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Deformation and Symmetry in the Inviscid SQG and the 3D Euler Equations

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Abstract The global regularity problem concerning the inviscid SQG and the 3D Euler equations remains an outstanding open question. This paper presents several geometric observations on solutions of these equations. One observation stems from a relation between what we call Eulerian and Lagrangian deformations and reflects the alignment of the stretching directions of these deformations and the tangent direction of the level curves for the SQG equation. Various spatial symmetries in solutions to the 3D Euler equations are exploited. In addition, two observations on the curvature of the level curves of the SQG equation are also included.

Keywords 3D Euler equation · Surface quasi-geostrophic equation · Geometric property

Mathematics Subject Classification (2000) 35Q35 · 35B35 · 35B65 · 76D03

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1 Introduction

The inviscid surface quasi-geostrophic (SQG) equation we are concerned with here assumes the form

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} \psi, \qquad -\Lambda \psi = \theta, \end{cases} \tag{1.1}$$

where $\theta = \theta(x,t)$ is a scalar function of $x \in \mathbb{R}^2$ and $t \geq 0$, u denotes the 2D velocity field, $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator, defined by its Fourier transform, $\widehat{\Lambda f}(\xi) = |\xi| \widehat{f}(\xi)$. Often u is written in terms of the Riesz transforms of θ,

$$u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \tag{1.2}$$

where the $R_j = \partial_{x_j} \Lambda^{-1}$, $j = 1, 2$, denote the 2D Riesz transforms (see, e.g., Stein 1970). The inviscid SQG equation was derived from the 3D quasi-geostrophic equations modeling geophysical fluids (see, e.g., Constantin et al. 1994; Gill 1982; Pedlosky 1987). In addition, the SQG equation is an important example of active scalar equations and serves as a test bed for certain turbulence theory (see, e.g., Blumen 1978; Held et al. 1995). We are mainly concerned with the fundamental issue of whether its classical solution corresponding to given initial data

$$\theta(x,0) = \theta_0(x)$$

is global in time. This problem has been studied extensively. Local well-posedness and regularity criteria in various functional settings have been established (see, e.g., (Constantin et al. 1994; Wu 2005)). The global regularity problem for the general case remains open. Numerical simulations and geometric approaches appear to be particularly enlightening (see, e.g. Chae 2008b; Constantin et al. 1994, 2012; Córdoba and Fefferman 2001, 2002a, 2002b; Córdoba et al. 2005; Deng et al. 2006; Ohkitani and Yamada 1997). We study this issue by examining the behavior of the level curves of the solutions. We present several observations on the geometric quantities associated with the level curves including the tangent direction and curvature.

We are also concerned with the global regularity problem on the 3D Euler equations of incompressible fluids

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0, \end{cases} \tag{1.3}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ represents the velocity of the fluid and $p = p(x,t)$ the scalar pressure. The 3D Euler equations and the inviscid SQG equation are closely related. As pointed out in Constantin et al. (1994), the SQG equation can be treated as a model of the 3D Euler equations and they share many parallel quantities and properties. For example, the 3D vorticity $\omega = \nabla \times u$ and $\nabla^{\perp}\theta$ both satisfy the equation

$$\partial_t W + u \cdot \nabla W = W \cdot \nabla u,$$

where W denotes either ω or $\nabla^{\perp}\theta$. There are numerous studies on the Euler equations, and interested readers can consult books and recent review papers (see, e.g., Chae 2008c; Majda and Bertozzi 2002). Our attention is focused on the potentially singular behavior of solutions with special spatial symmetries.

As is well known, the rate of deformation tenor $S = \frac{1}{2}(\nabla u + (\nabla u)^*)$ plays a crucial role in determining the global regularity of solutions to the SQG and the 3D Euler equations. In fact, if we know that

$$\int_0^T \left\| S(\cdot, t) \right\|_{L^\infty} \mathrm{d}t < \infty,$$

then the corresponding solution preserves its regularity on $[0, T]$ (see, e.g., Beale et al. 1984; Constantin et al. 1994). Physically S controls the deformation of the flow. The flow element stretches in the eigenvector directions associated with positive eigenvalues and contracts in the directions with negative eigenvalues (Batchelor 1967). Therefore, the eigenvalues and eigenvectors of S play crucial roles in understanding the behavior of solutions to the SQG and the 3D Euler equations. In a parallel way the eigenvalues and eigenvectors of two matrices represented in terms of the particle trajectory of the flow are also very important. More precisely, if we denote by $X = X(a, t)$ the particle trajectory and by $A = A(x, t)$ the inverse map or the back-to-label map, then these positive definite matrices are given by $\widetilde{S} \equiv (\nabla_a X)^* \nabla_a X$ and $M = (\nabla_x A)^* \nabla_x A$ are especially important. In contrast to S, $\widetilde{S}$ represents the deformation of the current flow with respect to the initial status while M represents the reverse deformation. Intuitively S reflects the deformation at a fixed time and $\widetilde{S}$ the accumulative deformation. Therefore we call S the Eulerian deformation tensor, while $\tilde{S}$ and M are labeled as Lagrangian deformations.

It is of practical and theoretical importance to relate the eigenvalues and eigenvectors of S and $\widetilde{S}$. For some special flows the relationship between them is really simple. For example, for the flow determined by the velocity field $u = (-x_2, x_1)$, we have

$$\widetilde{\lambda}_i(x, t) = \exp\left(2 \int_0^t \lambda_i\big(X(a, \tau), \tau\big) \, \mathrm{d}\tau\right), \quad i = 1, 2,$$

where $\lambda_1 \geq \lambda_2$ denote the eigenvalues of S and $\widetilde{\lambda_1} \geq \widetilde{\lambda_2}$ the eigenvalues of $\widetilde{S}$. For more general flows, these simple relationships do not really hold. An explicit example is presented in the Appendix to illustrate this point. We remark that this example itself is significant as a simple model of actual physical phenomena and can easily be expanded into a family of special solutions to the SQG equation, which may be used in testing numerical methods. Nevertheless we can show that their eigenvalues do obey the following bounds:

$$\begin{aligned}
&\exp\left(2 \int_0^t \lambda_n\big(X(a, \tau), \tau\big) \, \mathrm{d}\tau\right) \\
&\quad \leq \widetilde{\lambda}_n\big(X(a, t), t\big) \\
&\quad \leq \widetilde{\lambda}_1\big(X(a, t), t\big) \leq \exp\left(2 \int_0^t \lambda_1\big(X(a, \tau), \tau\big) \, \mathrm{d}\tau\right).
\end{aligned}$$

The precise statement is presented in Theorem 2.1 of Sect. 2.

The positive definite matrix M appears very naturally in the study of flows transported by a differentiable mapping. Here a flow W is said to be transported by a differentiable mapping if

$$W\big(X(a,t),t\big)=\nabla_a X(a,t)W_0(a).$$

Indeed M gauges the growth of $|W(x,t)|$ in time (see (2.10)). As derived in Sect. 2.3 below, if $\mu_1,\dots,\mu_n$ are the eigenvalues of M and $\varrho_1,\dots,\varrho_n$ the direction angles of W of the eigenvectors of M, these are linked to those of S by

$$\begin{aligned}&\mu_1(x,t)\cos^2\varrho_1(x,t)+\cdots+\mu_n(x,t)\cos^2\varrho_n(x,t)\\&\quad=\exp\left\{-2\int_0^t\big(\lambda_1\cos^2\varphi_1+\cdots+\lambda_n\cos^2\varphi_n\big)\big(A(x,t-\tau),\tau\big)\,\mathrm{d}\tau\right\}.\end{aligned}\tag{1.4}$$

This general relationship, when applied to the special flows such as the vorticity of the 3D Euler equations or $\nabla^\perp\theta$ of the SQG equation, reveals useful geometric information on their solutions.

If θ is a smooth solution of the SQG equation (1.1), then $\nabla^\perp\theta$ is a special example of flows transported by a differentiable map. Aiming at the issue of whether $|\nabla^\perp\theta|$ can blow up in a finite time, we examine the key quantities associated with the level curves of θ such as the tangent direction and the curvature. When (1.4) is applied to $W=\nabla^\perp\theta$, we conclude that, in a finite-time blow-up scenario, the stretching directions of S and M align with the tangent direction of the level curves. Although this result is essentially obtained in Chae (2008b), we prove it directly using the formula (1.4). The precise statement of this result is presented in Theorem 3.1. To relate the growth in $|\nabla^\perp\theta|$ to the unite tangent direction $\eta=\nabla^\perp\theta/|\nabla^\perp\theta|$ and the curvature κ, explicit evolution equations obeyed by η and κ are derived. Although these equations elude a definite conclusion, there is an indication that curvature of the level curves may turn out to be small in the regions where $|\nabla^\perp\theta|$ is large. Another geometric observation is that there is a uniform global (in time) bound for the signed mean curvature of a closed level curve (see Theorem 3.3). In addition, we conclude that if $|\nabla^\perp\theta|$ blows up in a finite time, the curvature of the curves orthogonal to the levels must also blow up in a finite time (see Theorem 3.4).

