (4.3) leads to

$$\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu\|\partial_1\widetilde{u}\|_{L^2}^2+\eta\|\partial_2\widetilde{\theta}\|_{L^2}^2 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big). \tag{4.8}$$

Applying ∇ to (4.2) yields

$$\begin{cases}\partial_t\nabla\widetilde{u}+\nabla(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla(\widetilde{u_2}\partial_2\overline{u})-\nu\partial_1^2\nabla\widetilde{u}+\nabla\nabla\widetilde{p}=\nabla(\widetilde{\theta}e_2)\,,\\ \partial_t\nabla\widetilde{\theta}+\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})+\nabla(\widetilde{u_2}\partial_2\overline{\theta})-\eta\partial_2^2\nabla\widetilde{\theta}+\nabla\widetilde{u_2}=0.\end{cases} \tag{4.9}$$

Taking the L^2-inner product of (4.9) with $(\nabla\widetilde{u},\nabla\widetilde{\theta})$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2\\ &=-\int\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{u})\cdot\nabla\widetilde{u}dx\\ &\quad-\int\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx\\ &:=B_1+B_2+B_3+B_4.\end{aligned} \tag{4.10}$$

According to Lemma 2.1, we write B_1 explicitly into the following four integrals,

$$\begin{aligned}B_1&:=-\int\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx\\ &=-\int\nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx+\underbrace{\int\nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx}_{=0}\\ &=-\int\partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx-\int\partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\ &\quad-\int\partial_2u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}dx-\int\partial_2u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}dx\\ &:=B_{11}+B_{12}+B_{13}+B_{14}.\end{aligned} \tag{4.11}$$

We start with B_{11}. Due to Lemma 2.1 and the inequality in (2.3),

$$\begin{aligned}B_{11}&:=-\int\partial_1u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx=-\int\partial_1\widetilde{u_1}\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}dx\\ &\le c\|\partial_1\partial_1\widetilde{u}\|_{L^2}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}\\ &\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2.\end{aligned} \tag{4.12}$$

Similarly,

$$\begin{aligned}B_{12}&:=-\int\partial_1u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx=-\int\partial_1\widetilde{u_2}\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}dx\\ &\le c\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}\\ &\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2.\end{aligned} \tag{4.13}$$

By (2.3),

$$
\begin{aligned}
B_{13} &:= -\int \partial_2 u_1 \partial_1 \widetilde{u} \cdot \partial_2 \widetilde{u} dx \\
&\le c\|\partial_1\partial_2\widetilde{u}\|_{L^2}\|\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.14)
\end{aligned}
$$

According to the divergence-free condition of u, Lemma 2.1 and (2.3),

$$
\begin{aligned}
B_{14} &:= -\int \partial_2 u_2 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u} dx = \int \partial_1 u_1 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u} dx \\
&= \int \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \partial_2 \widetilde{u} dx \\
&\le c\|\partial_1\partial_2\widetilde{u}\|_{L^2}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.15)
\end{aligned}
$$

In view (4.11), collecting the estimates (4.12), (4.13), (4.14) and (4.15) gives

$$
B_1 \le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.16)
$$

We write B_2 explicitly,

$$
\begin{aligned}
B_2 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{u}) \cdot \nabla\widetilde{u} dx \\
&= -\int \partial_1\widetilde{u_2}\partial_2\overline{u} \cdot \partial_1\widetilde{u} dx - \int \partial_2\widetilde{u_2}\partial_2\overline{u} \cdot \partial_2\widetilde{u} dx \\
&\quad - \int \widetilde{u_2}\partial_1\partial_2\overline{u} \cdot \partial_1\widetilde{u} dx - \int \widetilde{u_2}\partial_2\partial_2\overline{u} \cdot \partial_2\widetilde{u} dx \\
&:= B_{21} + B_{22} + B_{23} + B_{24}. \qquad (4.17)
\end{aligned}
$$

We start with B_{21}. By (2.3) and Lemma 2.1,

$$
\begin{aligned}
B_{21} &:= -\int \partial_1\widetilde{u_2}\partial_2\overline{u} \cdot \partial_1\widetilde{u} dx \\
&\le c\|\partial_1\partial_1\widetilde{u}\|_{L^2}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.18)
\end{aligned}
$$

