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Lower Bounds for an Integral Involving Fractional Laplacians and the Generalized Navier-Stokes Equations in Besov Spaces

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Abstract: When estimating solutions of dissipative partial differential equations in L^p-related spaces, we often need lower bounds for an integral involving the dissipative term. If the dissipative term is given by the usual Laplacian $-\Delta$, lower bounds can be derived through integration by parts and embedding inequalities. However, when the Laplacian is replaced by the fractional Laplacian $(-\Delta)^\alpha$, the approach of integration by parts no longer applies. In this paper, we obtain lower bounds for the integral involving $(-\Delta)^\alpha$ by combining pointwise inequalities for $(-\Delta)^\alpha$ with Bernstein's inequalities for fractional derivatives. As an application of these lower bounds, we establish the existence and uniqueness of solutions to the generalized Navier-Stokes equations in Besov spaces. The generalized Navier-Stokes equations are the equations resulting from replacing $-\Delta$ in the Navier-Stokes equations by $(-\Delta)^\alpha$.

1. Introduction

This paper is concerned with the generalized incompressible Navier-Stokes (GNS) equations

$$\partial_t u + u \cdot \nabla u + \nabla P = -\nu(-\Delta)^\alpha u, \quad \nabla \cdot u = 0, \tag{1.1}$$

where $\nu > 0$ and $\alpha > 0$ are real parameters, and the fractional Laplacian $(-\Delta)^\alpha$ is defined in terms of the Fourier transform

$$\widehat{(-\Delta)^\alpha u}(\xi) = (2\pi|\xi|)^{2\alpha}\widehat{u}(\xi).$$

We accomplish two major goals. First, we obtain lower bounds for the integral

$$D(f) \equiv \int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^\alpha f \, dx, \tag{1.2}$$

where $p \geq 2$ and $\alpha \geq 0$. Second, we apply these lower bounds to establish the existence and uniqueness of solutions to the GNS equations in homogeneous Besov spaces. We shall now explain in some detail our major results together with background information necessary for understanding these results.

When $\alpha = 1$, the GNS equations (1.1) reduce to the usual Navier-Stokes equations. One advantage of working with the GNS equations is that they allow simultaneous consideration of their solutions corresponding to a range of α's. For example, the 3-D GNS equations with any $\alpha \geq \frac{5}{4}$ always possess global classical solutions [15]. For the general d-D GNS equations, we have shown in [17] that $\alpha \geq \frac{1}{2} + \frac{d}{4}$ guarantees global regularity. In this paper, we consider the GNS equations with a general fraction $\alpha \geq 0$ and one of the difficulties is how to obtain a lower bound for $D(f)$ defined in (1.2). The quantity $D(f)$ arises very naturally in the process of bounding solutions of the GNS equations in L^p-related spaces. In the special case when $\alpha = 1$, lower bounds for $D(f)$ are often derived through integrating by parts ([1, 2, 16]). However, for a general fraction $\alpha \geq 0$, $(-\Delta)^\alpha$ is a nonlocal operator and this approach fails. In this paper, we establish lower bounds for $D(f)$ with a general fraction $\alpha \geq 0$ by combining the pointwise inequalities for $(-\Delta)^\alpha$ and Bernstein's inequalities for fractional derivatives.

In [10] and [11], A. Córdoba and D. Córdoba showed that for any $0 \leq \alpha \leq 1$ and any $f \in C^2(\mathbb{R}^d)$ that decays sufficiently fast at infinity, the pointwise inequality

$$2\, f(x)\, (-\Delta)^\alpha f\,(x) \geq (-\Delta)^\alpha f^2\,(x), \quad x \in \mathbb{R}^d \tag{1.3}$$

holds. By modifying the proof in [10], N. Ju proved in [14] that if $p \geq 0$ and f is, in addition, nonnegative, then

$$(p+1)\, f^p(x)\, (-\Delta)^\alpha f(x) \geq (-\Delta)^\alpha f^{p+1}(x).$$

We obtain here the inequality in the general form

$$(p_1 + p_2)\, f^{p_1}(x)\, (-\Delta)^\alpha f^{p_2}(x) \geq p_2\, (-\Delta)^\alpha f^{p_1+p_2}(x), \tag{1.4}$$

where $p_1 = \frac{k_1}{l_1}$ and $p_2 = \frac{k_2}{l_2}$ with l_1 and l_2 being odd and $k_1 l_2 + k_2 l_1$ being even (see Proposition 3.2 for more details). Another type of generalization of (1.3) was considered by P. Constantin, who established an identity for $(-\Delta)^\alpha$ acting on the product of two functions. This identity allowed him to obtain a calculus inequality involving fractional derivatives [7]. In this paper, we combine suitable pointwise inequalities with Bernstein's inequalities for fractional derivatives to derive several lower bounds for $D(f)$. In particular, we have

$$D(f) \geq C\, 2^{2\alpha j}\, \|f\|_{L^p}^p, \tag{1.5}$$

which is valid for any $f \in C^2(\mathbb{R}^d)$ satisfying

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1\, 2^j \leq |\xi| \leq K_2\, 2^j\}.$$

The precise statement of this result is provided in Theorem 3.4.

As an application of these lower bounds, we study the solutions of the GNS equations in the homogeneous Besov space $\mathring{B}^r_{p,q}(\mathbb{R}^d)$ with general indices and establish several existence and uniqueness results. In particular, it is shown that the GNS equations possess a unique global solution in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$ with $r = 1 - 2\alpha + \frac{d}{p}$ for any initial datum u_0 that is comparable to ν in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$. This result holds for any $1 \leq q \leq \infty$ and for either $\frac{1}{2} < \alpha$ and $p = 2$ or $\frac{1}{2} < \alpha \le 1$ and $2 < p < \infty$. We defer the exact statement to Theorem 6.1. The proof of this theorem is based on the contraction mapping principle and two major a priori inequalities. The first one states that any solution u of the GNS equations satisfies

$$\frac{d}{dt}\|u\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu\|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le C\,q\|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}}\|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}, \tag{1.6}$$

where $0 < \alpha \le 1, r \in \mathbb{R}$ and $1 \le q < \infty$. The inequality for the case $q = \infty$ is slightly different (see Theorem 4.1 for details). The second inequality

$$\frac{d}{dt}\|F\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu\|F\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le C\,q\left(\|v\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}}\|w\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} + \|w\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}}\|v\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}\right) \tag{1.7}$$

bounds solutions of the equation

$$\partial_t F + \nu(-\Delta)^\alpha F = \mathbb{P}(v \cdot \nabla w), \tag{1.8}$$

where $\mathbb{P}$ is the matrix operator projecting onto the divergence free vector fields. For arbitrarily large initial datum $u_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^d)$, the local existence and uniqueness of solutions in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$ is established. The proof of the local existence result requires different a priori bounds and they are provided in Theorems 5.2 and 5.4 of Sect. 5.

These results for the GNS equations together with those in [4, 8, 9, 17–20] contribute significantly to understanding how the general fractional dissipation effects the regularity of solutions to dissipative partial differential equations. The results of this paper have two important special consequences. First, the 3-D Navier-Stokes equations have a unique global solution for any initial datum u_0 comparable to ν in $\mathring{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, and a unique local solution for any large datum in this space, where $1 \le p < \infty$ and $1 \le q \le \infty$. Solutions of the 3-D Navier-Stokes equations in $\mathring{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ have previously been studied in [5] and [13]. Second, these existence and uniqueness results also hold for solutions of the GNS equations in the usual Sobolev spaces $\mathring{W}^{r,p}(\mathbb{R}^d)$ with $r = 1 - 2\alpha + \frac{d}{p}$. This is because $\mathring{B}^r_{p,q}$ reduces to $\mathring{W}^{r,p}$ when $p = q$.

The rest of this paper is divided into five sections. Section 2 provides the definition of the homogeneous Besov spaces and states Bernstein's inequalities for both integer and fractional derivatives. Section 3 presents the general form of the pointwise inequality (1.4) and the lower bound (1.5). Section 4 derives the a priori bounds (1.6) and (1.7). Section 5 establishes a priori bounds for the GNS equations and for Eq. (1.8) in $L^q((0,T); \mathring{B}^r_{p,q})$ and $\widetilde{L^q}((0,T); \mathring{B}^r_{p,q})$. Section 6 proves the existence and uniqueness results.

2. Besov Spaces

In this section, we provide the definition of the homogeneous Besov space and state the Bernstein inequalities for integer and fractional derivatives.

We first fix some notation. Let $\mathcal{S}$ be the usual Schwarz class and $\mathcal{S}'$ the space of tempered distributions. The Fourier transform $\widehat{f}$ of a L^1-function f is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi\, i\, x\cdot\xi}\, dx. \tag{2.1}$$

For $f \in \mathcal{S}'$, the Fourier transform of f is obtained by

$$(\widehat{f}, g) = (f, \widehat{g})$$

for any $g \in \mathcal{S}$. The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded. For this reason, the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in \mathbb{R}$ can be defined through its Fourier transform, namely,

$$\widehat{(-\Delta)^\alpha f}(\xi) = (2\pi\, |\xi|)^{2\alpha}\, \widehat{f}(\xi).$$

For notational convenience, we sometimes write Λ for $(-\Delta)^{\frac{1}{2}}$.

We use $\mathcal{S}_0$ to denote the following subset of $\mathcal{S}$,

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S},\ \int_{\mathbb{R}^d} \phi(x) x^\gamma\, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}.$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of multinomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a multinomial.

We now introduce a dyadic partition of $\mathbb{R}^d$. For each $j \in \mathbb{Z}$, we define

$$A_j = \{\xi \in \mathbb{R}^d \,:\, 2^{j-1} < |\xi| < 2^{j+1}\}.$$

Now, we choose $\phi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\operatorname{supp} \phi_0 = \{\xi \,:\, 2^{-1} \le |\xi| \le 2\} \quad \text{and } \phi_0 > 0 \text{ on } A_0.$$

Then, we set

$$\phi_j(\xi) = \phi_0(2^{-j}\xi)$$

and define $\Phi_j \in \mathcal{S}$ by

$$\widehat{\Phi}_j(\xi) = \frac{\phi_j(\xi)}{\sum_j \phi_j(\xi)}.$$

It is clear that $\widehat{\Phi}_j$ and Φ_j satisfy

$$\widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi), \quad \operatorname{supp} \widehat{\Phi}_j \subset A_j, \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x).$$

Furthermore,

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi) = \begin{cases} 1 \text{ if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 \text{ if } \xi = 0. \end{cases}$$

Thus, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

But, if $\psi \in \mathcal{S}_0$, then

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * \psi = \psi$$

and hence

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f \tag{2.2}$$

in the weak* topology of $\mathcal{S}'_0$ for any $f \in \mathcal{S}'_0$.