Finally, we investigate the potential finite-time singular behavior of solutions to the 3D Euler equations with various spatial symmetries. These symmetries are special cases of the general rotation symmetry (see, e.g., Majda and Bertozzi 2002, p. 3). The attention is focused on the reduced Euler equations at the spatial points that are invariant under these symmetries. We obtain regularity criteria for these reduced equations. These are the generalizations of the observations made in Chae (2008a, 2010). These results are presented in Sect. 4.

2 Lagrangian and Eulerian Deformations

This section intends to relate the Lagrangian and Eulerian deformations. We establishes several connections that are useful in the geometric study of solutions to hydrodynamics equations.

2.1 Basic Concepts

We first make these concepts more precise. Let D be a domain in $\mathbb{R}^n$, and $T \in (0, \infty]$. Suppose that for all $t \in [0, T)$ the mapping $a \to X(a, t)$ is a diffeomorphism on D. We denote by $A(\cdot, t)$ the inverse mapping of $X(\cdot, t)$, satisfying

$$A\big(X(a,t),t\big) = a, \qquad X\big(A(x,t),t\big) = x \quad \forall a, x \in D,\ \forall t \in [0, T).$$

In the applications to hydrodynamics the mapping $\{X(\cdot, t)\}$ is defined by a smooth velocity field $u(x, t)$ through the system of the ordinary differential equations:

$$\frac{\partial X(a,t)}{\partial t} = u\big(X(a,t),t\big); \quad X(a,0) = a \in D \subset \mathbb{R}^n. \tag{2.1}$$

In such a case we say the 'particle trajectory' map $X(\cdot, t)$ and its inverse, the 'back-to-label' map $A(\cdot, t)$, are generated by the fluid velocity field $u(x, t)$.

Assume u is divergence free, $\nabla \cdot u = 0$, and denote by S the rate of the Eulerian deformation matrix, namely

$$S = \frac{1}{2}\big[\nabla u + (\nabla u)^*\big],$$

where $(\nabla u)^*$ stands for the transpose of ∇u. Note that S is symmetric and its trace is zero. Let $\{(\lambda_j, \mathbf{e}_j)\}_1^n$ be the eigenvalue and eigenvector pairs of S with

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

and $\{\mathbf{e}_j\}_1^n$ forming an orthonormal basis for $\mathbb{R}^n$. Denote by O the orthogonal matrix with $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ as its columns and $F = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Clearly

$$SO = OF.$$

One way to characterize the Lagrangian deformation is by the symmetric, positive definite matrix $\widetilde{S} = (\nabla_a X)^* \nabla_a X$. Since $\nabla \cdot u = 0$, $\det \widetilde{S} = 1$. Let $\{(\widetilde{\lambda}_j, \widetilde{\mathbf{e}}_j)\}_1^n$ be the eigenvalue and eigenvector pairs of $\widetilde{S}$ with

$$\widetilde{\lambda}_1 \cdot \widetilde{\lambda}_2 \cdots \widetilde{\lambda}_n = 1, \qquad \widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_n$$

and with $\{\widetilde{\mathbf{e}}_j\}_1^n$ forming an orthonormal basis for $\mathbb{R}^n$. Again denote by $\widetilde{O}$ the orthogonal matrix with $\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \ldots, \widetilde{\mathbf{e}}_n$ as its columns and $\widetilde{F} = \operatorname{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n)$. Clearly

$$\widetilde{S}\widetilde{O} = \widetilde{O}\widetilde{F}.$$

2.2 Bounds Between the Eigenvalues of S and $\widetilde{S}$

We intend to understand the relations between the eigenvalues and eigenvectors of S and $\widetilde{S}$. Intuitively, $\widetilde{S}$ contains deformation information from the past to the present, while S represents the instantaneous deformation. Therefore, the eigenvalues of $\widetilde{S}$ should be related to the time integrals of the eigenvalues of S. This intuition can be verified by some simple flows. For example, consider the 2D velocity $u = (u_1, u_2) = (x_1, -x_2)$. Clearly,

$$\widetilde{\lambda}_i(x,t) = \exp\biggl(2\int_0^t \lambda_i\bigl(X(a,\tau),\tau\bigr)\,\mathrm{d}\tau\biggr), \quad i = 1, 2.$$

Of course, such relations are in general not true and an interesting example is provided in the appendix. Nevertheless, we are able to obtain the following bound.

Theorem 2.1 *Let u be a smooth divergence free vector field from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$. Let $\mathbf{c} \in \mathbb{S}^{n-1}$ be a unit vector and define*

$$\xi_j(a,t) := \mathbf{e}_j \cdot \frac{\nabla_a X(a,t)\mathbf{c}}{|\nabla_a X(a,t)\mathbf{c}|}, \qquad \tilde{\xi}_j(a,t) := \tilde{\mathbf{e}}_j \cdot \mathbf{c}.$$

Then we have

$$\widetilde{\xi_1}^2\tilde{\lambda}_1 + \cdots + \widetilde{\xi_n}^2\widetilde{\lambda}_n = \exp\biggl(2\int_0^t \bigl(\xi_1^2\lambda_1 + \cdots + \xi_n^2\lambda_n\bigr)\,\mathrm{d}\tau\biggr), \tag{2.2}$$

and therefore

$$\begin{aligned}
&\exp\biggl(2\int_0^t \lambda_n\bigl(X(a,\tau),\tau\bigr)\,\mathrm{d}\tau\biggr)\\
&\quad\le \widetilde{\lambda}_n\bigl(X(a,t),t\bigr)\\
&\quad\le \widetilde{\lambda}_1\bigl(X(a,t),t\bigr) \le \exp\biggl(2\int_0^t \lambda_1\bigl(X(a,\tau),\tau\bigr)\,\mathrm{d}\tau\biggr).
\end{aligned} \tag{2.3}$$

Proof It follows from (2.1) that

$$\frac{\partial}{\partial t}\bigl((\nabla_a X)^*\nabla_a X\bigr) = 2(\nabla_a X)^* S \nabla_a X. \tag{2.4}$$

Let us define $\mathbf{b}(a,t) := \nabla_a X(a,t)\mathbf{c}$. We multiply (2.4) from the left by $\mathbf{c}^*$ and from the right by $\mathbf{c}$, respectively. Then we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\bigl|\mathbf{b}(a,t)\bigr|^2 &= 2\mathbf{b}^*(a,t)S\mathbf{b}(a,t) = 2\bigl(O^*\mathbf{b}(a,t)\bigr)^* O^* S O\bigl(O^*\mathbf{b}(a,t)\bigr)\\
&= 2\bigl(O^*\mathbf{b}(a,t)\bigr)^* \operatorname{diag}(\lambda_1,\ldots,\lambda_n) O^*\mathbf{b}(a,t)\\
&= 2\bigl(\xi_1^2\lambda_1 + \cdots + \xi_n^2\lambda_n\bigr)\bigl|\mathbf{b}(a,t)\bigr|^2,
\end{aligned}$$

where O denotes the orthogonal matrix with $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ as its columns. Hence,

$$\bigl|\mathbf{b}(a,t)\bigr|^2 = \exp\biggl(2\int_0^t \bigl(\xi_1^2\lambda_1 + \cdots + \xi_n^2\lambda_n\bigr)\,\mathrm{d}\tau\biggr). \tag{2.5}$$

On the other hand, we also have

$$\bigl|\mathbf{b}(a,t)\bigr|^2 = \mathbf{c}^*\bigl(\nabla_a X(a,t)\bigr)^*\nabla_a X(a,t)\mathbf{c} = \mathbf{c}^*\widetilde{S}\mathbf{c}$$

$$= (\widetilde{O}^*\mathbf{c})^* \widetilde{O}^* \widetilde{S} \widetilde{O} \widetilde{O}^* \mathbf{c}$$
$$= (\widetilde{O}^*\mathbf{c})^* \operatorname{diag}(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_n) \widetilde{O}^* \mathbf{c}$$
$$= \widetilde{\xi}_1^2 \widetilde{\lambda}_1 + \cdots + \widetilde{\xi}_n^2 \widetilde{\lambda}_n, \tag{2.6}$$

where $\widetilde{O}$ denotes the orthogonal matrix with $\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \dots, \widetilde{\mathbf{e}}_n$ as its columns. Combining (2.5) with (2.6) we obtain (2.2). We now prove the first inequality of (2.3) from (2.2). The proof of the third inequality is similar. We observe

$$\exp\left(2 \int_0^t \lambda_n\big(X(a,\tau), \tau\big)\right) \le \widetilde{\xi}_1^2 \widetilde{\lambda}_1 + \cdots + \widetilde{\xi}_n^2 \widetilde{\lambda}_n. \tag{2.7}$$

Since $\mathbf{c}$ is arbitrary in $\mathbb{S}^{n-1}$, one can optimize (2.7) so that $\widetilde{\xi}_n = 1$, and $\widetilde{\xi}_j = 0$ for all $j \neq n$. Thus we obtain the first inequality of (2.3). Theorem 2.1 is now proven. □

2.3 Lagrangian and Eulerian Deformations for Flows Transported by Differentiable Mapping

The goal of this subsection is to derive an explicit relation between Lagrangian and Eulerian deformations for a special class of flow, flows transported by differentiable mapping.

