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Inviscid Models Generalizing the Two-dimensional Euler and the Surface Quasi-geostrophic Equations

Dongho Chae, Peter Constantin & Jiahong Wu

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Abstract

Any classical solution of the two-dimensional incompressible Euler equation is global in time. However, it remains an outstanding open problem whether classical solutions of the surface quasi-geostrophic (SQG) equation preserve their regularity for all time. This paper studies solutions of a family of active scalar equations in which each component u_j of the velocity field u is determined by the scalar θ through $u_j = \mathcal{R}\Lambda^{-1}P(\Lambda)\theta$, where $\mathcal{R}$ is a Riesz transform and $\Lambda = (-\Delta)^{1/2}$. The two-dimensional Euler vorticity equation corresponds to the special case $P(\Lambda) = I$ while the SQG equation corresponds to the case $P(\Lambda) = \Lambda$. We develop tools to bound $\|\nabla u\|_{L^\infty}$ for a general class of operators P and establish the global regularity for the Loglog-Euler equation for which $P(\Lambda) = (\log(I + \log(I - \Delta)))^\gamma$ with $0 \leqq \gamma \leqq 1$. In addition, a regularity criterion for the model corresponding to $P(\Lambda) = \Lambda^\beta$ with $0 \leqq \beta \leqq 1$ is also obtained.

1. Introduction and statements of the main results

This paper studies solutions of the active scalar equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad x\in\mathbb{R}^d,\ t>0, \\ u=(u_j), \quad u_j = \mathcal{R}_l\Lambda^{-1}P(\Lambda)\theta, \quad 1\leqq j,\ l\leqq d, \end{cases} \tag{1.1}$$

where $\theta = \theta(x,t)$ is a scalar function of $x \in \mathbb{R}^d$ and $t \geqq 0$, u denotes a velocity field with its component u_j ($1 \leqq j \leqq d$) given by a Riesz transform $\mathcal{R}_l$ applied to $\Lambda^{-1}P(\Lambda)\theta$. To avoid any confusion, we remark that the notation $a_j = b_l$ simply means each a_j given by b_l for some l. Here the operators $\Lambda = (-\Delta)^{\frac{1}{2}}$, $P(\Lambda)$ and $\mathcal{R}_l$ are defined through their Fourier transforms, namely

$$\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi), \quad \widehat{P(\Lambda)f}(\xi) = P(|\xi|)\widehat{f}(\xi), \quad \widehat{\mathcal{R}_l f}(\xi) = \frac{i\xi_l}{|\xi|}\widehat{f}(\xi),$$

where $1 \leqq l \leqq d$ is an integer, $\widehat{f}$ or $\mathcal{F}(f)$ denotes the Fourier transform

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

Our consideration is restricted to P satisfying the following assumption.

Assumption 1.1. The symbol $P = P(|\xi|)$ assumes the following properties:

(1) P is continuous on $\mathbb{R}^d$ and $P \in C^\infty(\mathbb{R}^d \setminus \{0\})$;
(2) P is radially symmetric;
(3) $P = P(|\xi|)$ is nondecreasing in $|\xi|$;
(4) There exist two constants C and C_0 such that

$$\sup_{2^{-1} \leqq |\eta| \leqq 2} |(I - \Delta_\eta)^n P(2^j|\eta|)| \leqq C P(C_0 2^j)$$

for any integer j and $n = 1, 2, \ldots, 1 + [\frac{d}{2}]$.

We remark that (4) in Assumption 1.1 is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin–Hörmander Multiplier Theorem (see for example [84, p. 96]). For notational convenience, we also assume that $P \geqq 0$. Some special examples of P are

$$\begin{aligned}
&P(\xi) = (\log(1+|\xi|^2))^\gamma \quad \text{with } \gamma \geqq 0,\\
&P(\xi) = (\log(1+\log(1+|\xi|^2)))^\gamma \quad \text{with } \gamma \geqq 0,\\
&P(\xi) = |\xi|^\beta \quad \text{with } \beta \geqq 0,\\
&P(\xi) = (\log(1+|\xi|^2))^\gamma |\xi|^\beta \quad \text{with } \gamma \geqq 0 \text{ and } \beta \geqq 0.
\end{aligned}$$

A particularly important case of (1.1) is the two-dimensional active scalar equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta = 0, \quad x \in \mathbb{R}^2,\ t > 0,\\ u = \nabla^\perp \psi \equiv (-\partial_{x_2}\psi, \partial_{x_1}\psi), \quad -\Lambda^2\psi = P(\Lambda)\,\theta \end{cases} \tag{1.2}$$

which generalizes the two-dimensional Euler vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega = 0,\\ u = \nabla^\perp\psi, \quad \Delta\psi = \omega \end{cases} \tag{1.3}$$

and the SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta = 0,\\ u = \nabla^\perp\psi, \quad -\Lambda\psi = \theta. \end{cases} \tag{1.4}$$

The two-dimensional Euler equation has been extensively studied and its global regularity has long been established (see for example [16,61,67]). The SQG equation and its dissipative counterpart have recently attracted a lot of attention and numerous efforts have been devoted to the global regularity and related issues concerning their solutions (see for example [1–3,5–15,17–24,26–56,58–66,68–83,86–102]).

The goal of this paper is to understand the global regularity issue concerning solutions of (1.1) with a given initial datum

$$\theta(x,0)=\theta_0(x), \quad x\in\mathbb{R}^d. \tag{1.5}$$

The key quantity involved in this issue is $\|\nabla u\|_{L^\infty}$. Tools are developed here to bound $\|\Delta_j \nabla u\|_{L^p}$ and $\|S_N \nabla u\|_{L^p}$ when a vector field $u:\mathbb{R}^d\to\mathbb{R}^d$ is related to a scalar function θ by

$$(\nabla u)_{jk}=\mathcal{R}_l\mathcal{R}_m P(\Lambda)\theta$$

where $1\leqq j,k,l,m\leqq d$, $(\nabla u)_{jk}$ denotes the (j,k)th entry of ∇u and $\mathcal{R}_l$ and $\mathcal{R}_m$ denote the Riesz transforms. Here Δ_j with $j\geqq -1$ denotes the Fourier localization operator and

$$S_N=\sum_{j=-1}^{N-1}\Delta_j.$$

The precise definitions of Δ_j and S_N are provided in Appendix A. The assumption that u is divergence-free is not used in deriving these bounds. The bounds obtained here are summarized in the following theorem.

Theorem 1.2. *Let $u:\mathbb{R}^d\to\mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by*

$$(\nabla u)_{jk}=\mathcal{R}_l\mathcal{R}_m P(\Lambda)\theta,$$

where $1\leqq j,k,l,m\leqq d$, $(\nabla u)_{jk}$ denotes the (j,k)th entry of ∇u, $\mathcal{R}_l$ denotes the Riesz transform, and P obeys Assumption 1.1. Then, for any integers $j\geqq 0$ and $N\geqq 0$,

$$\|S_N\nabla u\|_{L^p}\leqq C_{p,d}P(C_0 2^N)\|S_N\theta\|_{L^p}, \quad 1<p<\infty, \tag{1.6}$$

$$\|\Delta_j\nabla u\|_{L^q}\leqq C_d P(C_0 2^j)\|\Delta_j\theta\|_{L^q}, \quad 1\leqq q\leqq\infty, \tag{1.7}$$

$$\|S_N\nabla u\|_{L^\infty}\leqq C_d\|\theta\|_{L^1\cap L^\infty}+C_d N P(C_0 2^N)\|S_{N+1}\theta\|_{L^\infty}, \tag{1.8}$$

where $C_{p,d}$ is a constant depending on p and d only and C_ds' depend on d only.

We remark that, in general, the constant $C_{p,d}$ grows linearly with respect to p and thus (1.6) does not follow for $p=\infty$. With these bounds at our disposal, we are able to establish global regularity results covering two special cases of P. The first result is for (1.1) with $P(|\xi|)=(\log(1+\log(1+|\xi|^2)))^\gamma$. For the simplicity of our presentation here, we state the result for the two-dimensional case of (1.1), namely

$$\begin{cases}\partial_t\theta+u\cdot\nabla\theta=0,\\ u=\nabla^\perp\psi, \quad \Delta\psi=(\log(1+\log(1-\Delta)))^\gamma\theta,\end{cases} \tag{1.9}$$

which we call the Loglog-Euler equation. Although any classical solution θ of (1.9) obeys the global a priori bound

$$\|\theta(\cdot,t)\|_{L^p}\leqq\|\theta(\cdot,0)\|_{L^p} \quad \text{for any } 1\leqq p\leqq\infty,$$

the regularity of the velocity u recovered from the relation

$$u = \nabla^{\perp}\psi, \quad \Delta\psi = (\log(1+\log(1-\Delta)))^{\gamma}\theta$$

is worse than in the case of the two-dimensional Euler equation. Nevertheless we are able to obtain global regularity for (1.9) with $0 \leqq \gamma \leqq 1$.

