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UNIQUENESS OF WEAK SOLUTIONS TO THE BOUSSINESQ EQUATIONS WITHOUT THERMAL DIFFUSION*

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Abstract. This paper focuses on the general d-dimensional ($d\geq 2$) Boussinesq equations with the fractional dissipation $(-\Delta)^\alpha u$ and without thermal diffusion. Our primary goal here is the uniqueness of weak solutions to this partially dissipated system in the weakest possible setting. The issue of the uniqueness of weak solutions is very important and can be quite difficult as in the case of the Leray-Hopf weak solutions to the 3D Navier-Stokes equations. We present two main results. The first is the global existence and uniqueness of weak solutions which assesses the global existence of L^2-weak solutions for any $\alpha>0$ and the uniqueness of the weak solutions when $\alpha\geq\frac{1}{2}+\frac{d}{4}$ for $d\geq 2$. Especially the 2D Boussinesq equations without thermal diffusion have unique and global L^2 weak solutions. The second result establishes the zero thermal diffusion limit with an explicit convergence rate for the aforementioned weak solutions. This convergence result appears to be the very first one on weak solutions of partially dissipated Boussinesq systems.

Keywords. Besov space; Boussinesq equation; Uniqueness; Weak solution.

AMS subject classifications. 35A05; 35Q35; 76D03.

1. Introduction

This paper concerns itself with the d-dimensional (d-D) Boussinesq system

$$\begin{cases}\partial_t u+u\cdot\nabla u=-\nu(-\Delta)^\alpha u-\nabla P+\theta\mathbf{e}_d, \quad x\in\mathbb{R}^d, t>0,\\ \partial_t\theta+u\cdot\nabla\theta=0, \quad x\in\mathbb{R}^d, t>0,\\ \nabla\cdot u=0, \quad x\in\mathbb{R}^d, t>0,\\ (u,\theta)|_{t=0}=(u_0,\theta_0), \quad x\in\mathbb{R}^d,\end{cases} \tag{1.1}$$

where u, P and θ represent the velocity field, the pressure and the temperature, respectively, and $\nu>0$ denotes the kinematic viscosity and $\mathbf{e}_d=(0,0,\cdots,1)$ is the unit vector in the vertical direction. Here the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha>0$ is defined via the Fourier transform,

$$\mathcal{F}((-\Delta)^\alpha f)(\xi)=(4\pi^2|\xi|^2)^\alpha\mathcal{F}(f)(\xi),$$

where $\mathcal{F}(f)(\xi)$ denotes the Fourier transform of f,

$$\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

We may sometimes use $\Lambda=(-\Delta)^{\frac{1}{2}}$. When $\theta=0$, (1.1) reduces to the generalized Navier-Stokes equations.

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The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see, e.g., the books by Gill [21], Pedlosky [42] and Majda [39]). In addition, the Boussinesq system also plays an important role in the study of the Rayleigh-Benard convection, one of the most commonly studied convection phenomena (see, e.g., [14,15,20]). The first equation in (1.1) reflects Newton's second law, with the left-hand side being the acceleration and the right-hand side being the forces due to viscosity, the pressure gradient and the buoyancy. The term $\theta\mathbf{e}_d$ models the buoyancy in the direction of gravitational force. The temperature difference generates density difference, which in turn generates the buoyancy force. The second equation in (1.1) simply states that the temperature is transported by the velocity field.

Although the diffusion process is normally modeled by the standard Laplacian operator, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian arise. Flows in the middle atmosphere travelling upwards undergo changes due to the changes in atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled using the space fractional Laplacian (see [12, 21]).

The Boussinesq system retains some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows [40]. Equation (1.1) is a partially dissipated system with no thermal diffusion. Fundamental issues on the Boussinesq system with partial or fractional dissipation such as the global existence, uniqueness and regularity problem have attracted enormous interests during the last fifteen years and significant progress has been made [1–5, 8–11, 13, 16–19, 23–31, 33–38, 46–49, 52, 54].

Our goal here is twofold. The first is to establish the global existence and uniqueness of weak solutions of (1.1) with initial data $u_0\in L^2(\mathbb{R}^d),\theta_0\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$. Our key point here is the uniqueness of solutions in a very weak setting for a partially dissipated system. Although the global regularity of this partially dissipated system in smoother functional settings has been extensively investigated, the issue concerned here is quite different. The uniqueness in the weakest possible functional settings is what we care about here. The issue of the uniqueness of weak solutions is very important and can be quite difficult as in the case of the Leray-Hopf weak solutions of the 3D Navier-Stokes equations. The Boussinesq system concerned here involves only partial dissipation and the solution space appears to be the weakest setting in which one can prove the uniqueness.

Our second goal is to understand the zero thermal diffusion limit of the fully dissipative Boussinesq equations

$$\begin{cases}\partial_t u^{(\eta)}+u^{(\eta)}\cdot\nabla u^{(\eta)}=-\nu(-\Delta)^{\alpha}u^{(\eta)}-\nabla P^{(\eta)}+\theta^{(\eta)}\mathbf{e}_d, \quad x\in\mathbb{R}^d, t>0,\\ \partial_t\theta^{(\eta)}+u^{(\eta)}\cdot\nabla\theta^{(\eta)}=\eta\Delta\theta^{(\eta)}, \quad x\in\mathbb{R}^d, t>0,\\ \nabla\cdot u^{(\eta)}=0, \quad x\in\mathbb{R}^d, t>0,\\ (u^{(\eta)},\theta^{(\eta)})|_{t=0}=(u_0^{(\eta)},\theta_0^{(\eta)}), \quad x\in\mathbb{R}^d\end{cases} \tag{1.2}$$

and show that the solution of (1.2) converges strongly to the corresponding solution of (1.1) with an explicit convergence rate as $\eta\to 0$. Due to the weak initial setup $u_0^{(\eta)}\in L^2(\mathbb{R}^d),\theta_0^{(\eta)}\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$, we resort to lower regularity quantities and the Yudovich approach. Our precise results are stated in the following theorems.

THEOREM 1.1. *Consider the d-D Boussinesq equations in (1.1).*

(1) Let $\alpha>0$ and $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$ with $\nabla\cdot u_0=0$. Let $T>0$ be arbitrarily fixed. Then (1.1) has a global weak solution (u,θ) on $[0,T]$ satisfying

$$u\in C_w([0,T];L^2)\cap L^2(0,T;H^\alpha), \quad \theta\in C_w([0,T];L^2)\cap L^\infty(0,T;L^2).$$

(2) Let $\alpha\ge\frac{1}{2}+\frac{d}{4}$. Assume $u_0\in L^2(\mathbb{R}^d)$ and $\theta_0\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$ with $\nabla\cdot u_0=0$. Then (1.1) has a unique and global weak solution (u,θ) satisfying

$$u\in C([0,T];L^2)\cap L^2(0,T;H^\alpha), \quad u\in \widetilde{L}^1(0,T;B^{1+\frac{d}{2}}_{2,2}),$$
$$\theta\in C_w([0,T];L^2)\cap L^\infty(0,T;L^2\cap L^{\frac{4d}{d+2}}),$$

where the definition of $\widetilde{L}^1(0,T;B^{1+\frac{d}{2}}_{2,2})$ can be found in Section 2. Especially, u satisfies

$$\sup_{q\ge2}\frac{1}{\sqrt{q}}\int_0^T\|\nabla u(t)\|_{L^q}\,dt<\infty.$$

Theorem 1.1 assesses the global existence of weak solutions for any $\alpha>0$ and any L^2 initial data, and the uniqueness when $\alpha\ge\frac{1}{2}+\frac{d}{4}$. A special consequence of Theorem 1.1 is the global existence of Leray-Hopf weak solutions to the d-D generalized Navier-Stokes equations with any $\alpha>0$ and $u_0\in L^2(\mathbb{R}^d)$, and the uniqueness of weak solutions of the d-D Navier-Stokes equations with $\alpha\ge\frac{1}{2}+\frac{d}{4}$. Even though there is no thermal diffusion, the crucial step of passing to the limit in the thermal convection term still goes through.

THEOREM 1.2. *Let $\alpha\ge\frac{1}{2}+\frac{d}{4}$. Assume $u_0,\theta_0,u_0^{(\eta)},\theta_0^{(\eta)}$ satisfy*

$$u_0,u_0^{(\eta)}\in L^2(\mathbb{R}^d), \quad \nabla\cdot u_0=0, \quad \nabla\cdot u_0^{(\eta)}=0, \quad \theta_0,\theta_0^{(\eta)}\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d).$$

Let (u,θ) and $(u^{(\eta)},\theta^{(\eta)})$ be the corresponding weak solutions of (1.1) and (1.2), respectively. Then the difference $(\widetilde{u},\widetilde{h})$ with

$$\widetilde{u}=u^{(\eta)}-u, \qquad \widetilde{h}=h^{(\eta)}-h, \qquad -\Delta h^{(\eta)}=\theta^{(\eta)}, \quad -\Delta h=\theta$$

satisfies, for any $t>0$,

$$\|(\widetilde{u},\nabla\widetilde{h})(t)\|_{L^2}^2\le C\,M^{(1-e^{-C_0t})}\left(\|\widetilde{u_0},\nabla\widetilde{h_0})\|_{L^2}^2+\eta t\right)^{e^{-C_0t}}, \tag{1.3}$$

where C is a pure constant, $M=\|\theta_0\|_{L^2}^2+\|\theta_0^{(\eta)}\|_{L^2}^2$ and

$$C_0=C\int_0^t\left(1+\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u\|_{L^2}^2+\frac{\|\nabla u^{(\eta)}\|_{L^p}}{p}\right)d\tau<\infty.$$

We summarize closely related previous results on the Boussinesq equations without thermal diffusion to clarify how our theorems are different. The study of the global well-posedness of the 2D Boussinesq equations, namely (1.1) with $d=2$ and $\alpha=1$ in the whole space was initiated in the papers of Hou-Li [25] and of Chae [13], in which the global and unique solutions were obtained for the initial data $(u_0,\theta_0)\in H^s(\mathbb{R}^2)$ with

$s>2$. The global existence and uniqueness of solutions to (1.1) with $d=3$ and $\alpha\geq\frac{5}{4}$ was investigated by several researchers (see, e.g., [32, 43, 50, 51, 53]). The regularity assumptions on the initial data in these papers are $(u_0,\theta_0)\in H^s(\mathbb{R}^3)$ with $s>\frac{5}{2}$ ($s>\frac{5}{4}$ in [32]). The 2D Boussinesq system in a bounded domain with the Dirichlet boundary condition was first studied by Lai, Pan and Zhao [33] and the global existence and uniqueness was obtained in the functional setting $(u_0,\theta_0)\in H^3(\mathbb{R}^2)$. Danchin and Paicu extended the Fujita-Kato result for the Naviver-Stokes equations to the Boussinesq system and, as a special consequence, obtained the well-posedness of the finite energy solutions for the 2D Boussinesq equations [18]. The paper of Larios, Lunasin and Titi [34] seriously sought the uniqueness of solutions of (1.1) in a weak setup. They were able to show, among many other results, that $u_0\in H^1(\mathbb{T}^2)$ and $\theta_0\in L^2(\mathbb{T}^2)$ lead to a unique and global strong solution of (1.1). Here $\mathbb{T}^2$ denotes the 2D periodic box. For the bounded domain Ω with Dirichlet boundary conditions, the work of He [22] further reduced the regularity assumption to $(u_0,\theta_0)\in L^2(\Omega)$ and still managed to show the uniqueness. There are many more interesting results on the existence and uniqueness of the solutions to (1.1) with intermediate regularity settings (see, e.g., [27–29]). The zero thermal diffusion limit does not appear to have been much studied, especially in the circumstance when the functional setting is weak.

