Received: 11 January 2020
DOI: 10.1002/mma.6740

RESEARCH ARTICLE

WILEY

Unique weak solutions of the d-dimensional micropolar equation with fractional dissipation

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Communicated by: H. Li

This article examines the existence and uniqueness of weak solutions to the d-dimensional micropolar equations ($d = 2$ or $d = 3$) with general fractional dissipation $(-\Delta)^{\alpha}u$ and $(-\Delta)^{\beta}w$. The micropolar equations with standard Laplacian dissipation model fluids with microstructure. The generalization to include fractional dissipation allows simultaneous study of a family of equations and is relevant in some physical circumstances. We establish that, when $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$, any initial data (u_0, w_0) in the critical Besov space $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)$ and $w_0 \in B_{2,1}^{1+\frac{d}{2}-2\beta}(\mathbb{R}^d)$ yields a unique weak solution. For $\alpha \geq 1$ and $\beta = 0$, any initial data $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)$ and $w_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ also leads to a unique weak solution as well. The regularity indices in these Besov spaces appear to be optimal and can not be lowered in order to achieve the uniqueness. Especially, the 2D micropolar equations with the standard Laplacian dissipation, namely, $\alpha = \beta = 1$, have a unique weak solution for $(u_0, w_0) \in B_{2,1}^{0}$. The proof involves the construction of successive approximation sequences and extensive a priori estimates in Besov space settings.

KEYWORDS

Littlewood-Paley, local solution, micropolar equations, uniqueness

MSC CLASSIFICATION

35A01; 35A02; 35Q35; 76D03

1 | INTRODUCTION

The micropolar equations were first proposed in 1965 by C.A Eringen to modal micropolar fluids that are fluids with microstructure (see previous studies[1-4]). These equations can model large number of complex fluids such as animal blood, suspensions, and liquid crystals. In this paper, we focus on the following d-dimensional ($d = 2$ or $d = 3$) incompressible micropolar equations with fractional dissipation

$$\begin{cases} \partial_t u + (\nu + k)(-\Delta)^{\alpha} u + u \cdot \nabla u + \nabla \Pi - 2k\nabla \times w = 0, \\ \partial_t w + 4kw + 2k\nabla \times u + u \cdot \nabla w + \gamma(-\Delta)^{\beta} w = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x)\ , \quad w(x,0) = w_0(x), \end{cases} \tag{1}$$

where $u = u(x,t) \in \mathbb{R}^d$ denotes the fluid velocity, $w = w(x,t) \in \mathbb{R}^d$ the field of microrotation representing the angular velocity of the rotation of the fluid particles, $\Pi = \Pi(x,t)$ the scalar pressure, and the parameter ν denotes the Newtonian kinematic viscosity, k the microrotation viscosity, and γ the angular viscosity. Here, the fractional Laplacian operator

$(-\Delta)^\alpha$ (which is also referred as the Riesz potential operator) is defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Besides their many physical applications, the micropolar equations are also of great interest in mathematics. Fundamental issues such as the well-posedness problem on (1) have recently attracted considerable interest, and an array of important results have been established (see, e.g., previous works [5-14]). More recent focuses have been on the micropolar equations with partial or fractional dissipation (see, e.g., previous works[15-20]). Investigations on nonlocal diffusion have now become a trend.[21] The study of fractionally dissipated micropolar equations allow us to simultaneously treat a family of equations including those with the standard Laplacian dissipation. The investigations on the micropolar equations with fractional dissipation help reveal how the global well-posedness problem is related to the fractional regularization.

The main goal of this study is to obtain the existence and uniqueness of solutions to (1) in a weakest possible functional setting for the largest possible ranges of α and β. Our main results can be stated as follows.

Theorem 1.1. *Consider (1) with $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$. Assume the initial data u_0 and w_0 satisfy*

$$\nabla\cdot u_0 = 0, \quad u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d), \quad w_0 \in B_{2,1}^{\frac{d}{2}+1-2\beta}(\mathbb{R}^d).$$

Then there exist $T > 0$ and a unique weak solution (u, w) of (1) on $[0, T]$ satisfying

$$u \in L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\alpha}\left(\mathbb{R}^d\right)\right) \cap L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\left(\mathbb{R}^d\right)\right), \tag{2}$$

$$w \in L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}(\mathbb{R}^d)\right) \cap L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\left(\mathbb{R}^d\right)\right). \tag{3}$$

Here, $B^r_{p,q}$ denotes the inhomogeneous Besov space. A review of the Besov spaces and related facts is provided in the following section. As a special consequence of Theorem 1.1, the two-dimensional (2D) micropolar equations with $\alpha = \beta = 1$, namely, the standard Laplacian dissipation, always possess a unique local solution (u, w) in the critical Besov space $L^\infty(0, T; B^0_{2,1}(\mathbb{R}^2))$. For the 3D micropolar equations with the standard Laplacian dissipation, the uniqueness is also attained in the critical Besov space $L^\infty\left(0, T; B_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)\right)$. Here the critical Besov spaces are the Besov space settings for which the solution of the differential equations and its scaling invariant counterparts share the same norm. In the general fractional dissipation cases, the regularity indices $1 + \frac{d}{2} - 2\alpha$ and $1 + \frac{d}{2} - 2\beta$ in the Besov spaces appear to be optimal and one may not be able to achieve the uniqueness when they are lowered.

Theorem 1.2. *Consider (1) with $\alpha \geq 1$ and $\beta = 0$. Assume the initial data (u_0, w_0) satisfies*

$$\nabla\cdot u_0 = 0, \quad u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d), \quad w_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d).$$

Then there exist $T > 0$ and a unique weak solution (u, w) of (1) on $[0, T]$ satisfying

$$u \in L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\alpha}\left(\mathbb{R}^d\right)\right) \cap L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\left(\mathbb{R}^d\right)\right), \tag{4}$$

$$w \in L^\infty\left(0, T, B_{2,1}^{\frac{d}{2}}\left(\mathbb{R}^d\right)\right). \tag{5}$$

Theorem 1.2 deals with the case when the equation of w involves no diffusion. The Besov space for u remains the same, but the setting for w needs to be in a more regular Besov space due to the lack of diffusion in the equation of w. For an inviscid equation, the regularity index $\frac{d}{2}$ in the Besov space $B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ cannot be lowered in order to obtain the uniqueness of solutions.

The proof for each of the theorems is naturally split into two parts: the existence and uniqueness parts. The existence part starts with the construction a successive approximation sequence that iteratively solves systems close to (1). This successive approximation sequence is then shown to be uniformly bounded in suitable Besov spaces via the method of mathematical induction. These bounds allow us to extract a subsequence, which converges weakly to a limit. Using the Aubin-Lions lemma, the weak limit is then shown to be the weak solution of (1). The main efforts are devoted to proving the uniform boundedness. This process involves various analysis tools and techniques. The uniqueness is established by analyzing the differences in the L^2 space.

The rest of this paper is divided into three sections. Section 2 serves as a preparation. It reviews the Besov space and related tools to be used in the subsequent sections. Section 3 proves Theorem 1.1. It is further divided into two subsections with one devoted to the existence and the other to the uniqueness. The last section provides the proof of Theorem 1.2. It is again split into two subsections, one for the proof of existence and one for the uniqueness.

2 | PREPARATIONS: BESOV SPACES

This section serves as a preparation. Materials presented here will be used in the proofs of Theorems 1.1 and 1.2. The definition of the Besov space and related simple facts can be found in Bahouri et al.[22] Lemma 2.6 is taken from Jiu et al.[23, Lemma A.5] In what follows, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class and $\mathcal{S}'(\mathbb{R}^d)$ the tempered distribution.

Definition 2.1 (Inhomogenous Besov space $B^s_{p,q}$). $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^s_{p,q}$ with $s \in \mathbb{R}$ and $1 \le p \le q \le \infty$ if

$$\|f\|_{B^s_{p,q}} \equiv \|2^{sj}\|\Delta_j f\|_{L^p}\|_{l^q} = \begin{cases} \left(\sum\limits_{j=-1}^{+\infty} (2^{sj}\|\Delta_j f\|_{L^p})^q\right)^{\frac{1}{q}} & \text{if} \quad q < \infty, \\ \sup\limits_{j \ge -1} \ 2^{sj}\|\Delta_j f\|_{L^p} & \text{if} \quad q = \infty \end{cases}$$

is finite.

