Two regularity criteria for the 3D MHD equations

Chongsheng Cao [a], Jiahong Wu [b,*]

[a] *Department of Mathematics, Florida International University, Miami, FL 33199, USA*
[b] *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA*

ARTICLE INFO

ABSTRACT

This work establishes two regularity criteria for the 3D incompressible MHD equations. The first one is in terms of the derivative of the velocity field in one direction while the second one requires suitable boundedness of the derivative of the pressure in one direction.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

This paper is concerned with the global regularity of solutions to the 3D incompressible magneto-hydrodynamical (MHD) equations

$$u_t + u \cdot \nabla u = \nu \Delta u - \nabla p + b \cdot \nabla b, \quad x \in \mathbf{R}^3, \ t > 0, \tag{1.1}$$

$$b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \quad x \in \mathbf{R}^3, \ t > 0, \tag{1.2}$$

$$\nabla \cdot u = 0, \quad x \in \mathbf{R}^3, \ t > 0, \tag{1.3}$$

$$\nabla \cdot b = 0, \quad x \in \mathbf{R}^3, \ t > 0, \tag{1.4}$$

* Corresponding author.
E-mail addresses: caoc@fiu.edu (C. Cao), jiahong@math.okstate.edu (J. Wu).

where u is the fluid velocity, b the magnetic field, p the pressure, ν the viscosity and η the magnetic diffusivity. Without loss of generality, we set $\nu = \eta = 1$ in the rest of the paper. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas. (1.1) reflects the conservation of momentum, (1.2) is the induction equation and (1.3) specifies the conservation of mass. Besides their physical applications, the MHD equations are also mathematically significant. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research and many interesting results have been obtained (see, e.g., [2,7,11, 13,16–19,23,26–28,34–36,38–40,43,45,47–52,55]).

Attention here is focused on the global regularity of solutions to the initial-value problem (IVP) of (1.1), (1.2), (1.3) and (1.4) with a given initial data

$$u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x), \quad x \in \mathbf{R}^3. \tag{1.5}$$

It is currently unknown whether solutions of this IVP can develop finite time singularities even if (u_0, b_0) is sufficiently smooth. This work presents new regularity criteria under which the regularity of the solution is preserved for all time. The global regularity issue has been thoroughly investigated for the 3D Navier–Stokes equations and many important regularity criteria have been established (see, e.g., [3–6,8–10,12,14,15,20,21,25,29–32,37,41,42,44,46,53,54]). Some of these criteria can be extended to the 3D MHD equations by making assumptions on both u and b (see, e.g., [7,47]). Realizing the dominant role played by the velocity field in the regularity issue, He and Xin were able to derive criteria in terms of the velocity field u alone [27,28]. They showed that, if u satisfies

$$\int_0^T \left\| \nabla u(\cdot, t) \right\|_\alpha^\beta dt < \infty \quad \text{with } \frac{3}{\alpha} + \frac{2}{\beta} = 2 \text{ and } 1 < \beta \leqslant 2, \tag{1.6}$$

then the solution (u, b) is regular on $[0, T]$. This assumption was weakened in [51] with L^α-norm replaced by norms in Besov spaces and further improved by Chen, Miao and Zhang in [17]. As pointed out in [27], the regularity criteria in terms of the velocity field alone are consistent with the numerical simulations in [40] and with the observations of space and laboratory plasmas in [24].

This paper presents two new regularity criteria. The first one assumes

$$\int_0^T \left\| u_z(\cdot, t) \right\|_\alpha^\beta dt < \infty \quad \text{with } \alpha \geqslant 3 \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leqslant 1 \tag{1.7}$$

and the second requires the pressure satisfy

$$\int_0^T \left\| p_z(\tau) \right\|_\alpha^\beta d\tau < \infty \quad \text{with } \alpha \geqslant \frac{12}{7} \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leqslant \frac{7}{4}. \tag{1.8}$$

That is, any solution (u, b) of the 3D MHD equations is regular if the derivative of u in one direction, say along the z-axis, is bounded in $L^\beta([0, T]; L^\alpha)$ with (α, β) satisfying (1.7) or if the derivative of p in one direction satisfies (1.8). The proof of the first criterion is accomplished through two stages with the first controlling the time integrals of $\|\nabla u_z\|_2$ and $\|\nabla b_z\|_2$ in terms of the $L^\beta([0, T]; L^\alpha)$-norm of u_z and the second bounding $\|\nabla u\|_2$ and $\|\nabla b\|_2$ by the time integrals of $\|\nabla u_z\|_2$ and $\|\nabla b_z\|_2$. The details are presented in the second section. The criterion in terms of p_z and its proof are provided in the third section.