For B_{22}, we use the divergence-free condition of u, (2.3) and Lemma 2.1,

$$
\begin{aligned}
B_{22} &:= -\int \partial_2\widetilde{u_2}\partial_2\overline{u} \cdot \partial_2\widetilde{u} dx = \int \partial_1\widetilde{u_1}\partial_2\overline{u} \cdot \partial_2\widetilde{u} dx \\
&\le c\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.19)
\end{aligned}
$$

Due to the definition of $\overline{u}$,

$$
B_{23} := -\int \widetilde{u_2}\partial_1\partial_2\overline{u} \cdot \partial_1\widetilde{u} dx = 0. \qquad (4.20)
$$

To estimate B_{24}, we make use of (2.3) and the divergence- free condition of u,

$$\begin{aligned} B_{24} &:= -\int \widetilde{u_2}\partial_2\partial_2\overline{u}\cdot\partial_2\widetilde{u}dx \\ &\le c\|\partial_1\partial_2\widetilde{u}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\partial_2\overline{u}\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.21}$$

Combining (4.18), (4.19), (4.20), (4.21) and (4.17), we obtain

$$B_2 \le c\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \tag{4.22}$$

By the definition of $\overline{u}$, we can split B_3 into four integrals,

$$\begin{aligned} B_3 &:= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx \\ &= -\int \nabla(u\cdot\nabla\widetilde{\theta})\cdot\nabla\widetilde{\theta}dx + \underbrace{\int \overline{\nabla(u\cdot\nabla\widetilde{\theta})}\cdot\nabla\widetilde{\theta}dx}_{=0} \\ &= -\int \partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx - \int \partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx \\ &\quad -\int \partial_1\widetilde{\theta}\partial_2 u_1\partial_2\widetilde{\theta}dx - \int \partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx \\ &:= B_{31}+B_{32}+B_{33}+B_{34}. \end{aligned} \tag{4.23}$$

Integrating by parts and using Lemma 2.1 and Young's inequality, we have

$$\begin{aligned} B_{31} &:= -\int \partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx = \int \partial_2\widetilde{u_2}(\partial_1\widetilde{\theta})^2dx = -2\int \widetilde{u_2}\partial_2\partial_1\widetilde{\theta}\partial_1\widetilde{\theta}dx \\ &\le c\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12} \\ &\le c\|\partial_2\widetilde{\theta}\|_{H^1}^{\frac32}\|\partial_1\widetilde{u}\|_{H^1}^{\frac12}\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12} \\ &\le c\|u\|_{H^2}^{\frac12}\|\theta\|_{H^2}^{\frac12}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.24}$$

To deal with B_{32}, we use Lemma 2.4,

$$\begin{aligned} B_{32} &:= -\int \partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx \\ &\le c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.25}$$

For B_{33}, we invoke the decomposition $u_1=\overline{u_1}+\widetilde{u_1}$ to write it into two integrals

$$\begin{aligned} B_{33} &:= -\int \partial_1\widetilde{\theta}\partial_2 u_1\partial_2\widetilde{\theta}dx \\ &= -\int \partial_1\widetilde{\theta}\partial_2\widetilde{u_1}\partial_2\widetilde{\theta}dx - \int \partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_2\widetilde{\theta}dx \\ &:= B_{331}+B_{332}. \end{aligned} \tag{4.26}$$

By integration by parts, Hölder's inequality, and Lemmas 2.1 and 2.4,

$$\begin{aligned}
B_{331} &:= -\int \partial_1\widetilde{\theta}\partial_2\widetilde{u_1}\partial_2\widetilde{\theta}dx \\
&= \int \widetilde{\theta}\partial_1\partial_2\widetilde{u_1}\partial_2\widetilde{\theta}dx + \int \widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_2\widetilde{\theta}\|_{L^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\widetilde{\theta}\|_{L^\infty} + c\|\widetilde{\theta}\|_{L^\infty}\|\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} + c\|\theta\|_{H^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned} \tag{4.27}$$