Lemma 2.2. *Let $B(0,r)$ and $C(0,r_1,r_2)$ denote the standard ball and the annulus, respectively,*

$$B(0,r) = \left\{\xi \in \mathbb{R}^d, \quad |\xi| \le r\right\}, \quad C(0,r_1,r_2) = \left\{\xi \in \mathbb{R}^d, \quad r_1 \le |\xi| \le r_2\right\}.$$

There are two compactly supported smooth radial functions ϕ and ψ satisfying

$$\begin{aligned} &supp\ \phi \subset B\left(0, \frac{4}{3}\right), \quad supp\ \psi \subset C\left(0, \frac{3}{4}, \frac{8}{3}\right), \\ &\phi(\xi) + \sum_{j \ge 0} \psi(2^{-j}\xi) = 1 \quad \text{for all } \ \xi \in \mathbb{R}^d. \end{aligned} \tag{6}$$

The proof of Lemma 2.2 can be found in Bahouri et al.[22, p59]

Notations 2.2.1 *We use $\tilde{h}$ and h to denote the inverse Fourier transforms of ϕ and ψ, respectively,*

$$\tilde{h} = \mathcal{F}^{-1}\phi, \quad h = \mathcal{F}^{-1}\psi.$$

We write $\psi_j(\xi) = \psi(2^{-j}\xi)$. By a simple property of the Fourier transform,

$$h_j(x) := \mathcal{F}^{-1}\psi_j(x) = 2^{dj}h(2^j x).$$

Definition 2.3. The inhomogeneous dyadic block operator Δ_j are defined as

$$\begin{aligned}
\Delta_j f &= 0 && \text{for } j \le -2,\\
\Delta_{-1} f &= \tilde{h} * f = \int_{\mathbb{R}^d} f(x-y)\tilde{h}(y)\, dy, &&\\
\Delta_j f &= h_j * f = 2^{dj}\int_{\mathbb{R}^d} f(x-y)h(2^j y)\, dy && \text{for } j \ge 0.
\end{aligned}$$

The corresponding inhomogeneous low frequency cut-off operator S_j is defined by

$$S_j f = \sum_{k\le j-1} \Delta_k f.$$

Remarks 2.3.1. For any function *f* in the usual Schwarz class $\mathcal{S}$, (6) implies

$$\hat{f}(\xi) = \phi(\xi)\hat{f}(\xi) + \sum_{j\ge 0} \psi(2^{-j}\xi)\hat{f}(\xi),$$

or in terms of the inhomogenous dyadic block operators

$$f = \sum_{j\ge -1} \Delta_j f \quad \text{or} \quad Id = \sum_{j\ge -1} \Delta_j,$$

where *Id* denotes the identity operator. For generality, for any *F* in the space of tempered distributions $\mathcal{S}'$,

$$F = \sum_{j\ge -1} \Delta_j F \quad \text{or} \quad Id = \sum_{j\ge -1} \Delta_j \quad \text{in} \quad \mathcal{S}'. \tag{7}$$

(7) is referred to as the Littlewood-Paley decomposition for tempered distributions.

Definition 2.4. In terms of inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely,

$$FG = \sum_{|j-k|<2} S_{k-1}F\Delta_k G + \sum_{|j-k|<2} \Delta_k F S_{k-1} G + \sum_{k\ge j-1} \Delta_k F \tilde{\Delta}_k G,$$

where $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. This is the so-called Bony decomposition.

Lemma 2.5. *Let* $\alpha \ge 0$. *Let* $1 \le p \le q \le \infty$.

1. *If f satisfies*

$$supp\ \tilde{f} \subset \left\{\xi \in \mathbb{R}^d, \quad |\xi| \le K2^j\right\},$$

for some integer j and a constant $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le c_1 2^{2\alpha j + jd\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

2. *If f satisfies*

$$supp\ \tilde{f} \subset \left\{\xi \in \mathbb{R}^d, \quad K_1 2^j \le |\xi| \le K_2 2^j\right\},$$

for some integer j and constants $0 < K_1 \le K_2$, *then*

$$c_1 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le c_2 2^{2\alpha j + jd\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)},$$

where c_1, c_2 *are constants depending only on* α, p, q.

Below, we state bounds for the triple products involving Fourier localized functions. These bounds will be used in the proofs of Theorems 1.1 and 1.2. We refer the reader to lemma A.5 in Jiu et al.[23] for a detailed proof of the following lemma.

Lemma 2.6. *Let $j \geq 0$ be an integer. Let Δ_j be the inhomogeneous Littlewood-Paley-localization operator. For any vectors field F, G, H with $\nabla \cdot F = 0$, we have*

$$\left|\int_{\mathbb{R}^d} \Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx\right| \leq c\|\Delta_j H\|_{L^2}\left(2^j\sum_{m\leq j-1}2^{\frac{d}{2}m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\leq 2}\|\Delta_k G\|_{L^2}\right.$$
$$\left.+\sum_{|j-k|\leq 2}\|\Delta_k F\|_{L^2}\sum_{m\leq j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m G\|_{L^2}+\sum_{k\geq j-1}2^j 2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\tilde{\Delta}_k G\|_{L^2}\right)$$

and

$$\left|\int_{\mathbb{R}^d} \Delta_j(F\cdot\nabla G)\cdot\Delta_j G\,dx\right| \leq c\|\Delta_j G\|_{L^2}\left(\sum_{m\leq j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\leq 2}\|\Delta_k G\|_{L^2}\right.$$
$$\left.+\sum_{|j-k|\leq 2}\|\Delta_k F\|_{L^2}\sum_{m\leq j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m G\|_{L^2}+\sum_{k\geq j-1}2^j 2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\tilde{\Delta}_k G\|_{L^2}\right).$$

3 | PROOF OF THEOREM 1.1

3.1 | Existence of a weak solution

This subsection proves the existence part of Theorem 1.1. The approach is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1) in the weak sense.

Proof for the existence part of Theorem 1.1. We consider a successive approximation $\left\{(u^{(n)}, w^{(n)})\right\}$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0, \qquad w^{(1)} = S_2 w_0, \\ \partial_t u^{(n+1)} + (\nu+k)(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)}\cdot\nabla u^{(n+1)}) + 2k\nabla\times w^{(n)}, \\ \partial_t w^{(n+1)} + \gamma(-\Delta)^\beta w^{(n+1)} = -4kw^{(n+1)} - 2k\nabla\times u^{(n)} - u^{(n)}\cdot\nabla w^{(n+1)}, \\ u^{(n+1)}(x,0) = S_{n+1}u_0, \qquad w^{(n+1)}(x,0) = S_{n+1}w_0, \end{cases} \tag{8}$$

where $\mathbb{P} = I - \nabla(-\Delta)^{-1}\mathrm{div}$ is the standard Leray Projection. For

$$M = 2\left(\|u_0\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \|w_0\|_{B^{1+\frac{d}{2}-2\beta}_{2,1}}\right),$$

$T > 0$ sufficiently small and $0 < \delta < 1$ (to be specified later), we set

$$Y \equiv \left\{(u,w)\ \middle|\ \|u\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \leq M,\ \|w\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1}\right)} \leq M,\right.$$
$$\left.\|u\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \leq \delta,\ \|w\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \leq \delta\right\}. \tag{9}$$

We show that $\left\{(u^{(n)}, w^{(n)})\right\}$ has a subsequence that converges to the weak solution of (1). This process consists of three main steps. The first step is to show that $\left\{(u^{(n)}, w^{(n)})\right\}$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions lemma. While the last step is to show that the limit is indeed a weak solution of (1).

To show the uniform bound for $\left\{(u^{(n)}, w^{(n)})\right\}$ in Y, we prove by induction. Clearly,

$$\|u^{(1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} = \|S_2 u_0\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \le M,$$
$$\|w^{(1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} = \|S_2 w_0\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} \le M.$$

If $T > 0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le T\|S_2 u_0\|_{B_{2,1}^{1+\frac{d}{2}}} \le T\ c\ \|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \le \delta,$$

$$\|w^{(1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le T\|S_2 w_0\|_{B_{2,1}^{1+\frac{d}{2}}} \le T\ c\ \|w_0\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \le \delta.$$

Assuming that $(u^{(n)}, w^{(n)})$ obeys the bounds defined in Y, namely,

$$\|u^{(n)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \le M, \quad \|w^{(n)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} \le M,$$
$$\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le \delta, \quad \|w^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le \delta,$$

we prove that $\left\{(u^{(n+1)}, w^{(n+1)})\right\}$ obeys the same bound for suitably selected $T > 0, M > 0$, and $\delta > 0$. For the sake of clarity, the proof of the four bounds is achieved in the following four steps. □

3.1.1 | The estimate of $u^{(n+1)}$ in $B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)$

Let $j \ge 0$ be an integer. Applying Δ_j to the second equation in (8) and then dotting with $\Delta_j u^{(n+1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2}^2 + (\nu + k)\|\Lambda^\alpha \Delta u^{(n+1)}\|_{L^2}^2 = A_1 + A_2, \tag{10}$$

where

$$A_1 = \int_{\mathbb{R}^d} 2k\Delta_j(\nabla \times w^{(n)}) \cdot \Delta_j u^{(n+1)}\ dx,$$
$$A_2 = -\int_{\mathbb{R}^d} \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)})\Delta_j u^{(n+1)}\ dx.$$

We remark that the projection operator $\mathbb{P}$ has been eliminated due to the divergence-free condition $\nabla \cdot u^{(n+1)} = 0$. The dissipative part admits a lower bound

$$(\nu + k)\|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 \ge c_0\ 2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2}^2,$$

where $c_0 > 0$ is a constant. By Hölder's inequality and Bernstein's inequality

$$\begin{aligned} |A_1| &= \left|\int_{\mathbb{R}^d} 2k\ \Delta_j(\nabla \times w^{(n)}) \cdot \Delta_j u^{(n+1)}\ dx\right| \\ &\le 2k\ \|\Delta_j(\nabla \times w^{(n)})\|_{L^2}\|\Delta_j u^{(n+1)}\|_{L^2} \\ &\le c\ 2^j\ \|\Delta_j w^{(n)}\|_{L^2}\|\Delta_j u^{(n+1)}\|_{L^2}. \end{aligned}$$