Theorem 1.3. *Consider the initial-value problem* (1.9) *and* (1.5) *with* γ *and* θ_0 *satisfying*

$$0 \leqq \gamma \leqq 1, \quad \theta_0 \in L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2) \cap B^s_{q,\infty}(\mathbb{R}^2) \tag{1.10}$$

where $2 < q \leqq \infty$ *and* $s > 1$. *Then the initial-value problem* (1.9) *and* (1.5) *has a unique global solution* θ *satisfying, for any* $T > 0$,

$$\theta \in L^{\infty}([0,T]; B^s_{q,\infty}(\mathbb{R}^2)), \quad \nabla u \in L^{\infty}([0,T]; B^{1+s_1}_{q,\infty}(\mathbb{R}^2)),$$

where $s_1 < s$.

The general version of Theorem 1.3, namely the global regularity result for (1.1), will be stated in Section 3. Here $B^s_{q,\infty}$ denotes an inhomogeneous Besov space. The definition of a general Besov space $B^s_{p,q}$ is provided in Appendix A. Even though $\theta_0 \in B^s_{q,\infty}$ implies $\theta_0 \in L^{\infty}$, the condition on θ_0 is written as in (1.10) to emphasize the importance of L^{∞} assumption. The global regularity stated in the Besov space setting in Theorem 1.3 can be converted into a global regularity statement in Sobolev spaces. Combining Theorem 1.3 and the embedding relations

$$W^r_q \hookrightarrow B^r_{q,\infty} \hookrightarrow B^{r_1}_{q,\min\{2,q\}} \hookrightarrow W^{r_1}_q, \quad r > r_1,$$

we can conclude that any initial data in W^r_q with $2 < q \leqq \infty$ and $r > 1$ would yield a global solution in $W^{r_1}_q$ for any $r_1 < r$.

Theorem 1.3 is proven by combining the Besov space techniques and the following extrapolation inequality.

Proposition 1.4. *Let* $u : \mathbb{R}^d \to \mathbb{R}^d$ *be a vector field. Assume that* u *is related to a scalar* θ *by*

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m (\log(I+\log(I-\Delta)))^{\gamma}\theta \tag{1.11}$$

where $\gamma \geqq 0$, $1 \leqq j,k,l,m \leqq d$, $(\nabla u)_{jk}$ *denotes the* (j,k)*th entry of* ∇u *and* $\mathcal{R}_l$ *and* $\mathcal{R}_m$ *denote the Riesz transforms. Then, for any* $1 \leqq q \leqq \infty$ *and* $s > d/q$,

$$\|\nabla u\|_{L^{\infty}} \leqq \|\theta\|_{L^1\cap L^{\infty}} + C\|\theta\|_{L^{\infty}} \log(1+\|\theta\|_{B^s_{q,\infty}}) \left(\log(1+\log(1+\|\theta\|_{B^s_{q,\infty}}))\right)^{\gamma}$$

where C *is a constant that depends on* d, q *and* s *only.*

The second special case studied here is when $P(|\xi|) = |\xi|^\beta$ with $0 \leqq \beta \leqq 1$. Our aim is to understand how the parameter β affects the regularity of solutions to the initial-value problem

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0 \\ u = \nabla^\perp \psi, \quad -\Lambda^2 \psi = \Lambda^\beta \theta, \end{cases} \tag{1.12}$$

where $0 \leqq \beta \leqq 1$. The evolution of patch-like initial data under (1.12) has previously been studied in [33]. Clearly (1.12) bridges the two-dimensional Euler and the SQG equations. It is hoped that this study would shed light on the global regularity issue concerning the SQG equation.

It is unknown if all classical solutions of (1.12) conserve their regularity for all time except in the case of the two-dimensional Euler equation. In order to deal with global regularity for (1.12), it suffices to obtain a suitable bound for $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$. Intuitively, the relation

$$u = -\nabla^\perp \Lambda^{-2+\beta} \theta$$

implies that $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$ can be bounded more or less by a bound for $\Lambda^\beta \theta$. In fact, this intuitive idea can be made rigorous and is reflected in the following logarithmic Hölder inequality

$$\|S\|_{L^\infty} \leqq C\|\theta\|_{C^\beta} \log(1 + \|\theta\|_{C^\sigma}) + C\|\theta\|_{L^q}, \quad \sigma > \beta,\ q > 1,$$

where S denotes the symmetric part of ∇u and C^β the Hölder space. This inequality, together with a bound for the back-to-labels map determined by u, allows us to obtain the following regularity criterion.

Theorem 1.5. *Consider* (1.12) *with* $0 \leqq \beta \leqq 1$. *Let* θ *be a solution of* (1.12) *corresponding to the data* $\theta_0 \in C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ *with* $\sigma > 1$ *and* $q > 1$. *Let* $T > 0$. *If* θ *satisfies*

$$\int_0^T \|\theta(\cdot, t)\|_{C^\beta(\mathbb{R}^2)} \, \mathrm{d}t < \infty,$$

then θ *remains in* $C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ *on the time interval* $[0, T]$.

This criterion, especially, establishes global regularity for the two-dimensional Euler equation and reduces to the well-known criterion for the SQG equation when $\beta = 1$ (see [23]).

The rest of this paper is organized as follows. Section 2 is devoted to the bounds in Theorem 1.2 and Proposition 1.4. Theorem 1.3 and its general version, the global regularity result for (1.1) are stated and proven in Section 3. Section 4 details the proof of Theorem 1.5. Appendix A provides the definition of Besov spaces and some related facts.

2. Bounds for $\|\Delta_j \nabla u\|_{L^q}$, $\|S_N \nabla u\|_{L^q}$ and $\|\nabla u\|_{L^\infty}$

This section derives the bounds stated in Theorem 1.2 and proves the logarithmic interpolation inequality presented in Proposition 1.4.

We make use of a Mihlin and Hörmander Multiplier Theorem (see [84, p. 96]) in the proof of (1.6). This theorem is recalled first.

Theorem 2.1. *Suppose that $Q(\xi)$ is of class C^k in the complement of the origin of $\mathbb{R}^d$, where $k > \frac{d}{2}$ is an integer. Assume also that*

$$|D^\alpha Q(\xi)| \leqq B|\xi|^{-|\alpha|}, \quad \text{whenever } |\alpha| \leqq k. \tag{2.1}$$

Then $Q \in \mathcal{M}_q$, $1 < q < \infty$. That is, $\|T_Q f\|_{L^q} \leqq C_q \|f\|_{L^q}$, where T_Q is defined by

$$\widehat{T_Q f}(\xi) = Q(\xi)\widehat{f}(\xi).$$

For further reference, we rewrite (1.6) as a proposition.

Proposition 2.2. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by*

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda)\theta \tag{2.2}$$

where $1 \leqq j, k, l, m \leqq d$, $(\nabla u)_{jk}$ denotes the (j,k)th entry of ∇u, $\mathcal{R}_l$ denotes the Riesz transform and P obeys Assumption 1.1. Then, for any integer $N \geqq 0$,

$$\|S_N \nabla u\|_{L^p} \leqq C_{p,d} P(C_0 2^N) \|S_N \theta\|_{L^p}, \quad 1 < p < \infty, \tag{2.3}$$

where $C_{p,d}$ is a constant depending on p and d only.

Proof. As detailed in Appendix A, the symbol of S_N is $\psi(\xi/2^N)$ with ψ satisfying

$$\psi \in C_0^\infty(\mathbb{R}^d), \quad \text{supp}\psi \subset B\left(0, \frac{11}{12}\right), \quad \psi(\xi) = 1 \text{ for } |\xi| \leqq \frac{3}{4}.$$

It follows from (2.2) that

$$(\widehat{S_N \nabla u})_{jk}(\xi) = Q(\xi) P(C_0 2^N) \widehat{S_N \theta}(\xi)$$

where $Q(\xi)$ is supported on $|\xi| \leqq (11/12)2^N$ and, for $|\xi| \leqq (11/12)2^N$,

$$Q(\xi) = -\frac{\xi_l \xi_m}{|\xi|^2} \frac{P(|\xi|)}{P(C_0 2^N)}.$$

To apply Theorem 2.1, we verify (2.1). Clearly, for any α with $|\alpha| = 0, 1, \ldots, 1 + [\frac{d}{2}]$,

$$\left| D^\alpha \frac{\xi_l \xi_m}{|\xi|^2} \right| \leqq C|\xi|^{-|\alpha|}.$$