The proof of Theorem 1.1 starts with the global existence of weak solutions for any $\alpha>0$ and $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$. This process starts with showing the global existence of smooth solutions $(u^{(n)},\theta^{(n)})$ to a sequence of approximate systems. It is then followed by establishing global uniform bounds on this sequence and obtaining the strong L^2 convergence of $u^{(n)}$. It finishes with passing to the limit. Due to the lack of thermal diffusion, there is no strong convergence in $\theta^{(n)}$. However, we can still pass to the limit in the thermal convection term due to the strong L^2 convergence of $u^{(n)}$. When $\alpha\geq\frac{1}{2}+\frac{d}{4}$, the weak solution is unique. Due to the weak regularity setting of the solutions, u is not Lipschitz and the corresponding vorticity is not necessarily bounded. The proof makes use of the following smoothing property of the velocity

$$\|u\|_{\widetilde{L}^1\left(0,T;B_{2,2}^{1+\frac{d}{2}}\right)}\leq C\left(T,\|u_0\|_{L^2},\|\theta_0\|_{L^2\cap L^{\frac{4d}{d+2}}}\right), \tag{1.4}$$

and a special consequence of (1.4). The definition of the Besov related space $\widetilde{L}^1(0,T;B_{2,2}^{1+\frac{d}{2}})$ is provided in Section 2. Equation (1.4) is proven via the Littlewood-Paley decomposition and Besov spaces techniques. The proof for the coincidence of two weak solutions $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ is based on the bounds for the L^2-norms of the differences

$$\|u^{(1)}-u^{(2)}\|_{L^2}+\|\nabla h^{(1)}-\nabla h^{(2)}\|_{L^2},$$

where $h^{(1)}$ and $h^{(2)}$ satisfy

$$-\Delta h^{(i)}=\theta^{(i)},\quad i=1,2.$$

Due to the lack of thermal diffusion and the weak regularity of θ, it is not possible to bound the difference $\|\theta^{(1)}-\theta^{(2)}\|_{L^2}$. The introduction of $h^{(1)}$ and $h^{(2)}$ reduce the regularity requirements and helps facilitate the proof.

To prove Theorem 1.2 and compare the solutions $(u^{(\eta)},\theta^{(\eta)})$ of (1.2) and (u,θ) of (1.1), we make use of the lower regularity quantities $h^{(\eta)}$ and h satisfying

$$-\Delta h^{(\eta)}=\theta^{(\eta)},\qquad -\Delta h=\theta$$

and estimate the difference

$$\|(u^{(\eta)}-u)(t)\|_{L^2}^2+\|(\nabla h^{(\eta)}-\nabla h)(t)\|_{L^2}^2$$

via Yudovich techniques.

The rest of this paper is divided into three sections and an appendix. Section 2 provides the definitions of the Littlewood-Paley decomposition as well as the functional settings associated with the Besov spaces and related facts. Section 3 proves Theorem 1.1. Due to the length of the proof for the global existence of weak solutions, the proof of this part is given in the Appendix. Section 4 proves Theorem 1.2.

2. Preliminaries

This section provides the definitions of the Littlewood-Paley decomposition, functional settings associated with the Besov spaces and related facts. In addition, an Osgood-type inequality is also stated here for the convenience of readers. More details can be found in several books and many papers (see, e.g., [6,7,41,44,45]).

To introduce the Besov spaces, we start with a few notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0=\left\{\phi\in\mathcal{S}:\int_{\mathbb{R}^d}\phi(x)x^\gamma\,dx=0,|\gamma|=0,1,2,\cdots\right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0'=\mathcal{S}'/\mathcal{S}_0^\perp=\mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials. For each $j\in\mathbb{Z}$, we write

$$A_j=\left\{\xi\in\mathbb{R}^d:\,2^{j-1}\le|\xi|<2^{j+1}\right\}. \tag{2.1}$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}}\in\mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j\subset A_j,\qquad \widehat{\Phi}_j(\xi)=\widehat{\Phi}_0(2^{-j}\xi)\quad\text{or}\quad\Phi_j(x)=2^{jd}\Phi_0(2^jx),$$

and

$$\sum_{j=-\infty}^{\infty}\widehat{\Phi}_j(\xi)=\begin{cases}1\,, & \text{if}\,\xi\in\mathbb{R}^d\setminus\{0\},\\ 0\,, & \text{if}\,\xi=0.\end{cases}$$

Therefore, for a general function $\psi\in\mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty}\widehat{\Phi}_j(\xi)\widehat{\psi}(\xi)=\widehat{\psi}(\xi)\quad\text{for }\xi\in\mathbb{R}^d\setminus\{0\}.$$

In addition, if $\psi\in\mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty}\widehat{\Phi}_j(\xi)\widehat{\psi}(\xi)=\widehat{\psi}(\xi)\quad\text{for any }\xi\in\mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0' \tag{2.2}$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f = 2^{jd} \int \Phi_0(2^j(x-y)) f(y)\, dy, \qquad j \in \mathbb{Z}. \tag{2.3}$$

The homogeneous Littlewood-Paley decomposition (2.2) can then be written as

$$f = \sum_{j=-\infty}^{\infty} \mathring{\Delta}_j f, \qquad f \in \mathcal{S}_0'.$$

DEFINITION 2.1. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js} \|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.4}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{2.5}$$

The inhomogeneous Littlewood-Paley decomposition (2.4) can then be written as

$$f = \sum_{j=-1}^{\infty} \Delta_j f, \qquad f \in \mathcal{S}'.$$

Definition 2.2. *The inhomogeneous Besov space* $B^s_{p,q}$ *with* $1\le p,q\le\infty$ *and* $s\in\mathbb{R}$ *consists of functions* $f\in\mathcal{S}'$ *satisfying*

$$\|f\|_{B^s_{p,q}}\equiv\|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q}<\infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s\in(0,1)$ and $1\le p,q\le\infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}}=\left(\int_{\mathbb{R}^d}\frac{(\|f(x+t)-f(x)\|_{L^p})^q}{|t|^{d+sq}}dt\right)^{1/q},$$

$$\|f\|_{B^s_{p,q}}=\|f\|_{L^p}+\left(\int_{\mathbb{R}^d}\frac{(\|f(x+t)-f(x)\|_{L^p})^q}{|t|^{d+sq}}dt\right)^{1/q}.$$

When $q=\infty$, the expressions are interpreted in the normal way. We will also use the space-time spaces introduced by Chemin-Lerner (see, e.g., [6]).

Definition 2.3. *For* $t>0$, $s\in\mathbb{R}$ *and* $1\le p,q,r\le\infty$, *the space-time spaces* $\widetilde{L}^r_t\mathring{B}^s_{p,q}$ *and* $\widetilde{L}^r_t B^s_{p,q}$ *are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t\mathring{B}^s_{p,q}}\equiv\|2^{js}\|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}}\equiv\|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

Here L^r_t is the abbreviation for $L^r(0,t)$. These spaces are related to the classical space-time spaces $L^r_t\mathring{B}^s_{p,q}$ and $L^r_t B^s_{p,q}$ via the Minkowski inequality, if $r\ge q$,

$$\widetilde{L}^r_t\mathring{B}^s_{p,q}\subseteq L^r_t\mathring{B}^s_{p,q},\qquad \widetilde{L}^r_t B^s_{p,q}\subseteq L^r_t B^s_{p,q}$$

and, if $r<q$,

$$\widetilde{L}^r_t\mathring{B}^s_{p,q}\supset L^r_t\mathring{B}^s_{p,q},\qquad \widetilde{L}^r_t B^s_{p,q}\supset L^r_t B^s_{p,q}$$

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 2.1. *For any* $s\in\mathbb{R}$,

$$\mathring{H}^s\sim\mathring{B}^s_{2,2},\quad H^s\sim B^s_{2,2}.$$

For any $s\in\mathbb{R}$ *and* $1<q<\infty$,

$$\mathring{B}^s_{q,\min\{q,2\}}\hookrightarrow\mathring{W}^s_q\hookrightarrow\mathring{B}^s_{q,\max\{q,2\}}.$$

In particular, $\mathring{B}^0_{q,\min\{q,2\}}\hookrightarrow L^q\hookrightarrow\mathring{B}^0_{q,\max\{q,2\}}$.

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. There will be no confusion if we keep in mind that $\mathring{\Delta}_j$ associated with the homogeneous Besov spaces is defined in (2.3) while those associated with the inhomogeneous Besov spaces are defined in (2.5). Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (2.5). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j and

$$S_j f(x) = 2^{dj}\Psi(2^j x) * f(x) = 2^{dj}\int \Psi(2^j(x-y)) f(y)\, dy.$$

The operators Δ_j and S_j defined above satisfy the following properties:

$$\Delta_j \Delta_k f = 0 \quad \text{if } |k-j| \geq 2 \text{ and } \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \quad \text{if } |k-j| \geq 3.$$

Bernstein's inequalities are useful tools on Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein-type inequalities for fractional derivatives.

PROPOSITION 2.2. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*
1) If f satisfies

$$\mathit{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$\mathit{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We shall also use Bony's notion of paraproducts to decompose a product into three parts

$$f g = T_f g + T_g f + R(f,g),$$

where

$$T_f g = \sum_j S_{j-1} f \Delta_j g, \qquad R(f,g) = \sum_j \sum_{k \geq j-1} \Delta_k f \widetilde{\Delta}_k g$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Finally, we state an Osgood-type inequality to be used in the subsequent sections (see, e.g., [6]).

LEMMA 2.1. *Let $a > 0$ and $0 \leq t_0 < T$. Let ρ be a measurable function from $[t_0, T]$ to $[0, a]$. Let $\gamma(t) > 0$ be a locally integrable function on $[t_0, T]$. Let $\phi \geq 0$ be a continuous and non-decreasing function on $[0, a]$. Assume that ρ satisfies, for some constant c*

$$\rho(t) \leq c + \int_{t_0}^t \gamma(s)\phi(\rho(s))\, ds \quad \text{for a.e. } t \in [t_0, T].$$

Then, if $c>0$*, we have, for a.e.* $t\in[t_0,T]$,

$$-\mathcal{M}(\rho(t))+\mathcal{M}(c)\le\int_{t_0}^t\gamma(\tau)d\tau,$$

where

$$\mathcal{M}(x)=\int_x^a\frac{dr}{\phi(r)}.$$

If $c=0$ *and*

$$\int_0^a\frac{dr}{\phi(r)}=\infty,$$

then $\rho(t)=0$ *a.e.* $t\in[t_0,T]$.

3. Proof of Theorem 1.1

This section proves Theorem 1.1. Naturally the proof is divided into two main parts. The first part is the proof of the global existence of weak solutions of (1.1) with any $\alpha>0$. This is accomplished in Proposition 3.1, which is stated in this section here. Since the proof of Proposition 3.1 is lengthy, we leave it to the Appendix. The second part is the proof of the uniqueness of weak solutions of (1.1) when $\alpha\ge\frac{1}{2}+\frac{d}{4}$. In order to prove the uniqueness, we first prove a major smoothing estimate for the velocity field in Proposition 3.2.

We start with the definition of weak solutions of (1.1) with any $\alpha>0$.