Due to Lemma 2.2 and Hölder's inequality,

$$\begin{aligned}
B_{332} &:= -\int \widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx = \int_{\mathbb{R}} \partial_2\overline{u_1}\int_{\mathbb{T}} \widetilde{\theta}\partial_1\partial_2\widetilde{\theta}dx_1dx_2 \\
&\le c\int_{\mathbb{R}} |\partial_2\overline{u_1}|\ \|\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}dx_2 \\
&\le c\|\partial_2\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{\theta}\|_{L^2_{x_1}L^2_{x_2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}L^2_{x_2}} \\
&\le c\|\partial_2\overline{u_1}\|_{H^1}\|\widetilde{\theta}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned} \tag{4.28}$$

Combining (4.27), (4.28) and (4.26), we obtain,

$$B_{33} \le c\|(u,\theta)\|_{H^2}\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\partial_1\widetilde{u}\|_{H^1}^2\Big). \tag{4.29}$$

According to the divergence-free condition of u and Lemma 2.4,

$$\begin{aligned}
B_{34} &:= -\int \partial_2\widetilde{\theta}\partial_2\widetilde{u_2}\partial_2\widetilde{\theta}dx = \int \partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned} \tag{4.30}$$

Inserting the estimates (4.24), (4.25), (4.26) and (4.30) in (4.23) yields

$$B_3 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \tag{4.31}$$

It remains to bound B_4. By integration by parts, we write B_4 into four terms as follows,

$$\begin{aligned} B_4 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx \\ &= -\int \partial_1(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_1\widetilde{\theta}dx - \int \partial_2(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_2\widetilde{\theta}dx \\ &= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx - \int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx \\ &\quad -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx - \int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &:= B_{41}+B_{42}+B_{43}+B_{44}. \end{aligned} \tag{4.32}$$

We start with B_{41}. To bound B_{41}, we use integration by parts and Hölder's inequality

$$\begin{aligned} B_{41} &:= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx = \int \partial_1\partial_1\widetilde{u_2}\ \partial_2\overline{\theta}\ \widetilde{\theta}dx \\ &\le c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\widetilde{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big). \end{aligned} \tag{4.33}$$

Due to the definition of the horizontal average $\overline{\theta}$,

$$B_{42} := -\int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx = 0. \tag{4.34}$$

By the divergence-free condition of u and Lemma 2.4,

$$\begin{aligned} B_{43} &:= -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx = \int \partial_1\widetilde{u_1}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &\le c\|\partial_1\widetilde{u_1}\|_{L^2}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\overline{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.35}$$

Thanks to Lemmas 2.1 and 2.4,

$$\begin{aligned} B_{44} &:= -\int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &\le c\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_2\overline{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.36}$$

Inserting the estimates (4.33), (4.34), (4.35) and (4.36) in (4.32) gives

$$B_4 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big). \tag{4.37}$$

Combining (4.16), (4.22), (4.31) and (4.37) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2\\ \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}(t)\|_{H^1}^2+\|\partial_2\widetilde{\theta}(t)\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big).\quad(4.38)\end{aligned}$$

In order to control the norm $\|\widetilde{\theta}\|_{L^2}$ appearing in (4.8) and (4.38), we need to add the following term,

$$-\frac{d}{dt}\Big(\delta(\widetilde{u_2},\widetilde{\theta})\Big)=-\delta(\partial_t\widetilde{u_2},\widetilde{\theta})-\delta(\widetilde{u_2},\partial_t\widetilde{\theta}),$$

where $\delta>0$ is a small constant to be fixed in the end of the proof. The inclusion of this term will generate an extra regularization term to help bound $\|\widetilde{\theta}\|_{L^2}$. Clearly this stabilizing term comes from the interaction between $\widetilde{u}$ and $\widetilde{\theta}$. By Hölder's inequality, one easily sees that, for sufficiently small $\delta>0$,

$$\|(\widetilde{u},\widetilde{\theta})\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\ge 0.$$