According to Lemma 2.6,

$$
\begin{aligned}
|A_2| &= \left| -\int_{\mathbb{R}^d} \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j u^{(n+1)}\, dx \right| \\
&\leq c\, \|\Delta_j u^{(n+1)}\|_{L^2}^2 \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2} \\
&\quad + c\, \|\Delta_j u^{(n+1)}\|_{L^2} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2} \\
&\quad + c \sum_{k\geq j-1} 2^j 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_j u^{(n+1)}\|_{L^2}.
\end{aligned}
$$

Inserting the estimates above in (10) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from the both sides, we obtain

$$
\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2} + c_0 2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2} \leq J_1 + J_2 + J_3 + J_4, \tag{11}
$$

where

$$
\begin{aligned}
J_1 &= c\, \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2}, \\
J_2 &= c\, \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2}, \\
J_3 &= c\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2}, \\
J_4 &= c\, 2^j \|\Delta_j w^{(n)}\|_{L^2}.
\end{aligned}
$$

Integrating (11) in time yields

$$
\|\Delta_j u^{(n+1)}\|_{L^2} \leq e^{-c_0 2^{2\alpha j} t}\|\Delta_j u_0^{(n+1)}\|_{L^2} + \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)}(J_1 + \ldots + J_4)\, d\tau. \tag{12}
$$

Multiplying (12) by $2^{\left(1+\frac{d}{2}-2\alpha\right)j}$ and summing over j, we obtain

$$
\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \leq \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)}(J_1 + \ldots + J_4)\, d\tau. \tag{13}
$$

The terms on the right hand side of (13) can be estimated as follows using the simple bound

$$
e^{-c_0 2^{2\alpha j}(t-\tau)} \leq 1.
$$

Recalling the definition of J_1 above and using the inductive assumption on $u^{(n)}$, we have for any $t \leq T$,

$$
\begin{aligned}
&\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_1\, d\tau \\
&\quad \leq c \int_0^t \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} |\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\quad \leq c \int_0^t \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}}\, d\tau \\
&\quad \leq c\, \|u^{(n+1)}\|_{L^\infty\left(0,t,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,t,B^{1+\frac{d}{2}}_{2,1}\right)} \\
&\quad \leq c\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \leq c\, \delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}.
\end{aligned}
$$

The term involving J_2 admits the same bound. In fact, by Young's inequality for series convolution,

$$\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_2\, d\tau$$

$$\le c\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j} 2^{2\alpha(m-j)} 2^{\left(1+\frac{d}{2}-2\alpha\right)m}\|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\, d\tau$$

$$\le c\int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}}\|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau$$

$$\le c\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le c\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}.$$

The estimate for the term with J_3 is also similar,

$$\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_3\, d\tau$$

$$= \int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\sum_{k\ge j-1} c\, 2^j 2^{\frac{d}{2}k}\|\tilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\, d\tau$$

$$= c\int_0^t \sum_{j\ge -1}\sum_{k\ge j-1} 2^{\left(2+\frac{d}{2}-2\alpha\right)(j-k)} 2^{\left(1+\frac{d}{2}\right)k}\|\Delta_k u^{(n)}\|_{L^2} 2^{\left(1+\frac{d}{2}-2\alpha\right)k}\|\tilde{\Delta}_k u^{(n+1)}\|_{L^2}\, d\tau$$

$$\le c\int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}}\|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau$$

$$\le c\,\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le c\;\;\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}.$$

It remains to bound the term with J_4,

$$\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_4\, d\tau$$

$$= \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} c\, 2^j\|\Delta_j w^{(n)}\|_{L^2}\, d\tau$$

$$\le \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t c\, 2^j\, \|\Delta_j w^{(n)}\|_{L^2}\, d\tau = c\int_0^t \sum_{j\ge -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j}\|\Delta_j w^{(n)}\|_{L^2}\, d\tau$$

$$\underset{\text{since }\alpha\ge\frac{1}{2}}{\le} c\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j w^{(n)}\|_{L^2}\, d\tau = c\,\|w^{(n)}\|_{L^1\left(0,t,B^{1+\frac{d}{2}}_{2,1}\right)} \le c\,\delta.$$

Collecting the bounds above and inserting them in (13), we find for any $t\;\le\;T$

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + c\,\delta\,\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} + c\,\delta.$$

Therefore,

$$\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + c\,\delta\,\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} + c\,\delta.$$

Choosing δ such that $c\,\delta \le \min\left(\frac{1}{4},\frac{M}{4}\right)$, we get

$$\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le \frac{M}{2} + \frac{1}{4}\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} + \frac{M}{4},$$

which implies

$$\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \leq M.$$

3.1.2 | The estimate of $\|u^{(n+1)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}\right)}$

We multiply (12) by $2^{\left(1+\frac{d}{2}\right)j}$, sum over j and integrate in time to obtain

$$\begin{aligned}\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \\ + \int_0^T \int_0^s \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}(s-\tau)}(J_1 + \ldots + J_4)\, d\tau\, ds.\end{aligned} \tag{14}$$

We estimate the terms on the right hand side of (14) and start with the first term.

$$\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{n+1}\|_{L^2}\, dt = c\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2}.$$

Since $u_0 \in B_{2,1}^{\left(1+\frac{d}{2}-2\alpha\right)}$, then by dominated convergence theorem

$$\lim_{T\to 0}\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \leq \frac{\delta}{4}.$$

Applying Young's inequality for the time convolution, we have

$$\begin{aligned}&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1\, d\tau\, ds \\ &= c\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(m)}\|_{L^2}\, d\tau\, ds \\ &\leq c\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(m)}\|_{L^2}\, d\tau \int_0^T e^{-c_0 2^{2\alpha j}s}\, ds\end{aligned}$$

Using the fact that there exists $c_2 > 0$ satisfying for $j \geq 0$,

$$\int_0^s e^{-c_0 2^{2\alpha j}s} ds \leq c\, 2^{-2\alpha j}(1-e^{-c_2 T}), \tag{15}$$

we get

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau \, ds \\
&\quad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_j 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(m)}\|_{L^2} \, d\tau \\
&\quad \leq c\,(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \\
&\quad \leq c\,\delta\,(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}$$

The terms with J_2 and J_3 can be similarly estimated and obey the same bound.

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&= c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Owing to (15) and the above inequality,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\quad \leq c(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \\
&\quad \leq c(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\quad \leq c(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \\
&\quad \leq c(1-e^{-c_2 T})\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\quad \leq c \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^T \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Then, due to (15),

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \|\tilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\tilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \\
&\qquad \leq c\,(1-e^{-c_2 T})\, \delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}
$$

Now, for the term with J_4 we write

$$
\begin{aligned}
\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \, ds &= c \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^T \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds \\
&\leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \\
&\underset{\text{since } \alpha\geq\frac{1}{2}}{\leq} c\,(1-e^{-c_2 T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau}_{=\|w^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \delta} \\
&\leq c\,(1-e^{-c_2 T})\delta.
\end{aligned}
$$

Collecting the estimates above leads to

$$
\begin{aligned}
\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} &\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T})\|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} + c\,(1-e^{-c_2 T})\delta \\
&\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T})M + c\,(1-e^{-c_2 T})\delta.
\end{aligned}
$$

Choosing T sufficiently small such that $c\,(1-e^{-c_2 T}) \leq \min\left(\frac{1}{4M}, \frac{1}{2}\right)$, we get

$$
\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta.
$$

3.1.3 | The estimate of $w^{(n+1)}$ in $L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}\left(\mathbb{R}^d\right)\right)$

We apply Δ_j to the third equation in (8) and then dotting with $\Delta_j w^{(n+1)}$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2}^2 &+ (c_1 2^{2\beta j} + 4k)\|\Delta_j w^{(n+1)}\|_{L^2}^2 \\
&\leq -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx \\
&\quad - \int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx \\
&= B_1 + B_2,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
B_1 &= -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx, \\
B_2 &= -\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx.
\end{aligned}$$

By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}
|B_1| &= \left|-2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx\right| \\
&\leq 2k\,\|\Delta_j(\nabla\times u^{(n)})\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2} \\
&\leq c\,2^j\,\|\Delta_j u^{(n)}\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2}.
\end{aligned}$$

By Lemma 2.6,

$$\begin{aligned}
|B_2| &= \left|-\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx\right| \\
&\leq c\,\|\Delta_j w^{(n+1)}\|_{L^2}^2\sum_{m\leq j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2} \\
&\quad + c\,\|\Delta_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2} \\
&\quad + c\,\|\Delta_j w^{(n+1)}\|_{L^2}2^j\sum_{k\geq j-1}2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}$$

Inserting the estimates above in (16) and eliminating $\|\Delta_j w^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2} + (c_1 2^{2\beta j} + 4k)\|\Delta_j w^{(n+1)}\|_{L^2} \leq K_1 + K_2 + K_3 + K_4, \tag{17}$$

where

$$\begin{aligned}
K_1 &= c\,2^j\,\|\Delta_j u^{(n)})\|_{L^2}, \\
K_2 &= c\,\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\leq j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}, \\
K_3 &= c\,\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2}, \\
K_4 &= c\sum_{k\geq j-1}2^j 2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}$$