To define the homogeneous Besov space, we set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \cdots . \tag{2.3}$$

For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, we say that $f \in \mathring{B}^s_{p,q}$ if $f \in \mathcal{S}'_0$ and

$$\sum_{j=-\infty}^{\infty} \left(2^{js} \|\Delta_j f\|_{L^p}\right)^q < \infty \quad \text{for } q < \infty,$$

$$\sup_{-\infty<j<\infty} 2^{js} \|\Delta_j f\|_{L^p} < \infty \quad \text{for } q = \infty.$$

$\mathring{B}^s_{p,q}$ is a Banach space when equipped with the norm

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|\Delta_j f\|_{L^p}\right)^q\right)^{1/q} \quad \text{for } q < \infty,$$

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \sup_{-\infty<j<\infty} 2^{js} \|\Delta_j f\|_{L^p} \quad \text{for } q = \infty.$$

$\mathring{B}^s_{p,q}$ with this norm will be referred to as the homogeneous Besov space. The usual homogeneous Sobolev space $\mathring{W}^{s,p}$ defined by

$$\mathring{W}^{s,p} = \Lambda^{-s} L^p$$

is a special type of the homogenous Besov space. That is, $\mathring{B}^s_{p,p} = \mathring{W}^{s,p}$.

The homogenous Besov spaces obey the inclusion relations stated in the following proposition (see [3]). Part 2) of this proposition will be referred to as the Besov embedding.

Proposition 2.1. *Assume that $\beta \in \mathbb{R}$ and $p, q \in [1, \infty]$.*

1) If $1 \le q_1 \le q_2 \le \infty$, then $\mathring{B}^{\beta}_{p,q_1}(\mathbb{R}^d) \subset \mathring{B}^{\beta}_{p,q_2}(\mathbb{R}^d)$.
2) If $1 \le p_1 \le p_2 \le \infty$ and $\beta_1 = \beta_2 + d(\frac{1}{p_1} - \frac{1}{p_2})$, then $\mathring{B}^{\beta_1}_{p_1,q}(\mathbb{R}^d) \subset \mathring{B}^{\beta_2}_{p_2,q}(\mathbb{R}^d)$.

We now turn to Bernstein's inequalities. When the Fourier transform of a function is supported on a ball or an annulus, the L^p-norms of the derivatives of the function can be bounded in terms of the L^p-norms of the function itself. Inequalities of this nature are referred to as Bernstein's inequalities. The classical Bernstein's inequalities only allow integer derivatives.

Proposition 2.2. *Let $k \ge 0$ be an integer and $1 \le p \le q \le \infty$.*

1) If supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le K 2^j\}$ for some $K > 0$ and integer j, then

$$\sup_{|\gamma|=k} \|D^\gamma f\|_{L^q} \le C\, 2^{jk+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}.$$

2) If supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$ for some $K_1, K_2 > 0$ and integer j, then

$$C\, 2^{jk+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p} \le \sup_{|\gamma|=k} \|D^\gamma f\|_{L^q} \le \tilde{C}\, 2^{jk+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p},$$

where C's and $\tilde{C}$ are constants independent of j.

Bernstein's inequalities can actually be extended to involve fractional derivatives.

Proposition 2.3. *Let $\alpha \ge 0$ and $1 \le p \le q \le \infty$.*

1) If supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le K\, 2^j\}$ for some $K > 0$ and integer j, then

$$\|\Lambda^\alpha f\|_{L^q} \le C\, 2^{j\alpha+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}.$$

2) If supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$ for some $K_1, K_2 > 0$ and integer j, then

$$C\, 2^{j\alpha+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p} \le \|\Lambda^\alpha f\|_{L^q} \le \tilde{C}\, 2^{j\alpha+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}.$$

Proposition 2.3 is a simple extension of Proposition 2.2. I am indebted to David Ullrich who communicated Proposition 2.3 to me. We remark that a lemma in Danchin' work [12, p. 632] implies a special case of 2) of Proposition 2.3, which states that for $s > 0$ and for any even integer p, f satisfying

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$

implies

$$\|\Lambda^s (f^{\frac{p}{2}})\|_{L^2} \ge C\, 2^{sj} \|f^{\frac{p}{2}}\|_{L^2}.$$

We now point out several simple facts concerning the operators Δ_j:

$$\Delta_j \Delta_k = 0, \ \text{ if } |j-k| \ge 2; \tag{2.4}$$

$$S_j \equiv \sum_{k=-\infty}^{j-1} \Delta_k \to I, \text{ as } j \to \infty; \tag{2.5}$$

$$\Delta_j (S_{k-1} f\, \Delta_k f) = 0, \text{ if } |j-k| \geq 3. \tag{2.6}$$

I in (2.5) denotes the identity operator and (2.5) is simply another way of writing (2.2). Finally, we provide here two elementary inequalities involving summations over infinite terms: Hölder' inequality and Minkowski's inequality.

Lemma 2.4. *Let* $p \in [1, \infty]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\{a_k\} \in l^p$ *and* $\{b_k\} \in l^q$, *then*

$$\left| \sum_{k=1}^{\infty} a_k\, b_k \right| \leq \left[\sum_{k=1}^{\infty} |a_k|^p \right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} |b_k|^q \right]^{\frac{1}{q}}. \tag{2.7}$$

Lemma 2.5. *Let* $p \in [1, \infty]$. *If* $a_k \geq 0$ *and* $b_k \geq 0$ *for* $k = 1, 2, 3, \ldots,$ *then*

$$\left[\sum_{k=1}^{\infty} (a_k + b_k)^p \right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^{\infty} a_k^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^{\infty} b_k^p \right]^{\frac{1}{p}}.$$

3. Lower Bounds

In this section, we extend the pointwise inequality of Córdoba and Córdoba to a more general form and prove the lower bound (1.5) for $D(f)$. The extended pointwise inequality is given in Proposition 3.2 and the lower bound in Theorem 3.4.

In [10] and [11], Córdoba and Córdoba proved the following pointwise inequality.

Proposition 3.1. *Let* $0 \leq \alpha \leq 1$ *and assume* $f \in C^2(\mathbb{R}^d)$ *decays sufficiently fast at infinity. Then, for any* $x \in \mathbb{R}^d$,

$$2\, f(x)\, (-\Delta)^\alpha f(x) \geq (-\Delta)^\alpha f^2(x).$$

As they pointed out, the condition that $f \in C^2(\mathbb{R}^d)$ can be weakened. This proposition actually holds for any f such that f, $(-\Delta)^\alpha f$ and $(-\Delta)^\alpha f^2$ are defined, and are, respectively, the limits of f_m, $(-\Delta)^\alpha f_m$ and $(-\Delta)^\alpha f_m^2$ for $f_m \in C^2(\mathbb{R}^d)$ that decays sufficiently fast at infinity.

For later applications, we extend the inequality in Proposition 3.1 to a general form stated in the following proposition.

Proposition 3.2. *Let* $0 \leq \alpha \leq 1$. *Let* $p_1 = \frac{k_1}{l_1} \geq 0$ *and* $p_2 = \frac{k_2}{l_2} \geq 1$ *be rational numbers with* l_1 *and* l_2 *being odd, and with* $k_1 l_2 + k_2 l_1$ *being even. Then, for any* $x \in \mathbb{R}^d$ *and any function* $f \in C^2(\mathbb{R}^d)$ *that decays sufficiently fast at infinity,*

$$(p_1 + p_2)\, f^{p_1}(x)\, (-\Delta)^\alpha f^{p_2}(x) \geq p_2\, (-\Delta)^\alpha f^{p_1+p_2}(x). \tag{3.1}$$

We make several remarks. First, this theorem also applies to functions that are not necessarily in $C^2(\mathbb{R}^d)$. The estimate still holds if f, $(-\Delta)^\alpha f^{p_2}$ and $(-\Delta)^\alpha f^{p_1+p_2}$ are defined, and are respective limits of f_m, $(-\Delta)^\alpha f_m^{p_2}$ and $(-\Delta)^\alpha f_m^{p_1+p_2}$ for a sequence $f_m \in C^2(\mathbb{R}^d)$ that decays sufficiently fast at infinity. Secondly, the condition that $k_1 l_2 + k_2 l_1$ is even can not be removed.

Proof. When $\alpha = 0$ or $\alpha = 1$, (3.1) can be directly verified. When p_1 and p_2 are integers, (3.1) can be proven by slightly modifying the proof of Córdoba and Córdoba for Proposition 3.1. As shown in [11], $(-\Delta)^\alpha f$ can be represented as the integral

$$(-\Delta)^\alpha f(x) = C_\alpha \,\text{P.V.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+2\alpha}}\, dy, \quad x \in \mathbb{R}^d,$$

where C_α is a constant depending on α only and P.V. means the principal value. Therefore,

$$\begin{aligned} & f^{p_1}(x)\,(-\Delta)^\alpha f^{p_2}(x) \\ &\quad = C_\alpha \,\text{P.V.} \int_{\mathbb{R}^d} \frac{f(x)^{p_1+p_2} - f(x)^{p_1} f(y)^{p_2}}{|x-y|^{d+2\alpha}}\, dy \\ &\quad = C_\alpha \,\text{P.V.} \int_{\mathbb{R}^d} \frac{p_1 f(x)^{p_1+p_2} - (p_1+p_2) f(x)^{p_1} f(y)^{p_2} + p_2 f(y)^{p_1+p_2}}{(p_1+p_2)|x-y|^{d+2\alpha}}\, dy \\ &\qquad + \frac{p_2}{p_1+p_2} C_\alpha \,\text{P.V.} \int_{\mathbb{R}^d} \frac{f(x)^{p_1+p_2} - f(y)^{p_1+p_2}}{|x-y|^{d+2\alpha}}\, dy. \end{aligned} \tag{3.2}$$

When $k_1 l_2 + k_2 l_1$ is even, $p_1 + p_2$ is even and we have Young's inequality

$$p_1 f(x)^{p_1+p_2} - (p_1+p_2) f(x)^{p_1} f(y)^{p_2} + p_2 f(y)^{p_1+p_2} \ge 0.$$

Consequently, the first integral on the right of (3.2) is nonnegative. The second integral is simply the integral representation of $\frac{p_2}{p_1+p_2}(-\Delta)^\alpha f^{p_1+p_2}(x)$. Therefore, (3.2) implies (3.1).