Definition 3.1. *Consider (1.1) with* $\alpha>0$ *and* $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$ *and* $\nabla\cdot u_0=0$*. Let* $T>0$ *be arbitrarily fixed. A pair* (u,θ) *satisfying*

$$u\in C_w([0,T];L^2)\cap L^2(0,T;\mathring{H}^\alpha),\quad \nabla\cdot u=0,$$
$$\theta\in C_w([0,T];L^2)\cap L^\infty(0,T;L^2)$$

is a weak solution of (1.1) on $[0,T]$ *if (a) and (b) below hold:*

(a) For any $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$ *with* $\nabla\cdot\phi=0$,

$$-\int_0^T\int_{\mathbb{R}^d}u\cdot\partial_t\phi\,dx\,dt-\int_{\mathbb{R}^d}u_0(x)\cdot\phi(x,0)\,dx-\int_0^T\int_{\mathbb{R}^d}u\cdot\nabla\phi u\,dx\,dt$$
$$+\int_0^T\int_{\mathbb{R}^d}(-\Delta)^{\alpha/2}u\cdot(-\Delta)^{\alpha/2}\phi\,dx\,dt=\int_0^T\int_{\mathbb{R}^d}\theta\boldsymbol{e}_d\cdot\phi\,dx\,dt. \tag{3.1}$$

(b) For any $\psi\in C_0^\infty(\mathbb{R}^d\times[0,T))$

$$-\int_0^T\int_{\mathbb{R}^d}\partial_t\psi\theta\,dx\,dt-\int_{\mathbb{R}^d}\theta_0(x)\psi(x,0)\,dx=\int_0^T\int_{\mathbb{R}^d}u\cdot\nabla\psi\theta\,dx\,dt. \tag{3.2}$$

For any $\alpha>0$ and $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$, (1.1) always has a global weak solution. In the special case when $\theta\equiv0$, this result assesses the global existence of weak solutions of the generalized Navier-Stokes equations with any $\alpha>0$ and $u_0\in L^2(\mathbb{R}^d)$.

Proposition 3.1. *Consider (1.1) with* $\alpha>0$ *and* $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$ *and* $\nabla\cdot u_0=0$*. Let* $T>0$ *be arbitrarily fixed. Then (1.1) has a global weak solution* (u,θ) *as given in Definition 3.1 satisfying*

$$\|\theta(t)\|_{L^2}\le\|\theta_0\|_{L^2},$$

$$\|u(t)\|_{L^2}^2+2\nu\int_0^t\|\Lambda^\alpha u(\tau)\|_{L^2}^2\,d\tau\le(\|u_0\|_{L^2}+t\|\theta_0\|_{L^2})^2.$$

The proof of Proposition 3.1 is long and the details will be provided in the Appendix. Next we establish a smoothing estimate for the weak solution shown in Proposition 3.1.

PROPOSITION 3.2. *Let $d\ge 2$. Consider (1.1) with $\alpha\ge\frac{1}{2}+\frac{d}{4}$. Assume (u_0,θ_0) satisfies*

$$u_0\in L^2(\mathbb{R}^d),\quad \nabla\cdot u_0=0,\quad \theta_0\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d).$$

Let (u,θ) be the corresponding global weak solution of (1.1). Then, for any $0<t\le T$,

$$\|u\|_{\widetilde{L}_t^1B_{2,2}^{1+\frac{d}{2}}}\le C(t,\|u_0\|_{L^2},\|\theta_0\|_{L^2}).\tag{3.3}$$

As a special consequence,

$$\sup_{p\ge 2}\int_0^t\frac{\|\nabla u(\tau)\|_{L^p}}{\sqrt{p}}\,d\tau\le C(t,\|u_0\|_{L^2},\|\theta_0\|_{L^2}).\tag{3.4}$$

Proposition 3.2 is proven via the Littlewood-Paley decomposition and Besov space techniques. The proof for the 2D case is partially different from that for the general d-D case with $d\ge 3$. We need a lemma for the 2D case.

LEMMA 3.1. *Assume $(u_0,\theta_0)\in L^2(\mathbb{R}^d)$ with $\nabla\cdot u_0=0$. Consider the 2D Boussinesq equation in (1.1) with $\alpha=1$. Let (u,θ) be the corresponding weak solution. Then u satisfies*

$$\int_0^T\|u(t)\|_{L^\infty}^2\,dt\le C(T,\|u_0\|_{L^2},\|\theta_0\|_{L^2}).\tag{3.5}$$

Proof. Applying Δ_j to the velocity equation and then dotting with $\Delta_j u$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2}^2+\nu 2^{2j}\|\Delta_j u\|_{L^2}^2&=-\int_{\mathbb{R}^2}\Delta_j u\cdot\Delta_j(u\cdot\nabla u)\,dx+\int_{\mathbb{R}^2}\Delta_j u\cdot\Delta_j(\theta e_2)\,dx\\&\le\|\Delta_j u\|_{L^2}\|\Delta_j(u\cdot\nabla u)\|_{L^2}+\|\Delta_j u\|_{L^2}\|\Delta_j\theta\|_{L^2}.\end{aligned}$$

Eliminating $\|\Delta_j u\|_{L^2}$ from each side and integrating in time yield

$$\begin{aligned}&\|\Delta_j u(t)\|_{L^2}+\nu\int_0^t 2^{2j}\|\Delta_j u\|_{L^2}\,d\tau\\&\le\|\Delta_j u_0\|_{L^2}+\int_0^t\|\Delta_j(u\cdot\nabla u)\|_{L^2}\,d\tau+\int_0^t\|\Delta_j\theta\|_{L^2}\,d\tau.\end{aligned}$$

Taking the l^2-norm and identifying H^s with $B_{2,2}^s$ for $s\ge 0$, we have

$$\begin{aligned}&\Big\|\sup_{0\le\tau\le t}\|\Delta_j u(\tau)\|_{L^2}\Big\|_{l^2}+\nu\|u\|_{\widetilde{L}^1(0,t;H^2)}\\&\le\|u_0\|_{L^2}+\int_0^t\|u\cdot\nabla u\|_{L^2}\,d\tau+\int_0^t\|\theta(\tau)\|_{L^2}\,d\tau\\&\le\|u_0\|_{L^2}+\|\nabla u\|_{L_t^2L^2}\|u\|_{L_t^2L^\infty}+\|\theta\|_{L_t^1L^2}.\end{aligned}\tag{3.6}$$

According to the Littlewood-Paley decomposition and by Bernstein's inequality,

$$\begin{aligned}\int_0^t \|u(\tau)\|_{L^\infty}^2\,d\tau &\le \int_0^t \sum_{j=-1}^{\infty}\sum_{k=-1}^{\infty} 2^j\,2^k\,\|\Delta_j u\|_{L^2}\,\|\Delta_k u\|_{L^2}\,d\tau\\ &:= H_1+H_2,\end{aligned}\tag{3.7}$$

where

$$H_1=\int_0^t \sum_{|j-k|\le N}\cdots,\qquad H_2=\int_0^t \sum_{|j-k|>N}\cdots.$$

Due to $|j-k|\le N$, the summation in H_1 includes the diagonal entries $j=k$ and $2N$ sub-diagonal entries. Therefore,

$$\begin{aligned}H_1=&\int_0^t \sum_{j=-1}^{\infty} 2^j\,\|\Delta_j u\|_{L^2}\,(2^{j-N}\|\Delta_{j-N}u\|_{L^2}+2^{j-N+1}\|\Delta_{j-N+1}u\|_{L^2}\\ &\qquad +\cdots+2^{j+N}\|\Delta_{j+N}u\|_{L^2})\,d\tau\\ \le&\frac{1}{2}\int_0^t \sum_{j=-1}^{\infty}\Big(2^{2j}\|\Delta_j u\|_{L^2}^2+2^{2(j-N)}\|\Delta_{j-N}u\|_{L^2}^2+\cdots\\ &\qquad +2^{2j}\|\Delta_j u\|_{L^2}^2+2^{2(j+N)}\|\Delta_{j+N}u\|_{L^2}^2\Big)\,d\tau\\ \le&\, CN\int_0^t \sum_{j=-1}^{\infty} 2^{2j}\|\Delta_j u\|_{L^2}^2\,d\tau\\ =&\, CN\|\nabla u\|_{L_t^2L^2}^2,\end{aligned}\tag{3.8}$$

where C is a pure constant independent of N. The summation in H_2 contains two identical parts and thus

$$\begin{aligned}H_2=&\,2\int_0^t \sum_{j-k>N} 2^j\,\|\Delta_j u\|_{L^2}\,2^k\,\|\Delta_k u\|_{L^2}\,d\tau\\ =&\,2\int_0^t \sum_{j=N}^{\infty} 2^j\|\Delta_j u\|_{L^2}\sum_{m=-1}^{j-N-1} 2^m\|\Delta_m u\|_{L^2}\,d\tau\\ \le&\,2^{-N+1}\sum_{j=N}^{\infty}\int_0^t 2^{2j}\|\Delta_j u\|_{L^2}\,d\tau\sum_{m=-1}^{j-N-1}2^{m+N-j}\sup_{0\le\tau\le t}\|\Delta_m u(\tau)\|_{L^2}\\ \le&\,2^{-N+1}\left\|\int_0^t 2^{2j}\|\Delta_j u\|_{L^2}\,d\tau\right\|_{l^2}\left\|\sum_{m=-1}^{j-N-1}2^{m+N-j}\sup_{0\le\tau\le t}\|\Delta_m u(\tau)\|_{L^2}\right\|_{l^2}\\ \le&\,2^{-N+1}\|u\|_{\widetilde{L}^1(0,t;H^2)}\left\|\sup_{0\le\tau\le t}\|\Delta_j u(\tau)\|_{L^2}\right\|_{l^2},\end{aligned}\tag{3.9}$$

where we have used Young's inequality for sequence convolution. Combining (3.6), (3.7), (3.8), (3.9) and Proposition 3.1 yields

$$\int_0^t \|u(\tau)\|_{L^\infty}^2\,d\tau\le CN\|\nabla u\|_{L_t^2L^2}^2$$

$$+C2^{-N+1}\left(\|u_0\|_{L^2}+\|\nabla u\|_{L^2_tL^2}\|u\|_{L^2_tL^\infty}+\|\theta\|_{L^1_tL^2}\right)^2$$
$$\le\frac{1}{2}\|u\|^2_{L^2_tL^\infty}+C(t,\|\nabla u\|^2_{L^2_tL^2},\|u_0\|_{L^2},\|\theta_0\|_{L^2}), \tag{3.10}$$

where we have chosen N such that

$$C2^{-N+1}\|\nabla u\|_{L^2_tL^2}\le\frac{1}{2}.$$

(3.10) then yields the desired global bound in (3.5). This completes the proof of Lemma 3.1. ☐

Proof. **(Proof of Proposition 3.2.)** Let $j\in\mathbb{Z}$ and $j\ge 0$. Applying Δ_j to the first equation of (1.1) yields

$$\partial_t\Delta_j u+\nu(-\Delta)^\alpha\Delta_j u=-\Delta_j\nabla P+\Delta_j(\theta\mathbf{e}_d)-\Delta_j(u\cdot\nabla u).$$

Dotting with $\Delta_j u$, integrating by parts and using $\nabla\cdot u=0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|^2_{L^2}+\nu 2^{2\alpha j}\|\Delta_j u\|^2_{L^2}\le\|\Delta_j\theta\|_{L^2}\|\Delta_j u\|_{L^2}+I, \tag{3.11}$$

where

$$I=-\int_{\mathbb{R}^d}\Delta_j(u\cdot\nabla u)\cdot\Delta_j u\,dx.$$

We estimate I. By the notion of paraproducts provided in Section 2,

$$\begin{aligned}I=&-\sum_{|j-k|\le 3}\int_{\mathbb{R}^d}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k u)\cdot\Delta_j u\,dx\\&-\sum_{|j-k|\le 3}\int_{\mathbb{R}^d}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}u)\cdot\Delta_j u\,dx\\&-\sum_{k\ge j-1}\int_{\mathbb{R}^d}\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k u)\cdot\Delta_j u\,dx.\end{aligned}$$

In order to shift one spatial derivative to $S_k u$ via a commutator, we further write the first term above into three terms to obtain

$$I=I_1+I_2+I_3+I_4+I_5, \tag{3.12}$$

where

$$\begin{aligned}I_1&=-\sum_{|j-k|\le 3}\int_{\mathbb{R}^d}[\Delta_j,S_{k-1}u\cdot\nabla]\Delta_k u\cdot\Delta_j u\,dx,\\I_2&=-\sum_{|j-k|\le 3}\int_{\mathbb{R}^d}(S_{k-1}u-S_{j-1}u)\cdot\nabla\Delta_j\Delta_k u\cdot\Delta_j u\,dx,\\I_3&=-\int_{\mathbb{R}^d}S_{j-1}u\cdot\nabla\Delta_j u\cdot\Delta_j u\,dx,\\I_4&=-\sum_{|j-k|\le 3}\int_{\mathbb{R}^d}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}u)\cdot\Delta_j u\,dx,\end{aligned}$$

$$I_5 = -\sum_{k\ge j-1}\int_{\mathbb{R}^d}\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k u)\cdot\Delta_j u\,dx.$$