Integrating (17) in time yields, for any $t \leq T$,

$$\|\Delta_j w^{(n+1)}\|_{L^2} \leq e^{-(c_1 2^{2\beta j})t}\|\Delta_j w_0^{(n+1)}\|_{L^2} + \int_0^t e^{-(c_1 2^{2\beta j})(t-\tau)}(K_1 + \ldots + K_4)\,d\tau. \tag{18}$$

Multiplying (18) by $2^{\left(1+\frac{d}{2}-2\beta\right)j}$ and summing over j, we have

$$\|w^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \le \|w_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} + \sum_{j\ge -1}\int_0^t e^{-(c_1 2^{2\beta j})(t-\tau)} 2^{\left(1+\frac{d}{2}-2\beta\right)j}(K_1 + \ldots + K_4)\, d\tau. \tag{19}$$

The terms containing K_1 through K_4 on the right hand side of (19) can be bounded suitably as follows. We start with the term with K_1,

$$\begin{aligned}\sum_{j\ge -1}\int_0^t 2^{\left(1+\frac{d}{2}-2\beta\right)j} K_1\, d\tau &= \int_0^t \sum_{j\ge -1} c\, 2^{\left(2+\frac{d}{2}-2\beta\right)j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\, d\tau \\ &\underset{\text{since } \beta\ge\frac{1}{2}}{\le} c\underbrace{\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\, d\tau}_{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}\le\delta} \le c\,\delta.\end{aligned}$$

Similarly the term with K_2 is bounded by

$$\begin{aligned}\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}\int_0^t K_2\, d\tau &= c\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\le j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}\, d\tau \\ &\le c\,\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \\ &\le c\,\delta\,\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}.\end{aligned}$$

The terms related to K_3 and K_4 obey also the same bound,

$$\begin{aligned}\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}\int_0^t K_3\, d\tau &= c\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2}\, d\tau \\ &\le c\,\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \\ &\le c\,\delta\,\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}.\end{aligned}$$

For the term with K_4, we write

$$\begin{aligned}\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}\int_0^t K_4\, d\tau &= c\int_0^t \sum_{j\ge -1}\sum_{k\ge j-1} 2^{\left(2+\frac{d}{2}-2\beta\right)j} 2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\ &\le c\int_0^t \sum_{j\ge -1}\sum_{j\ge -1} 2^{\left(2+\frac{d}{2}-2\beta\right)j} 2^{\frac{d}{2}j}\|\tilde{\Delta}_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\, d\tau \\ &= c\int_0^t \sum_{j\ge -1}\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} 2^{\left(1+\frac{d}{2}\right)j}\|\tilde{\Delta}_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\, d\tau \\ &\le c\,\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \\ &\le c\,\delta\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}.\end{aligned}$$

Collecting the estimates and inserting them in (19), we obtain for any $t \le T$

$$\|w^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \le \|w_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} + c\,\delta + c\,\delta\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}.$$

Choosing $c\,\delta \le \min\left(\frac{1}{4}, \frac{M}{4}\right)$, we get

$$\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} \le \frac{M}{2} + \frac{M}{4} + \frac{1}{4}\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)},$$

which implies

$$\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} \le M.$$

3.1.4 | The estimate of $\|w^{(n+1)}(t)\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}$

We recall (18)

$$\|\Delta_j w^{(n+1)}\|_{L^2} \le e^{-(c_1 2^{2\beta j})t}\|\Delta_j w_0^{(n+1)}\|_{L^2} + \int_0^t e^{-c_1 2^{2\beta j}(t-\tau)}(K_1 + \ldots + K_4)\, d\tau.$$

We multiply by $2^{\left(1+\frac{d}{2}\right)j}$, sum over j and integrate in time to get

$$\begin{aligned}\|w^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le& \int_0^T \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_1 2^{2\beta j}t}\|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \\ &+ \int_0^T \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)}(K_1 + \ldots + K_4)\, d\tau\, ds. \end{aligned} \tag{20}$$

Clearly,

$$\int_0^T \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_1 2^{2\beta j}t}\|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt = c\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}(1 - e^{-c_1 2^{2\beta j}T})\|\Delta_j w_0^{(n+1)}\|_{L^2}.$$

Since $w_0 \in B_{2,1}^{1+\frac{d}{2}-2\beta}$, we have by the dominated convergence theorem,

$$\lim_{T\to 0}\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j}(1 - e^{-c_1 2^{2\beta j}T})\|\Delta_j w_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small

$$\int_0^T \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_1 2^{2\beta j}t}\|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \le \frac{\delta}{2}.$$

Applying Young's inequality for the time convolution, the term with K_1 is bounded by

$$\begin{aligned}\int_0^T \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)}K_1\, d\tau\, ds &\le c\int_0^T \sum_{j\ge -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)}\|\Delta_j u^{(n)}\|_{L^2}\, d\tau\, ds \\ &\le c\sum_{j\ge -1} 2^{\left(2+\frac{d}{2}\right)j}\int_0^T \|\Delta_j u^{(n)}\|_{L^2}\, d\tau \cdot \int_0^T e^{-c_1 2^{2\beta j}s}\, ds.\end{aligned}$$

Using the fact that there exists $c_3 > 0$ satisfying for all $j \geq 0$,

$$\int_0^T e^{-c_1 2^{2\beta j} s} ds \leq c\, 2^{-2\beta j}(1 - e^{-c_3 T}), \tag{21}$$

we get

$$\begin{aligned}
\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_1 \, d\tau \, ds &\leq c\,(1 - e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\beta\right)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \, d\tau \\
&\leq c\,(1 - e^{-c_3 T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \, d\tau}_{=\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \delta} \\
&\leq c\,(1 - e^{-c_3 T})\delta.
\end{aligned}$$

Similarly by applying Young's inequality for the time convolution, the term with K_2 is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_2 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} c\, \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \, ds \\
&\quad \leq c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \left(\int_0^T e^{-c_1 2^{2\beta j} s} ds\right) \\
&\quad \leq c\,(1 - e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \\
&\quad \leq c\,(1 - e^{-c_3 T}) \underbrace{\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}}_{\leq M} \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \\
&\quad \leq c\,(1 - e^{-c_3 T})\delta M.
\end{aligned}$$

The terms involving K_3 and K_4 obey also the same bound,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_3 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} c\, \|\Delta_j u^{(n)}(\tau)\|_{L^2} \left(\sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2}\right) d\tau ds \\
&\quad \leq c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2} \, d\tau \left(\int_0^T e^{-c_1 2^{2\beta j} s} ds\right)
\end{aligned}$$

Then, owing to (21)

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_3 \; d\tau \; ds \\
&\quad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2} \; d\tau \\
&\quad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \; d\tau \\
&\quad = c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \; d\tau \\
&\quad \leq c\,(1-e^{-c_3 T}) \underbrace{\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}}_{\leq M} \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \\
&\quad \leq c\,(1-e^{-c_3 T})\delta M.
\end{aligned}
$$

The term containing K_4 is bounded by

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_4 \; d\tau \; ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} 2^j 2^{\frac{d}{2}k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \; d\tau ds \\
&\quad \leq c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \sum_{k\geq j-1} 2^j 2^{\frac{d}{2}k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \; d\tau \left(\int_0^T e^{-c_1 2^{2\beta j} s} ds \right).
\end{aligned}
$$

Hence, due to (21)

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_4 \; d\tau \; ds \\
&\quad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\beta\right)j} 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \; d\tau \\
&\quad = c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} \sum_{k\geq j-1} 2^{\left(1+\frac{d}{2}-2\beta\right)(j-k)} 2^{\left(1+\frac{d}{2}\right)k} \|\Delta_k u^{(n)}\|_{L^2} 2^{\left(1+\frac{d}{2}-2\beta\right)k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \; d\tau \\
&\quad = c\,(1-e^{-c_3 T}) \int_0^T \|u^{(n)}\|_{B_{2,1}^{1+\frac{d}{2}}} \|w^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \; d\tau \\
&\quad \leq c\,(1-e^{-c_3 T}) \underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq \delta} \underbrace{\|w^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)}}_{\leq M} \\
&\quad \leq c\,(1-e^{-c_3 T})\delta M.
\end{aligned}
$$

Collecting the estimates above and inserting them in (20), we obtain

$$
\|w^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \frac{\delta}{2} + c\,(1-e^{-c_3 T})\delta + c\,(1-e^{-c_3 T})\delta M.
$$

Choosing T sufficiently small such that $c\,(1-e^{-c_3 T}) \le \min\left(\frac{1}{4M}, \frac{1}{4}\right)$, we get

$$\|w^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \le \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u, w) \in Y$ such that subsequence of (u^n, w^n) (still denoted by (u^n, w^n)) satisfies

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right), \\ w^n &\overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}\right). \end{aligned}$$

In order to show that (u, w) is a weak solution of (1), we need to further extract a subsequence that converges strongly to (u, w). We use the Aubin-Lions lemma. We can show by making use of the equation (8) that $(\partial_t u_n, \partial_t w_n)$ is uniformly bounded in

$$\begin{aligned} \partial_t u^n &\in L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right) \cap L^2\left(0, T, B_{2,1}^{\frac{d}{2}+1-3\alpha}\right), \\ \partial_t w^n &\in L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}\right) \cap L^2\left(0, T, B_{2,1}^{\frac{d}{2}+1-3\beta}\right). \end{aligned}$$

Since we are in this case in the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions lemma to show that a subsequence of a weakly convergent subsequence, still denoted by (u^n, w^n), has the following strongly convergent property

$$u^n \to u \quad \text{in} \quad L^2\left(0, T, B_{2,1}^{1+\frac{d}{2}-\gamma_1}(Q)\right), \quad w^n \to w \quad \text{in} \quad L^2\left(0, T, B_{2,1}^{1+\frac{d}{2}-\gamma_2}(Q)\right),$$

where $\alpha \le \gamma_1 \le 3\alpha$, $\beta \le \gamma_2 \le 3\beta$, and $Q \subset \mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u, w) is indeed a weak solution of (1). This completes the proof for the existence part of Theorem 1.1.