Now, we consider the case when $p_1 = \frac{k_1}{l_1} \ge 0$ and $p_2 = \frac{k_2}{l_2} \ge 1$ are rational numbers. For notational convenience, we write $F(x) = f(x)^{\frac{1}{l_1 l_2}}$. Since l_1 and l_2 are odd, $f \in C_0^2(\mathbb{R}^d)$ implies $F \in C_0^2(\mathbb{R}^d)$. Because (3.1) has been shown for integers, we have

$$\begin{aligned} f^{p_1}(x)\,(-\Delta)^\alpha f^{p_2}(x) &= F^{k_1 l_2}(x)\,(-\Delta)^\alpha F^{k_2 l_1}(x) \\ &\ge \frac{k_2 l_1}{k_1 l_2 + k_2 l_1} (-\Delta)^\alpha F^{k_1 l_2 + k_2 l_1}(x) \\ &= \frac{p_2}{p_1+p_2} (-\Delta)^\alpha f^{p_1+p_2}(x). \end{aligned}$$

That is, (3.1) holds in this case. This completes the proof of Proposition 3.2.

If we are willing to assume the function $f \ge 0$, then the assumption on the indices p_1 and p_2 can be reduced. The following proposition is due to N. Ju [14].

Proposition 3.3. *Let* $0 \le \alpha \le 1$ *and let* $p \ge 0$. *Then, for any* $f \in C^2(\mathbb{R}^d)$ *that decays sufficiently fast at infinity,* $f \ge 0$, *and any* $x \in \mathbb{R}^d$,

$$(p+1)\, f^p(x)\,(-\Delta)^\alpha f(x) \ge (-\Delta)^\alpha f^{p+1}(x).$$

We now derive the lower bound (1.5) for $D(f)$.

Theorem 3.4. *Assume either* $0 \le \alpha$ *and* $p = 2$ *or* $0 \le \alpha \le 1$ *and* $2 < p < \infty$. *If* $f \in C^2(\mathbb{R}^d)$ *decays sufficiently fast at infinity and satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\} \tag{3.3}$$

for some $K_1, K_2 > 0$ and some integer j, then

$$D(f) \equiv \int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^\alpha f \, dx \geq C \, 2^{2\alpha j} \, \|f\|^p_{L^p(\mathbb{R}^d)}, \tag{3.4}$$

where C is a constant depending on d, p, K_1 and K_2 only.

The restriction that $0 \leq \alpha \leq 1$ comes from applying the pointwise inequalities. When $p = 2$, (3.4) is a direct consequence of Plancherel's theorem and thus α is not required to satisfy $\alpha \leq 1$.

We also remark that the requirement $f \in C^2(\mathbb{R}^d)$ can be reduced. In fact, this theorem applied to any function f with the property that f and $(-\Delta)^\alpha f^j$ $(j = 1, 2, \dots, \frac{p}{2})$ are defined, and are, respectively, limits of f_m and $(-\Delta)^\alpha f_m^j$ with each $f_m \in C^2(\mathbb{R}^d)$. This also explains why we do not assume the functions are in $C^2(\mathbb{R}^d)$ when we apply this theorem in the subsequent sections.

Proof. When $p = 2$, the lower bound in (3.4) is a direct consequence of Plancherel's theorem. When $p > 2$, Proposition 3.3 implies

$$D(f) \equiv \int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^\alpha f \, dx \geq C \int_{\mathbb{R}^d} \left| \Lambda^\alpha (f^{\frac{p}{2}}) \right|^2 dx = C \, \|\Lambda^\alpha (f^{\frac{p}{2}})\|^2_{L^2}.$$

It then follows from 2) of Proposition 2.3 that

$$D(f) \geq C \, 2^{2\alpha j} \, \|f^{\frac{p}{2}}\|^2_{L^2} \geq C \, 2^{2\alpha j} \, \|f\|^p_{L^p}.$$

Other useful lower bounds for $D(f)$ can also be established using the pointwise inequality in Proposition 3.3. These lower bounds which do not require the support of $\widehat{f}$ satisfy the condition (3.3).

Theorem 3.5. *Assume either $0 \leq \alpha$ and $p = 2$ or $0 \leq \alpha \leq 1$ and $2 < p < \infty$. Then $D(f)$ can be bounded as follows.*

(a) If $p = 2$ and $2\alpha = d$, then

$$D(f) \geq C \, \|f\|^2_{L^q(\mathbb{R}^d)} \tag{3.5}$$

for any $q \in [2, \infty)$ and some constant depending on q only.

(b) If $p = 2$ and $2\alpha < d$, then

$$D(f) \geq C \, \|f\|^2_{L^{\frac{2d}{d-2\alpha}}(\mathbb{R}^d)}$$

for some constant C depending on α and d only.

(c) If $p > 2$ and $2\alpha = d$, then, for any $f \in L^2(\mathbb{R}^d)$,

$$D(f) \geq C \, \|f\|^{(1+\beta)p}_{L^p(\mathbb{R}^d)} \, \|f\|^{-\beta p}_{L^2(\mathbb{R}^d)},$$

where $\beta = \frac{2q-4}{pq-2q}$ for any $q \in [p, \infty)$, and C is a constant depending on p and q only.

(d) If $p > 2$ and $2\alpha < d$, then for any $f \in L^2(\mathbb{R}^d)$,

$$D(f) \geq C \, \|f\|^{(1+\gamma)p}_{L^p(\mathbb{R}^d)} \, \|f\|^{-\gamma p}_{L^2(\mathbb{R}^d)},$$

where $\gamma = \frac{4\alpha}{d(p-2)}$ and C is a constant depending on d, α and p only.

The proof of this theorem is left to the appendix.

4. A Priori Estimates in $\mathring{B}^r_{p,q}$

This section derives two a priori bounds in $\mathring{B}^r_{p,q}$: one for solutions of the GNS equations

$$\partial_t u + u \cdot \nabla u + \nabla P = -\nu(-\Delta)^\alpha u, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{4.1}$$

$$\nabla \cdot u = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{4.2}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{4.3}$$

and one for solutions of the equation

$$\partial_t F + \nu(-\Delta)^\alpha F = \mathbb{P}(v \cdot \nabla w), \quad x \in \mathbb{R}^d, \quad t > 0, \tag{4.4}$$

$$F(x, 0) = F_0(x), \quad x \in \mathbb{R}^d, \tag{4.5}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the matrix operator projecting onto divergence free vector fields and I is the identity matrix. The bounds are given in Theorems 4.1 and 4.3.

We work with the following form of the GNS equations:

$$\partial_t u + \nu(-\Delta)^\alpha u \;=\; -\mathbb{P}(u \cdot \nabla u), \tag{4.6}$$

which is equivalent to (4.1) and (4.2). This form of the GNS equations can be seen as a special case of (4.4). Before stating and proving the theorems, we first briefly introduce the spaces $L^\rho((a, b); \mathring{B}^s_{p,q})$ and $\widetilde{L^\rho}((a, b); \mathring{B}^s_{p,q})$.

For $\rho, p, q \in [1, \infty]$ and $s, a, b \in \mathbb{R}$, $L^\rho((a, b); \mathring{B}^s_{p,q})$ is defined in the standard fashion. That is, $L^\rho((a, b); \mathring{B}^s_{p,q})$ denotes the space of L^ρ-integrable functions from (a, b) to $\mathring{B}^s_{p,q}$. The space $\widetilde{L^\rho}((a, b); \mathring{B}^s_{p,q})$ was introduced in [6] and later used in [5] and [13]. We say that $f \in \widetilde{L^\rho}((a, b); \mathring{B}^s_{p,q})$ if

$$\{\mu_j\} \in l^q,$$

where μ_j is defined by

$$\mu_j = 2^{js} \left(\int_a^b \|\Delta_j f(\cdot, t)\|^\rho_{L^p} dt \right)^{\frac{1}{\rho}} \quad \text{for } \rho < \infty,$$
$$\mu_j = 2^{js} \sup_{t \in (a,b)} \|\Delta_j f(\cdot, t)\|_{L^p} \quad \text{for } \rho = \infty.$$

The norm in $\widetilde{L^\rho}((a, b); \mathring{B}^s_{p,q})$ is given by

$$\|f\|_{\widetilde{L^\rho}((a,b);\mathring{B}^s_{p,q})} = \|\mu_j\|_{l^q}.$$

For notational convenience, we sometimes write $\widetilde{L^\rho_t}(\mathring{B}^s_{p,q})$ for $\widetilde{L^\rho}((0, t); \mathring{B}^s_{p,q})$.

We now state the first major theorem of this section.

Theorem 4.1. *Let* $r \in \mathbb{R}$ *and* $1 \le q \le \infty$. *Assume either* $0 < \alpha$ *and* $p = 2$ *or* $0 < \alpha \le 1$ *and* $2 < p < \infty$. *Then any solution* u *of the GNS equations (4.1) and (4.2) or of (4.6) satisfies*

$$\frac{d}{dt}\|u\|^q_{\mathring{B}^r_{p,q}} + C\, q\nu \|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le C\, q \|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \quad \text{for } q \in [1,\infty) \tag{4.7}$$

and, for $q = \infty$,

$$\|u(\cdot,t)\|_{\mathring{B}^r_{p,\infty}} + C\nu \|u\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})} \le \|u_0\|_{\mathring{B}^r_{p,\infty}} + C \sup_{0\le\tau\le t} \|u(\cdot,\tau)\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}} \|u\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})}, \tag{4.8}$$

where C*'s are constants depending on* d, α, r *and* p *only.*

The inequality in (4.7) contains rich information about solutions of the GNS equations. For example, if we know u is comparable to ν in $\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}$, then u is bounded in $\mathring{B}^r_{p,q}$ for any $r \in \mathbb{R}$. In particular, when $r = 1 - 2\alpha + \frac{d}{p}$, (4.7) becomes a "closed" inequality. We state the result in this special case as a corollary.

Corollary 4.2. *Let* $1 \le q \le \infty$. *Assume either* $0 < \alpha$ *and* $p = 2$ *or* $0 < \alpha \le 1$ *and* $2 < p < \infty$. *Then any solution* u *of (4.1) and (4.2) or of (4.6) satisfies*

$$\frac{d}{dt}\|u\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} + C\, q\nu \|u\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}+\frac{2\alpha}{q}}_{p,q}} \le C\, q \|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|u\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}+\frac{2\alpha}{q}}_{p,q}} \quad \text{for } q \in [1,\infty)$$

and, for $q = \infty$,

$$\begin{aligned}\|u(\cdot,t)\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}} + C\nu \|u\|_{\widetilde{L}^1_t(\mathring{B}^{1+\frac{d}{p}}_{p,\infty})} \le{}& \|u_0\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}} \\ &+ C \sup_{0\le\tau\le t} \|u(\cdot,\tau)\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}} \|u\|_{\widetilde{L}^1_t(\mathring{B}^{1+\frac{d}{p}}_{p,\infty})},\end{aligned}$$

where C*'s are constants depending on* d, α *and* p *only. A special indication of this inequality is that any solution of the GNS equations will remain in the Besov space* $\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}$ *for all time if the corresponding initial datum* u_0 *is in this Besov space and is comparable to* ν.