We immediately get $I_3=0$ because of $\nabla\cdot u=0$. To bound I_1, we write the commutator into the integral form via the kernel functions defined in Section 2,

$$\begin{aligned}&[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k u\\ &=\int_{\mathbb{R}^d}\Phi_j(x-y)S_{k-1}u(y)\cdot\nabla\Delta_k u(y)dy - S_{k-1}u(x)\cdot\int_{\mathbb{R}^d}\Phi_j(x-y)\nabla\Delta_k u(y)\,dy\\ &=\int_{\mathbb{R}^d}\Phi_j(x-y)(S_{k-1}u(y)-S_{k-1}u(x))\cdot\nabla\Delta_k u(y)\,dy,\end{aligned}$$

Taking the L^2-norm of the commutator,

$$\begin{aligned}&\|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k u\|_{L^2}\\ &\le \|\nabla S_{k-1}u\|_{L^\infty}\left\|\int_{\mathbb{R}^d}|\Phi_j(x-y)||x-y||\nabla\Delta_k u(y)|dy\right\|_{L^2}\\ &\le \|\nabla S_{k-1}u\|_{L^\infty}\|x\Phi_j(x)\|_{L^1}\|\nabla\Delta_k u\|_{L^2}\\ &\le C2^{-j}\|\nabla S_{k-1}u\|_{L^\infty}\|\nabla\Delta_k u\|_{L^2},\end{aligned}$$

where we have invoked the fact

$$\|x\Phi_j(x)\|_{L^1}=\int_{\mathbb{R}^d}|x|2^{dj}|\Phi_0(2^j x)|dx\le C2^{-j}.$$

Therefore, by Hölder's inequality,

$$\begin{aligned}|I_1|&\le\sum_{|j-k|\le 3}\|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k u\|_{L^2}\|\Delta_j u\|_{L^2}\\ &\le C\|\Delta_j u\|_{L^2}\sum_{|j-k|\le 3}2^{-j}\|\nabla S_{k-1}u\|_{L^\infty}\|\nabla\Delta_k u\|_{L^2}\\ &\le C\|\Delta_j u\|_{L^2}\sum_{|j-k|\le 3}\|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k u\|_{L^2},\end{aligned}$$

where we have used Bernstein's inequality,

$$2^{-j}\|\nabla\triangle_k u\|_{L^2}\le C2^{k-j}\|\triangle_k u\|_{L^2}\le C\|\triangle_k u\|_{L^2}.$$

Since the summation above is over k for $|j-k|\le 3$, it suffices to deal with the representative term with $j=k$. Therefore, without loss of generality,

$$|I_1|\le C\|\Delta_j u\|_{L^2}^2\|\nabla S_j u\|_{L^\infty}. \tag{3.13}$$

The estimate for I_2 is easy. Since $S_{k-1}u-S_{j-1}u$ contains the terms of the form $\Delta_m u$ with m between $k-2$ and $j-2$, we can use Bernstein's inequality to shift one derivative from $\nabla\Delta_k u$ to $S_{k-1}u-S_{j-1}u$. Therefore,

$$|I_2|\le C\|\Delta_j u\|_{L^2}\sum_{|j-k|\le 3}\ \sum_{\min\{k-2,j-2\}\le m\le\max\{k-2,j-2\}}\|\Delta_m\nabla u\|_{L^\infty}\|\Delta_k u\|_{L^2}$$

Since the summation above is over a finite number of terms, it suffices to keep the bound for the representative term, for the sake of conciseness. Therefore,

$$|I_2| \le C\|\nabla\Delta_j u\|_{L^\infty}\|\Delta_j u\|_{L^2}^2. \tag{3.14}$$

The estimate of I_4 is direct. Again we only keep the bound for the representative term,

$$\begin{aligned}|I_4| &\le \sum_{|j-k|\le 3}\|\Delta_j u\|_{L^2}\|\Delta_j(\Delta_k u\cdot\nabla S_{k-1}u)\|_{L^2}\\ &\le C\|\Delta_j u\|_{L^2}^2\|\nabla S_j u\|_{L^\infty}.\end{aligned} \tag{3.15}$$

By the fact that $\nabla\cdot u=0$,

$$I_5 = -\sum_{k\ge j-1}\int_{\mathbb{R}^d}\Delta_j\nabla\cdot(\Delta_k u\otimes\widetilde{\Delta}_k u)\cdot\Delta_j u\,dx.$$

By Hölder's inequality,

$$\begin{aligned}|I_5| &\le C\|\Delta_j u\|_{L^2}2^j\sum_{k\ge j-1}\|\Delta_j(\Delta_k u\otimes\widetilde{\Delta}_k u)\|_{L^2}\\ &\le C\|\Delta_j u\|_{L^2}2^j\sum_{k\ge j-1}\|\Delta_k u\|_{L^2}\|\widetilde{\Delta}_k u\|_{L^\infty}.\end{aligned} \tag{3.16}$$

Inserting the bounds of (3.13), (3.14), (3.15) and (3.16) in (3.12) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2}^2 + C_0 2^{2\alpha j}\|\Delta_j u\|_{L^2}^2\\ &\le \|\Delta_j\theta\|_{L^2}\|\Delta_j u\|_{L^2} + C\|\Delta_j u\|_{L^2}^2\|\nabla S_j u\|_{L^\infty}\\ &\quad + C\|\nabla\Delta_j u\|_{L^\infty}\|\Delta_j u\|_{L^2}^2 + C2^j\|\Delta_j u\|_{L^2}\sum_{k\ge j-1}\|\Delta_k u\|_{L^2}\|\widetilde{\Delta}_k u\|_{L^\infty}.\end{aligned} \tag{3.17}$$

We further treat the right-hand side of (3.17) as follows. We distinguish between $d=2$ and $d\ge 3$. In the case when $d=2$,

$$\|\nabla S_j u\|_{L^\infty(\mathbb{R}^2)} \le 2^j\|S_j u(t)\|_{L^\infty(\mathbb{R}^2)}, \quad \|\nabla\Delta_j u\|_{L^\infty(\mathbb{R}^2)} \le C2^j\|\Delta_j u(t)\|_{L^\infty(\mathbb{R}^2)}.$$

Inserting the bounds above in (3.17), eliminating $\|\Delta_j u\|_{L^2}$ from each side and integrating in time, we have

$$\begin{aligned}&\|\Delta_j u(t)\|_{L^2} + C_0 2^{2\alpha j}\int_0^t\|\Delta_j u(\tau)\|_{L^2}d\tau \le \|\Delta_j u_0\|_{L^2}\\ &+\int_0^t\|\Delta_j\theta(\tau)\|_{L^2}d\tau + C\int_0^t 2^j\|\Delta_j u\|_{L^2}\|S_j u(\tau)\|_{L^\infty}d\tau\\ &+C\int_0^t 2^j\|\Delta_j u(\tau)\|_{L^2}\|\Delta_j u(\tau)\|_{L^\infty}d\tau + C\int_0^t 2^j\sum_{k\ge j-1}\|\Delta_k u(\tau)\|_{L^2}\|\widetilde{\Delta}_k u\|_{L^\infty}d\tau.\end{aligned}$$

Taking the l^2-norm of the sequence above and identifying $B^0_{2,2}$ with L^2, we obtain, after recalling the bound for $\|u\|_{L^2_t L^\infty}$ in Lemma 3.1,

$$\|u(t)\|_{L^2} + C_0\left\|2^{2\alpha j}\int_0^t\|\Delta_j u(\tau)\|_{L^2}d\tau\right\|_{l^2}$$

$$
\begin{aligned}
&\le 2\|u_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2}\, d\tau + C\int_0^t \|u(\tau)\|_{L^\infty}\|\nabla u(\tau)\|_{L^2}\, d\tau \\
&\quad + C\int_0^t \|u(\tau)\|_{L^\infty}\left\| 2^j \sum_{k\ge j-1} \|\Delta_k u(\tau)\|_{L^2}\right\|_{l^2} d\tau \\
&\le 2\|u_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2}\, d\tau + C\|\nabla u\|_{L^2_t L^2}\|u\|_{L^2_t L^\infty} < \infty,
\end{aligned}
$$

where we have used the bound, by Young's inequality for sequence convolution,

$$
\begin{aligned}
\left\| 2^j \sum_{k\ge j-1} \|\Delta_k u(\tau)\|_{L^2}\right\|_{l^2} &= \left\| \sum_{k\ge j-1} 2^{j-k}\, 2^k \|\Delta_k u(\tau)\|_{L^2}\right\|_{l^2} \\
&\le C\|2^k\|\Delta_k u(\tau)\|_{L^2}\|_{l^2} = C\|\nabla u(\tau)\|_{L^2}.
\end{aligned}
$$

We thus have obtained (3.3) for the case $d=2$. When $d\ge 3$, we bound some of the terms on the right of (3.17) differently. By Bernstein's inequality and Sobolev's inequality,

$$
\begin{aligned}
&\|\nabla S_j u\|_{L^\infty(\mathbb{R}^d)} \le C\, 2^{(\frac12+\frac d4)j}\|\nabla S_j u\|_{L^{\frac{4d}{d+2}}(\mathbb{R}^d)} \le C\, 2^{(\frac12+\frac d4)j}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}, \\
&\|\nabla \Delta_j u\|_{L^\infty(\mathbb{R}^d)} \le C\, 2^{(\frac12+\frac d4)j}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}, \\
&\|\widetilde{\Delta}_k u\|_{L^\infty(\mathbb{R}^d)} \le C\, 2^{(\frac d4-\frac12)k}\|\widetilde{\Delta}_k u\|_{L^{\frac{4d}{d-2}}(\mathbb{R}^d)} \le C\, 2^{(\frac d4-\frac12)k}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}.
\end{aligned}
$$

Inserting the estimates above in (3.17), eliminating $\|\Delta_j u\|_{L^2}$ from each side and integrating in time lead to

$$
\begin{aligned}
&\|\Delta_j u(t)\|_{L^2} + C_0\, 2^{2\alpha j}\int_0^t \|\Delta_j u(\tau)\|_{L^2}\, d\tau \\
&\le \|\Delta_j u_0\|_{L^2} + \int_0^t \|\Delta_j \theta(\tau)\|_{L^2}\, d\tau + C\int_0^t \|\Delta_j u\|_{L^2} 2^{(\frac12+\frac d4)j}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}\, d\tau \\
&\quad + C\int_0^t 2^j \sum_{k\ge j-1}\|\Delta_k u(\tau)\|_{L^2} 2^{(\frac d4-\frac12)k}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}\, d\tau.
\end{aligned}
$$