In addition, for any $\xi \neq 0$, there is an integer j such that $\xi = 2^j \eta$ with $2^{-1} \leqq |\eta| \leqq 2$. Trivially, for ξ in the support of Q, $j \leqq N$. It is easy to see that Condition (4) in Assumption 1.1 implies that

$$\sup_{2^{-1} \leqq |\eta| \leqq 2} |(-\Delta_\eta)^n P(2^j |\eta|)| \leqq C P(C_0 2^j)$$

for $n = 0, 1, \ldots, 1 + [\frac{d}{2}]$. Then,

$$\begin{aligned} \left| (-\Delta_\xi)^n \frac{P(|\xi|)}{P(C_0 2^N)} \right| &= \left| (-\Delta_\eta)^n \frac{2^{-2nj} P(2^j |\eta|)}{P(C_0 2^N)} \right| \\ &\leqq |\eta|^{2n} |2^j \eta|^{-2n} \frac{P(C_0 2^j)}{P(C_0 2^N)} \\ &\leqq |\eta|^{2n} |\xi|^{-2n}. \end{aligned} \tag{2.4}$$

A similar bound can be shown for any α with $|\alpha| = 0, 1, \ldots, 1 + [\frac{d}{2}]$. For example, to show the bound for $|\alpha| = 1$, we infer from Condition (4) in Assumption 1.1 and

$$\Delta_\eta P(2^j |\eta|) = 2^{2j} P''(2^j |\eta|) + (d-1)/|\eta| 2^j P'(2^j |\eta|)$$

that $|2^j P'(2^j |\eta|)| \leqq C P(C_0 2^j)$. Then a similar estimate as in (2.4) yields

$$\left| D_\xi^\alpha \frac{P(|\xi|)}{P(C_0 2^N)} \right| \leqq |\eta|^{|\alpha|} |\xi|^{-|\alpha|}.$$

This verifies (2.1). (2.3) then follows as a consequence of Theorem 2.1. □

For the sake of clarity, we restate (1.7) in Theorem 1.2 as a proposition.

Proposition 2.3. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by*

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda) \theta$$

where $1 \leqq j, k, l, m \leqq d$, $(\nabla u)_{jk}$ denotes the (j, k)th entry of ∇u and $\mathcal{R}_l$ denotes the Riesz transform. Here P obeys Assumption 1.1. Then, for any integer $N \geqq 0$,

$$\|\Delta_N \nabla u\|_{L^q} \leqq C_d P(C_0 2^N) \|\Delta_N \theta\|_{L^q}, \quad 1 \leqq q \leqq \infty. \tag{2.5}$$

where C_d is a constant depending on d only.

Remark 2.4. This proposition is invalid in the case when $N = -1$. The proof requires the symbol of Δ_N is supported away from the origin.

Proof of Proposition 2.3. Clearly,

$$(\Delta_N \nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda) \Delta_N \theta$$

and

$$(\widehat{\Delta_N \nabla u})_{jk}(\xi) = -\frac{\xi_l \xi_m}{|\xi|^2} P(|\xi|) \widehat{\Delta_N \theta}(\xi).$$

As defined in Appendix A, $\widehat{\Delta_N\theta}(\xi) = \phi(\xi/2^N)\widehat{\theta}(\xi)$ with $\phi(\xi/2^N)$ supported in the annulus $(3/4)2^N \leqq |\xi| \leqq (11/6)2^N$. We define a smooth function $\widetilde{\phi}_0$ satisfying

$$\widetilde{\phi}_0 \equiv 1 \text{ for } 3/4 \leqq |\xi| \leqq 11/6 \quad \text{and} \quad \operatorname{supp} \widetilde{\phi}_0 \subset \{\xi : 2^{-1} \leqq |\xi| \leqq 2\}$$

and set $\widetilde{\phi}_N(\xi) = \widetilde{\phi}_0(\xi/2^N)$. Then

$$(\widehat{\Delta_N \nabla u})_{jk}(\xi) = -\frac{\xi_l \xi_m}{|\xi|^2} P(|\xi|)\widetilde{\phi}_N(\xi)\widehat{\Delta_N\theta}(\xi)$$

or

$$(\Delta_N \nabla u)_{jk} = g * \Delta_N \theta,$$

where g denotes the inverse Fourier transform

$$g(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \left(-\frac{\xi_l \xi_m}{|\xi|^2} P(|\xi|)\widetilde{\phi}_N(\xi) \right) \mathrm{d}\xi$$

Clearly, $g(x) = 2^{Nd} g_1(2^N x)$, where

$$g_1(x) = -\frac{1}{(2\pi)^{d/2}} \int_{2^{-1} \leqq |\eta| \leqq 2} \mathrm{e}^{ix\cdot\eta} \frac{\eta_l \eta_m}{|\eta|^2} P(2^N |\eta|)\widetilde{\phi}_0(\eta)\, \mathrm{d}\eta$$

To show $g \in L^1(\mathbb{R}^d)$, it suffices to show $g_1 \in L^1(\mathbb{R}^d)$. Since

$$(1 + |x|^2)^n g_1(x) = -\frac{1}{(2\pi)^{d/2}} \int_{2^{-1} \leqq |\eta| \leqq 2} \mathrm{e}^{ix\cdot\eta} (I - \Delta_\eta)^n \frac{\eta_l \eta_m}{|\eta|^2} P(2^N |\eta|)\widetilde{\phi}_0(\eta)\, \mathrm{d}\eta,$$

we obtain, by (4) in Assumption 1.1,

$$(1 + |x|^2)^n |g_1(x)| \leqq C P(C_0 2^N).$$

where C is a constant independent of N. Equation (2.5) then follows from Young's inequality. □

We now prove (1.8) of Theorem 1.2. In fact, we have the following proposition.

Proposition 2.5. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by*

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda)\theta,$$

where $1 \leqq j, k, l, m \leqq d$, $(\nabla u)_{jk}$ denotes the (j, k)th entry of ∇u and $\mathcal{R}_l$ denotes the Riesz transform. Here P obeys Assumption 1.1. Then, for any integer $N \geqq 0$,

$$\|S_N \nabla u\|_{L^\infty} \leqq C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N P(C_0 2^N) \|S_{N+1}\theta\|_{L^\infty}, \tag{2.6}$$

where C_d depends on d only.

Proof. Splitting S_N into two parts and applying Proposition 2.3 with $q = \infty$, we have

$$\begin{aligned}\|\nabla S_N u\|_{L^\infty} &\leqq \|\nabla \Delta_{-1} u\|_{L^\infty} + \sum_{j=0}^{N-1} \|\nabla \Delta_j u\|_{L^\infty} \\ &\leqq C_d \|\Delta_{-1}\theta\|_{L^2} + \sum_{j=0}^{N-1} C_d P(C_0 2^j) \|\Delta_j \theta\|_{L^\infty} \end{aligned} \tag{2.7}$$

Since P is nondecreasing according to Assumption 1.1 and the simple fact that

$$\|\Delta_j \theta\|_{L^\infty} \leqq C \|S_{N+1}\theta\|_{L^\infty}, \quad j = 0, 1, \ldots, N-1,$$

we have

$$\|\nabla S_N u\|_{L^\infty} \leqq C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N P(C_0 2^N) \|S_{N+1}\theta\|_{L^\infty},$$

which is (2.6). □

We now prove Proposition 1.4, in which P assumes the special form

$$P(\Lambda) = (\log(I + \log(I - \Delta)))^\gamma.$$

Proof of Proposition 1.4. For any integer $N \geqq 0$, we have

$$\|\nabla u\|_{L^\infty} \leqq \|\Delta_{-1} \nabla u\|_{L^\infty} + \sum_{k=0}^{N-1} \|\Delta_k \nabla u\|_{L^\infty} + \sum_{k=N}^{\infty} \|\Delta_k \nabla u\|_{L^\infty}.$$

By Bernstein's inequality and Proposition 2.3, we have

$$\begin{aligned}\|\nabla u\|_{L^\infty} \leqq\ & C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N (\log(1 + \log(1 + 2^{2(N-1)})))^\gamma \|\theta\|_{L^\infty} \\ & + C_d \sum_{k=N}^{\infty} (2^k)^{\frac{d}{q}} \|\nabla \Delta_k u\|_{L^q}.\end{aligned}$$

Since $\log(1 + 2^{2(N-1)}) = (\log_2 e)^{-1} \log_2(1 + 2^{2(N-1)}) \leqq 2N$, we apply Proposition 2.3 again to obtain

$$\begin{aligned}\|\nabla u\|_{L^\infty} \leqq\ & C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N (\log(1 + N))^\gamma \|\theta\|_{L^\infty} \\ & + C_d \sum_{k=N}^{\infty} (2^k)^{\frac{d}{q}} (\log(1 + k))^\gamma \|\Delta_k \theta\|_{L^q}.\end{aligned}$$

By the definition of Besov space $B^s_{q,\infty}$ (see Appendix A),

$$\|\Delta_k \theta\|_{L^q} \leqq 2^{-sk} \|\theta\|_{B^s_{q,\infty}}.$$

Therefore,

$$\begin{aligned}\|\nabla u\|_{L^\infty} \leqq\ & C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N (\log(1 + N))^\gamma \|\theta\|_{L^\infty} \\ & + C_d \|\theta\|_{B^s_{q,\infty}} \sum_{k=N}^{\infty} (2^k)^{\left(\frac{d}{q} - s\right)} (\log(1 + k))^\gamma.\end{aligned}$$

Since $d/q - s < 0$, we obtain for large N,

$$\|\nabla u\|_{L^\infty} \leqq C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N (\log(1+N))^\gamma \|\theta\|_{L^\infty} + C_{d,q,s} \|\theta\|_{B^s_{q,\infty}} (2^N)^{\left(\frac{d}{q} - s\right)} (\log(1+N))^\gamma.$$

If we choose N to be the largest integer satisfying

$$N \leqq \frac{1}{s - d/q} \log_2 \left(1 + \|\theta\|_{B^s_{q,\infty}}\right),$$

we then obtain the desired result in Proposition 1.4. $\square$

3. Global regularity for (1.1) with $P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma$

This section establishes the global existence and uniqueness of solutions to (1.1) with $P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma$. The divergence-free condition on the velocity field u is not necessary if we are willing to assume that θ is bounded in $L^1 \cap L^\infty$ for all time. Of course when u is indeed divergence-free, the bound is then a trivial consequence. In the two-dimensional case, this general theorem reduces to Theorem 1.3 stated in the introduction.