We will use the following elementary inequalities:

$$\|\phi\|_\gamma \leqslant C \|\phi_x\|_\lambda^{\frac{1}{3}} \|\phi_y\|_\lambda^{\frac{1}{3}} \|\phi_z\|_\mu^{\frac{1}{3}}, \tag{1.9}$$

where the parameters μ, λ and γ satisfy

$$1 \leqslant \mu, \lambda < \infty, \qquad \frac{1}{\mu} + \frac{2}{\lambda} > 1 \quad \text{and} \quad 1 + \frac{3}{\gamma} = \frac{1}{\mu} + \frac{2}{\lambda}$$

and

$$\|\phi\|_r \leqslant C(r) \|\phi\|_2^{\frac{6-r}{2r}} \|\phi_x\|_2^{\frac{r-2}{2r}} \|\phi_y\|_2^{\frac{r-2}{2r}} \|\phi_z\|_2^{\frac{r-2}{2r}}, \quad 2 \leqslant r \leqslant 6. \tag{1.10}$$

These inequalities may be found in [1,22,33]. For the convenience of the readers, the proofs of these inequalities are provided in Appendix A. Throughout the rest of this paper the L^p-norm of a function f is denoted by $\|f\|_p$, the H^s-norm by $\|f\|_{H^s}$ and the norm in the Sobolev space $W^{s,p}$ by $\|f\|_{W^{s,p}}$.

2. Criterion in terms of u_z

This section establishes the regularity criteria in terms of u_z.

Theorem 2.1. *Assume* $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of the* 3D *MHD equations* (1.1), (1.2), (1.3) *and* (1.4). *If* u *satisfies*

$$M(T) \equiv \int_0^T \|u_z(\cdot, t)\|_\alpha^\beta \, dt < \infty \quad \text{with } \alpha \geqslant 3 \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leqslant 1 \tag{2.1}$$

for some $T > 0$, *then* (u, b) *can be extended to the time interval* $[0, T + \epsilon)$ *for some* $\epsilon > 0$.

The proof of this theorem is divided into two major parts. The first part establishes bounds for $\|u_z\|_2$, $\|b_z\|_2$ and the time integrals of $\|\nabla u_z\|_2^2$ and $\|\nabla b_z\|_2^2$ while the second controls $\|\nabla u\|_2$ and $\|\nabla b\|_2$ in terms of the time integrals of $\|\nabla u_z\|_2^2$ and $\|\nabla b_z\|_2^2$.

2.1. Bounds for $\|u_z\|_2$ and $\|b_z\|_2$

This subsection bounds $\|u_z\|_2$ and $\|b_z\|_2$ in terms of M in (2.1).

Proposition 2.2. *Assume* $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of the* 3D *MHD equations* (1.1), (1.2), (1.3) *and* (1.4). *Suppose* (2.1) *holds. Then, for any* $t \leqslant T$,

$$\|u_z(t)\|_2^2 + \|b_z(t)\|_2^2$$
$$\leqslant C e^{(\|u_0\|_2^2 + \|b_0\|_2^2)} e^{M(t)} \big[\big(\|u_z(0)\|_2^2 + \|b_z(0)\|_2^2 \big)^{\frac{3}{2\alpha - 3}} + C \big(\|u_0\|_2^2 + \|b_0\|_2^2 + M(t) \big) \big]^{\frac{2\alpha - 3}{3}} \tag{2.2}$$

and

$$\int_0^t \big(\|\nabla u_z(\tau)\|_2^2 + \|\nabla b_z(\tau)\|_2^2 \big) \, d\tau \leqslant F\big(M(t) \big) < \infty, \tag{2.3}$$

where $F(M(t))$ *is an explicit function of* $M(t)$.

Proof. It is easy to see that (u, b) satisfies

$$\left\| u(t) \right\|_2^2 + \left\| b(t) \right\|_2^2 + 2 \int_0^t \left(\left\| \nabla u(\tau) \right\|_2^2 + \left\| \nabla b(\tau) \right\|_2^2 \right) d\tau \leqslant \|u_0\|_2^2 + \|b_0\|_2^2. \tag{2.4}$$

Adding the inner products of u_z with ∂_z of (1.1) and of b_z with ∂_z of (1.2), we obtain, after integration by parts,

$$\frac{1}{2} \frac{d(\|u_z\|_2^2 + \|b_z\|_2^2)}{dt} + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2$$

$$= - \int \left[(u_z \cdot \nabla u) \cdot u_z - (b_z \cdot \nabla b) \cdot u_z + (u_z \cdot \nabla b) \cdot b_z - (b_z \cdot \nabla u) \cdot b_z \right] dx \, dy \, dz$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

To bound I_1, we integrate by parts and apply Hölder's inequality to obtain

$$|I_1| = \left| \int (u_z \cdot \nabla u_z) \cdot u \right| \leqslant C \|\nabla u_z\|_2 \|u_z\|_r \|u\|_{3\alpha},$$

where we have omitted $dx \, dy \, dz$ in the integral for notational convenience and r satisfies

$$2 \leqslant r \leqslant 6, \qquad \frac{1}{r} + \frac{1}{3\alpha} = \frac{1}{2}. \tag{2.5}$$