Taking the l^2-norm of the sequence above, we have

$$
\begin{aligned}
&\|u(t)\|_{L^2} + C_0\left\| 2^{2\alpha j}\int_0^t \|\Delta_j u(\tau)\|_{L^2}\, d\tau\right\|_{l^2} \\
&\le 2\|u_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2}\, d\tau + J_1 + J_2,
\end{aligned}
\tag{3.18}
$$

where

$$
\begin{aligned}
J_1 &= C\left\|\int_0^t \|\Delta_j u\|_{L^2} 2^{(\frac12+\frac d4)j}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}\, d\tau\right\|_{l^2} \\
&\le C\int_0^t \|2^{(\frac12+\frac d4)j}\|\Delta_j u\|_{L^2}\|_{l^2}\|\Lambda^{\frac12+\frac d4}u\|_{L^2}\, d\tau \\
&= C\int_0^t \|\Lambda^{\frac12+\frac d4}u(\tau)\|_{L^2}^2\, d\tau
\end{aligned}
$$

and

$$\begin{aligned} J_2 &= C\left\|\int_0^t 2^j \sum_{k\ge j-1} \|\Delta_k u(\tau)\|_{L^2} 2^{(\frac{d}{4}-\frac12)k}\, \|\Lambda^{\frac12+\frac{d}{4}} u\|_{L^2}\, d\tau\right\|_{l^2} \\ &= C\left\|\int_0^t \sum_{k\ge j-1} 2^{j-k}\, 2^{(\frac{d}{4}+\frac12)k} \|\Delta_k u(\tau)\|_{L^2}\, \|\Lambda^{\frac12+\frac{d}{4}} u\|_{L^2}\, d\tau\right\|_{l^2} \\ &\le C\int_0^t \left\|\sum_{k\ge j-1} 2^{j-k}\, 2^{(\frac{d}{4}+\frac12)k} \|\Delta_k u(\tau)\|_{L^2}\right\|_{l^2} \|\Lambda^{\frac12+\frac{d}{4}} u\|_{L^2}\, d\tau \\ &\le C\int_0^t \|\Lambda^{\frac12+\frac{d}{4}} u(\tau)\|_{L^2}^2\, d\tau. \end{aligned}$$

Inserting the bounds for J_1 and J_2 in (3.18) leads to

$$\|u(t)\|_{L^2} + C_0\, \|u\|_{\widetilde{L}^1_t B^{1+\frac{d}{2}}_{2,2}} \le \|u_0\|_{L^2} + t\|\theta_0\|_{L^2} + C\int_0^t \|\Lambda^{\frac12+\frac{d}{4}} u(\tau)\|_{L^2}^2\, d\tau,$$

which is the desired global bound in (3.3). Next we show that (3.3) implies (3.4). By Bernstein's inequality,

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \le \sum_{j=-1}^{\infty} \|\nabla \Delta_j u\|_{L^p(\mathbb{R}^d)} \le C \sum_{j=-1}^{\infty} 2^j 2^{dj\,(\frac12-\frac1p)} \|\Delta_j u\|_{L^2(\mathbb{R}^d)},$$

where C is a constant independent of p. By Hölder's inequality for sequences,

$$\begin{aligned} \int_0^t \|\nabla u\|_{L^p} dt &\le \sum_{j=-1}^{\infty} 2^{-\frac{d}{p}j} \int_0^t 2^{(1+\frac{d}{2})j} \|\Delta_j u\|_{L^2} d\tau \\ &\le \left(\sum_{j=-1}^{\infty} 2^{-\frac{2d}{p}j}\right)^{\frac12} \left(\int_0^t 2^{(1+\frac{d}{2})j} \|\Delta_j u\|_{L^2} d\tau\right)_{l^2}. \end{aligned}$$

Since

$$\left(\sum_{j=-1}^{\infty} 2^{-\frac{2d}{p}j}\right)^{\frac12} \le C\left(\int_{-1}^{\infty} 2^{-\frac{2d}{p}(x-1)}\, dx\right)^{\frac12} = C\sqrt{p},$$

(3.3) then implies

$$\int_0^t \|\nabla u\|_{L^p} dt \le C\sqrt{p}\|u\|_{\widetilde{L}^1_t B^{1+\frac{d}{2}}_{2,2}} \le C\sqrt{p},$$

where C depends on T, $\|u_0\|_{L^2}$ and $\|\theta_0\|_{L^2}$ only. We thus have shown (3.4). This completes the proof of Proposition 3.2. ∎

We now prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Due to Proposition 3.1 and Proposition 3.2, it suffices to show the uniqueness of the weak solutions of (1.1). Suppose (1.1) has two weak solutions $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ with the same initial data (u_0,θ_0). We show that $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ must coincide. To do so, we consider the difference $(\widetilde{u},\widetilde{\theta})$ with

$$\widetilde{u}:=u^{(1)}-u^{(2)}, \qquad \widetilde{\theta}:=\theta^{(1)}-\theta^{(2)}.$$

Let $P^{(1)}$ and $P^{(2)}$ be the corresponding pressure terms and $\widetilde{P}:=P^{(1)}-P^{(2)}$. In addition, we introduce the lower regularity quantities $h^{(1)}$ and $h^{(2)}$ satisfying

$$-\Delta h^{(1)}=\theta^{(1)}, \qquad -\Delta h^{(2)}=\theta^{(2)}$$

and set

$$\widetilde{h}=h^{(1)}-h^{(2)}.$$

It follows from (1.1) that $(\widetilde{u},\widetilde{\theta})$ satisfies

$$\begin{cases} \partial_t\widetilde{u}+u^{(1)}\cdot\nabla\widetilde{u}+\widetilde{u}\cdot\nabla u^{(2)}+\nu(-\Delta)^{\alpha}\widetilde{u}+\nabla\widetilde{P}=\widetilde{\theta}\mathbf{e}_d,\\ \partial_t\widetilde{\theta}+u^{(1)}\cdot\nabla\widetilde{\theta}+\widetilde{u}\cdot\nabla\theta^{(2)}=0,\\ \nabla\cdot\widetilde{u}=0,\\ (\widetilde{u},\widetilde{\theta})|_{t=0}=0. \end{cases} \tag{3.19}$$

Dotting the first equation of (3.19) by $\widetilde{u}$ and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2+\nu\|\Lambda^{\alpha}\widetilde{u}\|_{L^2}^2 &= -\int\widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u}dx+\int\widetilde{\theta}\cdot(\mathbf{e}_d\cdot\widetilde{u})dx\\ &:=K_1+K_2. \end{aligned} \tag{3.20}$$

where we have invoked the fact that, for $\alpha\geq\frac{1}{2}+\frac{d}{4}$,

$$\int_{\mathbb{R}^d}u^{(1)}\cdot\nabla\widetilde{u}\cdot\widetilde{u}dx=0$$

due to $\nabla\cdot u^{(1)}=0$, $\nabla\cdot\widetilde{u}=0$ and

$$\int_0^T\int_{\mathbb{R}^d}|u^{(1)}\cdot\nabla\widetilde{u}\cdot\widetilde{u}|dxdt\leq\int_0^T\|u^{(1)}(t)\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}(t)\|_{L^2}^2dt<\infty.$$

By Hölder's and Sobolev's inequalities, for $d\geq3$,

$$\begin{aligned} |K_1| &\leq\|\widetilde{u}\|_{L^2}\|\nabla u^{(2)}\|_{L^{\frac{4d}{d+2}}}\|\widetilde{u}\|_{L^{\frac{4d}{d-2}}}\\ &\leq C\|\widetilde{u}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}\|_{L^2}\\ &\leq\frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}\|_{L^2}^2+C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}^2\|\widetilde{u}\|_{L^2}^2. \end{aligned} \tag{3.21}$$

For $d=2$,

$$\begin{aligned} |K_1| &\leq\|\widetilde{u}\|_{L^4}^2\|\nabla u^{(2)}\|_{L^2}\\ &\leq\|\widetilde{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2}\|\nabla u^{(2)}\|_{L^2}\\ &\leq\frac{\nu}{16}\|\nabla\widetilde{u}\|_{L^2}^2+C\|\nabla u^{(2)}\|_{L^2}^2\|\widetilde{u}\|_{L^2}^2. \end{aligned} \tag{3.22}$$

By integration by parts and an interpolation inequality,

$$\begin{aligned}|K_2| &= \int_{\mathbb{R}^d} |(-\Delta\widetilde{h})(\mathbf{e}_d\cdot\widetilde{u})|\,dx\\
&\le \|\nabla\widetilde{h}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2}\\
&\le C\|\nabla\widetilde{h}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac{d-2}{d+2}}\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2}^{\frac{4}{d+2}}\\
&\le C\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{u}\|_{L^2}+\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2})\\
&\le \frac{\nu}{16}\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2}^2 + C\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{u}\|_{L^2}+\|\nabla\widetilde{h}\|_{L^2}).\end{aligned} \tag{3.23}$$

Dotting the second equation in (3.19) with $\widetilde{h}$ yields

$$\frac12\frac{d}{dt}\|\nabla\widetilde{h}\|_{L^2}^2 = -\int_{\mathbb{R}^d} u^{(1)}\cdot\nabla\widetilde{\theta}\widetilde{h}\,dx - \int_{\mathbb{R}^d}\widetilde{u}\cdot\nabla\theta^{(2)}\widetilde{h}\,dx := K_3+K_4. \tag{3.24}$$

We estimate K_4 first. The case with $d=2$ is treated differently from $d\ge 3$. For $d\ge3$, by integration by parts, Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}|K_4| &\le \int_{\mathbb{R}^d}|\theta^{(2)}\widetilde{u}\cdot\nabla\widetilde{h}|\,dx\\
&\le \|\theta^{(2)}\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\widetilde{u}\|_{L^{\frac{4d}{d-2}}}\\
&\le \|\theta_0\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2}\\
&\le \frac{\nu}{16}\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2}^2 + C\|\theta_0\|_{L^{\frac{4d}{d+2}}}^2\|\nabla\widetilde{h}\|_{L^2}^2.\end{aligned} \tag{3.25}$$

For $d=2$, by Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}|K_4| &\le \|\theta^{(2)}\|_{L^2}\|\widetilde{u}\|_{L^{2p}}\|\nabla\widetilde{h}\|_{L^q}\\
&\le C\sqrt{p}\|\widetilde{u}\|_{L^2}^{1/p}\|\nabla\widetilde{u}\|_{L^2}^{1-1/p}\|\theta_0\|_{L^2}\|\nabla\widetilde{h}\|_{L^2}^{1-\frac1p}\|\Delta\widetilde{h}\|_{L^2}^{\frac1p}\\
&\le C\sqrt{p}(\|\widetilde{u}\|_{L^2}+\|\nabla\widetilde{u}\|_{L^2})\|\nabla\widetilde{h}\|_{L^2}^{1-\frac1p}\\
&\le \frac{\nu}{16}\|\nabla\widetilde{u}\|_{L^2}^2+\|\widetilde{u}\|_{L^2}^2 + Cp\|\theta_0\|_{L^2}^{\frac2p}\|\nabla\widetilde{h}\|_{L^2}^{2(1-\frac1p)},\end{aligned} \tag{3.26}$$

where $1<p,q<\infty$ satisfy

$$\frac1p+\frac2q=1$$

and we have used the fact that $\|\Delta\widetilde{h}\|_{L^2}\le\|\theta^{(1)}\|_{L^2}+\|\theta^{(2)}\|_{L^2}\le 2\|\theta_0\|_{L^2}$. Recalling $\widetilde{\theta}=-\Delta\widetilde{h}$ and integrating by parts, we have

$$\begin{aligned}K_3 &= \int_{\mathbb{R}^d}u^{(1)}\cdot\nabla\Delta\widetilde{h}\widetilde{h}\,dx\\
&= -\int_{\mathbb{R}^d}\partial_{x_k}u_j^{(1)}\partial_{x_j}\partial_{x_k}\widetilde{h}\widetilde{h}\,dx - \int_{\mathbb{R}^d}u_j^{(1)}\partial_{x_j}\partial_{x_k}\widetilde{h}\partial_{x_k}\widetilde{h}\,dx\\
&= \int_{\mathbb{R}^d}\partial_{x_k}u_j^{(1)}\partial_{x_k}\widetilde{h}\partial_{x_j}\widetilde{h}\,dx,\end{aligned}$$

where the repeated indices are summed and we have used $\nabla\cdot u^{(1)}=0$. By Hölder's inequality, for $p>\frac{d}{2}$ and $\frac{1}{p}+\frac{2}{q}=1$,

$$\begin{aligned}|K_3| &\le C\|\nabla u^{(1)}\|_{L^p}\|\nabla\widetilde{h}\|_{L^q}^2\\ &\le C\|\nabla u^{(1)}\|_{L^p}\|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}}\|\widetilde{\theta}\|_{L^2}^{\frac{d}{p}}\\ &\le C\|\nabla u^{(1)}\|_{L^p}\|\theta_0\|_{L^2}^{\frac{d}{p}}\|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}}.\end{aligned}\tag{3.27}$$