Proof of Theorem 4.1. Let $j \in \mathbb{Z}$. Applying Δ_j to (4.6) yields

$$\partial_t \Delta_j u + (u\cdot\nabla)\Delta_j u + \nu(-\Delta)^\alpha \Delta_j u = -[\mathbb{P}\Delta_j, u\cdot\nabla]u, \tag{4.9}$$

where the brackets represent the commutator operator, namely

$$[\mathbb{P}\Delta_j, u\cdot\nabla]u \equiv \mathbb{P}\Delta_j(u\cdot\nabla u) - u\cdot\nabla\,\mathbb{P}\Delta_j u.$$

We then dot both sides of (4.9) by $p|\Delta_j u|^{p-2}\Delta_j u$ and integrate over $\mathbb{R}^d$. Since

$$\int_{\mathbb{R}^d} (u\cdot\nabla)\Delta_j u \cdot |\Delta_j u|^{p-2}\Delta_j u\, dx = 0,$$

we obtain

$$\frac{d}{dt}\|\Delta_j u\|_{L^p}^p + p\,\nu \int_{\mathbb{R}^d} |\Delta_j u|^{p-2}\Delta_j u\cdot(-\Delta)^\alpha \Delta_j u dx$$
$$= -p \int_{\mathbb{R}^d} |\Delta_j u|^{p-2}\Delta_j u\cdot\, [\mathbb{P}\Delta_j, u\cdot\nabla]u\,dx. \tag{4.10}$$

According to Theorem 3.4, the dissipative part admits the following lower bound:

$$p\nu \int_{\mathbb{R}^d} |\Delta_j u|^{p-2}\Delta_j u\cdot(-\Delta)^\alpha \Delta_j u dx \geq C\,\nu 2^{2\alpha j}\|\Delta_j u\|_{L^p}^p. \tag{4.11}$$

We now estimate the nonlinear part

$$I \equiv -p \int_{\mathbb{R}^d} |\Delta_j u|^{p-2}\Delta_j u\cdot\, [\mathbb{P}\Delta_j, u\cdot\nabla]u\,dx.$$

By Hölder's inequality,

$$|I| \leq C\,\|\Delta_j u\|_{L^p}^{p-1}\,\|[\mathbb{P}\Delta_j, u\cdot\nabla]u\|_{L^p}.$$

To estimate $\|[\mathbb{P}\Delta_j, u\cdot\nabla]u\|_{L^p}$, we use Bony's notion of paraproduct to write

$$[\mathbb{P}\Delta_j, u\cdot\nabla]u = I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.12}$$

where

$$\begin{aligned}
I_1 &= \sum_{k\in\mathbb{Z}} \mathbb{P}\Delta_j((S_{k-1}u\cdot\nabla)\Delta_k u) - (S_{k-1}u\cdot\nabla)\mathbb{P}\Delta_j\Delta_k u,\\
I_2 &= \sum_{k\in\mathbb{Z}} \mathbb{P}\Delta_j((\Delta_k u\cdot\nabla)S_{k-1}u),\\
I_3 &= -\sum_{k\in\mathbb{Z}} (\Delta_k u\cdot\nabla)\Delta_j S_{k-1}u,\\
I_4 &= \sum_{|k-l|\leq 1} \mathbb{P}\Delta_j((\Delta_k u\cdot\nabla)\Delta_l u),\\
I_5 &= -\sum_{|k-l|\leq 1} (\Delta_k u\cdot\nabla)\Delta_j\Delta_l u.
\end{aligned}$$

We now estimate the L^p-norms of these terms. According to (2.6), the summation in I_1 is only over those k satisfying $|k-j|\leq 2$. Let $\tilde{\Phi}_j$ be the convolution kernel associated with the operator $\mathbb{P}\Delta_j$. Since each entry in $\mathbb{P}$ is the difference between 1 and a product of two Riesz transforms, $\tilde{\Phi}_j$ is smooth. Using $\tilde{\Phi}_j$, we write

$$I_1 = \sum_{|k-j|\leq 2} \int_{\mathbb{R}^d} \tilde{\Phi}_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k u(y)\,dy.$$

We integrate by parts and use the fact that $\nabla\cdot u = 0$ to obtain

$$I_1 = -\sum_{|k-j|\leq 2} \int_{\mathbb{R}^d} \nabla\tilde{\Phi}_j(x-y)\cdot(S_{k-1}u(y) - S_{k-1}u(x))\;\Delta_k u\,dy.$$

By Young's inequality,

$$\|I_1\|_{L^p} \le C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k u\|_{L^p} \int_{\mathbb{R}^d} |x||\nabla \tilde{\Phi}_j(x)|\, dx$$
$$= C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k u\|_{L^p}. \tag{4.13}$$

Similarly, the summations in I_2 and I_3 are also only over k satisfying $|k - j| \le 2$. Applying Young's inequality and using the fact that the norm of the operator $\mathbb{P}$ in L^p with $1 < p < \infty$ is 1 yield

$$\|I_2||_{L^p} \le C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k u\|_{L^p}, \tag{4.14}$$

$$\|I_3\|_{L^p} \le C \sum_{|k-j|\le 2} \|\Delta_k u\|_{L^p}\, \|\Delta_j \nabla S_{k-1}u\|_{L^\infty}$$
$$\le C \sum_{|k-j|\le 2} \|\Delta_k u\|_{L^p}\, \|\nabla S_{k-1}u\|_{L^\infty}. \tag{4.15}$$

The summation in I_4 is over k with $|k - j| \le 3$. The L^p norm of I_4 is bounded by

$$\|I_4\|_{L^p} \le C \sum_{|k-j|\le 3, |k-l|\le 1} \|\Delta_k u\|_{L^p}\, \|\nabla \Delta_l u\|_{L^\infty}. \tag{4.16}$$

The estimate of I_5 is similar, namely,

$$\|I_5\|_{L^p} \le C \sum_{|k-j|\le 2, |k-l|\le 1} \|\Delta_k u\|_{L^p}\, \|\nabla \Delta_l u\|_{L^\infty}. \tag{4.17}$$

Collecting the estimates (4.13), (4.14), (4.15), (4.16) and (4.17), we find that

$$\|[\Delta_j, u \cdot \nabla]u\|_{L^p} \le C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k u\|_{L^p}$$
$$+C \sum_{|k-j|\le 3, |k-l|\le 1} \|\Delta_k u\|_{L^p}\, \|\nabla \Delta_l u\|_{L^\infty}. \tag{4.18}$$

We emphasize that the summations in (4.18) are only over a finite number of k's. It suffices to consider the term with $k = j$ in the first summation and the term with $k = l = j$ in the second summation. By Bernstein's inequality,

$$\|\nabla \Delta_j u\|_{L^\infty} \le C\, 2^{(1+\frac{d}{p})j}\, \|\Delta_j u\|_{L^p},$$

$$\|\nabla S_{j-1}u\|_{L^\infty} \le \sum_{m<j-1} \|\nabla \Delta_m u\|_{L^\infty} \le C \sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p}.$$

Therefore,

$$\|[\Delta_j, u \cdot \nabla]u\|_{L^p} \le C\, \|\Delta_j u\|_{L^p} \sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p} + C\, 2^{(1+\frac{d}{p})j}\, \|\Delta_j u\|^2_{L^p}.$$

Consequently, we obtain the following bound for the nonlinear term in (4.10):

$$|I| \le C\|\Delta_j u\|_{L^p}^p \sum_{m<j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u\|_{L^p} + C\,2^{(1+\frac{d}{p})j}\,\|\Delta_j u\|_{L^p}^{p+1}. \tag{4.19}$$

Combining (4.10), (4.11) and (4.19), we find

$$\begin{aligned}&\frac{d}{dt}\|\Delta_j u\|_{L^p}^p + C\,\nu 2^{2\alpha j}\|\Delta_j u\|_{L^p}^p\\ &\quad\le C\|\Delta_j u\|_{L^p}^p \sum_{m<j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u\|_{L^p} + C\,2^{(1+\frac{d}{p})j}\,\|\Delta_j u\|_{L^p}^{p+1}.\end{aligned} \tag{4.20}$$

Equivalently,

$$\begin{aligned}&\frac{d}{dt}\|\Delta_j u\|_{L^p} + C\,\nu 2^{2\alpha j}\|\Delta_j u\|_{L^p}\\ &\quad\le C\|\Delta_j u\|_{L^p} \sum_{m<j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u\|_{L^p} + C\,2^{(1+\frac{d}{p})j}\,\|\Delta_j u\|_{L^p}^{2}.\end{aligned} \tag{4.21}$$

For $1 \le q < \infty$, we multiply both sides by $q2^{rjq}\,\|\Delta_j u\|_{L^p}^{q-1}$ and sum over $j \in \mathbb{Z}$ to get

$$\frac{d}{dt}\|u\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu\|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le J_1 + J_2, \tag{4.22}$$

where J_1 and J_2 are given by

$$J_1 \equiv C\,q\sum_j 2^{rjq}\,\|\Delta_j u\|_{L^p}^q \sum_{m<j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u\|_{L^p},$$

$$J_2 \equiv C\,q\sum_j 2^{(1+\frac{d}{p})j}\,\|\Delta_j u\|_{L^p}\,2^{rjq}\,\|\Delta_j u\|_{L^p}^q.$$

To estimate J_1, we write

$$J_1 = C\,q\sum_j 2^{(r+\frac{2\alpha}{q})jq}\|\Delta_j u\|_{L^p}^q \sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha)m}\|\Delta_m u\|_{L^p}\,2^{2\alpha(m-j)}.$$

Applying the elementary inequality (2.7), we have

$$\begin{aligned}&\sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha)m}\|\Delta_m u\|_{L^p}\,2^{2\alpha(m-j)}\\ &\quad\le \left[\sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha)mq}\|\Delta_m u\|_{L^p}^q\right]^{\frac{1}{q}}\left[\sum_{m<j-1} 2^{2\alpha(m-j)\bar{q}}\right]^{\frac{1}{\bar{q}}}\\ &\quad= C\,\|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}},\end{aligned} \tag{4.23}$$

where $\bar{q}$ is the conjugate of q, namely $\frac{1}{q} + \frac{1}{\bar{q}} = 1$. Therefore, J_1 is bounded by

$$J_1 \leq C\, q \|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

To bound J_2, we first write

$$J_2 = C\, q \sum_j 2^{(1-2\alpha+\frac{d}{p})j} \|\Delta_j u\|_{L^p}\, 2^{(r+\frac{2\alpha}{q})jq} \|\Delta_j u\|^q_{L^p}.$$

It is clear that

$$2^{(1-2\alpha+\frac{d}{p})j} \|\Delta_j u\|_{L^p} \leq \left[\sum_j 2^{(1-2\alpha+\frac{d}{p})jq} \|\Delta_j u\|^q_{L^p}\right]^{\frac{1}{q}} = \|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}}. \quad (4.24)$$

Therefore, J_2 is bounded by

$$J_2 \leq C\, q \|u\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

Finally, we insert the estimates for J_1 and J_2 in (4.22) to obtain (4.7).