Definition 2.2 Let D be a domain in $\mathbb{R}^n$. We say that a parameterized vector field $W(\cdot,\cdot) : D \times [0,T) \to \mathbb{R}^n$ is transported by a differentiable mapping $X(\cdot,t)$ from D into itself for all $t \in [0,T)$ if

$$W\big(X(a,t), t\big) = \nabla_a X(a,t) W_0(a) \tag{2.8}$$

holds for all $(a,t) \in D \times [0,T)$, where we set $W_0(x) = W(x,0)$.

For flows transported by differentiable mapping, the Lagrangian deformation can be most conveniently measured through the matrix $M(x,t) := (\nabla A(x,t))^* \nabla A(x,t)$. Clearly, M is symmetric, positive definite with $\det M \equiv 1$. This subsection derives a formula that relates the eigenvalues and eigenvectors of M and S. More precisely, we have the following theorem.

Theorem 2.3 *Let W be a flow transported by X. Let η denote the unit vector in the direction of W, namely $\eta = \frac{W}{|W|}$. Let $M(x,t) := (\nabla A(x,t))^* \nabla A(x,t)$. Denote by $\mu_1 > \mu_2 > \cdots > \mu_n$ the eigenvalues of M and by $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n$ the corresponding normalized eigenvectors. Let $(\varrho_1, \varrho_2, \dots, \varrho_n)$ be the direction angles of ξ on $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n$. Recall that $(\lambda_1, \mathbf{e}_1), (\lambda_2, \mathbf{e}_2), \dots, (\lambda_n, \mathbf{e}_n)$ denote the eigenvalue and eigenvector pairs of S. Let $(\varphi_1, \varphi_2, \dots, \varphi_n)$ be the direction angles of η on $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Then*

$$\mu_1(x,t) \cos^2 \varrho_1(x,t) + \cdots + \mu_n(x,t) \cos^2 \varrho_n(x,t)$$
$$= \exp\left\{-2 \int_0^t \big(\lambda_1 \cos^2 \varphi_1 + \cdots + \lambda_n \cos^2 \varphi_n\big)\big(A(x,t-\tau), \tau\big)\, \mathrm{d}\tau\right\}. \tag{2.9}$$

Remark We observe that, since both $\varrho_1, \ldots, \varrho_n$ and $\varphi_1, \ldots, \varphi_n$ are direction angles, the following equalities hold:

$$\cos^2 \varrho_1 + \cdots + \cos^2 \varrho_n = \cos^2 \varphi_1 + \cdots + \cos^2 \varphi_n = 1.$$

Moreover, if the mapping $X(\cdot, t)$ is volume preserving, or equivalently the associated velocity field is incompressible, then since $\det(M) = \exp\{\mathrm{Tr}(S)\} = 1$, we have

$$\mu_1 \cdots \mu_n = 1, \quad \text{and} \quad \lambda_1 + \cdots + \lambda_n = 0.$$

Proof It follows from (2.8) that

$$W_0(a) = \big(\nabla_a X(a,t)\big)^{-1} W\big(X(a,t), t\big)$$

or

$$W_0\big(A(x,t)\big) = \big(\nabla_a X\big(A(x,t), t\big)\big)^{-1} W(x,t).$$

Since $(\nabla_a X(A(x,t), t))^{-1} = \nabla A(x,t)$,

$$\begin{aligned} \big|W_0\big(A(x,t)\big)\big|^2 &= W^*(x,t)\big(\big(\nabla_a X\big(A(x,t),t\big)\big)^*\big)^{-1}\big(\nabla_a X\big(A(x,t),t\big)\big)^{-1} W(x,t) \\ &= W^*(x,t) M(x,t) W(x,t). \end{aligned} \tag{2.10}$$

Denoting by Q the orthonormal matrix with $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n$ as its column, we have

$$\begin{aligned} \frac{|W_0(A(x,t))|^2}{|W(x,t)|^2} &= \eta^*(x,t) Q(x,t) Q^*(x,t) M Q(x,t) Q^*(x,t) \eta(x,t) \\ &= \mu_1(x,t)\cos^2 \varrho_1(x,t) + \cdots + \mu_n(x,t)\cos^2 \varrho_n(x,t). \end{aligned} \tag{2.11}$$

On the other hand, from the evolution equation

$$(\partial_t + u \cdot \nabla) W\big(X(a,t), t\big) = (W \cdot \nabla u)\big(X(a,t), t\big),$$

which is equivalent to (2.8), we obtain

$$(\partial_t + u \cdot \nabla)\big|W\big(X(a,t), t\big)\big| = \big(\eta^* S \eta\big)|W|\big(X(a,t), t\big),$$

from which we derive

$$\big|W\big(X(a,t), t\big)\big| = \big|W_0(a)\big| \exp\bigg(\int_0^t \big(\eta^* S \eta\big)\big(X(a,\tau), \tau\big)\, \mathrm{d}\tau\bigg).$$

This can be written in terms of $A(x,t)$ as follows:

$$\begin{aligned} \big|W(x,t)\big| &= \big|W_0\big(A(x,t)\big)\big| \exp\bigg(\int_0^t \big(\eta^* S \eta\big)\big(A(x,t-\tau), \tau\big)\, \mathrm{d}\tau\bigg) \\ &= \big|W_0\big(A(x,t)\big)\big| \exp\bigg(\int_0^t \big(\eta^* O^* \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \cdot O \eta\big) \\ &\qquad \times \big(A(x,t-\tau), \tau\big)\, \mathrm{d}\tau\bigg) \end{aligned}$$

$$= \left|W_0\big(A(x,t)\big)\right| \exp\left(\int_0^t \big(\lambda_1 \cos^2 \varphi_1 + \cdots + \lambda_n \cos^2 \varphi_n\big) \times \big(A(x, t-\tau), \tau\big)\, d\tau\right). \tag{2.12}$$

Comparing (2.11) and (2.12), we obtain (2.9). This completes the proof of Theorem 2.3. □

3 The SQG Equation

This section presents several results concerning the geometric aspects of solutions to the inviscid SQG equation (1.1).

We first recall and fix a little notation. Let $X = X(a,t)$ be the particle trajectory determined by u and A the back-to-label map. Let η be the unit vector in the direction $\nabla^\perp\theta$. Let S denote the symmetric part of ∇u and $(\lambda, \mathbf{e}_1)$ and $(-\lambda, \mathbf{e}_2)$ the eigenvalue and eigenvector pairs of S. Let M be the matrix as defined in Theorem 2.3 and let $(\mu_1, \mathbf{g}_1)$ and $(\mu_2, \mathbf{g}_2)$ be the eigenvalue and eigenvector pairs of M with $\mu_1 \geq \mu_2$. Denote by φ_1 and φ_2 the direction angles of η with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$ and by ϱ_1 and ϱ_2 the direction angles of η with respect to $\mathbf{g}_1$ and $\mathbf{g}_2$.

Our first theorem, relating λ to $\nabla^\perp\theta$, provides a regularity criterion in terms of λ and φ and reveals the connections among λ, μ_1, μ_2, φ_1, φ_2, ϱ_1 and ϱ_2 in a possible finite-time singularity scenario.