Adding (3.20) and (3.24) and collecting the estimates in (3.21), (3.22), (3.23), (3.25), (3.26) and (3.27), we find that, for $\delta>0$,

$$G_\delta(t):=\|\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{h}(t)\|_{L^2}^2+\delta$$

obeys the differential inequality, when $d=2$,

$$\frac{d}{dt}G_\delta(t)\le C\left(1+\|\nabla u^{(2)}\|_{L^2}^2\right)G_\delta(t)+C\left(1+\frac{\|\nabla u^{(1)}\|_{L^p}}{p}\right)pM^{\frac{1}{p}}G_\delta(t)^{1-\frac{1}{p}}\tag{3.28}$$

and, for $d\ge 3$,

$$\frac{d}{dt}G_\delta(t)\le C\left(1+\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}^2\right)G_\delta(t)+C\frac{\|\nabla u^{(1)}\|_{L^p}}{p}pM^{\frac{d}{2p}}G_\delta(t)^{1-\frac{d}{2p}},\tag{3.29}$$

where we have written $M=\|\theta_0\|_{L^2}^2$. Optimizing the quantities $pM^{\frac{1}{p}}G_\delta(t)^{1-\frac{1}{p}}$ and $pM^{\frac{d}{2p}}G_\delta(t)^{1-\frac{d}{2p}}$ with respect to p, we obtain

$$pM^{\frac{1}{p}}G_\delta(t)^{1-\frac{1}{p}}\ge eG_\delta(t)(\ln M-\ln G_\delta),$$

with the minimum being reached at $p=\ln\frac{M}{G_\delta(t)}$ (in the case when $\ln\frac{M}{G_\delta(t)}<2$, p is taken to be $2+\ln\frac{M}{G_\delta(t)}$), and

$$pM^{\frac{d}{2p}}G_\delta(t)^{1-\frac{d}{2p}}\ge\frac{d}{2}eG_\delta(t)(\ln M-\ln G_\delta)$$

with the minimum being reached at $p=\frac{d}{2}\ln\frac{M}{G_\delta(t)}$. Then both (3.28) and (3.29) are reduced to the following form

$$G_\delta(t)\le G_\delta(0)+C\int_0^t\gamma(s)\phi(G_\delta(s))\,ds,$$

where

$$\gamma(t)=C+C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}^2+C\frac{\|\nabla u^{(1)}\|_{L^p}}{p},\quad \phi(r)=r+r(\ln M-\ln r).$$

It follows from Proposition 3.3 that

$$\int_0^T\gamma(t)\,dt<\infty.$$

Let

$$\Omega(x)=\int_x^1\frac{dr}{\phi(r)}=\int_x^1\frac{dr}{r+r(\ln M-\ln r)}=\ln(1+\ln M-\ln x)-\ln(1+\ln M).$$

It then follows from Lemma 2.1 that

$$-\Omega(G_\delta(t))+\Omega(G_\delta(0))\le\int_0^t\gamma(s)\,ds.$$

Therefore,

$$-\ln(1+\ln M-\ln G_\delta(t))+\ln(1+\ln M-\ln G_\delta(0))\le\int_0^t\gamma(s)\,ds.$$

Therefore, for $\widetilde{C}(t)=\int_0^t\gamma(s)\,ds$,

$$G_\delta(t)\le(eM)^{1-e^{-\widetilde{C}(t)}}G_\delta(0)^{e^{-\widetilde{C}(t)}}.$$

Letting $\delta\to0^+$ and noting that $G_0(0)=0$, we obtain

$$G_0(t):=\|\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{h}(t)\|_{L^2}^2=0.$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

This section provides the proof of Theorem 1.2.

Proof. Let (u,θ) and $(u^{(\eta)},\theta^{(\eta)})$ be the weak solutions of (1.1) and (1.2), respectively. Then the difference $(\widetilde{u},\widetilde{\theta})$ with

$$\widetilde{u}=u^{(\eta)}-u,\qquad\widetilde{\theta}=\theta^{(\eta)}-\theta$$

satisfies

$$\begin{cases}\partial_t\widetilde{u}+u^{(\eta)}\cdot\nabla\widetilde{u}+\widetilde{u}\cdot\nabla u+\nu(-\Delta)^\alpha\widetilde{u}+\nabla\widetilde{P}=\widetilde{\theta}\mathbf{e}_d,\\ \partial_t\widetilde{\theta}+u^{(\eta)}\cdot\nabla\widetilde{\theta}+\widetilde{u}\cdot\nabla\theta=\eta\Delta\widetilde{\theta}+\eta\Delta\theta,\\ \nabla\cdot\widetilde{u}=0,\\ (\widetilde{u},\widetilde{\theta})|_{t=0}=(\widetilde{u}_0,\widetilde{\theta}_0),\end{cases}\tag{4.1}$$

where $\widetilde{P}:=P^{(\eta)}-P$ with $P^{(\eta)}$ and P being the corresponding pressure terms of (1.1) and (1.2), respectively. We introduce the lower regularity quantities $h^{(\eta)}$ and h satisfying

$$-\Delta h^{(\eta)}=\theta^{(\eta)},\qquad-\Delta h=\theta$$

and set

$$\widetilde{h}=h^{(\eta)}-h.$$

Dotting the first equation of (4.1) by $\widetilde{u}$ and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2+\nu\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2&=-\int\widetilde{u}\cdot\nabla u\cdot\widetilde{u}dx+\int\widetilde{\theta}\cdot(\mathbf{e}_d\cdot\widetilde{u})dx\\&:=L_1+L_2.\end{aligned}\tag{4.2}$$

The two terms on the right of (4.2) can be bounded similarly in the proof of Theorem 1.1 and we have

$$|L_1|\le\frac{\nu}{16}\|\Lambda^{\frac12+\frac d4}\widetilde{u}\|_{L^2}^2+C\|\Lambda^{\frac12+\frac d4}u\|_{L^2}^2\|\widetilde{u}\|_{L^2}^2$$

and

$$|L_2| \le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}\|_{L^2}^2 + C\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{u}\|_{L^2} + \|\nabla\widetilde{h}\|_{L^2})$$
$$\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}\|_{L^2}^2 + C(\|\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{h}\|_{L^2}^2).$$

Dotting the second equation in (4.1) with $\widetilde{h}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{h}\|_{L^2}^2 + \eta\|\Delta\widetilde{h}\|_{L^2}^2 = L_3 + L_4 + L_5, \tag{4.3}$$

where

$$L_3 := \int_{\mathbb{R}^d} u^{(\eta)} \cdot \nabla\widetilde{\theta}\widetilde{h}\, dx,$$
$$L_4 := \int_{\mathbb{R}^d} \widetilde{u} \cdot \nabla\theta\widetilde{h}\, dx,$$
$$L_5 := -\eta\int_{\mathbb{R}^d} \Delta\theta\widetilde{h}\, dx.$$

As in the proof of Theorem 1.1, L_3 admits the following bound,

$$|L_3| \le C\|\nabla u^{(\eta)}\|_{L^p}\|\theta_0\|_{L^2}^{\frac{d}{p}}\|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}},$$

where $p > \frac{d}{2}$. L_4 can also be similarly bounded as K_4 in the proof of Theorem 1.1. For $d = 2$,

$$|L_4| \le \frac{\nu}{16}\|\nabla\widetilde{u}\|_{L^2}^2 + \|\widetilde{u}\|_{L^2}^2 + Cp\|\theta_0\|_{L^2}^{\frac{2}{p}}\|\nabla\widetilde{h}\|_{L^2}^{2(1-\frac{1}{p})}$$

and, for $d \ge 3$,

$$|L_4| \le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{u}\|_{L^2}^2 + C\|\theta_0\|_{L^{\frac{4d}{d+2}}}^2\|\nabla\widetilde{h}\|_{L^2}^2.$$

By integration by parts and Hölder's inequality,

$$|L_5| \le \eta\|\theta\|_{L^2}\|\Delta\widetilde{h}\|_{L^2} \le \frac{\eta}{2}\|\Delta\widetilde{h}\|_{L^2}^2 + \frac{\eta}{2}\|\theta\|_{L^2}^2.$$

Adding (4.2) and (4.3) and incorporating the bounds for L_1 through L_5, we find, for $\delta > 0$,

$$E_\delta(t) := \|\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 + \delta$$

satisfies, for $d = 2$,

$$\frac{d}{dt}E_\delta(t) \le \frac{\eta}{2}\|\theta\|_{L^2}^2 + C\left(1 + \|\Lambda u\|_{L^2}^2\right)E_\delta(t) + C\left(1 + \frac{\|\nabla u^{(\eta)}\|_{L^p}}{p}\right)pM^{\frac{1}{p}}E_\delta(t)^{1-\frac{1}{p}}$$

and, for $d \ge 3$,

$$\frac{d}{dt}E_\delta(t) \le \frac{\eta}{2}\|\theta\|_{L^2}^2 + C\left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}}u\|_{L^2}^2\right)E_\delta(t) + C\frac{\|\nabla u^{(\eta)}\|_{L^p}}{p}pM^{\frac{d}{2p}}E_\delta(t)^{1-\frac{d}{2p}}.$$

By following a similar procedure as in the proof of Theorem 1.1 and applying Lemma 2.1, we obtain

$$E_\delta(t) \le (eM)^{1-e^{-\widetilde{C}(t)}} \left(E_\delta(0)+\eta t\|\theta\|_{L^2}^2\right)^{e^{-\widetilde{C}(t)}} \tag{4.4}$$

where $M=\|\theta_0\|_{L^2}^2+\|\theta_0^{(\eta)}\|_{L^2}^2$ denotes the sum of the initial L^2-norms squared, and $\widetilde{C}(t)$ is the uniform bound (independent of η)

$$\widetilde{C}(t)=C\int_0^t \left(1+\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u\|_{L^2}^2+\frac{\|\nabla u^{(\eta)}\|_{L^p}}{p}\right)d\tau<\infty.$$

Even though $u^{(\eta)}$ is the solution of (1.2), the bound

$$\sup_{q\ge 2}\int_0^t \frac{\|\nabla u^{(\eta)}\|_{L^p}}{p}\,d\tau<\infty$$

is uniform in η since it only depends on $\|\theta^\eta\|_{L^2}$. Letting $\delta\to 0$ in (4.4) yields

$$\|\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{h}(t)\|_{L^2}^2 \le (eM)^{1-e^{-\widetilde{C}(t)}}\left(\|\widetilde{u}_0\|_{L^2}^2+\|\nabla\widetilde{h}_0\|_{L^2}^2+\eta t\|\theta\|_{L^2}^2\right)^{e^{-\widetilde{C}(t)}},$$

which is the desired bound (1.3). This completes the proof of Theorem 1.2. ◻

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Appendix. Global existence of weak solutions. This section provides the proof of Proposition 3.1. For readers' convenience, we first list several simple facts to be used in the proof. The first two lemmas are Picard's existence and extension results (see, e.g., [40]).

LEMMA 5.1 (Picard Existence and Uniqueness Theorem). *Let E be a Banach space. Let $O\subseteq E$ be an open subset. Let $F: O\longrightarrow E$ be a locally Lipschitz map. More precisely, for any $y\in O$, there is a neighborhood of y (denoted by $U(y)$) and $L=L(y,U)$ such that*

$$\|F(y)-F(z)\|_E\le L\|y-z\|_E,\quad \forall z\in U(y).$$

Then, for any $y_0\in O$, the ODE

$$\begin{cases}\frac{dy}{dt}=F(y),\\ y|_{t=0}=y_0\in O.\end{cases} \tag{5.1}$$

has a unique local solution, namely, there is $T>0$ and a unique solution $y=y(t)$ satisfying $y\in C^1(0,T;O)$.