We now deal with the case when $q = \infty$. We multiply (4.21) by 2^{rj}, integrate over $(0, t)$ and take the supremum over $j \in \mathbb{Z}$ to get

$$\|u(\cdot, t)\|_{\mathring{B}^r_{p,\infty}} + C\,\nu \|u\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})} \leq \|u_0\|_{\mathring{B}^r_{p,\infty}} + \widetilde{J}_1 + \widetilde{J}_2, \quad (4.25)$$

where $\widetilde{J}_1$ and $\widetilde{J}_2$ are given by

$$\widetilde{J}_1 = C \sup_j \int_0^t 2^{jr} \|\Delta_j u\|_{L^p} \sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p}\, d\tau,$$

$$\widetilde{J}_2 = C \sup_j \int_0^t 2^{jr} 2^{(1+\frac{d}{p})j} \|\Delta_j u\|^2_{L^p}\, d\tau.$$

To estimate $\widetilde{J}_1$, we rearrange its terms as

$$\widetilde{J}_1 = C \sup_j \int_0^t 2^{(r+2\alpha)j} \|\Delta_j u\|_{L^p} \sum_{m<j-1} 2^{2\alpha(m-j)}\, 2^{(1-2\alpha+\frac{d}{p})m} \|\Delta_m u\|_{L^p}\, d\tau.$$

It is then clear that

$$\begin{aligned}
\widetilde{J}_1 &\leq C \sup_j \int_0^t 2^{(r+2\alpha)j} \|\Delta_j u\|_{L^p}\, d\tau \\
&\quad \times \sup_j \sup_{0\leq\tau\leq t} \sum_{m<j-1} 2^{2\alpha(m-j)}\, 2^{(1-2\alpha+\frac{d}{p})m} \|\Delta_m u(\cdot,\tau)\|_{L^p} \\
&\leq C \sup_j \int_0^t 2^{(r+2\alpha)j} \|\Delta_j u\|_{L^p}\, d\tau \sup_{0\leq\tau\leq t} \sup_j 2^{(1-2\alpha+\frac{d}{p})j} \|\Delta_j u(\cdot,\tau)\|_{L^p} \\
&= C\, \|u\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})} \sup_{0\leq\tau\leq t} \|u(\cdot,\tau)\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}}. \qquad (4.26)
\end{aligned}$$

$\widetilde{J}_2$ can be bounded as follows.

$$\begin{aligned}\widetilde{J}_2 &= C \sup_j \int_0^t 2^{(1-2\alpha+\frac{d}{p})j} \, \|\Delta_j u\|_{L^p} \, 2^{(r+2\alpha)j} \|\Delta_j u\|_{L^p} \, d\tau \\ &\le C \sup_j \sup_{0\le\tau\le t} 2^{(1-2\alpha+\frac{d}{p})j} \, \|\Delta_j u\|_{L^p} \sup_j \int_0^t 2^{(r+2\alpha)j} \|\Delta_j u\|_{L^p} \, d\tau \\ &\le C \sup_{0\le\tau\le t} \|u(\cdot,\tau)\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty}} \|u\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})}. \end{aligned} \tag{4.27}$$

Combining the estimates in (4.25), (4.26) and (4.27) yields (4.8). This completes the proof of Theorem 4.1.

We now present the priori bound for solutions of the general equation (4.4). This type of estimates will be needed in Sect. 6 when we study the existence and uniqueness of the solutions to the GNS equations. To establish the estimate, we need to restrict to $\alpha > \frac{1}{2}$.

Theorem 4.3. *Let $r \in \mathbb{R}$ and $q \in [1,\infty]$. Assume either $\frac{1}{2} < \alpha$ and $p = 2$ or $\frac{1}{2} < \alpha \le 1$ and $2 < p < \infty$. Assume that v and w are in the class*

$$L^\infty([0,T); \mathring{B}^r_{p,q}) \cap L^q([0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$$

for $0 < T \le \infty$. Then any solution F of (4.4) satisfies

$$\begin{aligned}&\frac{d}{dt}\|F\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu\|F\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \\ &\quad \le C\,q\left(\|v\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|w\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} + \|w\|^q_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|v\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}\right), \end{aligned} \tag{4.28}$$

where C is a constant depending on d and p only.

For the sake of conciseness, we did not include in this theorem the inequality for the case when $q = \infty$. It can be stated and derived analogously as (4.28).

Proof. We apply Δ_j to (4.4) and then multiply by $p|\Delta_j F|^{p-2}\Delta_j F$. Bounding the dissipative part by the lower bound as in (4.11) and estimating the right-hand side by Hölder's inequality, we obtain

$$\frac{d}{dt}\|\Delta_j F\|^p_{L^p} + C\,p\nu 2^{2\alpha j}\|\Delta_j F\|^p_{L^p} \le C\,p\|\Delta_j (v\cdot\nabla)w\|_{L^p}\,\|\Delta_j F\|^{p-1}_{L^p}.$$

That is,

$$\frac{d}{dt}\|\Delta_j F\|_{L^p} + C\,\nu 2^{2\alpha j}\|\Delta_j F\|_{L^p} \le C\,\|\Delta_j (v\cdot\nabla)w\|_{L^p}. \tag{4.29}$$

To estimate the term on the right, we write

$$\Delta_j (v\cdot\nabla)w = K_1 + K_2 + K_3,$$

where K_1, K_2 and K_3 are given by

$$
\begin{aligned}
K_1 &= \sum_k \Delta_j\left((S_{k-1}v\cdot\nabla)\Delta_k w\right),\\
K_2 &= \sum_k \Delta_j\left((\Delta_k v\cdot\nabla)S_{k-1} w\right),\\
K_3 &= \sum_{|k-l|\le 1} \Delta_j\left((\Delta_k v\cdot\nabla)\Delta_l w\right).
\end{aligned}
$$

To estimate these terms, we first notice that the summations in K_1, K_2 and K_3 are only over k satisfying $|k-j|\le 2$. Therefore, it suffices to estimate the representative term with $k=j$. Applying Young's inequality and Bernstein's inequality, we obtain

$$
\begin{aligned}
\|K_1\|_{L^p} &\le C\|S_{j-1}v\|_{L^\infty}\,\|\nabla\Delta_j w\|_{L^p} \le C\sum_{m<j-1}\|\Delta_m v\|_{L^\infty}\,2^j\,\|\Delta_j w\|_{L^p}\\
&\le C\sum_{m<j-1}2^{\frac{d}{p}m}\,\|\Delta_m v\|_{L^p}\,2^j\,\|\Delta_j w\|_{L^p},
\end{aligned}
$$

$$
\|K_2\|_{L^p} \le C\,\|\nabla S_{j-1}w\|_{L^\infty}\,\|\Delta_j v\|_{L^p} \le C\sum_{m<j-1}2^{(1+\frac{d}{p})m}\,\|\Delta_m w\|_{L^p}\,\|\Delta_j v\|_{L^p},
$$

$$
\|K_3\|_{L^p} \le C\,\|\Delta_j v\|_{L^p}\,\|\nabla\Delta_j w\|_{L^\infty} \le C\,2^{(1+\frac{d}{p})j}\,\|\Delta_j v\|_{L^p}\,\|\Delta_j w\|_{L^p}.
$$

Inserting these estimates in (4.29), then multiplying by $q\,2^{rjq}\,\|\Delta_j F\|_{L^p}^{q-1}$ and summing over all j, we obtain

$$
\frac{d}{dt}\|F\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu\,\|F\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le L_1+L_2+L_3, \tag{4.30}
$$

where L_1, L_2 and L_3 are given by

$$
\begin{aligned}
L_1 &= C\,q\sum_j 2^{rjq}\,\|\Delta_j F\|_{L^p}^{q-1}\,2^j\,\|\Delta_j w\|_{L^p}\sum_{m<j-1}2^{\frac{d}{p}m}\,\|\Delta_m v\|_{L^p},\\
L_2 &= C\,q\sum_j 2^{rjq}\,\|\Delta_j F\|_{L^p}^{q-1}\,\|\Delta_j v\|_{L^p}\sum_{m<j-1}2^{(1+\frac{d}{p})m}\,\|\Delta_m w\|_{L^p},\\
L_3 &= C\,q\sum_j 2^{rjq}\,\|\Delta_j F\|_{L^p}^{q-1}\,2^{(1+\frac{d}{p})j}\,\|\Delta_j v\|_{L^p}\,\|\Delta_j w\|_{L^p}.
\end{aligned}
$$

To bound L_1, we write

$$
\begin{aligned}
L_1 = C\,q\sum_j & 2^{(r+\frac{2\alpha}{q})j(q-1)}\,\|\Delta_j F\|_{L^p}^{q-1}\,2^{(r+\frac{2\alpha}{q})j}\,\|\Delta_j w\|_{L^p}\\
&\times\sum_{m<j-1}2^{(1+\frac{d}{p}-2\alpha)m}\,\|\Delta_m v\|_{L^p}\,2^{(2\alpha-1)(m-j)}.
\end{aligned}
$$

When $\alpha > \frac{1}{2}$, $(2\alpha-1)(m-j) < 0$ and we obtain as in (4.23)

$$\sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha)m} \|\Delta_m v\|_{L^p}\, 2^{(2\alpha-1)(m-j)} \le C\, \|v\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}}.$$

After applying the elementary inequality (2.7), we find

$$L_1 \le C\, q\, \|v\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|w\|_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \|F\|^{q-1}_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

L_2 is bounded similarly, but we do not need $\alpha > \frac{1}{2}$,

$$L_2 \le C\, q \|w\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|v\|_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \|F\|^{q-1}_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

To bound L_3, we write

$$L_3 = C\, q \sum_j 2^{(r+\frac{2\alpha}{q})j(q-1)} \|\Delta_j F\|^{q-1}_{L^p}\, 2^{(r+\frac{2\alpha}{q})j} \|\Delta_j w\|_{L^p}\, 2^{(1+\frac{d}{p}-2\alpha)j} \|\Delta_j v\|_{L^p}.$$

Estimating $2^{(1+\frac{d}{p}-2\alpha)j} \|\Delta_j v\|_{L^p}$ as in (4.24), we obtain

$$L_3 \le C\, q \|v\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|w\|_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \|F\|^{q-1}_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

Combining these estimates for L_1, L_2 and L_3 with (4.30), we obtain

$$\begin{aligned}
&\frac{d}{dt} \|F\|^q_{\mathring{B}^r_{p,q}} + C\, q\nu\, \|F\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \\
&\quad \le C\, q \left(\|v\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|w\|_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} + \|w\|_{\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q}} \|v\|_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \right) \|F\|^{q-1}_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.
\end{aligned}$$

Applying Young's inequality then yields (4.28).