Theorem 3.1. *Consider the active scalar equation* (1.1) *with*

$$P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma, \quad 0 \leqq \gamma \leqq 1.$$

Assume that the initial data θ_0 satisfies

$$\theta_0 \in X \equiv L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap B^s_{q,\infty}(\mathbb{R}^d),$$

with

$$d < q \leqq \infty \quad \text{and} \quad s > 1.$$

Assume either u is divergence-free or θ is bounded in $L^1 \cap L^\infty$ for all time. Then (1.1) *has a unique global in time solution θ that satisfies, for any $T > 0$,*

$$\theta \in L^\infty([0, T]; B^s_{q,\infty}(\mathbb{R}^d)) \quad \text{and} \quad u \in L^\infty([0, T]; B^{1+s'}_{q,\infty}(\mathbb{R}^d))$$

for any $s' < s$.

Proof. The proof consists of two main components. The first component derives a global a priori bound while the second constructs a unique local-in-time solution through the method of successive approximation.

We start with the part on the global a priori bound, which is further divided into two steps. The first step shows that for any $d/q < \sigma < 1$ and any $T > 0$,

$$\|\theta(t)\|_{B^\sigma_{q,\infty}} \leqq C(T, \|\theta_0\|_X), \quad t \leqq T$$

and the second step establishes the global bound in $B^{\sigma_1}_{q,\infty}$ for some $\sigma_1 > 1$. A finite number of iterations then yields the global bound in $B^s_{q,\infty}$.

When u is divergence-free, $\theta_0 \in L^1 \cap L^\infty$ implies that the corresponding solution θ of (1.9) satisfies the *a priori* bound

$$\|\theta(\cdot, t)\|_{L^1\cap L^\infty} \leqq \|\theta_0\|_{L^1\cap L^\infty}, \quad t \geqq 0. \tag{3.1}$$

When u is not divergence-free, we assume that (3.1) holds. Of course, the bound does not have to be $\|\theta_0\|_{L^1\cap L^\infty}$. In the rest of the proof, we can completely avoid using the divergence-free condition on u. This explains why the divergence-free condition is not used in the estimates.

Let $j \geqq -1$ be an integer. Applying Δ_j to (1.9) and following a standard decomposition, we have

$$\partial_t \Delta_j \theta = J_1 + J_2 + J_3 + J_4 + J_5 \tag{3.2}$$

where

$$\begin{aligned}
J_1 &= -\sum_{|j-k|\leqq 2} [\Delta_j, S_{k-1}(u)\cdot\nabla]\Delta_k\theta,\\
J_2 &= -\sum_{|j-k|\leqq 2} (S_{k-1}(u) - S_j(u))\cdot\nabla\Delta_j\Delta_k\theta,\\
J_3 &= -S_j(u)\cdot\nabla\Delta_j\theta,\\
J_4 &= -\sum_{|j-k|\leqq 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}(\theta)),\\
J_5 &= -\sum_{k\geqq j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta)
\end{aligned}$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. We remark that the convention $S_k \equiv 0$ for $k \leqq -1$ is adopted here. This decomposition is now standard and can be found in many references (see for example [16] or [57]).

Multiplying (3.2) by $\Delta_j\theta|\Delta_j\theta|^{q-2}$, integrating in space, integrating by part in the term associated with J_3, and applying Hölder's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^q} \leqq \|J_1\|_{L^q} + \|J_2\|_{L^q} + \|\widetilde{J}_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}. \tag{3.3}$$

By a standard commutator estimate (see for example [16, p. 39; 94, p. 814–815]),

$$\|J_1\|_{L^q} \leqq C\sum_{|j-k|\leqq 2} \|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k\theta\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leqq C\|\nabla\widetilde{\Delta}_j u\|_{L^\infty}\|\Delta_j\theta\|_{L^q}.$$

We have especially applied the lower bound part in Bernstein's inequalities (see Proposition A.3). The purpose is to shift the derivative ∇ from θ to u. It is worth pointing out that the lower bound does not apply when $j = -1$. In the case when

$j = -1$, J_2 involves only low modes and there is no need to shift the derivative from θ to u. J_2 is bounded differently. When $j = -1$, J_2 becomes

$$J_2 = -S_0(u) \cdot \nabla \Delta_1 \Delta_{-1}\theta = -\Delta_{-1}u \cdot \nabla \Delta_1 \Delta_{-1}\theta,$$

whose L^q-norm can be bounded by

$$\|J_2\|_{L^q} \leqq C\|\Delta_{-1}u\|_{L^{q_1}}\|\Delta_{-1}\theta\|_{L^{q_2}} \quad \text{with } 1/q_1 + 1/q_2 = 1/q.$$

Here we have applied the upper bound part in Bernstein's inequalities to remove ∇. Recalling that each component of u is of the form $\mathcal{R}\Lambda^{-1}(\log(1+\log(1-\Delta)))^{\gamma}\theta$, we apply the Hardy–Littlewood–Sobolev inequality [84, p. 119] to obtain

$$\|J_2\|_{L^q} \leqq C\|(\log(1+\log(1-\Delta)))^{\gamma}\Delta_{-1}\theta\|_{L^{q_3}}\|\Delta_{-1}\theta\|_{L^{q_2}}$$

where $1/q_1 = 1/q_3 - 1/d$. It then follows from Theorem 2.1 that

$$\|J_2\|_{L^q} \leqq C\|\Delta_{-1}\theta\|_{L^{q_3}}\|\Delta_{-1}\theta\|_{L^{q_2}} \leqq C\|\theta_0\|^2_{L^1\cap L^\infty}.$$

Therefore the low modes can be bounded in terms of the $L^1 \cap L^\infty$-norm of θ_0. Similar estimates apply to the low modes in J_4 and J_5, but they will not be repeated when we bound J_4 and J_5 below.

After integration by parts, the term J_3 leads to a term $\widetilde{J_3} = \frac{1}{q}(\nabla \cdot S_j u)\Delta_j\theta$, and so

$$\|\widetilde{J_3}\|_{L^q} \leqq C\|\nabla \cdot S_j u\|_{L^\infty}\|\Delta_j\theta\|_{L^q}.$$

For J_4 and J_5, we have

$$\begin{aligned}
\|J_4\|_{L^q} &\leqq C \sum_{|j-k|\leqq 2} \|\Delta_k u\|_{L^\infty}\|\nabla S_{k-1}\theta\|_{L^q} \\
&\leqq C \sum_{|j-k|\leqq 2} \|\nabla\Delta_k u\|_{L^\infty} \sum_{m\leqq k-1} 2^{m-k}\|\Delta_m\theta\|_{L^q}, \\
\|J_5\|_{L^q} &\leqq C \sum_{k\geqq j-1} \|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k\nabla\theta\|_{L^q} \\
&\leqq C \sum_{k\geqq j-1} \|\nabla\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k\theta\|_{L^q}.
\end{aligned}$$