LEMMA 5.2 (Picard Extension Theorem). *Assume the conditions in Lemma 5.1 hold and Let $y=y(t)$ be the local solution. Then either $y(t)$ is global in time, namely, $T=\infty$, or for a finite $T_0>0$, $\lim_{t\to T_0} y(t)\notin O$.*

Lemma 5.3 (Hodge Decomposition in $\mathbb{R}^d$). *For every $v \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, there exist unique w and p satisfying*

$$v = w + \nabla p, \quad \nabla \cdot w = 0,$$

and $w \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d), \nabla p \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, and $\|v\|_{L^2}^2 = \|w\|_{L^2}^2 + \|\nabla p\|_{L^2}^2$.

We introduce a few notations. $\widehat{f}(\xi)$ represents the Fourier transform of f,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx.$$

For a positive integer n, we denote by $B(0,n)$ the ball centered at the origin with radius n, and define

$$\widehat{J_n f}(\xi) = \chi_{B(0,n)}(\xi) \widehat{f}(\xi).$$

In addition, we write

$$L_n^2 = \{f \in L^2(\mathbb{R}^d) : supp\ \widehat{f} \subset B(0,n)\},$$
$$L_{n,\sigma}^2 = \{f \in L_n^2(\mathbb{R}^d) : \nabla \cdot f = 0\}.$$

A special consequence of Lemma 5.3 is the following fact.

Corollary 5.1. *There exists a linear bounded operator $\mathbb{P} : L_n^2 \to L_{n,\sigma}^2$ satisfying:*

- *For any $f \in L_n^2$, $\|\mathbb{P} f\|_{L^2} \le \|f\|_{L^2}$.*
- *For any $f \in L_{n,\sigma}^2$, $\mathbb{P} f = f$.*
  Especially, for any $f \in L_n^2$, $\mathbb{P}^2 f = \mathbb{P} f$.

In addition, we will also need the following Lions-Aubin compactness lemma.

Lemma 5.4 (Lions-Aubin Compactness Lemma). *Let $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ be three Banach spaces with the first embedding being compact and the second being continuous. Let $T > 0$. For $1 \le p, q \le +\infty$, let*

$$W = \{u \in L^p(0,T;X_1), \partial_t u \in L^q(0,T;X_3)\}.$$

Then,

(i). If $p < +\infty$, then the embedding of W into $L^p(0,T;X_2)$ is compact;

(ii). If $p = +\infty$ and $q > 1$, then the embedding of W into $C(0,T;X_2)$ is compact.

Lemma 5.4 states that any bounded sequence in W has a convergent subsequence in $L^p(0,T;X_2)$.

We are now ready to prove Proposition 3.1.

Proof. **(Proof of Proposition 3.1.)** The proof is divided into several steps. The first step is to show the global existence of smooth solutions to a sequence of approximate systems. The second is to establish uniform bounds for this sequence of solutions and extract a strongly convergent subsequence. The third is to verify that the limit of the subsequence is actually the weak solution.

Step I: The global existence of smooth solutions to an approximate system.

Let $n\in\mathbb{N}$ and we seek a solution $(u^{(n)},\theta^{(n)})\in L^2_n$ satisfying

$$\begin{cases}\partial_t u^{(n)}+\mathbb{P}J_n(u^{(n)}\cdot\nabla u^{(n)})+\nu(-\Delta)^\alpha u^{(n)}=\mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d),\\ \partial_t\theta^{(n)}+J_n(u^{(n)}\cdot\nabla\theta^{(n)})=0,\\ \nabla\cdot u^{(n)}=0,\\ u^{(n)}(x,0)=J_nu_0,\ \ \theta^{(n)}(x,0)=J_n\theta_0.\end{cases}\tag{5.2}$$

We remark that functions in $L^2_n(\mathbb{R}^d)$ are smooth.

$$L^2_n\subseteq\cap_{m=0}^\infty\mathring{H}^m.$$

In fact, let $f\in L^2_n$,

$$\|f\|^2_{\mathring{H}^m}=\Sigma_{|\beta|=m}\|D^\beta f\|^2_{L^2}=\Sigma_{|\beta|=m}\|\widehat{D^\beta f}\|^2_{L^2}=\Sigma_{|\beta|=m}\|(2\pi i\xi)^\beta\widehat{f}\|^2_{L^2}\le(2\pi n)^{2\beta}\|f\|^2_{L^2}.$$

We use the Picard theorem to show that (5.2) has a unique global solution in L^2_n. To this end, we first apply Lemma 5.1 to show (5.2) has a local-in-time solution. We can rewrite (5.2) as

$$\frac{dy}{dt}=F(y),$$

with

$$\begin{aligned}Y&=(u^{(n)},\theta^{(n)})^T,\\ F(Y)&=(F_1(Y),F_2(Y))^T\\ &=(-\mathbb{P}J_n(u^{(n)}\cdot\nabla u^{(n)})-\nu(-\Delta)^\alpha u^{(n)}+\mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d),-J_n(u^{(n)}\cdot\nabla\theta^{(n)}))^T.\end{aligned}$$

We set $E=L^2_n$ and $O=E$. We verify that $F:E\longrightarrow E$ is locally Lipschitz. Assume $Y\in L^2_n$ and show $F(Y)\in L^2_n$. Clearly $F(Y)\in L^2(\mathbb{R}^d)$. In fact,

$$\begin{aligned}\|F_1(Y)\|_{L^2}&\le\|u^{(n)}\cdot\nabla u^{(n)}\|_{L^2}+\|\nu(-\Delta)^\alpha u^{(n)}\|_{L^2}+\|\theta^{(n)}\|_{L^2}\\ &\le\|u^{(n)}\|_{L^4}\|\nabla u^{(n)}\|_{L^4}+\nu\|u^{(n)}\|_{\mathring{H}^{2\alpha}}+\|\theta^{(n)}\|_{L^2}\\ &\le\|u^{(n)}\|_{\mathring{H}^{\frac{d}{4}}}\|u^{(n)}\|_{\mathring{H}^{1+\frac{d}{4}}}+\nu\|u^{(n)}\|_{\mathring{H}^{2\alpha}}+\|\theta^{(n)}\|_{L^2}\\ &\le(2\pi n)^{2(1+\frac{d}{4})}\|u^{(n)}\|^2_{L^2}+\nu(2\pi n)^{2\alpha}\|u^{(n)}\|_{L^2}+\|\theta^{(n)}\|_{L^2}.\end{aligned}$$

That is $F_1(Y)\in L^2(\mathbb{R}^d)$. Similarly, $F_2(Y)\in L^2(\mathbb{R}^d)$. Obviously,

$$\text{supp }\widehat{F_1(Y)},\quad\text{supp }\widehat{F_2(Y)}\subseteq B(0,n).$$

Therefore, $F(Y)\in L^2_n(\mathbb{R}^d)$. Next we show $F(Y)$ is locally Lipschitz. Let $Y=(u^{(n)},\theta^{(n)})^T\in L^2_n$ and $Z=(v^{(n)},\rho^{(n)})^T\in L^2_n$ and consider

$$\begin{aligned}&\|F_2(Y)-F_2(Z)\|_{L^2}\\ &=\|-J_n(u^{(n)}\cdot\nabla\theta^{(n)})+J_n(v^{(n)}\cdot\nabla\rho^{(n)})\|_{L^2}\\ &=\|-J_n((u^{(n)}-v^{(n)})\cdot\nabla\theta^{(n)})-J_n(v^{(n)}\cdot\nabla(\theta^{(n)}-\rho^{(n)}))\|_{L^2}\\ &\le\|(u^{(n)}-v^{(n)})\cdot\nabla\theta^{(n)}\|_{L^2}+\|v^{(n)}\cdot\nabla(\theta^{(n)}-\rho^{(n)})\|_{L^2}\\ &\le\|u^{(n)}-v^{(n)}\|_{L^2}\|\nabla\theta^{(n)}\|_{L^\infty}+\|v^{(n)}\|_{L^\infty}\|\nabla(\theta^{(n)}-\rho^{(n)})\|_{L^2}\\ &\le\|u^{(n)}-v^{(n)}\|_{L^2}\|\theta^{(n)}\|_{\mathring{H}^{1+\frac{d}{2}+\epsilon}}+\|v^{(n)}\|_{\mathring{H}^{\frac{d}{2}+\epsilon}}\|\theta^{(n)}-\rho^{(n)}\|_{\mathring{H}^1}\end{aligned}$$

$$\leq (2\pi n)^{1+\frac{d}{2}+\epsilon}\|\theta^{(n)}\|_{L^2}\|u^{(n)}-v^{(n)}\|_{L^2}+(2\pi n)^{1+\frac{d}{2}+\epsilon}\|v^{(n)}\|_{L^2}\|\theta^{(n)}-\rho^{(n)}\|_{L^2}$$
$$\leq L\|Y-Z\|_{L^2},$$

where $\epsilon>0$ is a small parameter and $L=2(2\pi n)^{1+\frac{d}{2}+\epsilon}(\|Y\|_{L^2}+r)$ for $\|Z-Y\|_{L^2}\leq r$. Therefore $F_2(Y)$ is locally Lipschitz. Similarly, $F_1(Y)$ is locally Lipschitz. Lemma 5.1 implies (5.2) has a unique local-in-time solution in L^2_n.

Next we use the Picard Extension Theorem, Lemma 5.2 to show that the solution is global in time. It suffices to show that for any $t\leq T$, $\|(u^{(n)},\theta^{(n)})\|_{L^2}<+\infty$. This is done by the energy method. Dotting (5.2) by $(u^{(n)},\theta^{(n)})$ yields

$$\frac{1}{2}\frac{d}{dt}(\|u^{(n)}\|_{L^2}^2+\|\theta^{(n)}\|_{L^2}^2)+\nu\|\Lambda^\alpha u^{(n)}\|_{L^2}^2=M_1+M_2+M_3,$$

where $<f,g>=\int_{\mathbb{R}^d}f(x)g(x)\,dx$ and

$$M_1=-\int_{\mathbb{R}^d}\mathbb{P}J_n(u^{(n)}\cdot\nabla u^{(n)})\cdot u^{(n)}\,dx,$$
$$M_2=\int_{\mathbb{R}^d}\mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d)\cdot u^{(n)}\,dx,$$
$$M_3=-\int_{\mathbb{R}^d}J_n(u^{(n)}\cdot\nabla\theta^{(n)})\cdot\theta^{(n)}\,dx.$$

We note that

$$M_1=-\int\mathbb{P}J_n(u^{(n)}\cdot\nabla u^{(n)})\cdot u^{(n)}dx=-\int J_n(u^{(n)}\cdot\nabla u^{(n)})\cdot\mathbb{P}u^{(n)}dx$$
$$=-\int J_n(u^{(n)}\cdot\nabla u^{(n)})\cdot u^{(n)}dx=-\int(u^{(n)}\cdot\nabla u^{(n)})\cdot u^{(n)}dx=0.$$

Similarly, $M_3=0$. Clearly, $|M_2|\leq\|u^{(n)}\|_{L^2}\|\theta^{(n)}\|_{L^2}$. Therefore,

$$\frac{d}{dt}(\|u^{(n)}\|_{L^2}^2+\|\theta^{(n)}\|_{L^2}^2)+2\nu\|\Lambda^\alpha u^{(n)}\|_{L^2}^2\leq\|u^{(n)}\|_{L^2}\|\theta^{(n)}\|_{L^2}.$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\|\theta^{(n)}\|_{L^2}=0 \text{ or } \|\theta^{(n)}(t)\|_{L^2}=\|J_n\theta_0\|_{L^2}.$$

Consequently,

$$\|u^{(n)}(t)\|_{L^2}\leq\|J_nu_0\|_{L^2}+t\|J_n\theta_0\|_{L^2}\leq\|u_0\|_{L^2}+t\|\theta_0\|_{L^2}$$

and

$$\|u^{(n)}(t)\|_{L^2}^2+2\nu\int_0^t\|\Lambda^\alpha u^{(n)}\|_{L^2}^2d\tau\leq(\|u_0\|_{L^2}+t\|\theta_0\|_{L^2})^2.$$

Therefore, $(u^{(n)},\theta^{(n)})\in L^2_n$ for all time $t\leq T$. Then Lemma 5.2 allows us to conclude that $(u^{(n)},\theta^{(n)})$ is global in time.