Theorem 3.1 *Let $T \in (0,\infty]$ and let θ be a smooth solution of (1.1) on $[0,T)$. Let u be the corresponding velocity field given by (1.2). Then the following results hold*:

(a) *For any $t \in [0,T)$,*

$$\left\|\lambda(\cdot,t)\right\|_{L^2} = \frac{1}{2}\left\|\nabla^\perp\theta(\cdot,t)\right\|_{L^2} \quad \textit{and} \quad \left\|\lambda(\cdot,t)\right\|_{L^p} \leq C(p)\left\|\nabla^\perp\theta(\cdot,t)\right\|_{L^p},$$

where $1 < p < \infty$ and $C(p)$ is a constant depending on p only.

(b) *If*

$$\int_0^{T_1} \left\|\lambda(\cdot,t)\cos\big(2\varphi(\cdot,t)\big)\right\|_{L^\infty} dt < \infty,$$

then $\theta(\cdot,t)$ can be extended to the time interval $[0, T_1+\delta)$ for some $\delta > 0$.

(c) *Consider a blow-up scenario. Assume that there exist $\overline{x} \in \mathbb{R}^2$ and sequences $x_k \in \mathbb{R}^2$ and $t_k > 0$ such that $x_k \to \overline{x}$, $t_k \to T$ and*

$$\left|\nabla^\perp\theta(x_k,t_k)\right| \to \infty \quad \textit{as } x_k \to \overline{x} \textit{ and } t_k \to T.$$

Then

$$\lim_{k\to\infty} \mu_1(x_k,t_k) = \infty, \qquad \lim_{k\to\infty} \varrho_1(x_k,t_k) = \frac{\pi}{2},$$

$$\lim_{k\to\infty} \lambda(x_k,t_k) = \infty, \qquad \varphi(x_k,t_k) < \frac{\pi}{4}.$$

This theorem implies that if $\nabla^{\perp}\theta$ experiences a finite-time blow-up at $(\overline{x}, T)$, then the Lagrangian stretching direction lines up with the tangent direction and the angle between the Eulerian stretching direction and the tangent vector is less than $\pi/4$.

Proof We decompose $V = \nabla u$ into symmetric and skew-symmetric parts:

$$V = S + A, \quad A = \frac{1}{2}\big(V - V^*\big).$$

It is easy to check that the skew-symmetric part can be written as

$$A = \frac{1}{2}\begin{pmatrix} 0 & \Lambda\theta \\ -\Lambda\theta & 0 \end{pmatrix}.$$

Since

$$|\nabla u|^2 = \mathrm{tr}\big(VV^*\big) = \mathrm{tr}\big(S^2\big) - \mathrm{tr}\big(A^2\big) = 2\lambda^2 + \frac{1}{2}|\Lambda\theta|^2, \tag{3.1}$$

we have a pointwise representation of λ in terms of the function θ,

$$\lambda = \left\{\frac{1}{2}\big[(\mathcal{R}_1\partial_1\theta)^2 + (\mathcal{R}_1\partial_2\theta)^2 + (\mathcal{R}_2\partial_1\theta)^2 + (\mathcal{R}_2\partial_2\theta)^2\big] - \frac{1}{4}|\Lambda\theta|^2\right\}^{\frac{1}{2}}.$$

By Plancherel's identity,

$$\begin{aligned}\int_{\mathbb{R}^2} |\lambda|^2\,\mathrm{d}x &= \int_{\mathbb{R}^2}\left\{\frac{1}{2|\xi|^2}\big(\xi_1^4 + 2\xi_1^2\xi_2^2 + \xi_2^4\big) - \frac{|\xi|^2}{4}\right\}\big|\hat{\theta}(\xi)\big|^2\,\mathrm{d}\xi \\ &= \frac{1}{4}\int_{\mathbb{R}^2}|\xi|^2\big|\hat{\theta}(\xi)\big|^2\,\mathrm{d}\xi = \frac{1}{4}\int_{\mathbb{R}^2}\big|\nabla^{\perp}\theta\big|^2\,\mathrm{d}x. \end{aligned}\tag{3.2}$$

This proves the equation in (a). The inequality in (a) follows from (3.1) and the boundedness of Riesz transforms on L^p for $1 < p < \infty$.

To prove (b), we apply (2.12) with $W = \nabla^{\perp}\theta$, $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$ to find

$$\big|\nabla^{\perp}\theta\big(X(a,t),t\big)\big| = \big|\nabla^{\perp}\theta_0(a)\big|\exp\left[\int_0^t \lambda\big(X(a,s),s\big)\cos\big(2\varphi\big(X(a,s),s\big)\big)\,\mathrm{d}x\right].$$

Therefore, for any $1 \le p \le \infty$,

$$\big\|\nabla^{\perp}\theta(\cdot,t)\big\|_{L^p} \le \big\|\nabla^{\perp}\theta_0\big\|_{L^p}\exp\left[\int_0^t \big\|\lambda(\cdot,s)\cos\big(2\varphi(\cdot,s)\big)\big\|_{L^\infty}\,\mathrm{d}s\right].$$

When the condition in (b) holds, $\|\nabla^{\perp}\theta(\cdot,t)\|_{L^\infty} < \infty$ for $t \in [0, T_1]$ and therefore θ can be extended to $[0, T_1 + \delta)$ for some $\delta > 0$.

(c) is essentially a consequence of Theorem 2.3. In fact, (2.9) reduces to

$$\begin{aligned}&\mu_1(x,t)\cos^2\varrho_1(x,t) + \mu_2(x,t)\cos^2\varrho_2(x,t) \\ &\quad = \exp\left\{-2\int_0^t \big(\lambda_1\cos^2\varphi_1 + \lambda_2\cos^2\varphi_2\big)\big(A(x,t-\tau),\tau\big)\,\mathrm{d}\tau\right\}.\end{aligned}$$

When a finite-time blow-up scenario occurs, this equation easily leads to the conclusion in (c). This completes the proof of Theorem 3.1. □

The following theorem derives the evolution equations for two geometric quantities associated with the level curves of solutions to the SQG equation. They are the unit tangent vector η and the signed curvature κ, where κ is given by

$$\kappa\eta^{\perp} = \frac{\partial \eta}{\partial s} = (\eta \cdot \nabla)\eta \quad \text{or} \quad \kappa = (\eta \cdot \nabla)\eta \cdot \eta^{\perp}, \tag{3.3}$$

where $\frac{\partial}{\partial s}$ denotes the partial derivative with respect to the arc length. It is easily verified that

$$\kappa = \nabla \times \eta. \tag{3.4}$$

Theorem 3.2 *Let θ be a smooth solution of (1.1) on $[0, T)$ and let u be the corresponding velocity field. Let η be the unit tangent vector and let κ be the signed curvature of the level cures of θ. Then η and κ satisfy*

$$(\partial_t + u \cdot \nabla + \beta)\eta = \eta \cdot \nabla u \quad \textit{or} \quad (\partial_t + u \cdot \nabla)\eta = \gamma \eta^{\perp}, \tag{3.5}$$

$$(\partial_t + u \cdot \nabla + \beta)\kappa = \eta \cdot \nabla \gamma, \tag{3.6}$$

where $\beta = (\eta \cdot \nabla)u \cdot \eta$ and $\gamma = (\eta \cdot \nabla)u \cdot \eta^{\perp}$. Alternatively, γ can also be represented in terms of λ and φ,

$$\gamma = -\lambda \sin(2\varphi) + \frac{1}{2}\Lambda\theta. \tag{3.7}$$

Proof For notational convenience, we write $\frac{\mathrm{D}}{\mathrm{D}t}$ for $\partial_t + u \cdot \nabla$. The first equation in (3.5) can be verified from the equations for $\nabla^{\perp}\theta$ and $|\nabla^{\perp}\theta|$,

$$\frac{\mathrm{D}}{\mathrm{D}t}\nabla^{\perp}\theta = \nabla u \cdot \nabla^{\perp}\theta \quad \text{and} \quad \frac{\mathrm{D}}{\mathrm{D}t}\left|\nabla^{\perp}\theta\right| = \beta(x,t)\left|\nabla^{\perp}\theta\right|. \tag{3.8}$$

To obtain the second equation in (3.5), we notice that

$$(\eta \cdot \nabla u - \beta\eta) \cdot \eta = 0$$

or $\eta \cdot \nabla u - \beta\eta$ is parallel to $\eta^{\perp}$. Therefore, $\eta \cdot \nabla u - \beta\eta = \gamma\eta^{\perp}$. To prove (3.6), we apply $\frac{\mathrm{D}}{\mathrm{D}t}$ to (3.3) to get

$$\frac{\mathrm{D}}{\mathrm{D}t}(\eta \cdot \nabla\eta) = \frac{\mathrm{D}}{\mathrm{D}t}\left(\kappa\eta^{\perp}\right) = \frac{\mathrm{D}\kappa}{\mathrm{D}t}\eta^{\perp} + \kappa\frac{\mathrm{D}\eta^{\perp}}{\mathrm{D}t}.$$