By Proposition 1.4, for any $\sigma \in \mathbb{R}$,

$$\|J_1\|_{L^q} \leqq C \sum_{|j-k|\leqq 2} \|\nabla u\|_{L^\infty} 2^{-\sigma(k+1)}2^{\sigma(k+1)}\|\Delta_k\theta\|_{L^q} \tag{3.4}$$

$$\leqq C2^{-\sigma(j+1)}\|\theta\|_{B^\sigma_{q,\infty}}\|\nabla u\|_{L^\infty} \sum_{|j-k|\leqq 2} 2^{\sigma(j-k)} \tag{3.5}$$

$$\leqq C2^{-\sigma(j+1)}\|\theta\|_{B^\sigma_{q,\infty}}\|\nabla u\|_{L^\infty}, \tag{3.6}$$

where C is a constant depending on σ only. It is clear that $\|J_2\|_{L^q}$ and $\|\widetilde{J}_3\|_{L^q}$ obey the same bound. For any $\sigma < 1$, we have

$$\begin{aligned}
\|J_4\|_{L^q} &\leqq C\|\nabla u\|_{L^\infty} \sum_{|j-k|\leqq 2} \sum_{m<k-1} 2^{m-k} 2^{-\sigma(m+1)} 2^{\sigma(m+1)} \|\Delta_m \theta\|_{L^q} \\
&\leqq C\|\nabla u\|_{L^\infty} \|\theta\|_{B^\sigma_{q,\infty}} \sum_{|j-k|\leqq 2} \sum_{m<k-1} 2^{m-k} 2^{-\sigma(m+1)} \\
&= C 2^{-\sigma(j+1)} \|\theta\|_{B^\sigma_{q,\infty}} \|\nabla u\|_{L^\infty} \sum_{|j-k|\leqq 2} 2^{\sigma(j-k)} \sum_{m<k-1} 2^{(m-k)(1-\sigma)} \\
&\leqq C 2^{-\sigma(j+1)} \|\theta\|_{B^\sigma_{q,\infty}} \|\nabla u\|_{L^\infty}.
\end{aligned}$$

where C is a constant depending on σ only and the condition $\sigma < 1$ is used to guarantee that $(m-k)(1-\sigma) < 0$. For any $\sigma > 0$,

$$\begin{aligned}
\|J_5\|_{L^q} &\leqq C\|\nabla u\|_{L^\infty} 2^{-\sigma(j+1)} \sum_{k\geqq j-1} 2^{\sigma(j-k)} 2^{\sigma(k+1)} \|\widetilde{\Delta}_k \theta\|_{L^q} \\
&\leqq C 2^{-\sigma(j+1)} \|\theta\|_{B^\sigma_{q,\infty}} \|\nabla u\|_{L^\infty}.
\end{aligned}$$

Collecting these estimates, we obtain, for any $0 < \sigma < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \theta\|_{L^q} \leqq C 2^{-\sigma(j+1)} \|\theta\|_{B^\sigma_{q,\infty}} \|\nabla u\|_{L^\infty}.$$

Integrating in time yields

$$\|\theta(t)\|_{B^\sigma_{q,\infty}} \leqq (C\|\theta_0\|^2_{L^1\cap L^\infty} t + \|\theta_0\|_{B^\sigma_{q,\infty}}) + C\int_0^t \|\theta(\tau)\|_{B^\sigma_{q,\infty}} \|\nabla u(\tau)\|_{L^\infty}\, \mathrm{d}\tau. \tag{3.7}$$

Invoking the extrapolation inequality in Proposition 1.4, we obtain, for $d/q < \sigma < 1$,

$$\begin{aligned}
\|\theta(t)\|_{B^\sigma_{q,\infty}} \leqq\ & (C\|\theta_0\|^2_{L^1\cap L^\infty} t + \|\theta_0\|_{B^\sigma_{q,\infty}}) \\
& + C\int_0^t \|\theta(\tau)\|_{B^\sigma_{q,\infty}} [\|\theta\|_{L^1\cap L^\infty} + (1 + \|\theta\|_{L^\infty}) \\
& \times \log(1 + \|\theta\|_{B^\sigma_{q,\infty}}) (\log(1 + \log(1 + \|\theta\|_{B^\sigma_{q,\infty}})))^\gamma]\, \mathrm{d}\tau.
\end{aligned}$$

It then follows from Gronwall's inequality that, for any $T > 0$,

$$\|\theta(t)\|_{B^\sigma_{q,\infty}} \leqq C(T, \|\theta_0\|_X), \quad t \leqq T.$$

We now continue with the second step. Since $d < q \leqq \infty$, we can choose σ satisfying

$$\frac{d}{q} < \sigma < 1, \quad \sigma + 1 - \frac{d}{q} > 1$$

and then set σ_1 satisfying

$$1 < \sigma_1 < \sigma + 1 - \frac{d}{q}.$$

This step establishes the global bound for $\|\theta\|_{B^{\sigma_1}_{q,\infty}}$. J_1, J_2 and J_3 and J_5 can be bounded the same way as before, namely

$$\|J_1\|_{L^q},\, \|J_2\|_{L^q},\, \|\widetilde{J}_3\|_{L^q},\, \|J_5\|_{L^q} \leqq C2^{-\sigma_1(j+1)}\|\theta\|_{B^{\sigma_1}_{q,\infty}}\|\nabla u\|_{L^\infty}.$$

$\|J_4\|_{L^q}$ is estimated differently and bounded by the global bound in the first step. We start with the bound

$$\|J_4\|_{L^q} \leqq C \sum_{|j-k|\leqq 2} \|\nabla\Delta_k u\|_{L^\infty} \sum_{m<k-1} 2^{m-k}\|\Delta_m\theta\|_{L^q}.$$

By Bernstein's inequality and Proposition 2.3, we have

$$\begin{aligned}\|\nabla\Delta_k u\|_{L^\infty} &\leqq 2^{\frac{dk}{q}}\|\nabla\Delta_k u\|_{L^q}\\ &\leqq 2^{\frac{dk}{q}}(\log(2+k))^\gamma\|\Delta_k\theta\|_{L^q}.\end{aligned}$$

Clearly,

$$\begin{aligned}\sum_{m<k-1} 2^{m-k}\|\Delta_m\theta\|_{L^q} &= 2^{-\sigma k}\sum_{m<k-1} 2^{(m-k)(1-\sigma)}2^{\sigma m}\|\Delta_m\theta\|_{L^q}\\ &\leqq C2^{-\sigma k}\|\theta\|_{B^{\sigma}_{q,\infty}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|J_4\|_{L^q} &\leqq C\sum_{|j-k|\leqq 2} 2^{\frac{dk}{q}}(\log(2+k))^\gamma\|\Delta_k\theta\|_{L^q}2^{-\sigma k}\|\theta\|_{B^{\sigma}_{q,\infty}}\\ &= C2^{-\sigma_1(j+1)}\|\theta\|_{B^{\sigma}_{q,\infty}}\sum_{|j-k|\leqq 2} 2^{\sigma_1(j-k)}(\log(2+k))^\gamma 2^{(\sigma_1+\frac{d}{q}-\sigma)k}\|\Delta_k\theta\|_{L^q}\\ &= C2^{-\sigma_1(j+1)}\|\theta\|_{B^{\sigma}_{q,\infty}}\|\theta\|_{B^{\sigma_2}_{q,\infty}}\sum_{|j-k|\leqq 2} 2^{\sigma_1(j-k)}(\log(2+k))^\gamma 2^{(\sigma_1+\frac{d}{q}-\sigma-\sigma_2)k}\end{aligned}$$

where $\sigma_2 < 1$ is chosen very close to 1 and satisfies

$$\sigma_1 + \frac{2}{q} - \sigma - \sigma_2 < 0.$$

Then, by the global bound in the first step,

$$\|J_4\|_{L^q} \leqq C2^{-\sigma_1(j+1)}\|\theta\|_{B^{\sigma}_{q,\infty}}\|\theta\|_{B^{\sigma_2}_{q,\infty}} \leqq C(T, \|\theta_0\|_X)2^{-\sigma_1(j+1)}.$$

Collecting the estimates in this step, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^q} \leqq C2^{-\sigma_1(j+1)}\|\theta\|_{B^{\sigma_1}_{q,\infty}}\|\nabla u\|_{L^\infty} + C(T, \|\theta_0\|_X)2^{-\sigma_1(j+1)}.$$

By Proposition 1.4, for any $d/q < \sigma < 1$,

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqq \|\theta\|_{L^1\cap L^\infty} + (1+\|\theta\|_{L^\infty})\\ &\quad\times \log(1+\|\theta\|_{B^\sigma_{q,\infty}})(\log(1+\log(1+\|\theta\|_{B^\sigma_{q,\infty}})))^\gamma\\ &\leqq C(T,\|\theta_0\|_X).\end{aligned}$$

Therefore,

$$\|\theta(t)\|_{B^{\sigma_1}_{q,\infty}} \leqq \|\theta_0\|_{B^{\sigma_1}_{q,\infty}} + C(T,\|\theta_0\|_X)\left(1+\int_0^t \|\theta(\tau)\|_{B^{\sigma_1}_{q,\infty}}\,\mathrm{d}\tau\right).$$

Gronwall's inequality then yields the global bound $\|\theta(t)\|_{B^{\sigma_1}_{q,\infty}} \leqq C(T,\|\theta_0\|_X)$. If $s > \sigma_1$, we can repeat this step to achieve the desired regularity.