Step 2. Extraction of a strongly convergent subsequence.
This step extracts a subsequence of $u^{(n)}$ that converges strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ using the Aubin-Lions lemma. In order to use the Aubin-Lions method we show that

$$\partial_t u^{(n)}\in L^2(0,T;H^{-s}), \tag{5.3}$$

where $s=\max\{\alpha,1+\frac{d}{2}-\alpha\}$. Let $\phi\in H^s$. We take the L^2-inner product of ϕ and the velocity equation in (5.2) leads to

$$\int_{\mathbb{R}^d}\phi\cdot\partial_t u^{(n)}\;dx=Q_1+Q_2+Q_3,$$

with

$$\begin{aligned}Q_1&=-\int\phi\cdot\mathbb{P}J_n(u^{(n)}\cdot\nabla u^{(n)})\;dx,\\Q_2&=-\nu\int\phi\cdot(-\Delta)^\alpha u^{(n)}\;dx,\\Q_3&=\int\phi\cdot\mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d)\;dx.\end{aligned}$$

Integrating by parts, and applying Hölder's and Sobolev's inequalities yield

$$\begin{aligned}|Q_1|&\le\|u^{(n)}\|^2_{L^{\frac{2d}{d-\alpha}}}\|\nabla\mathbb{P}J_n\phi\|_{L^{\frac{d}{\alpha}}}\\&\le C\|u^{(n)}\|_{L^2}\|\Lambda^\alpha u^{(n)}\|_{L^2}\|\mathbb{P}J_n\phi\|_{H^{1+\frac{d}{2}-\alpha}}\\&\le C\|u^{(n)}\|_{L^2}\|\Lambda^\alpha u^{(n)}\|_{L^2}\|\phi\|_{H^{1+\frac{d}{2}-\alpha}}.\end{aligned}$$

Using integration by parts and Hölder's inequality, we have

$$|Q_2|\le\nu\|\Lambda^\alpha\phi\|_{L^2}\|\Lambda^\alpha u^{(n)}\|_{L^2}\le\nu\|\phi\|_{H^s}\|\Lambda^\alpha u^{(n)}\|_{L^2}.$$

Clearly,

$$|Q_3|\le\|\phi\|_{H^s}\|\theta^{(n)}\|_{L^2}.$$

Therefore,

$$\left|\int\phi\cdot\partial_t u^{(n)}\;dx\right|\le C\|\phi\|_{H^s}\Big(\|\Lambda^\alpha u^{(n)}\|_{L^2}(1+\|u^{(n)}\|_{L^2})+\|\theta^{(n)}\|_{L^2}\Big).$$

That is,

$$\|\partial_t u^{(n)}\|_{H^{-s}}\le C\Big(\|\Lambda^\alpha u^{(n)}\|_{L^2}(1+\|u^{(n)}\|_{L^2})+\|\theta^{(n)}\|_{L^2}\Big).$$

Squaring and integrating in time yield

$$\begin{aligned}&\int_0^T\|\partial_t u^{(n)}\|^2_{H^{-s}}\;dt\\&\le C\int_0^T\Big(1+\|u^{(n)}\|_{L^2}\Big)^2\|\Lambda^\alpha u^{(n)}\|^2_{L^2}\;dt+C\int_0^T\|\theta^{(n)}\|^2_{L^2}\;dt\\&\quad+C\int_0^T\Big(1+\|u^{(n)}\|_{L^2}\Big)\|\Lambda^\alpha u^{(n)}\|_{L^2}\|\theta^{(n)}\|^2_{L^2}\;dt\\&\le C\sup_{0\le t\le T}(1+\|u^{(n)}\|^2_{L^2})\int_0^T\|\Lambda^\alpha u^{(n)}\|^2_{L^2}\;dt+CT\sup_{0\le t\le T}\|\theta^{(n)}\|_{L^2}\\&\quad+C\Big(T\sup_{0\le t\le T}\|\theta^{(n)}\|_{L^2}\Big)\cdot\Big(\sup_{0\le t\le T}(1+\|u^{(n)}\|_{L^2})\Big)\int_0^T\|\Lambda^\alpha u^{(n)}\|^2_{L^2}\;dt\end{aligned}$$

$$<+\infty.$$

Thus we have obtained (5.3). Since we have

$$u^{(n)}\in L^2(0,T;H^\alpha(\mathbb{R}^d)),\quad \partial_t u^{(n)}\in L^2(0,T;H^{-s}(\mathbb{R}^d)),$$

and the facts that $H^\alpha(\mathbb{R}^d)\hookrightarrow L^2(\mathbb{R}^d)$ is locally compact and $L^2(\mathbb{R}^d)\hookrightarrow H^{-(1+\frac{d}{2}-\alpha)}$ is continuous, we can apply the Aubin-Lions Lemma to conclude that $u^{(n)}$ has a convergent subsequence in $L^2(0,T;L^2(\mathbb{R}^d))$. Let u be the limit of $u^{(n)}$ and θ be the weak limit of $\theta^{(n)}$. Clearly,

$$\theta\in L^\infty(0,T;L^2(\mathbb{R}^d)),\quad u\in L^\infty(0,T;L^2(\mathbb{R}^d))\cap L^2(0,T;H^\alpha(\mathbb{R}^d)).$$

Step 3. Passing to the limit.
This step shows that (u,θ) obtained in the previous step is a weak solution of (1.1). It is easy to see from (5.2) that, for any $\phi\in C_0^\infty(\mathbb{R}^d\times[0,T))$ with $\nabla\cdot\phi=0$, and for any $\psi\in C_0^\infty(\mathbb{R}^d\times[0,T))$,

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^d} u^{(n)}\cdot\partial_t\phi\,dx\,dt-\int_{\mathbb{R}^d} u_0^{(n)}\cdot\phi(x,0)\,dx-\int_0^T\int_{\mathbb{R}^d} u^{(n)}\cdot\nabla(J_n\phi)u^{(n)}\,dx\,dt\\
+\int_0^T\int_{\mathbb{R}^d}\Lambda^\alpha u^{(n)}\cdot\Lambda^\alpha\phi\,dx\,dt=\int_0^T\int_{\mathbb{R}^d}\theta^{(n)}\mathbf{e}_d\cdot J_n\phi\,dx\,dt,\\
-\int_0^T\int_{\mathbb{R}^d}\partial_t\psi\theta^{(n)}\,dx\,dt+\int_{\mathbb{R}^d}\theta_0^{(n)}\psi(x,0)\,dx=\int_0^T\int_{\mathbb{R}^d} u^{(n)}\cdot\nabla(J_n\psi)\theta^{(n)}\,dx\,dt.
\end{aligned}$$

The task is then to verify that, as $n\to\infty$, the terms above converge to the corresponding terms in the definition of the weak solution given in Definition 3.1. We need the strong convergence $u^{(n)}\to u$ in $L^2(0,T;L^2)$. It suffices to consider the convergence of the nonlinear terms. Let

$$\begin{aligned}
A&:=-\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla(J_n\phi)u\,dx\,dt,\\
A^{(n)}&:=-\int_0^T\int_{\mathbb{R}^d} u^{(n)}\cdot\nabla(J_n\phi)u^{(n)}\,dx\,dt,
\end{aligned}$$

and consider the difference

$$\begin{aligned}
A^{(n)}-A=&-\int_0^T\int_{\mathbb{R}^d}(u^{(n)}-u)\cdot\nabla(J_n\phi)u^{(n)}\,dx\,dt\\
&+\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla(J_n\phi-\phi)u^{(n)}\,dx\,dt\\
&+\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla\phi\cdot(u^{(n)}-u)\,dx\,dt\\
=&R_1+R_2+R_3.
\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}
|R_1|&\le\|u^{(n)}-u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\nabla J_n\phi\|_{L^\infty(\mathbb{R}^d\times[0,T])}\|u^{(n)}\|_{L^2(\mathbb{R}^d\times[0,T])}\\
&\le C\|u^{(n)}-u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\phi\|_{H^{2+\frac{d}{2}}}\|u_0\|_{L^2(\mathbb{R}^d\times[0,T])}\to 0\text{ as }n\to\infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}|R_2| &\leq \|u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\nabla(J_n\phi-\phi)\|_{L^\infty(\mathbb{R}^d\times[0,T])}\|u^{(n)}\|_{L^2(\mathbb{R}^d\times[0,T])}\\ &\leq C\|u_0\|_{L^2}\|J_n\phi-\phi\|_{H^{2+\frac{d}{2}}}\|u_0\|_{L^2}\to 0 \text{ as } n\to\infty\end{aligned}$$

and, as $n\to\infty$,

$$|R_3|\leq \|u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d\times[0,T])}\|u^{(n)}-u\|_{L^2(\mathbb{R}^d\times[0,T])}\to 0.$$

Therefore $|A^{(n)}-A|\to 0$ as $n\to\infty$. The convergence of the other nonlinear terms is slightly different. We do not have strong convergence in $\theta^{(n)}$. Define

$$\begin{aligned}B &:= -\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla(J_n\psi)\theta\,dx\,dt,\\ B^{(n)} &:= -\int_0^T\int_{\mathbb{R}^d} u^{(n)}\cdot\nabla(J_n\psi)\theta^{(n)}\,dx\,dt\end{aligned}$$

and consider the difference

$$\begin{aligned}B^{(n)}-B =& -\int_0^T\int_{\mathbb{R}^d}(u^{(n)}-u)\cdot\nabla(J_n\psi)\theta^{(n)}\,dx\,dt\\ &+\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla(J_n\psi-\psi)\theta^{(n)}\,dx\,dt\\ &+\int_0^T\int_{\mathbb{R}^d} u\cdot\nabla\psi\cdot(\theta^{(n)}-\theta)\,dx\,dt\\ =& W_1+W_2+W_3.\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}|W_1| &\leq \|u^{(n)}-u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\nabla J_n\psi\|_{L^\infty(\mathbb{R}^d\times[0,T])}\|\theta^{(n)}\|_{L^2(\mathbb{R}^d\times[0,T])}\\ &\leq C\|u^{(n)}-u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\psi\|_{H^{2+\frac{d}{2}}}\|\theta_0\|_{L^2(\mathbb{R}^d\times[0,T])}\to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}|W_2| &\leq \|u\|_{L^2(\mathbb{R}^d\times[0,T])}\|\nabla(J_n\psi-\psi)\|_{L^\infty(\mathbb{R}^d\times[0,T])}\|\theta^{(n)}\|_{L^2(\mathbb{R}^d\times[0,T])}\\ &\leq C\|u_0\|_{L^2}\|J_n\psi-\psi\|_{H^{2+\frac{d}{2}}}\|\theta_0\|_{L^2}\to 0 \text{ as } n\to\infty.\end{aligned}$$

W_3 is estimated differently from R_3 since we do not have strong convergence in $\theta^{(n)}$. Since L^2 functions can be approximated by smooth functions with compact support, $u\cdot\nabla\psi$ can be treated as a test function. Since $\theta^{(n)}$ converges weakly to θ, we thus have

$$W_3\to 0 \quad \text{as } n\to\infty.$$

Hence $|B^{(n)}-B|\to 0$ as $n\to\infty$. Therefore, (u,θ) is indeed a weak solution. This completes the proof of Proposition 3.1. ∎

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