Applying the Sobolev inequality

$$\|u_z\|_r \leqslant C \|u_z\|_2^{1-3\left(\frac{1}{2}-\frac{1}{r}\right)} \|\nabla u_z\|_2^{3\left(\frac{1}{2}-\frac{1}{r}\right)}$$

and bounding $\|u\|_{3\alpha}$ by (1.9), we find

$$|I_1| \leqslant C \|\nabla u_z\|_2^{1+3\left(\frac{1}{2}-\frac{1}{r}\right)} \|u_z\|_2^{1-3\left(\frac{1}{2}-\frac{1}{r}\right)} \|u_z\|_\alpha^{\frac{1}{3}} \|\nabla u\|_2^{\frac{2}{3}}.$$

By Young's inequality,

$$|I_1| \leqslant \frac{1}{4} \|\nabla u_z\|_2^2 + C \|u_z\|_2^2 \|u_z\|_\alpha^q \|\nabla u\|_2^{2q}$$

with

$$q = \frac{2}{3 - 9\left(\frac{1}{2} - \frac{1}{r}\right)} = \frac{2}{3\left(1 - \frac{1}{\alpha}\right)}. \tag{2.6}$$

When $\alpha \geqslant 3$, we have $2q \leqslant 2$ and another application of Young's inequality implies

$$|I_1| \leqslant \frac{1}{4} \|\nabla u_z\|_2^2 + C \|u_z\|_2^2 \left(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2 \right),$$

where

$$\gamma \equiv \frac{q}{1-q} = \frac{2}{1-\frac{3}{\alpha}} \quad \text{or} \quad \frac{3}{\alpha} + \frac{2}{\gamma} = 1.$$

We now bound I_2. By Hölder's, Sobolev's and Young's inequalities,

$$
\begin{aligned}
|I_2| &\leqslant C \|\nabla b\|_2 \|u_z\|_\alpha \|b_z\|_{\frac{2\alpha}{\alpha-2}} \\
&\leqslant C \|\nabla b\|_2 \|u_z\|_\alpha \|b_z\|_2^{1-\frac{3}{\alpha}} \|\nabla b_z\|_2^{\frac{3}{\alpha}} \\
&\leqslant \frac{1}{4} \|\nabla b_z\|_2^2 + C \|\nabla b\|_2^{\frac{2\alpha}{2\alpha-3}} \|u_z\|_\alpha^{\frac{2\alpha}{2\alpha-3}} \|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}} \\
&\leqslant \frac{1}{4} \|\nabla b_z\|_2^2 + C \big(\|\nabla b\|_2^2 + \|u_z\|_\alpha^\gamma \big) \|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}},
\end{aligned}
$$

where

$$\gamma = \frac{2\alpha}{\alpha-3} \quad \text{or} \quad \frac{3}{\alpha} + \frac{2}{\gamma} = 1.$$

I_3 can be bounded exactly as I_2. To bound I_4, we integrate by parts and apply Hölder's inequality,

$$I_4 = -\int \big[(b_z \cdot \nabla u) \cdot b_z \big] = \int \big[(b_z \cdot \nabla b_z) \cdot u \big] \leqslant \|\nabla b_z\|_2 \|b_z\|_r \|u\|_{3\alpha},$$

where $\frac{1}{r} + \frac{1}{3\alpha} = \frac{1}{2}$. Following the steps as in the bound of I_1, we have

$$|I_4| \leqslant \frac{1}{4} \|\nabla b_z\|_2^2 + C \big(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2 \big) \|b_z\|_2^2.$$

Combining the estimates for I_1, I_2, I_3 and I_4, we find

$$
\begin{aligned}
\frac{d(\|u_z\|_2^2 + \|b_z\|_2^2)}{dt} &+ \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2 \\
&\leqslant C \big(\|u_z\|_\alpha^\gamma + \|\nabla u\|_2^2 \big) \big(\|u_z\|_2^2 + \|b_z\|_2^2 \big) + C \big(\|\nabla b\|_2^2 + \|u_z\|_\alpha^\gamma \big) \|b_z\|_2^{\frac{2\alpha-6}{2\alpha-3}}.
\end{aligned}
\tag{2.7}
$$

(2.2) and (2.3) then follow from (2.4), (2.7) and Gronwall's inequality. $\square$

2.2. Bounds for $\|\nabla u\|_2$ and $\|\nabla b\|_2$

This subsection establishes bounds for $\|\nabla u\|_2$ and $\|\nabla b\|_2$.

Proposition 2.3. *Assume $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of the 3D MHD equations* (1.1), (1.2), (1.3) *and* (1.4). *Suppose* (2.1) *holds. Then, for any $t \leqslant T$,*

$$\big\| \nabla u(t) \big\|_2^2 + \big\| \nabla b(t) \big\|_2^2 + \int_0^t \big(\big\| \Delta u(\tau) \big\|_2^2 + \big\| \Delta b(\tau) \big\|_2^2 \big) \, d\tau \leqslant G\big(M(t)\big) < \infty,$$

where $G(M(t))$ denotes an explicit function of $M(t)$.