We now describe the process of constructing a local solution of (1.1). The solution is constructed through the method of successive approximation. Consider a successive approximation sequence $\{\theta^{(n)}\}$ satisfying

$$\begin{cases}\theta^{(1)} = S_2\theta_0,\\ u^{(n)} = (u_j^{(n)}),\quad u_j^{(n)} = \mathcal{R}_l\Lambda^{-1}P(\Lambda)\theta^{(n)},\\ \partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} = 0,\\ \theta^{(n+1)}(x,0) = S_{n+2}\theta_0,\end{cases} \tag{3.8}$$

where $P(\Lambda) = (\log(1+\log(1-\Delta)))^\gamma$. In order to show that $\{\theta^{(n)}\}$ converges to a solution of (1.1) , it suffices to prove the following properties of $\{\theta^{(n)}\}$:

(1) There exists $T_1 > 0$ such that $\theta^{(n)}$ is bounded uniformly in $B^s_{q,\infty}$ for any $t \in [0,T]$, namely

$$\|\theta^{(n)}(\cdot,t)\|_{B^s_{q,\infty}} \leqq C_1\|\theta_0\|_X,\quad t\in[0,T_1], \tag{3.9}$$

where C_1 is a constant independent of n.

(2) There exists $T_2 > 0$ such that $\eta^{(n+1)} = \theta^{(n+1)} - \theta^{(n)}$ is a Cauchy sequence in $B^{s-1}_{q,\infty}$,

$$\|\eta^{(n)}(\cdot,t)\|_{B^{s-1}_{q,\infty}} \leqq C_2 2^{-n},\quad t\in[0,T_2], \tag{3.10}$$

where C_2 is independent of n and depends on T_2 and $\|\theta_0\|_X$ only.

These two properties are established by following the ideas of the previous part and we provide some details for the proof of (3.9) and (3.10) at the end of this section. Let $T = \min\{T_1, T_2\}$. We conclude from these two properties that there exists θ satisfying

$$\begin{aligned}&\theta(\cdot,t)\in B^s_{q,\infty}\quad \text{for } 0\leqq t\leqq T,\\ &\theta^{(n)}(\cdot,t)\rightharpoonup\theta(\cdot,t)\quad\text{in } B^s_{q,\infty},\\ &\theta^{(n)}(\cdot,t)\to\theta(\cdot,t)\quad\text{in } B^{s-1}_{q,\infty}.\end{aligned}$$

Due to the interpolation inequality, for any $s-1 \leqq \widetilde{s} \leqq s$,

$$\|f\|_{B^{\widetilde{s}}_{q,\infty}} \leqq C\|f\|^{s-\widetilde{s}}_{B^{s-1}_{q,\infty}} \|f\|^{\widetilde{s}+1-s}_{B^{s}_{q,\infty}},$$

we deduce that

$$\theta^{(n)}(\cdot,t) \to \theta(\cdot,t) \quad \text{in} \quad B^{\widetilde{s}}_{q,\infty}. \tag{3.11}$$

In addition, by the relation $u_k^{(n)} = \mathcal{R}_l \Lambda^{-1} P(\Lambda)\theta^{(n)}$ and Proposition 2.3, we can easily check that

$$\nabla u^{(n)}, \quad \nabla u(\cdot,t) \in B^{s_1}_{q,\infty} \quad \text{for any } s_1 < s.$$

In order to pass to the limit in the nonlinear term, we write

$$u^{(n)}\cdot\nabla\theta^{(n+1)} - u\cdot\nabla\theta = u^{(n)}\cdot\nabla(\theta^{(n+1)}-\theta) + (u^{(n)}-u)\cdot\nabla\theta.$$

We can show that, for any $\sigma < s-1$,

$$u^{(n)}\cdot\nabla(\theta^{(n+1)}-\theta) \to 0, \quad (u^{(n)}-u)\cdot\nabla\theta \to 0 \quad \text{in } B^{\sigma}_{q,\infty}, \tag{3.12}$$

as $n \to \infty$. Again, these can be proven by following the ideas in the first part of this proof. Finally, uniqueness can be established by estimating the difference of any two solutions in $B^{s-1}_{q,\infty}$. An argument similar to that used in the proof of $\|\eta^{(n)}(\cdot,t)\|_{B^{s-1}_{q,\infty}} \leqq C_2 2^{-n}$ yields the conclusion that the difference must be zero. This completes the proof of Theorem 3.1. □

We now provide some details for the proof of (3.9) and (3.10). Equation (3.9) is proven by induction. Clearly,

$$\|\theta^{(1)}\|_{B^{s}_{q,\infty}} = \|S_2\theta_0\|_{B^{s}_{q,\infty}} \leqq C_1\|\theta_0\|_{B^{s}_{q,\infty}}.$$

We now make the ansatz that, for any $t \in [0, T_1]$,

$$\|\theta^{(n)}(\cdot,t)\|_{B^{s}_{q,\infty}} \leqq C_1\|\theta_0\|_X \tag{3.13}$$

and prove that

$$\|\theta^{(n+1)}(\cdot,t)\|_{B^{s}_{q,\infty}} \leqq C_1\|\theta_0\|_X. \tag{3.14}$$

Following the idea of the previous part, we first prove (3.14) for any σ satisfying $d/q < \sigma < 1$ and then iterate to get (3.14). As in the proof of the *a priori* bounds, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta^{(n+1)}\|_{L^q} \leqq \|K_1\|_{L^q} + \|K_2\|_{L^q} + \|\widetilde{K_3}\|_{L^q} + \|K_4\|_{L^q} + \|K_5\|_{L^q} \tag{3.15}$$

where K_1, K_2, $\widetilde{K_3}$, K_4 and K_5 are the counterparts of J_1, J_2, $\widetilde{J_3}$, J_4 and J_5, respectively, with u replaced by $u^{(n)}$ and θ by $\theta^{(n+1)}$. Similar estimates yield the counterpart of (3.7), namely

$$\begin{aligned}\|\theta^{(n+1)}(t)\|_{B^{\sigma}_{q,\infty}} &\leqq (C\|\theta_0\|^2_{L^1\cap L^\infty} t + \|S_{n+1}\theta_0\|_{B^{\sigma}_{q,\infty}}) \\ &\quad + C\int_0^t \|\theta^{(n+1)}(\tau)\|_{B^{\sigma}_{q,\infty}} \|\nabla u^{(n)}(\tau)\|_{L^\infty}\,\mathrm{d}\tau. \end{aligned}\tag{3.16}$$

Recalling the relation between $u^{(n)}$ and $\theta^{(n)}$ in (3.8), we have by applying Proposition 1.4 and the inductive ansatz (3.13),

$$\begin{aligned}\|\nabla u^{(n)}\|_{L^\infty} &\leqq \|\theta^{(n)}\|_{L^1\cap L^\infty} + (1+\|\theta^{(n)}\|_{L^\infty})\\ &\quad\times \log(1+\|\theta^{(n)}\|_{B^\sigma_{q,\infty}})(\log(1+\log(1+\|\theta^{(n)}\|_{B^\sigma_{q,\infty}})))^\gamma\\ &\leqq C(T_1, \|\theta_0\|_X). \end{aligned} \tag{3.17}$$

Inserting (3.17) in (3.16) and applying Gronwall's inequality would allow us to conclude (3.14) with $s = \sigma$, when the time interval $[0, T_1]$ is taken to be sufficiently small. (3.14) is then obtained through iteration, as in the previous part. We omit further details to avoid redundancy.

4. Generalized inviscid SQG equation

This section is devoted to the generalized inviscid SQG equation

$$\begin{cases} \partial_t\theta + (u\cdot\nabla)\theta = 0, \quad x\in\mathbb{R}^2,\ t>0,\\ u = \nabla^\perp\psi, \quad -\Lambda^{2-\beta}\psi = \theta, \quad x\in\mathbb{R}^2,\ t>0, \end{cases} \tag{4.1}$$

where $0 \leqq \beta \leqq 1$ is a parameter. (4.1) with $\beta = 0$ becomes the two-dimensional Euler vorticity equation while (4.1) with $\beta = 1$ is the SQG equation. Except in the case when $\beta = 0$, the global regularity issue for (4.1) remains open. This section presents a regularity criterion in terms of the norm of θ in the Hölder space $C^\beta(\mathbb{R}^2)$, which directly relates the regularity of θ to the parameter β. The precise conclusion has been stated in Theorem 1.5 and we reproduce it here.

Theorem 4.1. *Consider (4.1) with $0 \leqq \beta \leqq 1$. Let θ be a solution of (4.1) corresponding to the data $\theta_0 \in C^\sigma(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$ with $\sigma > 1$ and $q > 1$. Let $T > 0$. If θ satisfies*

$$\int_0^T \|\theta(\cdot,t)\|_{C^\beta(\mathbb{R}^2)}\,\mathrm{d}t < \infty, \tag{4.2}$$

then θ remains in $C^\sigma(\mathbb{R}^2)\cap L^q(\mathbb{R}^2)$ on the time interval $[0, T]$.

Some special consequences of this theorem are given in the following remark.

Remark 4.2. In the special case when $\beta = 0$, Theorem 4.1 re-establishes the global regularity for the two-dimensional Euler equation. In the special case when $\beta = 1$, (4.1) becomes the inviscid SQG equation and Theorem 4.1 reduces to a regularity criterion of [23] for the SQG equation.