5. A Priori Estimates in $L^q((0,T); \mathring{B}^s_{p,q})$ and in $\widetilde{L^q}((0,T); \mathring{B}^s_{p,q})$

In this section, we establish a priori estimates for solutions of the GNS equations and for those of (4.4) in two different type of spaces: $L^q((0,T); \mathring{B}^s_{p,q})$ and $\widetilde{L^q}((0,T); \mathring{B}^s_{p,q})$. The major results are stated in Theorems 5.1, 5.2, 5.3 and 5.4. We will need these estimates when we study solutions of the GNS equations in the next section.

The first theorem provides an a priori estimate for solutions of the GNS equations in $L^q([0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$. The derivation of this bound requires $q > 2$.

Theorem 5.1. *Let* $2 < q \le \infty$. *Assume either* $0 < \alpha$ *and* $p = 2$ *or* $0 < \alpha \le 1$ *and* $2 < p < \infty$. *Let* $r = 1 + \frac{d}{p} - 2\alpha + \frac{2\alpha}{q}$. *Let* $u_0 \in \mathring{B}^r_{p,q}$ *and let* u *be a solution of the GNS equations with the initial datum* u_0. *Set*

$$A(t) = \|u\|^q_{L^q((0,t); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})}.$$

Then, for any $T > 0$,

$$A(T) \le C\,\nu^{-1} \sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j u_0\|^q_{L^p} + C\,\nu^{-(q-2)} \int_0^T A^2(t)\,dt, \quad (5.1)$$

where $E_j(t) \equiv \exp(-C\,\nu 2^{2\alpha j} t)$ and C's are constants depending on d, p and q. In particular,

$$A(T) \le C\,\nu^{-1} \|u_0\|^q_{\mathring{B}^r_{p,q}} + C\,\nu^{-(q-2)} \int_0^T A^2(t)\,dt.$$

Proof. As derived in the proof of Theorem 4.1, u satisfies (4.21), namely,

$$\begin{aligned} &\frac{d}{dt}\|\Delta_j u\|_{L^p} + C\,\nu 2^{2\alpha j}\|\Delta_j u\|_{L^p} \\ &\quad \le C\|\Delta_j u\|_{L^p} \sum_{m\le j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u\|_{L^p} + C\,2^{(1+\frac{d}{p})j}\,\|\Delta_j u\|^2_{L^p}. \end{aligned}$$

We first convert it into the following integral form:

$$\begin{aligned} \|\Delta_j u(\cdot,t)\|_{L^p} \le C E_j(t)\|\Delta_j u_0\|_{L^p} &+ C\int_0^t E_j(t-s)\,2^{(1+\frac{d}{p})j}\,\|\Delta_j u(\cdot,s)\|^2_{L^p} ds \\ &+ C\int_0^t E_j(t-s)\,\|\Delta_j u(\cdot,s)\|_{L^p} \sum_{m<j-1} 2^{(1+\frac{d}{p})m}\|\Delta_m u(\cdot,s)\|_{L^p} ds, \end{aligned}$$

where $E_j(t) \equiv \exp(-C\,\nu 2^{2\alpha j} t)$. Multiplying both sides by $2^{(r+\frac{2\alpha}{q})j}$, raising them to the q^{th} power, summing over all j and integrating over $(0,T)$, we obtain

$$\|u\|^q_{L^q\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} \le M_1 + M_2 + M_3, \quad (5.2)$$

where M_1, M_2 and M_3 are given by

$$\begin{aligned} M_1 &\equiv C\int_0^T \sum_j E^q_j(t)\,2^{(r+\frac{2\alpha}{q})jq}\|\Delta_j u_0\|^q_{L^p} dt, \\ M_2 &\equiv C\int_0^T \sum_j 2^{(1+\frac{d}{p}+r+\frac{2\alpha}{q})jq}\, M_{21}(t)dt, \\ M_3 &\equiv C\int_0^T \sum_j 2^{(r+\frac{2\alpha}{q})jq}\, M_{31}(t)dt \end{aligned}$$

with

$$M_{21} = \left(\int_0^t E_j(t-s)\|\Delta_j u(\cdot,s)\|^2_{L^p} ds\right)^q \quad (5.3)$$

and

$$M_{31} = \left(\int_0^t E_j(t-s)\, \|\Delta_j u(\cdot,s)\|_{L^p} \sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u(\cdot,s)\|_{L^p} ds \right)^q. \quad (5.4)$$

We now estimate M_1, M_2 and M_3. Inserting

$$\int_0^T E_j^q(t)\, dt = C\,\nu^{-1} 2^{-2\alpha j}(1 - E_j(qT))$$

in M_1, we have

$$M_1 = C\nu^{-1} \sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j u_0\|_{L^p}^q. \quad (5.5)$$

In particular, $M_1 \le C\,\nu^{-1}\|u_0\|^q_{B^r_{p,q}}$. To bound M_2, we start with an estimate for M_{21}. For $q > 2$, Hölder's inequality implies

$$\begin{aligned} \left(\int_0^t E_j(t-s) \|\Delta_j u(\cdot,s)\|^2_{L^p} ds \right)^q &\le \left(\int_0^t E_j^{\frac{q}{q-2}}(t-s)\, ds \right)^{q-2} \left(\int_0^t \|\Delta_j u\|^q_{L^p} ds \right)^2 \\ &\le C\,\nu^{-(q-2)} 2^{-2\alpha j(q-2)} \left(1 - E\left(\frac{q}{q-2} t\right)\right)^{q-2} \left(\int_0^t \|\Delta_j u\|^q_{L^p} ds \right)^2. \end{aligned} \quad (5.6)$$

Therefore, M_2 is bounded by

$$M_2 \le C\,\nu^{-(q-2)} \int_0^T \sum_j 2^{(1+\frac{d}{p}-2\alpha+\frac{4\alpha}{q}+r+\frac{2\alpha}{q})jq} \left(\int_0^t \|\Delta_j u\|^q_{L^p} ds \right)^2 dt.$$

For $r = 1 + \frac{d}{p} - 2\alpha + \frac{2\alpha}{q}$, we have

$$\begin{aligned} M_2 &\le C\,\nu^{-(q-2)} \int_0^T \sum_j \left(\int_0^t 2^{(r+\frac{2\alpha}{q})jq} \|\Delta_j u\|^q_{L^p} ds \right)^2 dt \\ &\le C\,\nu^{-(q-2)} \int_0^T \left(\int_0^t \sum_j 2^{(r+\frac{2\alpha}{q})jq} \|\Delta_j u\|^q_{L^p} ds \right)^2 dt \\ &= C\,\nu^{-(q-2)} \int_0^T \|u\|^{2q}_{L^q\left((0,t); B^{r+\frac{2\alpha}{q}}_{p,q}\right)} dt. \end{aligned} \quad (5.7)$$

We now bound M_3. First, we estimate M_{31}. As in the estimate of (5.6), we obtain

$$\begin{aligned} M_{31} &\le C\,\nu^{-(q-2)} 2^{-2\alpha j(q-2)} \int_0^t \|\Delta_j u\|^q_{L^p} ds \int_0^t \left(\sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p} \right)^q ds \\ &= C\nu^{-(q-2)} \int_0^t \|\Delta_j u\|^q_{L^p} ds \int_0^t \left(\sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha+\frac{4\alpha}{q})m} \|\Delta_m u\|_{L^p} 2^{2\alpha(1-\frac{2}{q})(m-j)} \right)^q ds. \end{aligned}$$

For $q > 2$ and $r = 1 + \frac{d}{p} - 2\alpha + \frac{2\alpha}{q}$, we obtain by applying (2.7),

$$\sum_{m<j-1} 2^{(1+\frac{d}{p}-2\alpha+\frac{4\alpha}{q})m} \|\Delta_m u\|_{L^p} 2^{2\alpha(1-\frac{2}{q})(m-j)} \tag{5.8}$$
$$\le \left[\sum_{m<j-1} 2^{(r+\frac{2\alpha}{q})mq} \|\Delta_m u\|_{L^p}^q\right]^{\frac{1}{q}} \left[\sum_{m<j-1} 2^{2\alpha(1-\frac{2}{q})(m-j)\bar{q}}\right]^{\frac{1}{\bar{q}}} = C\, \|u\|_{B^{r+\frac{2\alpha}{q}}_{p,q}},$$

where $\bar{q}$ is the conjugate of q. Therefore,

$$\begin{aligned} M_{31} &\le C\, \nu^{-(q-2)} \int_0^t \|u\|^q_{B^{r+\frac{2\alpha}{q}}_{p,q}} ds \int_0^t \|\Delta_j u\|^q_{L^p} ds \\ &= C\, \nu^{-(q-2)}\, \|u\|^q_{L^q\left((0,t);B^{r+\frac{2\alpha}{q}}_{p,q}\right)} \int_0^t \|\Delta_j u\|^q_{L^p} ds. \end{aligned} \tag{5.9}$$

Inserting this bound in M_3, we obtain

$$\begin{aligned} M_3 &\le C\, \nu^{-(q-2)} \int_0^T \sum_j 2^{(r+\frac{2\alpha}{q})jq} \int_0^t \|\Delta_j u\|^q_{L^p} ds\ \|u\|^q_{L^q((0,t);B^{r+\frac{2\alpha}{q}}_{p,q})} dt \\ &= C\, \nu^{-(q-2)} \int_0^T \|u\|^{2q}_{L^q((0,t);B^{r+\frac{2\alpha}{q}}_{p,q})} dt. \end{aligned} \tag{5.10}$$

Combining (5.5), (5.7), (5.10) with (5.2) yields (5.1). This completes the proof of Theorem 5.1.

We now provide the estimate for solutions of the GNS equations in $\widetilde{L^q}([0,t); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$.