Due to the first equation of (4.2) and the fact that $\overline{u_2}=0$, we have

$$\partial_t\widetilde{u_2}+\widetilde{u\cdot\nabla\widetilde{u_2}}+\underbrace{\widetilde{u_2}\partial_2\overline{u_2}}_{=0}-\nu\partial_1^2\widetilde{u_2}+\partial_2\widetilde{p}=\widetilde{\theta}.\quad(4.39)$$

On the other hand, applying $\nabla\cdot$ to the first equation of (4.2), we get

$$\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+\Delta\widetilde{p}=\partial_2\widetilde{\theta}.\quad(4.40)$$

By (4.40), we can write

$$\widetilde{p}=-\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})-\Delta^{-1}\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+\Delta^{-1}\partial_2\widetilde{\theta}.$$

Hence,

$$\partial_2\widetilde{p}=-\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})-\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u_2}\partial_2\overline{u})+\partial_2\partial_2\Delta^{-1}\widetilde{\theta}.\quad(4.41)$$

Using (4.39) and the second equation of (4.2), we get

$$\begin{aligned}-\delta\frac{d}{dt}(\widetilde{u_2},\widetilde{\theta})&=-\delta(\partial_t\widetilde{u_2},\widetilde{\theta})-\delta(\widetilde{u_2},\partial_t\widetilde{\theta})\\&=-\delta(\widetilde{\theta}-\partial_2\widetilde{p}+\nu\partial_1^2\widetilde{u_2}-\widetilde{u\cdot\nabla\widetilde{u}_2},\widetilde{\theta})\\&\quad-\delta(\widetilde{u_2},-\widetilde{u_2}+\eta\partial_2^2\widetilde{\theta}-\widetilde{u_2}\partial_2\overline{\theta}-\widetilde{u\cdot\nabla\widetilde{\theta}})\\&=-\delta\|\widetilde{\theta}\|_{L^2}^2+\int\partial_2\widetilde{p}\widetilde{\theta}dx-\delta\nu\int\partial_1^2\widetilde{u_2}\widetilde{\theta}dx+\delta\int\widetilde{u\cdot\nabla\widetilde{u_2}}\widetilde{\theta}dx\\&\quad+\delta\|\widetilde{u_2}\|_{L^2}^2-\delta\eta\int\partial_2^2\widetilde{\theta}\widetilde{u_2}dx+\delta\int\widetilde{u_2}\widetilde{u_2}\partial_2\overline{\theta}dx+\delta\int\widetilde{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx\\&:=N_1+\cdots+N_8.\end{aligned}\quad(4.42)$$

We start with N_2. By (4.41), we have

$$
\begin{aligned}
N_2 &:= \delta \int \partial_2 \widetilde{p}\widetilde{\theta} dx \\
&= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \widetilde{\theta} dx - \delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx \\
&\quad + \delta \int \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta} \cdot \widetilde{\theta} dx \\
&:= N_{21} + N_{22} + N_{23}.
\end{aligned} \tag{4.43}
$$

By Hölder's inequality, the boundedness of the Riesz transform and Lemma 2.4,

$$
\begin{aligned}
N_{21} &:= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \widetilde{\theta} dx \\
&\le c\delta \|\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}})\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|\widetilde{u \cdot \nabla \widetilde{u}}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u \cdot \nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{L^\infty} \|\nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \|\partial_1 \nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big).
\end{aligned} \tag{4.44}
$$

By Hölder's inequality, the boundedness of the Riesz transform and Lemma 2.2,

$$
\begin{aligned}
N_{22} &:= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx \\
&\le c\delta \|\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u})\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|\widetilde{u_2} \partial_2 \overline{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|\partial_2 \overline{u}\|_{L^\infty_{x_2}} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|\partial_2 \overline{u}\|_{H^1} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \|\partial_1 \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big).
\end{aligned} \tag{4.45}
$$