To prove Theorem 4.1, we first establish two propositions. The first one bounds the back-to-labels map (the inverse map of the particle trajectory) in terms of the symmetric part of ∇u. The second proposition is a logarithmic Hölder space inequality.

Let $X(a,t)$ be the particle trajectory determined by the velocity u, namely

$$\begin{cases} \frac{dX(a,t)}{dt} = u(X(a,t),t), \\ X(a,0) = a. \end{cases} \tag{4.3}$$

Let $A(x,t)$ be the back-to-labels map or the inverse map of X. Then

$$A(X(a,t),t) = a \quad \text{for any } a \in \mathbb{R}^2. \tag{4.4}$$

Let S denote the symmetric part of ∇u, namely

$$S = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}), \tag{4.5}$$

where $(\nabla u)^{\mathrm{T}}$ denotes the transpose of ∇u. The following proposition bounds $\nabla_x A$ in terms of S.

Proposition 4.3. *Let u be a velocity field and let S be the strain tensor as defined in (4.5). Let A be the back-to-labels map. Then,*

$$\|\nabla_x A(\cdot,t)\|_{L^\infty} \leqq \exp\left(\int_0^t \|S(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

The second proposition bounds the L^∞-norm of S in terms of the logarithm of the Hölder-norm of θ.

Proposition 4.4. *Let $0 \leqq \beta \leqq 1$. Assume that u and θ are related by*

$$u = -\nabla^\perp \Lambda^{-2+\beta}\theta \tag{4.6}$$

If $\theta \in C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $\sigma > \beta$ and $q > 1$,

$$\|S\|_{L^\infty} \leqq C_1\|\theta\|_{C^\beta}\log(1 + \|\theta\|_{C^\sigma}) + C_2\|\theta\|_{L^q}, \tag{4.7}$$

where C_1 and C_2 are constants depending on β, σ and q only.

The rest of this section is arranged as follows. We prove Theorem 4.1 first and then provide the proofs of Propositions 4.3 and 4.4.

Proof of Theorem 4.1. Let X be the particle trajectory as defined in (4.3) and $A(x,t)$ be the back-to-labels map. The first equation in (4.1) implies that θ is conserved along the particle trajectory,

$$\theta(x,t) = \theta_0(A(x,t)), \quad x \in \mathbb{R}^2,\ t \geqq 0.$$

Therefore, for any $\sigma \leqq 1$,

$$\|\theta(\cdot,t)\|_{C^\sigma} = \sup_{x\neq y}\frac{|\theta(x,t)-\theta(y,t)|}{|x-y|^\sigma} \leqq \|\theta_0\|_{C^\sigma}\|\nabla_x A(\cdot,t)\|_{L^\infty}^\sigma.$$

By Proposition 4.3, we have

$$\|\theta(\cdot,t)\|_{C^\sigma} \leqq \|\theta_0\|_{C^\sigma}\exp\left(\sigma\int_0^t \|S(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Therefore,

$$\log(1+\|\theta(\cdot,t)\|_{C^\sigma}) \leqq \log(1+\|\theta_0\|_{C^\sigma}) + \sigma\int_0^t \|S(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau. \tag{4.8}$$

According to Proposition 4.4,

$$\int_0^t \|S(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau \leqq C_1\int_0^t \|\theta(\cdot,\tau)\|_{C^\beta}\log(1+\|\theta(\cdot,\tau)\|_{C^\sigma})\,\mathrm{d}\tau + C_2 t\|\theta_0\|_{L^q}. \tag{4.9}$$

Combining (4.8) and (4.9) and applying Gronwall's inequality yield

$$\log(1+\|\theta(\cdot,t)\|_{C^\sigma}) \leqq C\log(1+\|\theta_0\|_{C^\sigma}+\|\theta_0\|_{L^q})\exp\left(C\int_0^t \|\theta(\cdot,\tau)\|_{C^\beta}\,\mathrm{d}\tau\right).$$

In particular, taking $\sigma = 1$ yields a bound for $\|\nabla\theta\|_{L^\infty}$. The desired regularity $\theta\in C^\sigma$ with $\sigma>1$ then follows easily from the bound for $\|\nabla\theta\|_{L^\infty}$. This completes the proof of Theorem 4.1. □

Proof of Proposition 4.3. Differentiating the identity in (4.4) with respect to t, we obtain the equation for A,

$$\partial_t A + u\cdot\nabla A = 0.$$

Taking the gradient with respect to x, we find

$$\partial_t(\nabla_x A) + u\cdot\nabla(\nabla_x A) = \nabla u(\nabla_x A).$$

Taking (Euclidian) inner product of this equation with $\nabla_x A$, we find

$$\frac{1}{2}\frac{\mathrm{D}}{\mathrm{D}t}|\nabla_x A(x,t)|^2 = -\nabla u(\nabla_x A))\cdot(\nabla_x A).$$

Adopting the Einstein summation convention, we have

$$(\nabla u(\nabla_x A))\cdot(\nabla_x A) = \partial_{x_k}u_j\partial_{x_j}A_i\partial_{x_k}A_i = \partial_{x_j}u_k\partial_{x_k}A_i\partial_{x_j}A_i$$

and thus

$$(\nabla u(\nabla_x A))\cdot(\nabla_x A) = ((\nabla u)^{\mathrm{T}}(\nabla_x A))\cdot(\nabla_x A) = (S(\nabla_x A))\cdot(\nabla_x A).$$

Therefore

$$\frac{1}{2}\frac{\mathrm{D}}{\mathrm{D}t}|\nabla_x A|^2 \leqq |S(x,t)||\nabla_x A|^2 \leqq \|S(\cdot,t)\|_{L^\infty}|\nabla_x A|^2,$$

and integrating along the particle trajectory we obtain

$$|\nabla_x A(X(a,t),t)| \leqq \exp\left(\int_0^t \|S(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Proposition 4.3 follows from this immediately, taking the supremum over $a\in\mathbb{R}^2$. □

Proof of Proposition 4.4. The proof is divided into two cases: $\beta < 1$ and $\beta = 1$. The case $\beta = 1$ requires that $\sigma > 1$ and is handled differently from the case $\beta < 1$.

We first deal with the case when $\beta < 1$. Invoking the Riesz potential for the operator $\Lambda^{-2+\beta}$, the relation in (4.6) can be rewritten

$$u(x) = C_\beta \int \nabla^\perp \left(\frac{1}{|x-y|^\beta} \right) \theta(y)\,\mathrm{d}y = \int K_\beta(x-y)\theta(y)\,\mathrm{d}y$$

with

$$K_\beta(x) = C_\beta \frac{(-x_2, x_1)^{\mathrm{T}}}{|x|^{2+\beta}},$$

where C_β is a constant depending on β only. ∇u can be written as

$$\nabla u(x) = \text{p.v.} \int \nabla_x K(x-y)\theta(y)\,\mathrm{d}y,$$

where p.v. denotes the principal value and $\nabla_x K(x)$ can be explicitly written as

$$\nabla_x K(x) = C_\beta \frac{1}{|x|^{4+\beta}} \begin{pmatrix} x_1 x_2 & x_2^2 \\ -x_1^2 & -x_1 x_2 \end{pmatrix} + C_\beta \frac{1}{|x|^{2+\beta}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore the symmetric part of ∇u can be written as

$$S(x) = \text{p.v.} \int \Gamma(x-y)\theta(y)\,\mathrm{d}y$$

where

$$\Gamma(x) = C_\beta \frac{1}{|x|^{4+\beta}} \begin{pmatrix} 2x_1 x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1 x_2 \end{pmatrix}.$$

The property that $\Gamma(x)$ is homogenous of degree $-(2+\beta)$ and has zero mean on the unit circle is useful in the following estimate of S.

Let $\chi(x)$ be a standard smooth cutoff function with $\chi(x) = 1$ for $|x| \leqq \frac{1}{2}$ and $\chi(x) = 0$ for $|x| \geqq 1$. Let $0 < \rho \leqq R$. We divide S into three parts,

$$S(x,t) = L_1 + L_2 + L_3,$$

where

$$L_1 = \int \chi\left(\frac{|x-y|}{\rho} \right) \Gamma(x-y)(\theta(y) - \theta(x))\,\mathrm{d}y,$$

$$L_2 = \int_{|x-y| \leqq R} \left(1 - \chi\left(\frac{|x-y|}{\rho} \right) \right) \Gamma(x-y)(\theta(y) - \theta(x))\,\mathrm{d}y,$$

$$L_3 = \int_{|x-y| > R} \Gamma(x-y)\theta(y)\,\mathrm{d}y.$$

Since $\sigma > \beta$,

$$\begin{aligned} |L_1| &\leqq C_\beta \|\theta\|_{C^\sigma} \int_{|x-y|\leqq\rho} \frac{1}{|x-y|^{2+\beta-\sigma}}\, \mathrm{d}y \\ &= C_\beta \|\theta\|_{C^\sigma} \rho^{\sigma-\beta}. \end{aligned}$$

L_2 can be bounded as follows.

$$\begin{aligned} |L_2| &\leqq C_\beta \|\theta\|_{C^\beta} \int_{\frac{\rho}{2}\leqq|x-y|\leqq R} \frac{1}{|x-y|^2}\, \mathrm{d}y \\ &= C_\beta \|\theta\|_{C^\beta} \log\left(\frac{2R}{\rho}\right). \end{aligned}$$

By Hölder's inequality,

$$|L_3| \leqq C_{\beta,q} R^{-1-\beta} \|\theta\|_{L^q}$$

Setting $\rho = \log(1 + \|\theta\|_{C^\sigma})$ and $R = 1$ yields (4.7).