3.2 | Uniqueness of weak solutions

Proof. Assume that $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ are two solutions of (1) in the regularity class in (4) and (5). Their difference $(\tilde{u}, \tilde{w})$ with

$$\tilde{u} = u^{(2)} - u^{(1)} \quad \text{and} \quad \tilde{w} = w^{(2)} - w^{(1)}$$

satisfies

$$\begin{cases} \partial_t \tilde{u} + (\nu + k)(-\Delta)^\alpha \tilde{u} = -\mathbb{P}(u^{(2)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(1)}) + 2k\nabla \times \tilde{w}, \\ \partial_t \tilde{w} + \gamma(-\Delta)^\beta \tilde{w} = -4k\tilde{w} - 2k\nabla \times \tilde{u} - u^{(2)} \cdot \nabla \tilde{w} - \tilde{u} \cdot \nabla w^{(1)}, \\ \nabla \cdot \tilde{u} = 0, \\ \tilde{u}(x, 0) = 0, \quad \tilde{w}(x, 0) = 0. \end{cases} \tag{22}$$

We estimate the difference $(\tilde{u}, \tilde{w})$ in $L^2(\mathbb{R}^d)$. Dotting (22) by $(\tilde{u}, \tilde{w})$ and applying the divergence-free condition, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{w}\|_{L^2}^2\right) + (\nu + k)\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + \gamma\|\Lambda^\beta \tilde{w}\|_{L^2}^2 + 4k\|\tilde{w}\|_{L^2}^2 \\ &= -\int u^{(2)} \cdot \nabla \tilde{u} \cdot \tilde{u}\, dx - \int \tilde{u} \cdot \nabla u^{(1)} \cdot \tilde{u}\, dx \\ &- \int u^{(2)} \cdot \nabla \tilde{w} \cdot \tilde{w}\, dx - \int \tilde{u} \cdot \nabla w^{(1)} \cdot \tilde{w}\, dx \\ &= L_1 + L_2 + L_3 + L_4, \end{aligned}$$

where

$$L_1 = -\int u^{(2)} \cdot \nabla \tilde{u} \cdot \tilde{u}\, dx,$$

$$L_2 = -\int \tilde{u} \cdot \nabla u^{(1)} \cdot \tilde{u}\, dx,$$

$$L_3 = -\int u^{(2)} \cdot \nabla \tilde{w} \cdot \tilde{w}\, dx,$$

$$L_4 = -\int \tilde{u} \cdot \nabla w^{(1)} \cdot \tilde{w}\, dx.$$

Due to $\nabla \cdot u^{(2)} = 0$, we find $L_1 = L_3 = 0$ after integration by parts. In fact,

$$\begin{aligned} L_1 &= -\int u^{(2)} \cdot \nabla \tilde{u} \cdot \tilde{u}\, dx \\ &= -\int u^{(2)} \cdot \nabla \left(\frac{1}{2}|\tilde{u}|^2\right)\, dx \\ &= -\int \nabla \cdot (u^{(2)} \frac{1}{2}|\tilde{u}|^2)\, dx \\ &= 0. \end{aligned}$$

By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} |L_2| &= \left| -\int \tilde{u} \cdot \nabla u^{(1)} \cdot \tilde{u}\, dx \right| \\ &\leq \|\nabla u^{(1)}\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 \\ &\leq \sum_{j \geq -1} \|\Delta_j \nabla u^{(1)}\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 \\ &\leq \underbrace{c \sum_{j \geq -1} 2^{dj\left(\frac{1}{2} - \frac{1}{\infty}\right)} 2^j \|\Delta_j u^{(1)}\|_{L^2}}_{=\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}}} \|\tilde{u}\|_{L^2}^2 \leq c\, \|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}} \|\tilde{u}\|_{L^2}^2. \end{aligned} \tag{23}$$

To bound L_4, we set

$$\frac{1}{p} = \frac{1}{2} - \frac{\beta}{d}, \quad \frac{1}{q} = \frac{\beta}{d} \quad \left(or\ \frac{d}{q} = \beta \right)$$

By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} |L_4| &= \left| -\int \tilde{u} \cdot \nabla w^{(1)} \cdot \tilde{w}\, dx \right| \\ &\leq \|\tilde{u}\|_{L^2} \|\nabla w^{(1)}\|_{L^q} \|\tilde{w}\|_{L^p} \\ &\leq \sum_{j \geq -1} \|\Delta_j \nabla w^{(1)}\|_{L^q} \|\tilde{u}\|_{L^2} \|\tilde{w}\|_{L^p} \\ &\leq c \sum_{j \geq -1} 2^j 2^{dj\left(\frac{1}{2} - \frac{1}{q}\right)} \|\Delta_j w^{(1)}\|_{L^2} \|\tilde{u}\|_{L^2} \|\tilde{w}\|_{L^p} \\ &\leq \sum_{j \geq -1} 2^{j + \frac{dj}{2} - \beta j} \|\Delta_j w^{(1)}\|_{L^2} \|\tilde{w}\|_{L^p} \|\tilde{u}\|_{L^2} \\ &\leq c\, \|w^{(1)}\|_{B^{1+\frac{d}{2}-\beta}_{2,1}} \|\tilde{u}\|_{L^2} \|\Lambda^\beta \tilde{w}\|_{L^2} \\ &\leq \frac{\gamma}{2} \|\Lambda^\beta \tilde{w}\|_{L^2}^2 + c\, \|w^{(1)}\|^2_{B^{1+\frac{d}{2}-\beta}_{2,1}} \|\tilde{u}\|_{L^2}^2, \end{aligned}$$

where in the last inequality we have made use of

$$\|\tilde{w}\|_{L^p} \le c\ \|\Lambda^\alpha \tilde{w}\|_{L^2}.$$

Combining the estimates leads to

$$\begin{aligned}&\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{w}\|_{L^2}^2) + 2(\nu+k)\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + \gamma\ \|\Lambda^\beta \tilde{w}\|_{L^2}^2 + 8k\ \|\tilde{w}\|_{L^2}^2 \\ &\quad\le \left(c\ \|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}} + c\ \|w^{(1)}\|^2_{B_{2,1}^{1+\frac{d}{2}-\beta}} \right) \left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{w}\|_{L^2}^2\right).\end{aligned} \tag{24}$$

Since $u^{(1)} \in L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\right)$ and $w^{(1)} \in L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\right) \cap L^\infty\left(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}\right)$,

$$\begin{aligned}&\int_0^T \|u^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} dt < \infty, \\ &\int_0^T \|w^{(1)}(t)\|^2_{B_{2,1}^{1+\frac{d}{2}-2\beta}} dt \le \int_0^T \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}}\ \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} dt \\ &\qquad \le \|w^{(1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}\right)} \int_0^T \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}}\ dt < \infty.\end{aligned}$$

Applying Gronwall's inequality to (24) yields

$$\|\tilde{u}\|_{L^2} = \|\tilde{w}\|_{L^2} = 0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 1.1. □

4 | PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is similar to the one of Theorem 1.1. To avoid repetitions, we will refer next to some inequalities already showed in the proof of Theorem 1.1. In this proof, we consider the system of equations (1) with $\beta = 0$, that is,

$$\begin{cases} \partial_t u + (\nu+k)(-\Delta)^\alpha u + u\cdot\nabla u + \nabla\Pi - 2k\nabla\times w = 0, \\ \partial_t w + (4k+\gamma)w + 2k\nabla\times u + u\cdot\nabla w = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\quad w(x,0) = w_0(x). \end{cases} \tag{25}$$

4.1 | Existence of a weak solution

This subsection proves the existence part of Theorem 1.2. The approach is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (25) in the weak sense.

Proof for the existence part of Theorem 1.2. We consider a successive approximation $\left\{(u^{(n)}, w^{(n)})\right\}$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0, \qquad w^{(1)} = S_2 w_0 \\ \partial_t u^{(n+1)} + (\nu+k)(-\Delta^\alpha)u^{(n+1)} = \mathbb{P}(-u^{(n)}\cdot\nabla u^{(n+1)}) + 2k\nabla\times w^{(n)} \\ \partial_t w^{(n+1)} = -(4k+\gamma)w^{(n+1)} - 2k\nabla\times u^{(n)} - u^{(n)}\cdot\nabla w^{(n+1)} \\ u^{(n+1)}(x,0) = S_{n+1}u_0, \qquad w^{(n+1)}(x,0) = S_{n+1}w_0, \end{cases} \tag{26}$$

where $\mathbb{P} = I - \nabla(-\Delta)^{-1}\mathrm{div}$ is the standard Leray projection. For

$$M = 2\left(\|u_0\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \|w_0\|_{B^{\frac{d}{2}}_{2,1}}\right),$$

$T > 0$ being sufficiently small and $0 < \delta < 1$ (to be specified later), we set

$$Y \equiv \left\{ (u,w) \mid \quad \|u\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le M\,,\ \ \|w\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le M, \right.$$
$$\left. \|u\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \le \delta\,,\ \ \|w\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le \delta \right\}. \tag{27}$$

We show that $\left\{(u^{(n)}, w^{(n)})\right\}$ has a subsequence that converges to the weak solution of (25). This process consists of three main steps. The first step is to show that $\left\{(u^{(n)}, w^{(n)})\right\}$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma. While the last step is to show that the limit is indeed a weak solution of (25).