Taking the dot product with $\eta^{\perp}$ and using the fact that $\eta^{\perp} \cdot \frac{\mathrm{D}\eta^{\perp}}{\mathrm{D}t} = 0$, we have

$$\frac{\mathrm{D}\kappa}{\mathrm{D}t} = \eta^{\perp} \cdot \frac{\mathrm{D}}{\mathrm{D}t}(\eta \cdot \nabla\eta). \tag{3.9}$$

To simplify the right-hand side, we write the vector operations in terms of components and adopt Einstein's summation convention. By (3.5),

$$\begin{aligned}
\eta_k^\perp \frac{\mathrm{D}}{\mathrm{D}t}(\eta_j \partial_j \eta_k) &= \eta_k^\perp \left(\frac{\mathrm{D}\eta_j}{\mathrm{D}t} \partial_j \eta_k + \eta_j \frac{\mathrm{D}}{\mathrm{D}t} \partial_j \eta_k \right) \\
&= \eta_k^\perp (-\beta \eta_j + \eta_l \partial_l u_j) \partial_j \eta_k + \eta_k^\perp \eta_j \left(\partial_j \frac{\mathrm{D}}{\mathrm{D}t} \eta_k - \partial_j u_m \partial_m \eta_k \right) \\
&= -\beta \eta_k^\perp \eta_j \partial_j \eta_k + \eta_k^\perp \eta_l \partial_l u_j \partial_j \eta_k + \eta_k^\perp \eta_j \partial_j \big(\gamma \eta_k^\perp\big) \\
&\quad - \eta_k^\perp \eta_j \partial_j u_m \partial_m \eta_k \\
&= -\beta \kappa + \eta_j \partial_j \big(\gamma \eta_k^\perp\big) \eta_k^\perp \\
&= -\beta \kappa + \eta_j \partial_j \gamma \eta_k^\perp \eta_k^\perp + \eta \gamma \partial_j \eta_k^\perp \eta_k^\perp \\
&= -\beta \kappa + \eta \cdot \nabla \gamma,
\end{aligned}$$

where we have used $\eta^\perp \cdot \eta^\perp = 1$ and $\partial_j \eta^\perp \cdot \eta^\perp = 0$. This establishes (3.6). To show (3.7), we notice that $\nabla u = S + A$ and

$$A = \frac{1}{2} \begin{pmatrix} 0 & \Lambda\theta \\ -\Lambda\theta & 0 \end{pmatrix}.$$

Then a direct computation yields

$$\gamma(x,t) = \eta \cdot \big(\nabla u \eta^\perp\big) = \eta \cdot \big(S\eta^\perp\big) + \eta \cdot \big(A\eta^\perp\big) = -\lambda \sin(2\varphi) + \frac{1}{2}\Lambda\theta.$$

This completes the proof of Lemma 3.2. □

We present two more observations on the level curves of the SQG equation. The first one provides an upper bound for the signed mean curvature of any closed level curve, while the second reveals the curvature behavior of the orthogonal trajectories to the level curves. Consider a flow W transported by X, namely

$$W\big(X(a,t),t\big) = \nabla_a X(a,t) W_0(a). \tag{3.10}$$

Let $I \subset \mathbb{R}$ be an interval and $\{\gamma_0(s)\}_{s\in I}$ be a curve in $\mathbb{R}^n$ such that

$$\frac{\partial \gamma_0(s)}{\partial s} = W_0\big(\gamma_0(s)\big).$$

If we set $\gamma(s,t) = X(\gamma_0(s),t)$, then

$$\frac{\partial \gamma(s,t)}{\partial s} = \nabla_a X\big(\gamma_0(s),t\big) \frac{\partial \gamma_0(s)}{\partial s} = \nabla_a X\big(\gamma_0(s),t\big) W_0\big(\gamma_0(s)\big). \tag{3.11}$$

On the other hand, setting $a = \gamma_0(s)$ in (3.10), we obtain

$$W\big(\gamma(s,t),t\big) = \nabla_a X\big(\gamma_0(a),t\big) W_0\big(\gamma_0(s)\big). \tag{3.12}$$

Comparing (3.11) with (3.12), we find that

$$W\big(\gamma(s,t),t\big)=\frac{\partial\gamma(s,t)}{\partial s}. \tag{3.13}$$

Thus, we have deduced the fact that the curves γ are transported by W. In the case of the SQG equation, $W=\nabla^{\perp}\theta$ and the level curves are transported by $\nabla^{\perp}\theta$.

We now focus on the SQG equation, namely $W(x,t)=\nabla^{\perp}\theta(x,t)$. Let $\Omega(t)\subset\mathbb{R}^2$ be a domain with its boundary $\partial\Omega(t)$ given by a closed, regular level curve $\gamma(s,t)$ of $\theta(x,t)$. We apply the theorems of the differential geometry to obtain an upper bound for the signed mean curvature. More precisely, we have the following theorem.

Theorem 3.3 *Let $T\in(0,\infty]$. Let θ be a smooth solution of (1.1) on $[0,T)$. Let $\gamma(s,t)$ be a closed level curve of θ and let $\Omega(t)$ be the domain enclosed by $\gamma(s,t)$, which we assume to be simply connected. Let κ denote the signed curvature of γ as defined in (3.3). Then the signed mean curvature has the uniform bound*

$$\big\langle\kappa(\gamma,t)\big\rangle:=\frac{\oint_{\partial\Omega(t)}\kappa(\gamma(s,t),t)|\nabla^{\perp}\theta(\gamma(s,t),t)|\,\mathrm{d}s}{\oint_{\partial\Omega(t)}|\nabla^{\perp}\theta(\gamma(s,t),t)|\,\mathrm{d}s}\le\sqrt{\frac{\pi}{|\Omega(0)|}}, \tag{3.14}$$

where $0<t<T$ and N_r denotes the winding number of the initial level curve $\gamma_0(s)$ with respect to a point in $\Omega(0)$.

The proof uses the fact that any topologically invariant quantities involving the level curves are preserved in time along the flows, since the flow map $X(\cdot,t)$ is a diffeomorphism as long as the smoothness of solutions persists.

Proof The isoperimetric inequality implies

$$\begin{aligned}\oint_{\partial\Omega(t)}\big|\nabla^{\perp}\theta\big(\gamma(s,t),t\big)\big|\,\mathrm{d}s&=\oint_{\partial\Omega(t)}\left|\frac{\partial\gamma(s,t)}{\partial s}\right|\mathrm{d}s\\&\ge 2\sqrt{\pi\big|\Omega(t)\big|}=2\sqrt{\pi\big|\Omega(0)\big|},\end{aligned} \tag{3.15}$$

where $|A|$ denotes the area of the measurable set A. For a closed level curve $\gamma(s,t)$ with the signed curvature κ of $\theta(x,t)$ the winding number N_γ is given by

$$\oint_{\partial\Omega(t)}k\big(\gamma(s,t),t\big)\big|\nabla^{\perp}\theta\big(\gamma(s,t),t\big)\big|\,\mathrm{d}s=2\pi N_\gamma, \tag{3.16}$$

which is a topologically invariant. Under our assumption of a simply connected domain $\Omega(t)$ we have $N_\gamma=1$. The theorem follows from (3.15) with (3.16). □

Another observation is on the curvature behavior of the curves orthogonal to the level curves. If a finite-time singularity occurs in the solution of the SQG equation, then the curvature of the orthogonal curves to the levels must also blow up simultaneously.

Theorem 3.4 *Let $T > 0$ and let θ be a smooth solution of* (1.1) *on* $[0, T)$. *Consider a finite-time singularity scenario*: *there exist a sequence* $\{x_k\} \in \mathbb{R}^2$ *converging to* $\overline{x}$ *and a sequence* $\{t_k\}$ *converging to* T *such that*

$$\left|\nabla^\perp \theta(x_k, t_k)\right| \to \infty \quad \text{as } k \to \infty.$$

Then

$$\lim_{k\to\infty} \widetilde{\kappa}(x_k, t_k) = \infty \tag{3.17}$$

where $\widetilde{\kappa}$ *denotes the curvature of the orthogonal curves to the levels*.