We now turn to the case when $\beta = 1$. This case corresponds to the SQG equation. Then $\sigma > \beta = 1$. It follows from the relation in (4.6) that

$$\nabla u(x) = \text{p.v.} \int \hat{y} \otimes \nabla\theta(x+y) \frac{\mathrm{d}y}{|y|^2}$$

where $\hat{y}$ denotes the unit vector in the direction of y and $a \otimes b$ denotes the tensor product of two vectors a and b. Therefore,

$$S(x) = \text{p.v.} \int \frac{1}{2}(\hat{y} \otimes \nabla\theta(x+y) + \nabla\theta(x+y) \otimes \hat{y}) \frac{\mathrm{d}y}{|y|^2}.$$

The difference between this representation and the one in the case $\beta < 1$ is that this formula involves $\nabla\theta$ instead of just θ. $\|S\|_{L^\infty}$ can be bounded in a similar fashion as in the case $\beta < 1$. In fact, we again use a smooth cutoff function χ to decompose the integral into three parts and estimate each one of them as we did previously. For example,

$$\begin{aligned} L_1 = \text{p.v.} \int \chi\left(\frac{|y|}{\rho}\right) \frac{1}{2}(\hat{y} \otimes (\nabla\theta(x+y) - \nabla\theta(x)) \\ +(\nabla\theta(x+y) - \nabla\theta(x)) \otimes \hat{y}) \frac{\mathrm{d}y}{|y|^2} \end{aligned}$$

can be bounded by

$$\begin{aligned} |L_1| &\leqq \int_{|y|\leqq\rho} |\nabla\theta(x+y) - \nabla\theta(x)| \frac{\mathrm{d}y}{|y|^2} \\ &\leqq \|\nabla\theta\|_{C^{\sigma-1}} \rho^{\sigma-1} \leqq \|\theta\|_{C^\sigma} \rho^{\sigma-1}. \end{aligned}$$

We omit details for the estimates of the other parts. Putting the estimates together yield the same bound as in the case $\beta < 1$. This completes the proof of Proposition 4.4. □

Appendix A. Besov spaces and related facts

This appendix provides the definitions of Δ_j, S_j and inhomogeneous Besov spaces. Related useful facts such as the Bernstein inequality are also provided here. Materials presented in this appendix can be found in several books and papers (see for example [4,16] or [85]).

Let $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}^d)$ denote the Schwartz class and tempered distributions, respectively. The partition of unity states that there exist two nonnegative radial functions $\psi, \phi \in \mathcal{S}$ such that

$$
\begin{aligned}
&\operatorname{supp}\psi \subset B\left(0, \frac{11}{12}\right), \quad \operatorname{supp}\phi \subset A\left(0, \frac{3}{4}, \frac{11}{6}\right),\\
&\psi(\xi) + \sum_{j \geqq 0} \phi_j(\xi) = 1 \quad \text{for } \xi \in \mathbf{R}^d, \qquad \phi_j(\xi) = \phi(2^{-j}\xi),\\
&\operatorname{supp}\psi \cap \operatorname{supp}\phi_j = \emptyset \quad \text{if } j \geqq 1,\\
&\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_k = \emptyset \quad \text{if } |j-k| \geqq 2,
\end{aligned}
$$

where $B(0, r)$ denotes the ball centered at the origin with radius r and $A(0, r_1, r_2)$ the annulus centered at the origin with the inner radius r_1 and the outer radius r_2.

For any $f \in \mathcal{S}'$, set

$$
\begin{aligned}
\Delta_{-1} f &= \mathcal{F}^{-1}(\psi(\xi)\mathcal{F}(f)) = \Psi * f,\\
\Delta_j f &= \mathcal{F}^{-1}(\phi_j(\xi)\mathcal{F}(f)) = \Phi_j * f, \quad j = 0, 1, 2, \dots,\\
\Delta_j f &= 0 \quad \text{for } j \leqq -2,\\
S_j &= \sum_{k=-1}^{j-1} \Delta_k \quad \text{when } j \geqq 0,
\end{aligned}
$$

where we have used $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier and inverse Fourier transforms. respectively. Clearly,

$$
\Psi = \mathcal{F}^{-1}(\psi), \quad \Phi_0 = \Phi = \mathcal{F}^{-1}(\phi), \quad \Phi_j(x) = \mathcal{F}^{-1}(\phi_j)(x) = 2^{jd}\Phi(2^j x).
$$

In addition, we can write

$$
\mathcal{F}(S_j f) = \psi\left(\frac{\xi}{2^j}\right)\mathcal{F}(f).
$$

With these notation at our disposal, we now provide the definition of the inhomogeneous Besov space.

Definition A.1. For $s \in \mathbf{R}$ and $1 \leqq p, q \leqq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$
B^s_{p,q} = \{f \in \mathcal{S}' : \ \|f\|_{B^s_{p,q}} < \infty\},
$$

where

$$
\|f\|_{B^s_{p,q}} \equiv \begin{cases} \left(\sum_{j=-1}^{\infty} (2^{js}\|\Delta_j f\|_{L^p})^q\right)^{1/q}, & \text{if } q < \infty,\\ \sup_{-1 \leqq j < \infty} 2^{js}\|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases} \tag{A.1}
$$

The Besov spaces and the standard Sobolev spaces defined by

$$W_p^s = (1-\Delta)^{-s/2} L^p$$

obey the simple facts stated in the following lemma (see [4]).

Proposition A.2. *Assume that $s \in \mathbf{R}$ and $p, q \in [1, \infty]$.*

(1) *If $s_1 \leqq s_2$, then $B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$,*
(2) *If $1 \leqq q_1 \leqq q_2 \leqq \infty$, then $B^s_{p,q_1} \subset B^s_{p,q_2}$,*
(3) *If $1 \leqq p_1 \leqq p_2 \leqq \infty$, $1 \leqq q_1, q_2 \leqq \infty$, and $s_1 \geqq s_2 + d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then*

$$B^{s_1}_{p_1,q_1}(\mathbf{R}^d) \subset B^{s_2}_{p_2,q_2}(\mathbf{R}^d),$$

(4) *If $1 < p < \infty$, then*

$$B^s_{p,\min(p,2)} \subset W^s_p \subset B^s_{p,\max(p,2)}.$$

The following Bernstein type inequalities are very useful and have been used in the previous sections. These types of inequalities can be found in many references (see, for example [57, p. 32]).

Proposition A.3. *Let $\alpha \geqq 0$. Let $1 \leqq p \leqq q \leqq \infty$.*

(1) *If f satisfies*

$$\text{supp}\, \widehat{f} \subset \{\xi \in \mathbf{R}^d : \ |\xi| \leqq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\max_{|\beta|=k} \|D^\beta f\|_{L^q(\mathbf{R}^d)} \leqq C 2^{kj + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbf{R}^d)},$$

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbf{R}^d)} \leqq C 2^{2\alpha j + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbf{R}^d)}$$

for some constant C depending on K, p and q only.

(2) *If f satisfies*

$$\text{supp}\, \widehat{f} \subset \{\xi \in \mathbf{R}^d : \ K_1 2^j \leqq |\xi| \leqq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leqq K_2$, then

$$C 2^{kj} \|f\|_{L^q(\mathbf{R}^d)} \leqq \max_{|\beta|=k} \|D^\beta f\|_{L^q(\mathbf{R}^d)} \leqq C 2^{kj + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbf{R}^d)},$$

$$C 2^{2\alpha j} \|f\|_{L^q(\mathbf{R}^d)} \leqq \|(-\Delta)^\alpha f\|_{L^q(\mathbf{R}^d)} \leqq C 2^{2\alpha j + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbf{R}^d)},$$

where the constants C depend on K_1, K_2, p and q only.

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Department of Mathematics,
Sungkyunkwan University,
Suwon 440-746,
Korea.
e-mail: chae@skku.edu

and

Department of Mathematics,
University of Chicago,
5734 S. University Avenue,
Chicago, IL 60637,
USA.
e-mail: const@cs.uchicago.edu

and

Department of Mathematics,
Oklahoma State University,
401 Mathematical Sciences,
Stillwater, OK 74078,
USA.
e-mail: jiahong@math.okstate.edu

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