Proof. Adding the inner products of (1.1) with Δu and of (1.2) with Δb and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\big) + \|\Delta u\|_2^2 + \|\Delta b\|_2^2$$
$$= -\int u \cdot \nabla u \cdot \Delta u + \int b \cdot \nabla b \cdot \Delta u - \int u \cdot \nabla b \cdot \Delta b + \int b \cdot \nabla u \cdot \Delta b.$$

By further integrating by parts, we obtain

$$-\int u \cdot \nabla u \cdot \Delta u + \int b \cdot \nabla b \cdot \Delta u - \int u \cdot \nabla b \cdot \Delta b + \int b \cdot \nabla u \cdot \Delta b \leqslant \|\nabla u\|_3^3 + 3\|\nabla u\|_3\|\nabla b\|_3^2.$$

By (1.10),

$$\|\nabla u\|_3^3 \leqslant C\big(\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_h \nabla u\|_2^{\frac{1}{3}}\|\nabla u_z\|_2^{\frac{1}{6}}\big)^3,$$

where $\nabla_h \equiv (\partial_x, \partial_y)$. By Young's inequality,

$$\|\nabla u\|_3^3 \leqslant \frac{1}{4}\|\nabla_h \nabla u\|_2^2 + C\|\nabla u\|_2^3\|\nabla u_z\|_2 \leqslant \frac{1}{4}\|\nabla_h \nabla u\|_2^2 + C\big(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2\big)\|\nabla u\|_2^2.$$

Similarly,

$$\|\nabla u\|_3\|\nabla b\|_3^2 \leqslant \frac{1}{4}\|\nabla_h \nabla u\|_2^2 + \frac{1}{2}\|\nabla_h \nabla b\|_2^2 + C\big(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2\big)\|\nabla b\|_2^2.$$

Therefore,

$$\frac{d}{dt}\big(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\big) + \|\Delta u\|_2^2 + \|\Delta b\|_2^2 \leqslant C\big(\|\nabla u\|_2^2 + \|\nabla u_z\|_2^2 + \|\nabla b_z\|_2^2\big)\big(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\big).$$

Gronwall's inequality coupled with Proposition 2.2 then yields the desired bounds. □

3. Criterion in terms of p_z

This section presents the regularity criterion with an assumption on p_z.

Theorem 3.1. *Assume the initial data* $(u_0, b_0) \in H^1 \cap L^4$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of the* 3D *MHD equations* (1.1), (1.2), (1.3) *and* (1.4). *If the pressure p associated with the solution satisfies*

$$\int\limits_0^T \|p_z(\tau)\|_\alpha^\beta \, d\tau < \infty \quad \text{with } \alpha \geqslant \frac{12}{7} \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leqslant \frac{7}{4} \tag{3.1}$$

for some $T > 0$, *then* (u, b) *remains regular on* $[0, T]$, *namely* $(u, b) \in C([0, T]; H^1 \cap L^4)$.

Since higher-order Sobolev norms of (u, b) can be controlled by its H^1-norm (see e.g. [45]), a special consequence of this theorem is that (3.1) yields the global regularity of classical solutions. To prove this theorem, we establish the L^4-bound of (u, b) and the desired regularity then follows from the standard Serrin type criteria on the 3D MHD equations [48].

Proposition 3.2. *Assume the initial data* $(u_0, b_0) \in H^1 \cap L^4$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of the 3D MHD equations* (1.1), (1.2), (1.3) *and* (1.4). *If the pressure p satisfies* (3.1), *then* (u, b) *obeys the bound*

$$\|w^+\|_4^4 + \|w^-\|_4^4 + \int_0^t \left(\left\||\nabla|w^+|^2\right\|_2^2 + \left\||\nabla|w^-|^2\right\|_2^2 \right) d\tau$$

$$+ 4 \int_0^t \iint \left(|w^+|^2|\nabla w^+|^2 + |w^-|^2|\nabla w^-|^2 \right) dx\,dy\,dz\,d\tau < \infty$$

for any $t \leqslant T$, *where*

$$w^\pm = u \pm b.$$

Proof. We first convert the MHD equations into a symmetric form. Adding and subtracting (1.1) and (1.2), we find that w^+ and w^- satisfy

$$\partial_t w^+ + w^- \cdot \nabla w^+ = \Delta w^+ - \nabla p, \tag{3.2}$$

$$\partial_t w^- + w^+ \cdot \nabla w^- = \Delta w^- - \nabla p, \tag{3.3}$$

$$\nabla \cdot w^+ = 0, \qquad \nabla \cdot w^- = 0. \tag{3.4}$$

Adding the inner products of (3.2) with $w^+|w^+|^2$ and of (3.3) with $w^-|w^-|^2$ and integrating by parts, we find

$$\frac{1}{4}\frac{d}{dt}\left(\|w^+\|_4^4 + \|w^-\|_4^4\right) + \frac{1}{2}\left(\left\||\nabla|w^+|^2\right\|_2^2 + \left\||\nabla|w^-|^2\right\|_2^2\right) + \int \left(|w^+|^2|\nabla w^+|^2 + |w^-|^2|\nabla w^-|^2\right)$$

$$= J_1 + J_2, \tag{3.5}$$

where

$$J_1 = \int p w^+ \cdot \nabla |w^+|^2, \qquad J_2 = \int p w^- \cdot \nabla |w^-|^2.$$