Proof For each k, let $\gamma_k = \gamma_k(s, t_k)$ be the unique level curve of $\theta(x, t_k)$ that pass through x_k. Here s denotes the arc length and $x_k = \gamma_k(s_k, t_k)$. Using the identity

$$0 = \nabla \cdot \left(\nabla^\perp \theta\right) = \nabla \cdot \left(\eta\left|\nabla^\perp \theta\right|\right) = (\nabla \cdot \eta)\left|\nabla^\perp \theta\right| + \eta \cdot \nabla\left(\left|\nabla^\perp \theta\right|\right)$$

and the fact that $\eta \cdot \nabla = \frac{\partial}{\partial s}$, we have

$$\frac{\partial}{\partial s} \log\left|\nabla^\perp \theta\right|\left(\gamma_k(s, t_k), t_k\right) = -\nabla \cdot \eta\left(\gamma_k(s, t_k), t_k\right).$$

Integrating with respect to s from $\widetilde{s}$ to s_k, we find

$$\log\left|\nabla^\perp \theta\right|\left(\gamma_k(s_k, t_k), t_k\right) - \log\left|\nabla^\perp \theta\right|\left(\gamma(\widetilde{s}, t_k), t_k\right) = -\int_{\widetilde{s}}^{s_k} \nabla \cdot \eta\left(\gamma_k(\rho, t_k), t_k\right) \mathrm{d}\rho.$$

Realizing $\nabla \cdot \eta = \nabla \times \eta^\perp$ denotes the signed curvature of the orthogonal curves to the level curve, we obtain, by denoting that curvature by $\widetilde{\kappa}$,

$$\log\left|\nabla^\perp \theta\right|\left(\gamma_k(s_k, t_k), t_k\right) - \log\left|\nabla^\perp \theta\right|\left(\gamma(\widetilde{s}, t_k), t_k\right) = -\int_{\widetilde{s}}^{s_k} \widetilde{\kappa}\left(\gamma_k(\rho, t_k), t_k\right) \mathrm{d}\rho.$$

In the finite-time singularity scenario, the first term on the left tends to $-\infty$ as $k \to \infty$. When $\tilde{s}$ is taken sufficiently close to s_k, we obtain (3.17). This proves Theorem 3.4. □

4 The 3D Euler Equations with Spatial Symmetries

This section is focused on solutions of the 3D Euler equations with various reflection symmetries. The idea is to see if the global regularity issue can be better understood when there is a spatial symmetry. Seven different spatial symmetries are considered and in each case the simplified forms of the Euler equations on the invariant points of the symmetries are obtained. Regularity criteria based on these simplified equations are established. This section could be regarded as a generalized version of the results in Chae (2008a).

Given $(u(x,t), p(x,t))$ solving the 3D Euler equations (1.3), we introduce the 3×3 matrices

$$V_{ij} = \frac{\partial u_j}{\partial x_i}, \qquad S_{ij} = \frac{V_{ij}+V_{ji}}{2}, \qquad A_{ij} = \frac{V_{ij}-V_{ji}}{2}, \qquad P_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j},$$

with $i, j = 1, 2, 3$. Then we have the decomposition $V = (V_{ij}) = S + A$, where $S = (S_{ij})$ represents the rate of the deformation tensor of the fluid, and $A = (A_{ij})$ is related to the vorticity ω by

$$A_{ij} = \frac{1}{2}\sum_{k=1}^{3} \varepsilon_{ijk}\omega_k, \quad \omega_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_{jk}, \tag{4.1}$$

where ε_{ijk} is the skew-symmetric tensor with the normalization $\varepsilon_{123} = 1$. Note that $P = (P_{ij})$ is the Hessian of the pressure. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the set of eigenvalues of S. Computing partial derivatives $\partial/\partial x_k$ of (1.3) yields

$$\frac{\mathrm{D}V}{\mathrm{D}t} = -V^2 - P. \tag{4.2}$$

Taking the symmetric part of (4.2), we have

$$\frac{\mathrm{D}S}{\mathrm{D}t} = -S^2 - A^2 - P, \tag{4.3}$$

from which, using the formula (4.1), we derive

$$\frac{\mathrm{D}S_{ij}}{\mathrm{D}t} = -\sum_{k=1}^{3} S_{ik}S_{kj} + \frac{1}{4}\left(|\omega|^2\delta_{ij} - \omega_i\omega_j\right) - P_{ij}, \tag{4.4}$$

where δ_{ij} is the Kronecker delta defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. The antisymmetric part of (4.2) is

$$\frac{\mathrm{D}A}{\mathrm{D}t} = -SA - AS, \tag{4.5}$$

from which, using the formula (4.1) again, we easily obtain

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = S\omega, \tag{4.6}$$

which is the well-known vorticity evolution equation that could be derived also by taking the curl of (1.3). Taking the trace of (4.4), we have the identity

$$-\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) + \frac{1}{2}|\omega|^2 = \Delta p. \tag{4.7}$$

As a preparation, we define the reflection transforms

$$M_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \qquad M_2(x_1, x_2, x_3) = (x_1, -x_2, x_3),$$

$$
\begin{aligned}
&M_3(x_1,x_2,x_3)=(x_1,x_2,-x_3), \qquad M_{12}(x_1,x_2,x_3)=(-x_1,-x_2,x_3),\\
&M_{23}(x_1,x_2,x_3)=(x_1,-x_2,-x_3), \qquad M_{13}(x_1,x_2,x_3)=(-x_1,x_2,-x_3),\\
&M_{123}(x_1,x_2,x_3)=(-x_1,-x_2,-x_3).
\end{aligned}
$$

The rotation around the x_3-axis is represented by

$$R_\theta(x_1,x_2,x_3)=(\cos\theta x_1-\sin\theta x_2,\cos\theta x_2-\sin\theta x_1,x_3).$$

The rest of this section is divided into seven subsections with each one of them devoted to one of the reflection symmetries.

4.1 Reflection with Respect to the x_1x_2-Plane

Consider a solution (u,p) of (1.3) satisfying

$$
\begin{aligned}
&u_3(M_3x,t)=-u_3(x,t), \qquad u_j(M_3x,t)=u_j(x,t),\\
&p(M_3x,t)=p(x,t), \quad j=1,2.
\end{aligned}
\tag{4.8}
$$

We derive the reduced equations on the plane $x_3=0$:

$$
\begin{cases}
u_3(x_1,x_2,0,t)=\partial_3 p(x_1,x_2,0,t)=0,\\
\partial_t u_j+(u_1\partial_1+u_2\partial_2)u_j=-\partial_j p, \quad j=1,2,\\
\operatorname{div} u=0,
\end{cases}
$$

$$
S=\begin{pmatrix}
\partial_1u_1 & \frac{1}{2}(\partial_1u_2+\partial_2u_1) & 0\\
\frac{1}{2}(\partial_1u_2+\partial_2u_1) & \partial_2u_2 & 0\\
0 & 0 & \partial_3u_3
\end{pmatrix}, \qquad
\omega=\begin{pmatrix}0\\0\\\omega_3\end{pmatrix},
$$

$$
P=\begin{pmatrix}
\partial_1^2p & \partial_1\partial_2p & 0\\
\partial_1\partial_2p & \partial_2^2p & 0\\
0 & 0 & \partial_3^2p
\end{pmatrix}.
$$

The $(3,3)$ entry of the matrix equation (4.4) is

$$\partial_t\lambda_3+(u_1\partial_1+u_2\partial_2)\lambda_3=-\lambda_3^2-\partial_3^2p, \tag{4.9}$$

where $\lambda_3=\partial_3u_3$. The vorticity equation is

$$\partial_t\omega_3+(u_1\partial_1+u_2\partial_2)\omega_3=\lambda_3\omega_3.$$

Even for this reduced system of equations, it is not clear if all of their classical solutions are global in time. The following theorem explains a scenario for which a finite-time singularity develops.

Theorem 4.1 *Let $m > \frac{5}{2}$. Let $u_0 \in H^m(\mathbb{R}^3)$ and p_0 be given by*

$$p_0 = \sum_{j,k=1}^{3} \mathcal{R}_j \mathcal{R}_k (u_{0j} u_{0k}). \tag{4.10}$$

Assume that u_0 and p_0 obey the spatial symmetry as defined in (4.8). Define the set

$$S_0 = \left\{ a = (a_1, a_2, 0) \mid \partial_3 u_3(a) < 0, \partial_3^2 p_0(a) > 0 \right\}$$

and the turnover time

$$T_0(a) = \inf\left\{ t > 0 \mid \partial_3^2 p\big(X(a,t), t\big) < 0 \right\},$$

where $X(a,t)$ denotes the 2D particle trajectory $X = (X_1, X_2, 0)$ with

$$\frac{d}{dt} X_j(a,t) = u_j\big(X(a,t), t\big), \quad X_j(a,0) = a, \ a \in \mathbb{R}^2, \ j = 1, 2.$$

If there exists $a \in S_0$ such that

$$T_0(a) > -\frac{1}{\partial_3 u_3(a)}, \tag{4.11}$$

then $\lambda_3(X(a,t),t) = \partial_3 u_3(X(a,t),t)$ decreases to $-\infty$ in a finite time.