For N_{23}, we integrate by parts and use Plancherel's theorem

$$
\begin{aligned}
N_{23} &:= \delta \int \partial_2 \partial_2 \Delta^{-1} \widetilde{\theta} \cdot \widetilde{\theta} dx \\
&= \delta \int \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} \cdot \partial_2 \Delta^{-\frac{1}{2}} \widetilde{\theta} dx
\end{aligned}
$$

$$
\begin{aligned}
&= \delta \|\partial_2 \Lambda^{-1} \widetilde{\theta}\|_{L^2}^2 \\
&= \delta \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \int_{\mathbb{R}} \frac{\xi_2^2}{k^2+\xi_2^2} |\widehat{\widetilde{\theta}}(k,\xi_2)|^2 d\xi_2 \\
&\le \delta \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \int_{\mathbb{R}} \xi_2^2 |\widehat{\widetilde{\theta}}(k,\xi_2)|^2 d\xi_2 = \delta \|\partial_2 \widetilde{\theta}\|_{L^2}^2, \qquad (4.46)
\end{aligned}
$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and we have used the fact that the oscillation part has the horizontal mode equal to 0, or $\widehat{\widetilde{\theta}}(0,\xi_2) = 0$. Combining (4.44), (4.45), (4.46) and (4.43) yields

$$
N_2 \le c\delta \|(u,\theta)\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big) + \delta \|\partial_2 \widetilde{\theta}\|_{L^2}^2. \qquad (4.47)
$$

By Hölder's inequality,

$$
N_3 := -\delta\nu \int \partial_1^2 \widetilde{u_2} \widetilde{\theta} dx \le \delta\nu \|\partial_1^2 \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \le \delta\nu^2 \|\partial_1 \widetilde{u}\|_{H^1}^2 + \frac{\delta}{4} \|\widetilde{\theta}\|_{L^2}^2. \qquad (4.48)
$$

To bound N_4, we use Lemma 2.1, Hölder's inequality, and Lemmas 2.3 and 2.5,

$$
\begin{aligned}
N_4 &:= \delta \int \widetilde{u \cdot \nabla \widetilde{u_2}} \widetilde{\theta} dx \\
&= \delta \int u \cdot \nabla \widetilde{u_2} \widetilde{\theta} dx - \underbrace{\delta \int \overline{u \cdot \nabla \widetilde{u_2}} \widetilde{\theta} dx}_{=0} \\
&\le c\delta \|u \cdot \nabla \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{L^\infty} \|\nabla \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \|\partial_1 \nabla \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2} \\
&\le c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \qquad (4.49)
\end{aligned}
$$

By Lemma 2.5,

$$
N_5 := \delta \|\widetilde{u_2}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{u_2}\|_{L^2}^2 \le c\delta \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.50)
$$

Due to Hölder's inequality and Lemma 2.5,

$$
\begin{aligned}
N_6 &:= -\delta\eta \int \partial_2^2 \widetilde{\theta} \widetilde{u_2} dx \\
&\le c\delta \|\partial_2^2 \widetilde{\theta}\|_{L^2} \|\widetilde{u_2}\|_{L^2} \\
&\le c\delta \|\partial_2 \widetilde{\theta}\|_{H^1} \|\partial_1 \widetilde{u_2}\|_{L^2} \\
&\le c\delta \|\partial_2 \widetilde{\theta}\|_{H^1} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\le c\delta \Big(\|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\partial_1 \widetilde{u}\|_{H^1}^2 \Big). \qquad (4.51)
\end{aligned}
$$

Using Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}
N_7 &:= \delta \int \widetilde{u_2}\widetilde{u_2}\partial_2\overline{\theta}dx \\
&\leq c\delta\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2} \\
&\leq c\delta\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.52)
\end{aligned}$$

To deal with N_8, we first split it into three terms using Lemma 2.1,

$$\begin{aligned}
N_8 &:= \delta \int \widetilde{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx \\
&= \delta \int u\cdot\nabla\widetilde{\theta}\widetilde{u_2}dx - \underbrace{\delta \int \overline{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx}_{=0} \\
&= \delta \int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta \int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta \int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx \\
&:= N_{81} + N_{82} + N_{83}. \qquad (4.53)
\end{aligned}$$