By Hölder's inequality,

$$J_1 \leqslant C\|p\|_4 \|w^+\|_4 \left\||\nabla|w^+|^2\right\|_2.$$

We choose λ such that

$$\frac{1}{\alpha} + \frac{2}{\lambda} = \frac{7}{4} \quad \text{or} \quad \frac{3\lambda}{2 - \lambda(1 - \frac{1}{\alpha})} = 4.$$

It then follows from (1.9) that

$$\|p\|_4 \leqslant C\|p_z\|_\alpha^{\frac{1}{3}} \|\nabla p\|_\lambda^{\frac{2}{3}}.$$

To further bound $\|\nabla p\|_\lambda$, we take the divergence of (3.2) to obtain

$$\Delta p = -\nabla \cdot (w^- \cdot \nabla w^+).$$

By Hölder's inequality,

$$\|\nabla p\|_\lambda \leqslant C \|w^-\|_{\frac{2\lambda}{2-\lambda}} \|\nabla w^+\|_2.$$

Furthermore, by Sobolev's inequality,

$$\|w^-\|_{\frac{2\lambda}{2-\lambda}} = \left\||w^-|^2\right\|_{\frac{\lambda}{2-\lambda}}^{\frac{1}{2}} \leqslant C \left\||w^-|^2\right\|_2^{\frac{3}{\lambda}-\frac{7}{4}} \left\||\nabla|w^-|^2\right\|_2^{\frac{9}{4}-\frac{3}{\lambda}} = C \|w^-\|_4^{\frac{6}{\lambda}-\frac{7}{2}} \left\||\nabla|w^-|^2\right\|_2^{\frac{9}{4}-\frac{3}{\lambda}},$$

where we have used the fact that $\frac{12}{7} \leqslant \alpha$ and thus $\lambda \leqslant \frac{12}{7}$. Therefore,

$$\|p\|_4 \leqslant C \|p_z\|_\alpha^{\frac{1}{3}} \|\nabla w^+\|_2^{\frac{2}{3}} \|w^-\|_4^{\frac{4}{\lambda}-\frac{7}{3}} \left\||\nabla|w^-|^2\right\|_2^{\frac{3}{2}-\frac{2}{\lambda}}$$

and thus

$$J_1 \leqslant C \|p_z\|_\alpha^{\frac{1}{3}} \|\nabla w^+\|_2^{\frac{2}{3}} \|w^-\|_4^{\frac{4}{\lambda}-\frac{7}{3}} \left\||\nabla|w^-|^2\right\|_2^{\frac{3}{2}-\frac{2}{\lambda}} \|w^+\|_4 \left\||\nabla|w^+|^2\right\|_2.$$

By Young's inequality,

$$J_1 \leqslant \frac{1}{8} \left\||\nabla|w^+|^2\right\|_2^2 + \frac{1}{8} \left\||\nabla|w^-|^2\right\|_2^2 + C \|p_z\|_\alpha^{\frac{4\lambda}{3(4-\lambda)}} \|\nabla w^+\|_2^{\frac{8\lambda}{3(4-\lambda)}} \|w^-\|_4^{\frac{4(12-7\lambda)}{3(4-\lambda)}} \|w^+\|_4^{\frac{4\lambda}{4-\lambda}}.$$

Further applications of Young's inequality imply

$$\|p_z\|_\alpha^{\frac{4\lambda}{3(4-\lambda)}} \|\nabla w^+\|_2^{\frac{8\lambda}{3(4-\lambda)}} \leqslant \|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^+\|_2^2,$$

$$\|w^-\|_4^{\frac{4(12-7\lambda)}{3(4-\lambda)}} \|w^+\|_4^{\frac{4\lambda}{4-\lambda}} \leqslant \|w^+\|_4^4 + \|w^-\|_4^{\frac{2(12-7\lambda)}{3(2-\lambda)}}.$$

Since $\frac{2(12-7\lambda)}{3(2-\lambda)} < 4$, we obtain without loss of generality that

$$J_1 \leqslant \frac{1}{8} \left\||\nabla|w^+|^2\right\|_2^2 + \frac{1}{8} \left\||\nabla|w^-|^2\right\|_2^2 + C \left(\|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^+\|_2^2\right)\left(\|w^+\|_4^4 + \|w^-\|_4^4\right). \quad (3.6)$$

Similarly, we have

$$J_2 = \int p w^- \cdot \nabla |w^-|^2$$

$$\leqslant \frac{1}{8} \left\||\nabla|w^+|^2\right\|_2^2 + \frac{1}{8} \left\||\nabla|w^-|^2\right\|_2^2 + C \left(\|p_z\|_\alpha^{\frac{4\lambda}{12-7\lambda}} + \|\nabla w^-\|_2^2\right)\left(\|w^+\|_4^4 + \|w^-\|_4^4\right). \quad (3.7)$$

Inserting (3.6) and (3.7) in (3.5) and applying Gronwall's inequality, we obtain the desired result. $\quad\square$

Acknowledgments

Appendix A

This appendix provides the proofs of the inequalities (1.9) and (1.10). For the convenience of future references, we write these inequalities as lemmas.