Theorem 5.2. *Let $1 < q \le \infty$. Assume either $0 < \alpha$ and $p = 2$ or $0 < \alpha \le 1$ and $2 < p < \infty$. Let $r = 1 + \frac{d}{p} - 2\alpha$. Let $u_0 \in \mathring{B}^r_{p,q}$ and let u be a solution of the GNS equations with the initial datum u_0. Set*

$$B(t) = \|u\|^q_{\widetilde{L^q}\left((0,t);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}.$$

Then, for any $T > 0$,

$$B(T) \le C\, \nu^{-1} \sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j u_0\|^q_{L^p} + C\, \nu^{-(q-1)} B^2(T). \tag{5.11}$$

In particular,

$$B(T) \le C\, \nu^{-1} \|u_0\|^q_{\mathring{B}^r_{p,q}} + C\, \nu^{-(q-1)}\, B^2(T).$$

Proof. Following a similar procedure as in the proof of Theorem 5.1, we obtain

$$\|u\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} \le N_1 + N_2 + N_3, \tag{5.12}$$

where N_1, N_2 and N_3 are given by

$$\begin{aligned}
N_1 &\equiv C \sum_j 2^{(r+\frac{2\alpha}{q})jq} \int_0^T E_j^q(t)\, \|\Delta_j u_0\|^q_{L^p} dt,\\
N_2 &\equiv C \sum_j 2^{(1+\frac{d}{p}+r+\frac{2\alpha}{q})jq} \int_0^T M_{21}(t) dt,\\
N_3 &\equiv C \sum_j 2^{(r+\frac{2\alpha}{q})jq} \int_0^T M_{31}(t) dt,
\end{aligned}$$

with M_{21} defined in (5.3) and M_{31} in (5.4). The bound for N_1 is the same as that for M_1, which is given in (5.5). To estimate N_2, we apply Young's inequality to obtain

$$\begin{aligned}
\int_0^T M_{21}(t)\, dt &\le \left(\int_0^T E_j^{\frac{q}{q-1}}(t)\, dt\right)^{q-1} \left(\int_0^T \|\Delta_j u\|^q_{L^p} dt\right)^2\\
&\le C\, \nu^{-(q-1)}\, 2^{-2\alpha j(q-1)} \left(\int_0^T \|\Delta_j u\|^q_{L^p} dt\right)^2.
\end{aligned}$$

If $r = 1 + \frac{d}{p} - 2\alpha$, then N_2 is bounded by

$$\begin{aligned}
N_2 &\le C\, \nu^{-(q-1)} \sum_j 2^{(1+\frac{d}{p}-2\alpha+\frac{2\alpha}{q}+r+\frac{2\alpha}{q})jq} \left(\int_0^T \|\Delta_j u\|^q_{L^p} dt\right)^2\\
&= C\, \nu^{-(q-1)} \sum_j \left(2^{(r+\frac{2\alpha}{q})jq} \int_0^T \|\Delta_j u\|^q_{L^p} dt\right)^2\\
&\le C\, \nu^{-(q-1)} \left(\sum_j 2^{(r+\frac{2\alpha}{q})jq} \int_0^T \|\Delta_j u\|^q_{L^p} dt\right)^2\\
&= C\, \nu^{-(q-1)} \|u\|^{2q}_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}.
\end{aligned} \tag{5.13}$$

To bound N_3, we again apply Young's inequality to get

$$\begin{aligned}
\int_0^T M_{31}\, dt &\le C\, \|E_j\|^q_{L^{\frac{q}{q-1}}(0,T)} \|\Delta_j u\|^q_{L^q(0,T)} \left\| \sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p} \right\|^q_{L^q(0,T)}\\
&= C\, \nu^{-(q-1)} 2^{-2\alpha j(q-1)} \int_0^T \|\Delta_j u\|^q_{L^p} dt \int_0^T \left(\sum_{m<j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p}\right)^q dt.
\end{aligned}$$

Since $r = 1 + \frac{d}{p} - 2\alpha$, the right-hand side can be written as

$$C\,\nu^{-(q-1)} \int_0^T \|\Delta_j u\|_{L^p}^q dt \int_0^T \left(\sum_{m<j-1} 2^{(r+\frac{2\alpha}{q})m} \|\Delta_m u\|_{L^p} 2^{2\alpha(1-\frac{1}{q})(m-j)} \right)^q dt.$$

For $q > 1$, we have, according to (2.7),

$$\sum_{m<j-1} 2^{(r+\frac{2\alpha}{q})m} \|\Delta_m u\|_{L^p}\ 2^{2\alpha(1-\frac{1}{q})(m-j)} \le C \left[\sum_{m<j-1} 2^{(r+\frac{2\alpha}{q})mq} \|\Delta_m u\|_{L^p}^q \right]^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned}\int_0^T M_{31}\, dt &\le C\,\nu^{-(q-1)} \int_0^T \|\Delta_j u\|_{L^p}^q dt \sum_m 2^{(r+\frac{2\alpha}{q})mq} \int_0^T \|\Delta_m u\|_{L^p}^q dt \\ &= C\,\nu^{-(q-1)} \int_0^T \|\Delta_j u\|_{L^p}^q dt\ \|u\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}.\end{aligned}$$

Therefore,

$$\begin{aligned}N_3 &\le C\,\nu^{-(q-1)} \sum_j 2^{(r+\frac{2\alpha}{q})jq} \int_0^T \|\Delta_j u\|_{L^p}^q dt\ \|u\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} \\ &= C\,\nu^{-(q-1)} \|u\|^{2q}_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}. \end{aligned} \tag{5.14}$$

Inserting the estimates (5.5), (5.13) and (5.14) in (5.12), we obtain (5.11).

The a priori estimates of Theorems 5.1 and 5.2 can be extended to solutions of the equations in (4.4). The bound in $L^q\left((0,T);\ \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)$ is stated in Theorem 5.3, while the bound in $\widetilde{L^q}\left((0,T);\ \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)$ is provided in Theorem 5.4. These theorems can be shown by combining the arguments in the proofs of Theorems 4.3, 5.1 and 5.2, so we omit the details.

Theorem 5.3. *Let* $2 < q \le \infty$. *Assume either* $\frac{1}{2} < \alpha$ *and* $p = 2$ *or* $\frac{1}{2} < \alpha \le 1$ *and* $2 < p < \infty$. *Let* $r = 1 + \frac{d}{p} - 2\alpha + \frac{2\alpha}{q}$. *Assume that* $F_0 \in \mathring{B}^r_{p,q}$ *and*

$$v,\ w \in L^q\left((0,T);\ \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)$$

for some $T > 0$. *Then any solution of (4.4) and (4.5) satisfies*

$$\begin{aligned}\|F\|^q_{L^q\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} &\le C\,\nu^{-1} \sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j F_0\|^q_{L^p} \\ &\quad + C\,\nu^{-(q-2)} \int_0^T \|v\|^q_{L^q\left((0,t);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} \|w\|^q_{L^q\left((0,t);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} dt.\end{aligned} \tag{5.15}$$

In particular, (5.15) holds when the first term on the right of (5.15) is replaced by $C\,\nu^{-1}\,\|F_0\|_{\mathring{B}^r_{p,q}}$.

Theorem 5.4. *Let* $1 < q \le \infty$. *Assume either* $\frac{1}{2} < \alpha$ *and* $p = 2$ *or* $\frac{1}{2} < \alpha \le 1$ *and* $2 < p < \infty$. *Let* $r = 1 + \frac{d}{p} - 2\alpha$. *Assume that* $F_0 \in \mathring{B}^r_{p,q}$ *and*

$$v,\ \ w \in \widetilde{L^q}\left((0,T);\, \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)$$

for some $T > 0$. *Then any solution of (4.4) and (4.5) satisfies*

$$\|F\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)} \le C\,\nu^{-1}\sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j F_0\|^q_{L^p}$$
$$+ C\,\nu^{-(q-1)}\,\|v\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}\, \|w\|^q_{\widetilde{L^q}\left((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)}. \tag{5.16}$$

In particular, (5.16) holds when the first term on the right of (5.16) is replaced by $C\,\nu^{-1}\,\|F_0\|_{\mathring{B}^r_{p,q}}$.

6. Existence and Uniqueness

This section presents three existence and uniqueness results for the GNS equations. The first one asserts the global existence and uniqueness of solutions in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$ with $r = 1 - 2\alpha + \frac{d}{p}$ for initial data that are comparable to ν in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$. The second one establishes the local existence and uniqueness of solutions in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$ for arbitrarily large data in $\mathring{B}^r_{p,q}(\mathbb{R}^d)$. The third one concerns solutions in $L^q((0,T);\, \mathring{B}^{s+\frac{2\alpha}{q}}_{p,q}(\mathbb{R}^d))$ with $s = 1 - 2\alpha + \frac{d}{p} + \frac{2\alpha}{q}$. The precise statements are given in Theorems 6.1, 6.2 and 6.3.

Theorem 6.1. *Let* $1 \le q \le \infty$. *Assume either* $\frac{1}{2} < \alpha$ *and* $p = 2$ *or* $\frac{1}{2} < \alpha \le 1$ *and* $2 < p < \infty$. *Let* $r = 1 - 2\alpha + \frac{d}{p}$. *Assume that* $u_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^d)$ *satisfies*

$$\|u_0\|_{\mathring{B}^r_{p,q}} \le C_0\nu$$

for some suitable constant C_0 *depending on* d *and* p *only. Then the GNS equations (4.1),(4.2) and (4.3) have a unique global solution* u *satisfying*

$$u \in C([0,\infty);\, \mathring{B}^r_{p,q}) \cap L^q((0,\infty);\, \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}) \ \text{ for } 1 \le q < \infty,$$
$$u \in C([0,\infty);\, \mathring{B}^r_{p,\infty}) \cap \widetilde{L^1}((0,\infty);\, \mathring{B}^{1+\frac{d}{p}}_{p,\infty}) \ \text{ for } q = \infty.$$

In addition, for any $t > 0$,

$$\|u(\cdot,t)\|_{\mathring{B}^r_{p,q}} + C\,\nu\|u\|_{L^q((0,t);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})} \le C_1\nu \ \text{ for } 1 \le q < \infty,$$
$$\|u(\cdot,t)\|_{\mathring{B}^r_{p,\infty}} + C\,\nu\|u\|_{\widetilde{L^1}((0,t);\mathring{B}^{1+\frac{d}{p}}_{p,\infty})} \le C_1\nu \ \text{ for } q = \infty,$$

for some constants C *and* C_1 *depending on* d *and* p *only.*

Although this theorem does not explicitly mention the case when $1 \leq p < 2$, we can easily extend it to cover any initial datum $u_0 \in \mathring{B}^r_{p,q}$ with $1 \leq p < 2$. In fact, by the Besov embedding (see Part 2) of Proposition 2.1),

$$\mathring{B}^{r_1}_{p_1,q}(\mathbb{R}^d) \subset \mathring{B}^{r_2}_{p_2,q}(\mathbb{R}^d)$$

for any $1 \leq q \leq \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ and $r_1 = r_2 + d(\frac{1}{p_1} - \frac{1}{p_2})$. Thus, if $u_0 \in \mathring{B}^r_{p,q}$ with $1 \leq p < 2$ and $r = 1 - 2\alpha + \frac{d}{p}$, then $u_0 \in \mathring{B}^{r_2}_{p_2,q}$ for any $p_2 > 2$ and $r_2 = 1 - 2\alpha + \frac{d}{p_2}$. Theorem 6.1 then implies that the GNS equations have a unique global solution associated with any $u_0 \in \mathring{B}^r_{p,q}$ with $1 \leq p < 2$ and $r = 1 - 2\alpha + \frac{d}{p}$.