By Lemma 2.4 and divergence free condition of u, we have

$$\begin{aligned}
N_{81} &:= \delta \int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\
&\leq c\delta\|\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \qquad (4.54)
\end{aligned}$$

By integration by parts, Hölder's inequality and Lemma 2.2,

$$\begin{aligned}
N_{82} &:= \delta \int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\
&= -\delta \int \overline{u_1}\widetilde{\theta}\partial_1\widetilde{u_2}dx \\
&\leq \delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{\theta}\partial_1\widetilde{u_2}\|_{L^1} \\
&\leq c\delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2} \\
&\leq c\delta\|u\|_{H^1}\|\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2} \\
&\leq c\delta\|u\|_{H^2}\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_1\widetilde{u}\|_{H^1}^2\Big). \qquad (4.55)
\end{aligned}$$

Due to (2.3), Lemma 2.5 and the divergence-free condition of u,

$$\begin{aligned}
N_{83} &:= \delta \int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx \\
&= \delta \int \widetilde{u_2}\partial_2\widetilde{\theta}\widetilde{u_2}dx \\
&\leq c\delta\|\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|\partial_1\widetilde{u}\|_{H^1}^2\|\theta\|_{H^2}. \qquad (4.56)
\end{aligned}$$

In view of (4.53), combining (4.54), (4.55) and (4.56), we get

$$N_8 \leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \tag{4.57}$$

Inserting (4.47), (4.48), (4.49), (4.50), (4.51), (4.52) and (4.57) in (4.42) leads to

$$\begin{aligned}
-\delta\frac{d}{dt}(\widetilde{u_2},\widetilde{\theta}) \leq{}& -\delta\|\widetilde{\theta}\|_{L^2}^2 + c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&+\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned} \tag{4.58}$$

Putting (4.8), (4.38) and (4.58) together, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2},\widetilde{\theta})\Big) + 2\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + 2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\
&\quad\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&\qquad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&\qquad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}$$

Now, by Theorem 1.1, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{L^2} + \|\theta_0\|_{L^2} \leq \varepsilon$, then $\|(u(t),\theta(t))\|_{H^2} \leq c\varepsilon$. Hence we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2},\widetilde{\theta})\Big) + 2\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + 2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\
&\quad\leq c\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&\qquad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&\qquad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}$$

Choosing $\epsilon > 0$ such that $c\epsilon \leq \min(\frac{1}{4},\frac{\delta}{4})$, we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2},\widetilde{\theta})\Big) + 2\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + 2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\
&\quad\leq \frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big) + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2\\
&\qquad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + \frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big)\\
&\qquad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big)\\
&\quad\leq -\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).
\end{aligned}$$

Choosing $\delta > 0$ such that $c\delta \leq \min(\nu,\eta,\frac{c}{2})$, we obtain

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2},\widetilde{\theta})\Big) + \nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 \leq 0. \tag{4.59}$$

Due to the choice of δ, we have

$$\frac{1}{2}\Big(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2\Big) - \delta(\widetilde{u_2}, \widetilde{\theta}) \geq 0.$$

or

$$\frac{1}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) \leq \|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta}) \leq \frac{3}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2).$$

For any $0 \leq s \leq t$, integrating (4.59) in time yields

$$\frac{1}{2}(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) + \int_s^t \Big(\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2\Big)\,d\tau$$
$$\leq \frac{3}{2}(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2).$$

Especially, for any $0 \leq s \leq t$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2) \tag{4.60}$$

and

$$\int_0^\infty (\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2)\,d\tau \leq C < \infty.$$

Combining with the time integral bounds from Theorem 1.1,

$$\int_0^\infty \|\partial_1 u\|_{H^2}^2\,dt < \infty, \quad \int_0^\infty \|\partial_1\theta\|_{L^2}^2\,dt < \infty \quad \textit{and} \quad \int_0^\infty \|\partial_2\theta\|_{H^2}^2\,dt < \infty,$$

we obtain

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\,dt < \infty. \tag{4.61}$$

Applying Lemma 2.6 to (4.60) and (4.61) yields

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \leq c(1+t)^{-1},$$

and the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

This completes the proof of Theorem 1.2. □

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