Lemma A.1. *Let μ, λ and γ be three parameters that satisfy*

$$1 \leqslant \mu, \lambda < \infty, \qquad \frac{1}{\mu} + \frac{2}{\lambda} > 1 \quad and \quad 1 + \frac{3}{\gamma} = \frac{1}{\mu} + \frac{2}{\lambda}.$$

Assume $\phi \in H^1(\mathbf{R}^3)$, $\phi_x, \phi_y \in L^\lambda(\mathbf{R}^3)$ and $\phi_z \in L^\mu(\mathbf{R}^3)$. Then, there exists a constant $C = C(\mu, \lambda)$ such that

$$\|\phi\|_\gamma \leqslant C \|\phi_x\|_\lambda^{\frac{1}{3}} \|\phi_y\|_\lambda^{\frac{1}{3}} \|\phi_z\|_\mu^{\frac{1}{3}}. \tag{A.1}$$

Especially, when $\lambda = 2$, there exists a constant $C = C(\mu)$ such that

$$\|\phi\|_{3\mu} \leqslant C \|\phi_x\|_2^{\frac{1}{3}} \|\phi_y\|_2^{\frac{1}{3}} \|\phi_z\|_\mu^{\frac{1}{3}}, \tag{A.2}$$

which holds for any $\phi \in H^1(\mathbf{R}^3)$ and $\phi_z \in L^\mu(\mathbf{R}^3)$ with $1 \leqslant \mu < \infty$.

Proof. Clearly,

$$\left|\phi(x, y, z)\right|^{1 + (1 - \frac{1}{\lambda})\gamma} \leqslant C \int_{-\infty}^{x} \left|\phi(t, y, z)\right|^{(1 - \frac{1}{\lambda})\gamma} \left|\partial_t \phi(t, y, z)\right| dt, \tag{A.3}$$

$$\left|\phi(x, y, z)\right|^{1 + (1 - \frac{1}{\lambda})\gamma} \leqslant C \int_{-\infty}^{y} \left|\phi(x, t, z)\right|^{(1 - \frac{1}{\lambda})\gamma} \left|\partial_t \phi(x, t, z)\right| dt, \tag{A.4}$$

$$\left|\phi(x, y, z)\right|^{1 + (1 - \frac{1}{\mu})\gamma} \leqslant C \int_{-\infty}^{z} \left|\phi(x, y, t)\right|^{(1 - \frac{1}{\mu})\gamma} \left|\partial_t \phi(x, y, t)\right| dt. \tag{A.5}$$

Therefore,

$$\left|\phi(x, y, z)\right|^\gamma \leqslant C \left[\int_{-\infty}^{\infty} \left|\phi(x, y, z)\right|^{(1 - \frac{1}{\lambda})\gamma} \left|\partial_x \phi(x, y, z)\right| dx\right]^{\frac{1}{2}}$$

$$\times \left[\int_{-\infty}^{\infty} \left|\phi(x, y, z)\right|^{(1 - \frac{1}{\lambda})\gamma} \left|\partial_y \phi(x, y, z)\right| dy\right]^{\frac{1}{2}}$$

$$\times \left[\int_{-\infty}^{\infty} \left|\phi(x, y, z)\right|^{(1 - \frac{1}{\mu})\gamma} \left|\partial_z \phi(x, y, z)\right| dz\right]^{\frac{1}{2}}.$$

Integrating with respect to x and applying Hölder's inequality, we have

$$\int_{-\infty}^{\infty} |\phi(x,y,z)|^\gamma \, dx \leqslant \left[\int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma} |\partial_x \phi(x,y,z)| \, dx \right]^{\frac{1}{2}}$$

$$\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma} |\partial_y \phi(x,y,z)| \, dx \, dy \right]^{\frac{1}{2}}$$

$$\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(x,y,z)|^{(1-\frac{1}{\mu})\gamma} |\partial_z \phi(x,y,z)| \, dx \, dz \right]^{\frac{1}{2}}.$$

Further integration with respect to y and z yields

$$\int_{\mathbf{R}^3} |\phi(x,y,z)|^\gamma \, dx \, dy \, dz \leqslant \left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma} |\partial_x \phi(x,y,z)| \, dx \, dy \, dz \right]^{\frac{1}{2}}$$

$$\times \left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\lambda})\gamma} |\partial_y \phi(x,y,z)| \, dx \, dy \, dz \right]^{\frac{1}{2}}$$

$$\times \left[\int_{\mathbf{R}^3} |\phi(x,y,z)|^{(1-\frac{1}{\mu})\gamma} |\partial_z \phi(x,y,z)| \, dx \, dy \, dz \right]^{\frac{1}{2}}.$$