A special consequence of this theorem is a finite-time singularity result provided the pressure satisfies $\partial_3^2 p(x_1, x_2, 0, t) < 0$.

Corollary 4.2 *Let $m > \frac{5}{2}$. Let $u_0 \in H^m(\mathbb{R}^3)$. Assume that u_0 and p_0 obey the spatial symmetry as defined in (4.8). Assume that there is $a = (a_1, a_2, 0)$ such that $\partial_3 u_3(a) < 0$, $\partial_3^2 p_0(a) > 0$ and $\partial_3^2 p(X(a,t),t) > 0$ for $t > 0$. Then $\partial_3 u_3(X(a,t),t) \to -\infty$ as t approaches a finite time.*

We now provide the proof of Theorem 4.1.

Proof of Theorem 4.1 Noticing $\lambda_3 = \partial_3 u_3$ and (4.9), we have, for any $t < T_0$,

$$\partial_t \lambda_3\big(X(a,t),t\big) \le -\lambda_3^2\big(X(a,t),t\big).$$

Therefore, for any $t < \min\{T_0(a), -\frac{1}{\lambda_0(a)}\}$,

$$\lambda\big(X(a,t),t\big) \le \frac{\lambda_0(a)}{1 + \lambda_0(a) t}.$$

Thanks to (4.11), we conclude that $\lambda(X(a,t),t) = \partial_3 u_3(X(a,t),t)$ becomes $-\infty$ for $t \le -\frac{1}{\partial_3 u_3(a)}$. This completes the proof of Theorem 4.1. □

4.2 Rotation About the x_3-Axis for 180 Degrees

Consider a solution (u, p) of (1.3) satisfying, for $j = 1, 2$,

$$u_j(M_{12}x, t) = -u_j(x, t), \qquad u_3(M_{12}x, t) = u_3(x, t), \qquad p(M_{12}x, t) = p(x, t). \tag{4.12}$$

We derive the reduced equation on the x_3 axis:

$$\begin{cases} u_1(0, 0, x_3, t) = u_2(0, 0, x_3, t) = \partial_1 p(0, 0, x_3, t) = \partial_2 p(0, 0, x_3, t) = 0, \\ \partial_t u_3 + u_3 \partial_3 u_3 = -\partial_3 p, \\ \operatorname{div} u = 0, \end{cases}$$

$$S = \begin{pmatrix} \partial_1 u_1 & \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) & 0 \\ \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) & \partial_2 u_2 & 0 \\ 0 & 0 & \partial_3 u_3 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix},$$

$$P = \begin{pmatrix} \partial_1^2 p & \partial_1 \partial_2 p & 0 \\ \partial_1 \partial_2 p & \partial_2^2 p & 0 \\ 0 & 0 & \partial_3^2 p \end{pmatrix}.$$

The $(3, 3)$ entry of the matrix equation (4.4) is

$$\partial_t \lambda_3 + u_3 \partial_3 \lambda_3 = -\lambda_3^2 - \partial_3^2 p,$$

where $\lambda_3 = \partial_3 u_3(0, 0, x_3, t)$. The vorticity equation is

$$\partial_t \omega_3 + u_3 \partial_3 \omega_3 = \lambda_3 \omega_3.$$

A result similar to Theorem 4.1 can be established.

Theorem 4.3 *Let $m > \frac{5}{2}$. Let $u_0 \in H^m(\mathbb{R}^3)$ and p_0 be given by (4.10). Assume that u_0 and p_0 obey the spatial symmetry as defined in (4.12). Define the set*

$$S_0 = \{a = (0, 0, a_3) \mid \partial_3 u_3(a) < 0, \partial_3^2 p_0(a) > 0\}$$

and the turnover time

$$T_0(a) = \inf\{t > 0 \mid \partial_3^2 p\big(X(a, t), t\big) < 0\},$$

where $X(a, t)$ denotes the 1D particle trajectory $X = (0, 0, X_3)$ with

$$\frac{d}{dt} X_3(a, t) = u_3\big(X(a, t), t\big), \quad X(a, 0) = a.$$

If there exists $a \in S_0$ such that

$$T_0(a) > -\frac{1}{\partial_3 u_3(a)},$$

then $\lambda_3(X(a, t), t) = \partial_3 u_3(X(a, t), t)$ decreases to $-\infty$ in a finite time.

4.3 Reflection About the Origin

Consider a solution (u, p) of (1.3) satisfying

$$u_j(M_{123}x, t) = -u_j(x, t), \qquad p(M_{123}x, t) = p(x, t), \quad j = 1, 2, 3.$$

At the origin $(0, 0, 0)$ the reduced equation becomes

$$\begin{cases} u_j(0, 0, 0, t) = \partial_j p(0, 0, 0, t) = 0 \quad \forall j = 1, 2, 3, \\ \operatorname{div} v = 0, \end{cases}$$

$$\partial_t S_{ij} = -\sum_{k=1}^{3} S_{ik}S_{kj} + \frac{1}{4}\big(|\omega|^2\delta_{ij} - \omega_i\omega_j\big) - P_{ij},$$

$$\partial_t \omega = S\omega,$$

where

$$S_{ij} = S_{ij}(0, 0, 0, t), \qquad \omega = \omega(0, 0, 0, t), \qquad P_{ij} = P_{ij}(0, 0, 0, t).$$

4.4 Reflections with Respect to Two Planes

Consider a solution (u, p) of (1.3) satisfying

$$\begin{aligned} &u_i(M_jx, t) = (-1)^{\delta_{ij}}u_i(x, t), \qquad u_3(M_jx, t) = u_3(x, t), \\ &p(M_jx, t) = p(x, t), \quad i, j = 1, 2. \end{aligned}$$

On the x_3-axis the reduced equations are

$$\begin{cases} u_j(0, 0, x_3, t) = \partial_j p(0, 0, x_3, t) = 0 \quad \forall j = 1, 2. \\ \partial_t u_3 + u_3\partial_3 u_3 = -\partial_3 p, \\ \operatorname{div} u = 0, \end{cases} \tag{4.13}$$

$$S = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \omega = 0, \qquad P = \operatorname{diag}\big(\partial_1^2 p, \partial_2^2 p, \partial_3^2 p\big)$$

where

$$\lambda_j = \partial_j u_j, \quad j = 1, 2, 3, \tag{4.14}$$

on the x_3-axis. The $(1, 1)$ and $(2, 2)$ entries of the matrix equation (4.4) reduce to

$$\partial_t\lambda_j + u_3\partial_3\lambda_j = -\lambda_j^2 - \partial_j^2 p, \quad j = 1, 2, 3.$$

We note that (4.7) reduces to

$$\Delta p = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2. \tag{4.15}$$

4.5 Reflection and Rotation About the x_3-Axis

Consider a solution (u, p) of (1.3) satisfying

$$u_j(M_{12}x, t) = -u_j(x, t), \qquad u_3(M_{12}x, t) = u_3(x, t),$$
$$u_j(M_3x, t) = u_j(x, t), \qquad u_3(M_3x, t) = -u_3(x, t),$$
$$p(M_{12}x, t) = p(x, t), \qquad p(M_3x, t) = p(x, t) \quad j = 1, 2.$$

The origin $(0, 0, 0)$ is the invariant point in this case. We have

$$\begin{cases} u_j(0, 0, 0, t) = \partial_j p(0, 0, 0, t) = 0 & \forall j = 1, 2, 3, \\ \operatorname{div} v = 0, \end{cases}$$
$$S = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \omega = 0, \qquad P = \operatorname{diag}\bigl(\partial_1^2 p, \partial_2^2 p, \partial_3^2 p\bigr) \tag{4.16}$$

where

$$\lambda_j = \partial_j u_j, \quad j = 1, 2, 3, \tag{4.17}$$

on the x_3-axis. The $(1, 1)$ and $(2, 2)$ entries of the matrix equation (4.4) reduce to

$$\partial_t \lambda_j = -\lambda_j^2 - \partial_j^2 p, \quad j = 1, 2, 3.$$

We note that (4.7) reduces to

$$\Delta p = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2. \tag{4.18}$$

4.6 Reflection About the Axes

Consider a solution (u, p) of (1.3) satisfying

$$u_i(M_jx, t) = (-1)^{\delta_{ij}} u_i(x, t), \qquad p(M_jx, t) = p(x, t), \quad i, j = 1, 2, 3.$$