Proof of Theorem 6.1. To apply the contraction mapping principle, we write the GNS equations in the integral form

$$u(t) = Gu(t) \equiv \exp(-\nu(-\Delta)^\alpha t)u_0 - \int_0^t \exp(-\nu(-\Delta)^\alpha (t-s))\, \mathbb{P}(u \cdot \nabla u)(s)ds,$$

where $\exp(-\nu(-\Delta)^\alpha t)$ for each $t \geq 0$ is a convolution operator with

$$\widehat{\exp(-\nu(-\Delta)^\alpha t)}(\xi) = \exp(-\nu(2\pi|\xi|)^{2\alpha} t).$$

We shall only provide the proof for the case when $1 \leq q < \infty$ since the proof for $q = \infty$ is analogous. To set up, we write

$$X = C([0,\infty); \mathring{B}^r_{p,q}), \quad Y = L^q\left((0,\infty); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right), \quad Z = X \cap Y.$$

For the norm in Z, we choose

$$\|u\|_Z = \max\{\|u\|_X + C\,\nu\,\|u\|_Y\},$$

where C is a suitable constant depending on d, p and q only. In addition, we use D to denote the subset

$$D = \{u \in Z : \|u\|_Z \leq C_1\,\nu\}.$$

We aim to show that G is a contractive map from D to D. For notational convenience, we write Gu as

$$Gu = u^0 - F(u,u),$$

where u^0 and $F(v,w)$ are given by

$$\begin{aligned} u^0 &= \exp(-\nu(-\Delta)^\alpha t)u_0, \\ F(v,w) &= \int_0^t \exp(-\nu(-\Delta)^\alpha (t-s))\mathbb{P}(v \cdot \nabla w)(s)ds. \end{aligned}$$

Obviously, u^0 satisfies the equation

$$\partial_t u^0 + \nu(-\Delta)^\alpha u^0 = 0, \qquad u^0(x,0) = u_0(x).$$

As shown in the proof of Theorem 4.1, we have

$$\|u^0(\cdot,t)\|^q_{\mathring{B}^r_{p,q}} + C\,q\nu \int_0^t \|u(\cdot,s)\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}\,ds \le \|u_0\|^q_{\mathring{B}^r_{p,q}}.$$

Consequently, $\|u^0\|_Z \le C_0\nu$. To bound F in D, we first notice that F satisfies

$$\partial_t F + \nu(-\Delta)^\alpha F = \mathbb{P}(u\cdot\nabla u), \qquad F(u,u)(x,0) = 0.$$

Applying the result of Theorem 4.3, we have

$$\frac{d}{dt}\|F\|^q_{\mathring{B}^r_{p,q}} + C\,\nu\|F\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} \le C\,\|u\|^q_{\mathring{B}^r_{p,q}}\;\|u\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}.$$

Therefore, for $u \in D$ and suitable C_1,

$$\|F(u,u)\|_Z \le C_1\nu.$$

Thus, G maps D to D. To see that G is contractive, we write the difference $Gu - Gv$ into two parts, namely,

$$Gu - Gv = -(F(u,u-v) + F(u-v,v)).$$

Since $F(u,u-v)$ satisfies

$$\partial_t F + \nu(-\Delta)^\alpha F = \mathbb{P}(u\cdot\nabla(u-v)), \quad F(u,u)(x,0) = 0.$$

Again the result of Theorem 4.3 shows

$$\|F(u,u-v)\|_Z \le C\|u\|_Z\|u-v\|_Z.$$

$F(u-v,v)$ admits a similar bound. Therefore,

$$\|Gu - Gv\|_Z \le C\,(\|u\|_Z + \|v\|_Z)\|u-v\|_Z.$$

For suitable C_1, $C\,(\|u\|_Z + \|v\|_Z) < 1$ and G is contractive. The result of Theorem 6.1 then follows from the contraction mapping principle.

We now state and prove the local existence and uniqueness result. For the sake of conciseness, the statement for the case when $q=\infty$ is omitted from the theorem.

Theorem 6.2. *Let $1 < q \le \infty$. Assume either $\frac{1}{2} < \alpha$ and $p = 2$ or $\frac{1}{2} < \alpha \le 1$ and $2 < p < \infty$. Let $r = 1 - 2\alpha + \frac{d}{p}$. Assume $u_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^d)$. Then there exists a $T > 0$ such that the GNS equations (4.1),(4.2) and (4.3) have a unique solution u on $[0,T)$ in the class*

$$C([0,T);\mathring{B}^r_{p,q}) \cap \widetilde{L^q}((0,T);\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}).$$

Proof. The proof combines the contraction mapping principle with the a priori bounds in Theorems 5.2 and 5.4. The operator G is the same as in the proof of the previous theorem, but the functional setting is different. For $T > 0$ and $R > 0$ to be selected, we define

$$Z_1 = \widetilde{L^q}((0,T); \mathring{B}_{p,q}^{r+\frac{2\alpha}{q}}), \quad D_1 = \left\{u \in Z_1 : \ \|u\|_{Z_1} \le R\right\}.$$

The goal is to show that G is a contractive map from D_1 to D_1. We again write $Gu = u^0 - F(u,u)$. Since u^0 satisfies

$$\partial_t u^0 + \nu(-\Delta)^\alpha u^0 = 0, \quad u^0(x,0) = u_0(x),$$

we can show as in the proof of Theorem 5.2,

$$\|u^0\|_{Z_1}^q \le C\,\nu^{-1} \sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j u_0\|_{L^p}^q,$$

where $E_j(t) \equiv \exp(-C\,\nu 2^{2\alpha j} t)$. Since $u_0 \in \mathring{B}_{p,q}^r$, this inequality especially implies that $\|u^0\|_{Z_1}$ is finite. In addition, by the Dominated Convergence Theorem,

$$\sum_j (1 - E_j(qT))\, 2^{rjq}\, \|\Delta_j u_0\|_{L^p}^q \to 0 \quad \text{as } T \to 0.$$

Therefore, $\|u^0\|_{Z_1}$ is small for small $T > 0$. Since $F(u,u)$ satisfies

$$\partial_t F + \nu(-\Delta)^\alpha F = \mathbb{P}(u \cdot \nabla u), \quad F(u,u)(x,0) = 0,$$

we have, according to Theorem 5.4,

$$\|F(u,u)\|_{Z_1} \le C\|u\|_{Z_1}^2 \le C\,R^2$$

for any $u \in D_1$. Thus for small $T > 0$ and suitable $R > 0$, $Gu \in D_1$. As in the proof of Theorem 6.1, we can show by applying Theorem 5.4 again that

$$\|Gu - Gv\|_{Z_1} \le C\,(\|u\|_{Z_1} + \|v\|_{Z_1})\|u - v\|_{Z_1} \le 2C\,R\,\|u - v\|_{Z_1},$$

which implies G is a contraction for suitable selected R. By applying the contraction mapping principle, we find that the GNS equations have a solution in Z_1, namely, in $\widetilde{L^q}((0,T); \mathring{B}_{p,q}^{r+\frac{2\alpha}{q}})$.

To show $u \in \mathring{B}_{p,q}^r$, we can derive similarly as in the proof of Theorem 5.2 the bound

$$\|u(\cdot,t)\|_{\mathring{B}_{p,q}^r} \le \|u_0\|_{\mathring{B}_{p,q}^r} + C\,\|u\|_{Z_1}^2$$

for any solution u of the GNS equations. Thus, for $t \in [0,T)$, $u(\cdot,t)$ is in $\mathring{B}_{p,q}^r$. This completes the proof of Theorem 6.2.

The space $\widetilde{L^q}((0,T); \mathring{B}_{p,q}^{r+\frac{2\alpha}{q}})$ was used in the proof of the previous theorem. If we use $L^q((0,T); \mathring{B}_{p,q}^{s+\frac{2\alpha}{q}})$ instead, we have the following local existence and uniqueness theorem. This result can be proven by the contraction mapping principle combined with the a priori estimates in Theorems 5.1 and 5.3. We omit details of the proof.

Theorem 6.3. *Let $2 < q \le \infty$. Assume either $\frac{1}{2} < \alpha$ and $p = 2$ or $\frac{1}{2} < \alpha \le 1$ and $2 < p < \infty$. Let $s = 1 - 2\alpha + \frac{d}{p} + \frac{2\alpha}{q}$. Assume $u_0 \in \mathring{B}^s_{p,q}(\mathbb{R}^d)$. Then there exists a $T > 0$ such that the GNS equations have a unique solution u on $[0, T)$ satisfying*

$$C([0,T); \mathring{B}^s_{p,q}) \cap L^q((0,T); \mathring{B}^{s+\frac{2\alpha}{q}}_{p,q}).$$

Appendix

We provide the proof of Proposition 3.5 in Sect. 3.

Proof of Proposition 3.5. (a) When $p = 2$, we have

$$D(f) = \int_{\mathbb{R}^d} f\,(-\Delta)^\alpha f\,dx = \int_{\mathbb{R}^d} |\Lambda^\alpha f|^2\,dx. \tag{A.1}$$

Since $\alpha = 1$ and $d = 2$, we obtain (3.5) by applying the Sobolev embedding

$$\mathrm{W}^{\frac{d}{r},r}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \tag{A.2}$$

valid for any $r \in (1,\infty)$ and $q \in [r,\infty)$. The inequality in (b) is a consequence of (A.1) and the Sobolev embedding

$$\mathrm{W}^{s,r}(\mathbb{R}^d) \subset L^{\frac{rd}{d-rs}}(\mathbb{R}^d) \tag{A.3}$$

for any $sr < d$. To prove (c), we first apply the pointwise inequality in Proposition 3.2 to obtain

$$D(f) \ge C \int_{\mathbb{R}^d} f^{p/2}\,(-\Delta)^\alpha \left(f^{p/2}\right) dx = C \int_{\mathbb{R}^d} \left|\Lambda^\alpha \left(f^{p/2}\right)\right|^2 dx.$$

By (A.2), we have for any $q \in [p,\infty)$,

$$D(f) \ge C\,\|f\|^p_{L^{\frac{pq}{2}}}.$$

Since $2 < p < \frac{pq}{2}$, we have the following interpolation inequality:

$$\|f\|_{L^p} \le C\,\|f\|_{L^2}^{\frac{2q-4}{pq-4}}\,\|f\|_{L^{\frac{pq}{2}}}^{\frac{pq-2q}{pq-4}}.$$

For any $f \in L^2(\mathbb{R}^d)$ with $\|f\|_{L^2} \ne 0$, this inequality is equivalent to

$$\|f\|^p_{L^{\frac{pq}{2}}} \ge C\|f\|_{L^p}^{(1+\beta)p}\,\|f\|_{L^2}^{-\beta p}$$

with $\beta = \frac{2q-4}{pq-2q}$. The proof of (d) is similar to that of (c) except that we use the Sobolev embedding (A.3) instead of (A.2). This completes the proof.

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