If Hölder's inequality is applied again, we have

$$\|\phi\|_\gamma^\gamma \leqslant C \|\phi\|_\gamma^{(1-\frac{1}{\lambda})\frac{\gamma}{2}} \|\partial_x \phi\|_\lambda^{\frac{1}{2}} \|\phi\|_\gamma^{(1-\frac{1}{\lambda})\frac{\gamma}{2}} \|\partial_y \phi\|_\lambda^{\frac{1}{2}} \|\phi\|_\gamma^{(1-\frac{1}{\mu})\frac{\gamma}{2}} \|\partial_z \phi\|_\mu^{\frac{1}{2}},$$

which leads to (A.1). $\quad\square$

Lemma A.2. *Let $2 \leqslant q \leqslant 6$ and assume $\phi \in H^1(\mathbf{R}^3)$. Then, there exists a constant $C = C(q)$ such that*

$$\|\phi\|_q \leqslant C \|\phi\|_2^{\frac{6-q}{2q}} \|\partial_x \phi\|_2^{\frac{q-2}{2q}} \|\partial_y \phi\|_2^{\frac{q-2}{2q}} \|\partial_z \phi\|_2^{\frac{q-2}{2q}}. \tag{A.6}$$

Proof. This inequality can be obtained by interpolating the trivial inequality $\|\phi\|_q \leqslant \|\phi\|_2^{\frac{6-q}{2q}} \|\phi\|_6^{\frac{3}{2}-\frac{3}{q}}$ and (A.2) with $\mu = 2$, namely

$$\|\phi\|_6 \leqslant C \|\phi_x\|_2^{\frac{1}{3}} \|\phi_y\|_2^{\frac{1}{3}} \|\phi_z\|_2^{\frac{1}{3}}. \quad\square$$

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] R. Agapito, M. Schonbek, Non-uniform decay of MHD equations with and without magnetic diffusion, Comm. Partial Differential Equations 32 (2007) 1791–1812.
[3] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.
[4] H. Beirão da Veiga, A new regularity class for the Navier–Stokes equations in R^n, Chinese Ann. Math. 16 (1995) 407–412.
[5] H. Beirão da Veiga, On the smoothness of a class of weak solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 2 (2000) 315–323.

[6] L.C. Berselli, G.P. Galdi, Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations, Proc. Amer. Math. Soc. 130 (2002) 3585–3595.

[7] R. Caflisch, I. Klapper, G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Comm. Math. Phys. 184 (1997) 443–455.

[8] C. Cao, Sufficient conditions for the regularity to the 3D Navier–Stokes equations, Discrete Contin. Dyn. Syst. (Special issue), in press.

[9] C. Cao, J. Qin, E. Titi, Regularity criterion for solutions of three-dimensional turbulent channel flows, Comm. Partial Differential Equations 33 (2008) 419–428.

[10] C. Cao, E. Titi, Regularity criteria for the three-dimensional Navier–Stokes equations, Indiana Univ. Math. J. 57 (2008) 2643–2662.

[11] C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, arXiv:0901.2908 [math.AP], 19 January 2009.

[12] D. Chae, On the regularity conditions for the Navier–Stokes and related equations, Rev. Mat. Iberoamericana 23 (2007) 371–384.

[13] D. Chae, Nonexistence of self-similar singularities in the viscous magnetohydrodynamics with zero resistivity, J. Funct. Anal. 254 (2008) 441–453.

[14] D. Chae, J. Lee, Regularity criterion in terms of pressure for the Navier–Stokes equations, Nonlinear Anal. 46 (2001) 727–735.

[15] D. Chae, H.J. Choe, Regularity of solutions to the Navier–Stokes equations, Electron. J. Differential Equations 5 (1999) 1–7.

[16] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford Univ. Press, 1961.

[17] Q. Chen, C. Miao, Z. Zhang, On the regularity criterion of weak solution for the 3D viscous magneto-hydrodynamics equations, Comm. Math. Phys. 284 (2008) 919–930.

[18] D. Córdoba, C. Marliani, Evolution of current sheets and regularity of ideal incompressible magnetic fluids in 2D, Comm. Pure Appl. Math. 53 (2000) 512–524.

[19] G. Duvaut, J.-L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, Arch. Ration. Mech. Anal. 46 (1972) 241–279.

[20] L. Escauriaza, G. Seregin, V. Šverák, Backward uniqueness for parabolic equations, Arch. Ration. Mech. Anal. 169 (2003) 147–157.

[21] L. Escauriaza, G. Seregin, V. Šverák, $L^{3,\infty}$-solutions of the Navier–Stokes equations and backward uniqueness, Russian Math. Surveys 58 (2003) 211–250.

[22] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vols. I, II, Springer-Verlag, New York, 1994.