To show the uniform bound for $\left\{(u^{(n)}, w^{(n)})\right\}$ in Y, we prove by induction. Clearly,

$$\|u^{(1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} = \|S_2 u_0\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le M,$$
$$\|w^{(1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} = \|S_2 w_0\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le M.$$

If $T > 0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \le T\|S_2 u_0\|_{B^{1+\frac{d}{2}}_{2,1}} \le T\, c\|u_0\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le \delta,$$

$$\|w^{(1)}\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le T\|S_2 w_0\|_{B^{\frac{d}{2}}_{2,1}} \le T\, c\|w_0\|_{B^{\frac{d}{2}}_{2,1}} \le \delta.$$

Assuming that $(u^{(n)}, w^{(n)})$ obeys the bounds defined in Y, namely,

$$\|u^{(n)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} \le M, \quad \|w^{(n)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le M,$$
$$\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \le \delta, \quad \|w^{(n)}\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \le \delta.$$

we prove that $\left\{(u^{(n+1)}, w^{(n+1)})\right\}$ obeys the same bound for suitably selected $T > 0, M > 0$, and $\delta > 0$. For sake of clarity, the proof of the four bounds is achieved in the following four steps. □

4.1.1 | The estimate of $u^{(n+1)}$ in $L^\infty\left(0, T, B^{1+\frac{d}{2}-2\alpha}_{2,1}\left(\mathbb{R}^d\right)\right)$

Following the same method as in the proof of the first step of Theorem 1.1, we write the inequality

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}$$
$$+ \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)}(J_1 + \ \dots\ + J_4)\, d\tau, \tag{28}$$

where

$$J_1 = c\,\|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2},$$
$$J_2 = c\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n+1)}\|_{L^2},$$
$$J_3 = c\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\|\tilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2},$$
$$J_4 = c\,2^j\|\Delta_j w^{(n)}\|_{L^2}.$$

The terms on the right hand side can be estimated as follows. Recalling the definition of J_1 above and using the inductive assumption on $u^{(n)}$, we have for any $t \le T$,

$$\begin{aligned}
&\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{c_0 2^{2\alpha j}(t-\tau)} J_1\,d\tau\\
&= c\int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\|\Delta_j u^{(n+1)}\|_{L^2}\sum_{m\le j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau\\
&\le c\int_0^t \underbrace{\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\|\Delta_j u^{(n+1)}\|_{L^2}}_{=\|u^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}}\underbrace{\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}}_{=\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}}\,d\tau\\
&\le c\,\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}\\
&\le c\,\delta\,\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}.
\end{aligned}$$

The terms with J_2 and J_3 can be similarly estimated and obey the same bound. In fact, by Young's inequality for series convolution,

$$\begin{aligned}
&\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_2\,d\tau\\
&\le c\int_0^t \sum_{j\le -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j} 2^{2\alpha(m-j)}2^{\left(1+\frac{d}{2}-2\alpha\right)m}\|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\,d\tau\\
&\le c\int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}}\|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\,d\tau\\
&\le c\,\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}\\
&\le c\,\delta\,\|u^{(n+1)}\|_{L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)}.
\end{aligned}$$

Similarly the term with J_3 is bounded by

$$\begin{aligned}
&\sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_3\,d\tau\\
&= \int_0^t \sum_{j\ge -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\sum_{k\ge j-1} 2^j 2^{\frac{d}{2}k}\|\tilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\,d\tau\\
&= c\int_0^t \sum_{j\ge -1}\sum_{k\ge j-1} 2^{\left(2+\frac{d}{2}-2\alpha\right)(j-k)}2^{\left(1+\frac{d}{2}\right)k}\|\Delta_k u^{(n)}\|_{L^2}2^{\left(1+\frac{d}{2}-2\alpha\right)k}\|\tilde{\Delta}_k u^{(n+1)}\|_{L^2}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq c \int_0^t \|u^{(n)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}}} \|u^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}\, d\tau \\
&\leq c\, \|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \\
&\leq c\, \delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}
$$

Now for the term with J_4, we write

$$
\begin{aligned}
\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \int_0^t e^{-c_0 2^{2\alpha j(t-\tau)}} J_4\, d\tau &= \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \int_0^t e^{-c_0 2^{2\alpha j(t-\tau)}} c 2^j \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\leq \sum_{j\geq -1} \int_0^t c\, 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\underbrace{\leq}_{\text{since } \alpha\geq 1} c\, \|w^{(n)}\|_{L^1\left(0,T,B_{2,1}^{\frac{d}{2}}\right)} \\
&\leq c\, \delta.
\end{aligned}
$$

Collecting the bounds above and inserting them in (28), we find for any $t \leq T$

$$
\|u^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + c\, \delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} + c\, \delta.
$$

Therefore

$$
\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \leq \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + c\, \delta \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} + c\, \delta.
$$

Choosing δ such that $c\, \delta \leq \min\left(\frac{1}{4}, \frac{M}{4}\right)$, we get

$$
\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \leq \frac{M}{2} + \frac{1}{4}\|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} + \frac{M}{4},
$$

which implies

$$
\|u^{(n+1)}(t)\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \leq M.
$$

4.1.2 | The estimate of $\|u^{(n+1)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}\right)}$

Following the same method as in the proof of the second step of Theorem 1.1, we write the inequality

$$
\begin{aligned}
\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \\
+ \int_0^T \int_0^s \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}(s-\tau)} (J_1 + \ldots + J_4)\, d\tau\, ds.
\end{aligned} \tag{29}
$$

We estimate the terms on the right and start with the first term.

$$
\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt = c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} (1 - e^{-c_0 2^{2\alpha j}T}) \|\Delta_j u_0^{(n+1)}\|_{L^2}.
$$

Since $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}$, then by the dominated convergence theorem

$$\lim_{T\to 0}\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2} \leq \frac{\delta}{4}.$$

Applying Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1\, d\tau\, ds\\
&\quad = c\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}\, d\tau\, ds\\
&\quad \leq c\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}\, d\tau \int_0^T e^{-c_0 2^{2\alpha j}s}\, ds.
\end{aligned}$$

Then, using the fact that there exists $c_2 > 0$ satisfying for $j \geq 0$,

$$\int_0^T e^{-c_0 2^{2\alpha j}s} ds \leq c\, 2^{-2\alpha j}(1-e^{-c_2 T}), \tag{30}$$

we get

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1\, d\tau\, ds\\
&\qquad \leq c\,(1-e^{-c_2 T})\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(m)}\|_{L^2}\, d\tau\\
&\qquad \leq c\,(1-e^{-c_2 T})\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}\|_{L^2}\, d\tau\\
&\qquad \leq c\,(1-e^{-c_2 T})\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}\underbrace{\|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)}}_{\leq\delta}\\
&\qquad \leq c\,\delta\,(1-e^{-c_2 T})\|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)},
\end{aligned}$$

The terms with J_2 and J_3 can be similarly estimated and obey the same bound.

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2\, d\tau\, ds\\
&\quad = c\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n+1)}\|_{L^2} d\tau\, ds\\
&\quad \leq c\sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\int_0^T \|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j}s}\, ds.
\end{aligned}$$

Hence due to (30)

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\leq c(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n+1)}\|_{L^2} d\tau \\
&\leq c(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} d\tau \\
&\leq c(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \\
&\leq c(1-e^{-c_2 T}) \ \delta \ \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&= c \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^T \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}
$$

By (30) and the inequality above,

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\leq c \ (1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \\
&\leq c \ (1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j} \|\tilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\leq c \ (1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}-2\alpha\right)j} \|\tilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\leq c \ (1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \|u^{(n)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \\
&\leq c \ (1-e^{-c_2 T}) \ \delta \ \|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}
$$

The term with J_4 is bounded by

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \, ds \\
&= c \int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}\right)j} \int_0^T \|\Delta_j w^{(n)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq c\,(1-e^{-c_2 T})\int_0^T \sum_{j\geq -1} 2^{\left(2+\frac{d}{2}-2\alpha\right)j}\|\Delta_j w^{(n)}\|_{L^2}\,d\tau\\
&\underset{\text{since }\alpha\geq 1}{\leq} c\,(1-e^{-c_2 T})\int_0^T \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j}\|\Delta_j w^{(n)}\|_{L^2}\,d\tau\\
&= c(1-e^{-c_2 T})\|w^{(n)}\|_{L^1\left(0,T,B_{2,1}^{\frac{d}{2}}\right)}.
\end{aligned}
$$

Collecting the estimates above and inserting them in (29) leads to

$$
\begin{aligned}
\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} &\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T})\|u^{(n+1)}\|_{L^\infty\left(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}\\
&\quad + c\,(1-e^{-c_2 T})\|w^{(n)}\|_{L^1\left(0,T,B_{2,1}^{\frac{d}{2}}\right)}\\
&\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T})M + c\,(1-e^{-c_2 T})\delta.
\end{aligned}
$$