The origin $(0, 0, 0)$ is the invariant point. The reduced equation at the origin is

$$\begin{cases} u_j(0, 0, 0, t) = \partial_j p(0, 0, 0, t) = 0 & \forall j = 1, 2, 3 \\ \operatorname{div} u = 0 \end{cases}$$
$$S = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \omega = 0, \qquad P = \operatorname{diag}\bigl(\partial_1^2 p, \partial_2^2 p, \partial_3^2 p\bigr) \tag{4.19}$$

where

$$\lambda_j = \partial_j u_j, \quad j = 1, 2. \tag{4.20}$$

The $(1, 1)$ and $(2, 2)$ entries of the matrix equation (4.4) reduce to

$$\partial_t \lambda_j = -\lambda_j^2 - \partial_j^2 p, \quad j = 1, 2, 3.$$

We note that (4.7) reduces to

$$\Delta p = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2. \tag{4.21}$$

4.7 Rotation About the x_3-Axis

Consider a solution (u, p) of (1.3) satisfying

$$(u_1, u_2)(R_\theta x, t) = R_\theta(u_1, u_2)(x, t), \quad u_3(M_3 x, t) = -u_3(x, t),$$

where $\theta \in [0, 2\pi)$. At the origin $(0, 0, 0)$ the reduced equation is

$$\begin{cases} u_j(0,0,0,t) = \partial_j p(0,0,0,t) = 0 \quad \forall j = 1, 2, 3, \\ 2\partial_r u_r + \partial_3, u_3 = 0, \end{cases}$$
$$S = \operatorname{diag}\left(-\frac{1}{2}\lambda_3, -\frac{1}{2}\lambda_3, \lambda_3\right), \qquad \omega = (0, 0, \omega_3), \qquad P = \operatorname{diag}\left(\partial_1^2 p, \partial_2^2 p, \partial_3^2 p\right) \tag{4.22}$$

where

$$\lambda_3 = \partial_3 u_3, \quad j = 1, 2, 3. \tag{4.23}$$

The $(1, 1)$ and $(2, 2)$ entries of the matrix equation (4.4) reduce to

$$\partial_t \lambda_3 = \frac{1}{2}\lambda_3^2 - \frac{1}{2}\omega_3^2 + 2\partial_r^2 p, \quad j = 1, 2, 3.$$

The vorticity equation is

$$\partial_t \omega = \lambda_3 \omega.$$

We note that (4.7) reduces to

$$\Delta p = -\frac{3}{2}\lambda_3^2 + \omega_3^2. \tag{4.24}$$

5 Conclusion

The global regularity issue on the inviscid SQG equation or the 3D Euler equations is extremely difficult and no effective analytic approach is currently available. Numerical simulations have revealed some very significant features of geometric quantities associated with the level curves of the SQG equation and the vortex tubes of the 3D Euler equations in potential finite-time singularity scenarios. Geometric regularity criteria have previously been derived to reflect the nature of the numerical observations. These criteria are mainly expressed in terms of the tangent directions (see, e.g., Constantin et al. 1994, 1996; Deng et al. 2005). This paper examines and exploits the relationship between what we call the Eulerian and the Lagrangian deformations and present geometric observations rigorously relating potential finite-time singularities to the curvatures and the stretching directions characterized through the Lagrangian and Euler deformations. In addition, solutions of the 3D Euler equations with special spatial symmetries are investigated and the regularity criteria obtained here reveal some potential finite-time singularities in the circumstances when the pressure obeys certain properties.

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Appendix: An Example of Lagrangian and Eulerian Deformations

This appendix presents a special steady solution of the SQG equation and explicit computations of the eigenvalues and eigenvectors of the associated Lagrangian and Eulerian deformations.

If ψ is a 2D radial function, $\psi(x) = \psi(r)$ with $r = |x|$ and $u = \nabla^\perp \psi$, then $\theta = \Lambda\psi$ is a steady solution of the SQG equation. Let X and A be the particle trajectory and let the back-to-label map be determined by u, respectively. Let S be the deformation tensor (or Eulerian deformation) and $\widetilde{S} = (\nabla_a X)^*(\nabla_a X)$ be the Lagrangian deformation. We compute and compare the eigenvalues and eigenvectors of S and M.

It is clear that $u(x,t) = r^{-1}\psi'(r)x^\perp$, where $x^\perp = (-x_2, x_1)$. Then

$$S = \frac{1}{2}\big((\nabla u)^* + \nabla u\big) = \tau \begin{pmatrix} -\hat{x}_1\hat{x}_2 & \frac{1}{2}(\hat{x}_1 - \hat{x}_2) \\ \frac{1}{2}(\hat{x}_1 - \hat{x}_2) & \hat{x}_1\hat{x}_2 \end{pmatrix}$$

where $\tau = r(\psi'(r)/r)'$ and $\hat{x} = (\hat{x}_1, \hat{x}_2) = r^{-1}x$ denotes the unit vector in the direction of x. Using the notation of the tensor product $a \otimes b = (a_i b_j)$, we can write S as

$$S = \frac{\tau}{2}\big(\hat{x} \otimes \hat{x}^\perp + \hat{x}^\perp \otimes \hat{x}\big)$$

and the eigenvalues and eigenvectors are

$$\lambda_1 = \frac{\tau}{2}, \qquad \lambda_2 = -\frac{\tau}{2}, \qquad \mathbf{w}_1 = \frac{1}{\sqrt{2}}\big(\hat{x} + \hat{x}^\perp\big), \qquad \mathbf{w}_2 = \frac{1}{\sqrt{2}}\big(\hat{x} - \hat{x}^\perp\big).$$

To compute $\widetilde{S}$, we notice that the Lagrangian map X satisfies

$$\partial_t X = u(X,t), \quad X(a,0) = a.$$

Since $u(x,t) = r^{-1}\psi'(r)x^\perp$, $r = |X| = |a|$ is conserved and

$$X(a,t) = R\big(t, |a|\big)a = a_1 e_1 + a_2 e_2$$

where $R(t,|a|)$ is the rotation of angle $\Theta = t\psi'(r)/r$ with the columns given by $e_1 = (\cos\Theta, \sin\Theta)^*$ and $e_2 = (-\sin\Theta, \cos\Theta)^*$. Computing the vectors $\partial_1 X = \frac{\partial X}{\partial a_1}$ and $\partial_2 X = \frac{\partial X}{\partial a_2}$, we obtain

$$\partial_1 X = (1 - s\hat{a}_1\hat{a}_2)e_1 + s\hat{a}_1^2 e_2, \qquad \partial_2 X = -s\hat{a}_2^2 e_1 + (1 + s\hat{a}_1\hat{a}_2)e_2,$$

where s is the nondimensional time $s = r\Theta' = r(\psi'(r)/r)'t$ and $\hat{a} = |a|^{-1}a$. The entry $\widetilde{S}_{ij}$ of $\widetilde{S}$ is given by $\partial_i X \cdot \partial_j X$ and thus

$$\widetilde{S} = \begin{pmatrix} 1 - 2s\hat{a}_1\hat{a}_2 + s^2\hat{a}_1^2 & s(\hat{a}_1^2 - \hat{a}_2^2) + s^2\hat{a}_1\hat{a}_2 \\ s(\hat{a}_1^2 - \hat{a}_2^2) + s^2\hat{a}_1\hat{a}_2 & 1 + 2s\hat{a}_1\hat{a}_2 + s^2\hat{a}_2^2 \end{pmatrix}.$$

The eigenvalues of $\widetilde{S}$ are

$$\mu_1 = 1 + \frac{s^2}{2} + \frac{s}{2}\sqrt{s^2+4}, \qquad \mu_2 = 1 + \frac{s^2}{2} - \frac{s}{2}\sqrt{s^2+4}$$

and the corresponding eigenvectors are given by

$$\mathbf{v}_1 = \hat{a} - \frac{1}{2}\big(s - \sqrt{s^2+4}\big)\hat{a}^{\perp}, \qquad \mathbf{v}_2 = \hat{a} - \frac{1}{2}\big(s + \sqrt{s^2+4}\big)\hat{a}^{\perp}.$$

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