[23] J.D. Gibbon, K. Ohkitani, Evidence for singularity formation in a class of stretched solutions of the equations for ideal MHD, in: Tubes, Sheets and Singularities in Fluid Dynamics, Zakopane, 2001, in: Fluid Mech. Appl., vol. 71, Kluwer Acad. Publ., Dordrecht, 2002, pp. 295–304.

[24] A. Hasegawa, Self-organization processed in continuous media, Adv. Phys. 34 (1985) 1–42.

[25] C. He, New sufficient conditions for regularity of solutions to the Navier–Stokes equations, Adv. Math. Sci. Appl. 12 (2002) 535–548.

[26] C. He, Y. Wang, On the regularity criteria for weak solutions to the magnetohydrodynamic equations, J. Differential Equations 238 (2007) 1–17.

[27] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations 213 (2005) 235–254.

[28] C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, J. Funct. Anal. 227 (2005) 113–152.

[29] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations, Math. Z. 242 (2002) 251–278.

[30] H. Kozono, N. Yatsu, Extension criterion via two-components of vorticity on strong solution to the 3D Navier–Stokes equations, Math. Z. 246 (2003) 55–68.

[31] I. Kukavica, M. Ziane, One component regularity for the Navier–Stokes equations, Nonlinearity 19 (2006) 453–469.

[32] I. Kukavica, M. Ziane, Regularity of the Navier–Stokes equation in a thin periodic domain with large data, Discrete Contin. Dyn. Syst. 16 (2006) 67–86.

[33] O.A. Ladyzhenskaya, Mathematical Theory of Viscous Incompressible Flow, 2nd ed., Gordon and Breach, New York, 1969, English translation.

[34] Z. Lei, Y. Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, arXiv:0901.2683 [math.AP], 19 January 2009.

[35] C. Miao, B. Yuan, Well-posedness of the ideal MHD system in critical Besov spaces, Methods Appl. Anal. 13 (2006) 89–106.

[36] C. Miao, B. Yuan, B. Zhang, Well-posedness for the incompressible magneto-hydrodynamic system, Math. Methods Appl. Sci. 30 (2007) 961–976.

[37] J. Neustupa, P. Penel, Regularity of a suitable weak solution to the Navier–Stokes equations as a consequence of regularity of one velocity component, in: Applied Nonlinear Analysis, Kluwer/Plenum, New York, 1999, pp. 391–402.

[38] M. Núñez, Estimates on hyperdiffusive magnetohydrodynamics, Phys. D 183 (2003) 293–301.

[39] K. Ohkitani, A note on regularity conditions on ideal magnetohydrodynamic equations, Phys. Plasmas 13 (2006) 044504, 3 pp.

[40] H. Politano, A. Pouquet, P.L. Sulem, Current and vorticity dynamics in three dimensional magnetohydrodynamic turbulence, Phys. Plasmas 2 (1995) 2931–2939.

[41] P. Penel, M. Pokorný, Some new regularity criteria for the Navier–Stokes equations containing gradient of the velocity, Appl. Math. 49 (2004) 483–493.

[42] M. Pokorný, On the result of He concerning the smoothness of solutions to the Navier–Stokes equations, Electron. J. Differential Equations 10 (2003) 1–8.

[43] M.E. Schonbek, T.P. Schonbek, E. Süli, Large-time behaviour of solutions to the magnetohydrodynamics equations, Math. Ann. 304 (1996) 717–756.

[44] G. Seregin, V. Šverák, Navier–Stokes equations with lower bounds on the pressure, Arch. Ration. Mech. Anal. 163 (2002) 65–86.

[45] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math. 36 (1983) 635–664.

[46] J. Serrin, On the interior regularity of weak solutions of the Navier–Stokes equations, Arch. Ration. Mech. Anal. 9 (1962) 187–195.

[47] J. Wu, Viscous and inviscid magnetohydrodynamics equations, J. Anal. Math. 73 (1997) 251–265.

[48] J. Wu, Bounds and new approaches for the 3D MHD equations, J. Nonlinear Sci. 12 (2002) 395–413.

[49] J. Wu, Generalized MHD equations, J. Differential Equations 195 (2003) 284–312.

[50] J. Wu, Regularity results for weak solutions of the 3D MHD equations, Discrete Contin. Dyn. Syst. 10 (2004) 543–556.

[51] J. Wu, Regularity criteria for the generalized MHD equations, Comm. Partial Differential Equations 33 (2008) 285–306.

[52] B. Yuan, Regularity criterion of weak solutions to the MHD system based on vorticity and electric current in negative index Besov spaces, Adv. Math. (China) 37 (2008) 451–458.

[53] Z. Zhang, Q. Chen, Regularity criterion via two components of vorticity on weak solutions to the Navier–Stokes equations in R^3, J. Differential Equations 216 (2005) 470–481.

[54] Y. Zhou, Regularity criteria in terms of pressure for the 3-D Navier–Stokes equations in a generic domain, Math. Ann. 328 (2004) 173–192.

[55] Y. Zhou, Regularity criteria for the generalized viscous MHD equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007) 491–505.