Choosing T sufficiently small such that $c(1-e^{-c_2 T}) \leq \min\left(\frac{1}{4M},\frac{1}{2}\right)$, we get

$$
\|u^{(n+1)}\|_{L^1\left(0,T,B_{2,1}^{1+\frac{d}{2}}\right)} \leq \frac{\delta}{4}+\frac{\delta}{4}+\frac{\delta}{2} = \delta.
$$

4.1.3 | The estimate of $w^{(n+1)}$ in $L^\infty\left(0,T,B_{2,1}^{\frac{d}{2}}\left(\mathbb{R}^d\right)\right)$

Applying Δ_j to the third equation in (26) and then dotting with $\Delta_j w^{(n+1)}$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2}^2 + (4k+\gamma)\|\Delta_j w^{(n+1}\|_{L^2}^2 &= -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}dx\\
&\quad -\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}dx\\
&= B_1 + B_2,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
B_1 &= -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}dx,\\
B_2 &= -\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}dx.
\end{aligned}
$$

By Hölder's inequality and Bernstein's inequality

$$
\begin{aligned}
|B_1| &= \left|-2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}dx\right|\\
&\leq 2k\,\|\Delta_j(\nabla\times u^{(n)})\|_{L^2}\|\Delta_j w^{(n+1)}dx\|_{L^2}\\
&\leq c\,2^j\,\|\Delta_j u^{(n)}\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2}.
\end{aligned}
$$

By Lemma 2.6,

$$
\begin{aligned}
|B_2| &= \left|-\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx\right|\\
&\le c\,\|\Delta_j w^{(n+1)}\|_{L^2}^2\sum_{m\le j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}\\
&\quad + c\,\|\Delta_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2}\\
&\quad + c\,\|\Delta_j w^{(n+1)}\|_{L^2}2^j\sum_{k\le j-1}2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}
$$

Inserting the estimates above in (31) and eliminating $\|\Delta_j w^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2} + (8k+2\gamma)\|\Delta_j w^{(n+1)}\|_{L^2} \le{}& c\,2^j\|\Delta_j u^{(n)}\|_{L^2}\\
&+ c\,\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2}\\
&+ c\,\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2}\\
&+ c\,2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}
\tag{32}
$$

Integrating (32) in time yields, for any $t \le T$,

$$
\|\Delta_j w^{(n+1)}\|_{L^2} \le e^{-(8k+2\gamma)t}\|\Delta_j w_0^{(n+1)}\|_{L^2} + \int_0^t e^{-(8k+2\gamma)(t-\tau)}(K_1+\ \dots\ +K_4)d\tau, \tag{33}
$$

where

$$
\begin{aligned}
K_1 &= c\,2^j\|\Delta_j u^{(n)}\|,\\
K_2 &= c\,\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m u^{(n)}\|_{L^2},\\
K_3 &= c\,\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j}2^{\left(1+\frac{d}{2}\right)m}\|\Delta_m w^{(n+1)}\|_{L^2},\\
K_4 &= c\,2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}
$$

We multiply (33) by $2^{\left(\frac{d}{2}\right)j}$ and sum over j to get

$$
\|w^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \le \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + \sum_{j\ge -1}\int_0^t e^{-(8k+2\gamma)(t-\tau)}2^{\left(\frac{d}{2}\right)j}(K_1+\ \dots\ +K_4)\,d\tau. \tag{34}
$$

The term with K_1 is bounded by

$$
\begin{aligned}
\sum_{j\ge -1}2^{\frac{d}{2}j}\int_0^t e^{-(8k+2\gamma)(t-\tau)}K_1\,d\tau &\le \int_0^t\sum_{j\ge -1}c\,2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\,d\tau\\
&\le c\,\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}.
\end{aligned}
$$

The terms with K_2 through K_4 can be bounded suitably and obey the same bound. In fact, for the term involving K_2, we write

$$\begin{aligned}
&\sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \int_0^t e^{-(8k+2\gamma)(t-\tau)} c\, \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\quad \leq c \int_0^t \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} d\tau \\
&\quad \leq c \int_0^t \|w^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \|u^{(n)}\|_{B^{1+\frac{d}{2}}_{2,1}}\, d\tau \\
&\quad \leq c\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}.
\end{aligned}$$

Similarly the term with K_3 is bounded by

$$\begin{aligned}
&\sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \int_0^t e^{-(8k+2\gamma)(t-\tau)} c\, \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}\|_{L^2} d\tau \\
&\quad \leq c \int_0^t \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j w^{(n+1)}\|_{L^2}\, d\tau \\
&\quad \leq c\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}.
\end{aligned}$$

For the term with K_4, we write

$$\begin{aligned}
&\sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \int_0^t c\, 2^j \sum_{k\geq j-1} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} 2^{\frac{dk}{2}}\, d\tau \\
&\quad \leq c \int_0^t \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \|\tilde{\Delta}_j w^{(n+1)}\|_{L^2} d\tau \\
&\quad \leq c\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}.
\end{aligned}$$

Collecting the estimates and inserting them in (34), we obtain for any $t \leq T$

$$\begin{aligned}
\|w^{(n+1)}(t)\|_{B^{\frac{d}{2}}_{2,1}} &\leq \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + c\, \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} + c\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \\
&\leq \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + c\,\delta + c\,\delta\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)}.
\end{aligned}$$

Therefore

$$\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + c\,\delta + c\,\delta\, \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)}.$$

Choosing $c\,\delta \leq \min\left(\frac{1}{4}, \frac{M}{4}\right)$, we get

$$\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq \frac{M}{2} + \frac{M}{4} + \frac{1}{4}\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)},$$

which implies

$$\|w^{(n+1)}(t)\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq M.$$

4.1.4 | The estimate of $\|w^{(n+1)}(t)\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)}$

We multiply (33) by $2^{\left(\frac{d}{2}\right)j}$, sum over j and integrate in time to get

$$\|w^{(n+1)}\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t} \|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \\ + \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} (K_1 + \ldots + K_4)\, d\tau\, ds. \tag{35}$$

Clearly,

$$\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t} \|\Delta_j w_0^{(n+1)}\|_{L^2} dt = c \sum_{j\geq -1} 2^{\frac{d}{2}j} (1 - e^{-(8k+2\gamma)T}) \|\Delta_j w_0^{(n+1)}\|_{L^2}.$$

Since $w_0 \in B^{\frac{d}{2}}_{2,1}$, it follows from the dominated convergence theorem that

$$\lim_{T\to 0} \sum_{j\geq -1} 2^{\frac{d}{2}j} (1 - e^{-(8k+2\gamma)T}) \|\Delta_j w_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t} \|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \leq \frac{\delta}{2}.$$

Applying the Young's inequality for the time convolution, the term with K_1 is bounded by

$$\begin{aligned} &\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_1\, d\tau\, ds \\ &= c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2}\, d\tau\, ds \\ &\leq \left(c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2}\, d\tau \right) \left(\int_0^T e^{-(8k+2\gamma)s} ds \right) \\ &\leq c\, (1 - e^{-(8k+2\gamma)T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2}\, d\tau}_{=\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \leq \delta} \\ &\leq c\, (1 - e^{-(8k+2\gamma)T}) \delta. \end{aligned}$$

Similarly by applying Young's inequality for the time convolution the term with K_2 is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_2 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2} d\tau ds \\
&\quad \leq \left(c \sum_{j\leq -1} 2^{\frac{d}{2}j} \int_0^T \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \right) \left(\int_0^T e^{-(8k+2\gamma)s} ds \right) \\
&\quad \leq c(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\leq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \\
&\quad \leq c\,(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\quad \leq c\,(1-e^{-(8k+2\gamma)T}) \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \\
&\quad \leq c(1-e^{-(8k+2\gamma)T})\delta M.
\end{aligned}$$

Applying Young's inequality for the time convolution, the term with K_3 is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_3 \, d\tau ds \\
&= c \left(\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2} \right) \left(\sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}\|_{L^2} \right) d\tau \, ds \\
&\leq \left(c \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}\|_{L^2} \, d\tau \right) \left(\int_0^T e^{-(8k+2\gamma)s} ds \right) \\
&\leq c\,(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{\left(1+\frac{d}{2}\right)m} \|\Delta_m w^{(n+1)}\|_{L^2} \, d\tau \\
&\leq c\,(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2} \, d\tau \\
&\leq c(1-e^{-(8k+2\gamma)T}) \|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)} \\
&\leq c\,(1-e^{-(8k+2\gamma)T})\delta M.
\end{aligned}$$

Similarly the term with K_4 is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\left(\frac{d}{2}\right)j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_4 \, d\tau ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau ds \\
&\quad \leq \left(c \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j} \int_0^T \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\tilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau \right) \left(\int_0^T e^{-(8k+2\gamma)s} ds \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq c(1-e^{-(8k+2\gamma)T})\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\sum_{k\geq j-1} 2^{\frac{d}{2}k}\|\tilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}d\tau\\
&\leq c(1-e^{-(8k+2\gamma)T})\int_0^T \sum_{j\geq -1} 2^{\left(1+\frac{d}{2}\right)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{j\geq -1} 2^{\frac{d}{2}j}\|\tilde{\Delta}_j w^{(n+1)}\|_{L^2}d\tau\\
&\leq c(1-e^{-(8k+2\gamma)T})\|w^{(n+1)}\|_{L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right)}\|u^{(n)}\|_{L^1\left(0,T,B^{1+\frac{d}{2}}_{2,1}\right)}\\
&\leq c(1-e^{-(8k+2\gamma)T})\delta M.
\end{aligned}
$$

Collecting the estimates above and inserting them in (35), we obtain

$$
\|w^{(n+1)}\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq \frac{\delta}{2} + c(1-e^{-(8k+2\gamma)T})\delta + c(1-e^{-(8k+2\gamma)T})\delta M.
$$

Choosing T sufficiently small such that $c(1-e^{-(8k+2\gamma)T}) \leq \min\left(\frac{1}{4M},\frac{1}{4}\right)$, we get

$$
\|w^{(n+1)}\|_{L^1\left(0,T,B^{\frac{d}{2}}_{2,1}\right)} \leq \frac{\delta}{2}+\frac{\delta}{4}+\frac{\delta}{4} = \delta.
$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u,w) \in Y$ such that a subsequence of (u^n, w^n) (still denoted by (u^n, w^n)) satisfies

$$
\begin{aligned}
&u^n \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right),\\
&w^n \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty\left(0,T,B^{\frac{d}{2}}_{2,1}\right).
\end{aligned}
$$

In order to show that (u,w) is a weak solution of (25), we need to further extract a subsequence that converges strongly to (u,w). We use the Aubin-Lions Lemma. We can show by making use of the equation (26) that $(\partial_t u^n, \partial_t w^n)$ is uniformly bounded in

$$
\begin{aligned}
&\partial_t u^n \in L^1\left(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)\cap L^2\left(0,T,B^{1+\frac{d}{2}-3\alpha}_{2,1}\right),\\
&\partial_t w^n \in L^1\left(0,T,B^{\frac{d}{2}-2\alpha}_{2,1}\right)\cap L^2\left(0,T,B^{\frac{d}{2}-\alpha}_{2,1}\right).
\end{aligned}
$$

Since we are in this case in the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of a weakly convergent subsequence, still denoted by (u^n, w^n), has the following strongly convergent property

$$
u^n \to u \quad \text{in} \quad L^2\left(0,T,B^{1+\frac{d}{2}-\gamma_3}_{2,1}(Q)\right), \quad w^n \to w \quad \text{in} \quad L^2\left(0,T,B^{\frac{d}{2}-\gamma_4}_{2,1}(Q)\right),
$$

where $\alpha \leq \gamma_3 \leq 3\alpha$, $0 \leq \gamma_4 \leq 2\alpha$, and $Q \subset \mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u,w) is indeed a weak solution of (25). This completes the proof for the existence part of Theorem 1.2.

4.2 | Uniqueness of weak solutions

Proof. Assume that $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ are two solutions of (25) in the regularity class in (4) and (5). Their difference $(\tilde{u}, \tilde{w})$ with

$$
\tilde{u} = u^{(2)} - u^{(1)} \quad \text{and} \quad \tilde{w} = w^{(2)} - w^{(1)}
$$

satisfies

$$
\begin{cases}
\partial_t \tilde{u} + (\nu+k)(-\Delta)^\alpha \tilde{u} = -\mathbb{P}(u^{(2)}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla u^{(1)}) + 2k\nabla\times\tilde{w},\\
\partial_t \tilde{w} = -(4k+\gamma)\tilde{w} - 2k\nabla\times\tilde{u} - u^{(2)}\cdot\nabla\tilde{w} - \tilde{u}\cdot\nabla w^{(1)},\\
\nabla\cdot\tilde{u} = 0,\\
\tilde{u}(x,0) = 0, \quad \tilde{w}(x,0) = 0.
\end{cases}
\tag{36}
$$

We estimate the difference $(\tilde{u}, \tilde{w})$ in $L^2(\mathbb{R}^d)$. Dotting (36) by $(\tilde{u}, \tilde{w})$ and applying the divergence-free condition, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2+\|\tilde{w}\|_{L^2}^2\right)+(\nu+k)\|\Lambda^\alpha \tilde{u}\|_{L^2}^2+(4k+\gamma)\|\tilde{w}\|_{L^2}^2\\
&=-\int u^{(2)}\cdot\nabla\tilde{u}\cdot\tilde{u}\,dx-\int \tilde{u}\cdot\nabla u^{(1)}\cdot\tilde{u}\,dx\\
&-\int u^{(2)}\cdot\nabla\tilde{w}\cdot\tilde{w}\,dx-\int \tilde{u}\cdot\nabla w^{(1)}\cdot\tilde{w}\,dx\\
&=L_1+L_2+L_3+L_4,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1&=-\int u^{(2)}\cdot\nabla\tilde{u}\cdot\tilde{u}\,dx,\\
L_2&=-\int \tilde{u}\cdot\nabla u^{(1)}\cdot\tilde{u}\,dx,\\
L_3&=-\int u^{(2)}\cdot\nabla\tilde{w}\cdot\tilde{w}\,dx,\\
L_4&=-\int \tilde{u}\cdot\nabla w^{(1)}\cdot\tilde{w}\,dx.
\end{aligned}
$$

Due to $\nabla\cdot u^{(2)}=0$, we find $L_1=L_3=0$ after integration by parts. As in (23),

$$
|L_2|\le c\|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}}\|\tilde{u}\|_{L^2}^2.
$$

To bound L_4, we set

$$
\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{d},\quad \frac{1}{q}=\frac{\alpha}{d}\quad\left(or\ \frac{d}{q}=\alpha\right).
$$

By Hölder's inequality,

$$
\begin{aligned}
|L_4|&=\left|-\int \tilde{u}\cdot\nabla w^{(1)}\cdot\tilde{w}\,dx\right|\\
&\le\|\tilde{w}\|_{L^2}\|\nabla w^{(1)}\|_{L^q}\|\tilde{u}\|_{L^p}\\
&\le\sum_{j\ge-1}\|\Delta_j\nabla w^{(1)}\|_{L^q}\|\tilde{w}\|_{L^2}\|\tilde{u}\|_{L^p}\\
&\le c\sum_{j\ge-1}2^j2^{dj\left(\frac{1}{2}-\frac{1}{q}\right)}\|\Delta_j w^{(1)}\|_{L^2}\|\tilde{w}\|_{L^2}\|\tilde{u}\|_{L^p}\\
&=\sum_{j\ge-1}2^j2^{\frac{dj}{2}-\frac{d}{q}j}\|\Delta_j w^{(1)}\|_{L^2}\|\tilde{w}\|_{L^2}\|\tilde{u}\|_{L^p}\\
&\underset{\text{since }\alpha\ge1}{\le}c\sum_{j\ge-1}2^{\frac{dj}{2}}\|\Delta_j w^{(1)}\|_{L^2}\|\tilde{w}\|_{L^2}\|\tilde{u}\|_{L^p}\\
&\le c\,\|w^{(1)}\|_{B_{2,1}^{\frac{d}{2}}}\|\tilde{w}\|_{L^2}\|\Lambda^\alpha\tilde{u}\|_{L^2}\\
&\le\frac{(\nu+k)}{2}\|\Lambda^\alpha\tilde{u}\|_{L^2}^2+c\|w^{(1)}\|_{B_{2,1}^{\frac{d}{2}}}^2\|\tilde{w}\|_{L^2}^2,
\end{aligned}
$$

where in the last inequality, we make use of

$$
\|\tilde{u}\|_{L^p}\le c\,\|\Lambda^\alpha\tilde{u}\|_{L^2}.
$$

Combining the estimates leads to

$$
\begin{aligned}
&\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2+\|\tilde{w}\|_{L^2}^2)+(\nu+k)\|\Lambda^\alpha\tilde{u}\|_{L^2}^2+(8k+2\gamma)\|\tilde{w}\|_{L^2}^2\\
&\le\left(c\|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}}+c\,\|w^{(1)}\|_{B_{2,1}^{\frac{d}{2}}}^2\right)\left(\|\tilde{u}\|_{L^2}^2+\|\tilde{w}\|_{L^2}^2\right).
\end{aligned}
\tag{37}
$$

Since $u^{(1)} \in L^1\left(0, T, B_{2,1}^{1+\frac{d}{2}}\right)$ and $w^{(1)} \in L^1\left(0, T, B_{2,1}^{\frac{d}{2}}\right) \cap L^\infty\left(0, T, B_{2,1}^{\frac{d}{2}}\right)$,

$$\int_0^T \|u^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} dt < \infty \qquad \text{and} \qquad \int_0^T \|w^{(1)}(t)\|^2_{B_{2,1}^{\frac{d}{2}}} dt \leq T\|w^{(1)}(t)\|^2_{L^\infty\left(0,T,B_{2,1}^{\frac{d}{2}}\right)}.$$

Applying Gronwall's inequality to (37) yields

$$\|\tilde{u}\|_{L^2} = \|\tilde{w}\|_{L^2} = 0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 1.2. □

CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Ben Said O, Wu J. Unique weak solutions of the d-dimensional micropolar equation with fractional dissipation. *Math Meth Appl Sci*. 2021;44:345–377. https://doi.org/10.1002/mma.6740