



Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion

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HIGHLIGHTS

- We study the 2D Boussinesq equations without buoyancy diffusion.
- We establish the global-in-time existence and uniqueness of classical solutions.
- We study linear stability and instability of some stratified hydrostatic equilibria.
- Direct numerical simulations corroborate analytical results.

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ABSTRACT

We study the global well-posedness and stability/instability of perturbations near a special type of hydrostatic equilibrium associated with the 2D Boussinesq equations without buoyancy (a.k.a. thermal) diffusion on a bounded domain subject to stress-free boundary conditions. The boundary of the domain is not necessarily smooth and may have corners such as in the case of rectangles. We achieve three goals. First, we establish the global-in-time existence and uniqueness of large-amplitude classical solutions. Efforts are made to reduce the regularity assumptions on the initial data. Second, we obtain the large-time asymptotics of the full nonlinear perturbation. In particular, we show that the kinetic energy and the first order derivatives of the velocity field converge to zero as time goes to infinity, regardless of the magnitude of the initial data, and the flow stratifies in the vertical direction in a weak topology. Third, we prove the linear stability of the hydrostatic equilibrium $\mathcal{T}(y)$ satisfying $\mathcal{T}'(y) = \alpha > 0$, and the linear instability of periodic perturbations when $\mathcal{T}'(y) = \alpha < 0$. Numerical simulations are supplemented to corroborate the analytical results and predict some phenomena that are not proved.

The authors are pleased to dedicate this paper to Professor Edriss Saleh Titi on the occasion of his 60th birthday. Professor Titi's myriad research contributions – including contributions to the problem constituting the focus of this work – and leadership in the mathematical fluid dynamics community serve as an inspiration to his students, collaborators and colleagues.

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1. Introduction

1.1. Background

The two-dimensional (2D) incompressible Boussinesq equations describing the motion of buoyancy driven fluid flows are

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu_x \partial_{xx} \mathbf{u} + \nu_y \partial_{yy} \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa_x \partial_{xx} \theta + \kappa_y \partial_{yy} \theta, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (1.1)$$

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where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $t > 0$ and $\mathbf{u} = (u, v)^T$, P , θ denote, respectively, the velocity, pressure and buoyancy fields. The vertical unit vector is $\mathbf{e}_2 = (0, 1)^T$ and $\nu_x, \nu_y, \kappa_x, \kappa_y$ are the (real non-negative) viscosity and buoyancy diffusion coefficients. The acceleration term $\theta \mathbf{e}_2$ in the momentum equation, the first equation in (1.1), models the buoyancy due to density variations in presence of gravity. Such density variations are often due to thermal expansion so we will interchangeably refer to θ as a temperature field and its evolution, the second equation in (1.1) as the temperature equation.

This system is routinely used to model flows across a tremendous range of length and time scales from microfluidics and biophysics to geodynamics and astrophysics. It plays an important

role in the study of turbulence and, suitably modified, in atmospheric and oceanographic situations where rotation and stratification play a dominant role [1–3]. It is also known for its close connection with fundamental models in mathematical fluid mechanics:

• **Analogy to 3D Euler Equations.** One important feature of the non-dissipative ($\nu_x = \nu_y = \kappa_x = \kappa_y = 0$) 2D Boussinesq system is its analogy to the 3D incompressible Euler equations. Indeed, the vorticity formulation of the 3D incompressible Euler equations for axisymmetric swirling flow in cylindrical coordinates, (r, ϕ, x_3) , reads

$$\begin{cases} \partial_t \left(\frac{\omega^\phi}{r} \right) + v^r \partial_r \left(\frac{\omega^\phi}{r} \right) + v^3 \partial_{x_3} \left(\frac{\omega^\phi}{r} \right) = -\frac{1}{r^4} \partial_{x_3} [(rv^\phi)^2], \\ \partial_t [(rv^\phi)^2] + v^r \partial_r [(rv^\phi)^2] + v^3 \partial_{x_3} [(rv^\phi)^2] = 0, \end{cases}$$

where ω^ϕ denotes the angular vorticity in cylindrical coordinates. On the other hand, the vorticity formulation of the 2D non-dissipative Boussinesq system in Cartesian coordinates takes the form

$$\begin{cases} \partial_t \omega + v \partial_y \omega + u \partial_x \omega = \partial_x \theta, \\ \partial_t \theta + v \partial_y \theta + u \partial_x \theta = 0, \end{cases}$$

where $\omega = \partial_x v - \partial_y u$ denotes the 2D vorticity in Cartesian coordinates. In view of the two systems we see the correspondence:

$$\begin{aligned} \frac{\omega^\phi}{r} &\longleftrightarrow \omega, & v^r &\longleftrightarrow v, & v^3 &\longleftrightarrow u, & r &\longleftrightarrow y, \\ x_3 &\longleftrightarrow x, & (rv^\phi)^2 &\longleftrightarrow -\theta. \end{aligned}$$

Therefore, away from the line of singularity ($r = 0$), one should expect that the qualitative behavior of the 3D incompressible Euler equations for axisymmetric swirling flow is identical to that of the 2D non-dissipative Boussinesq equations.

• **Vortex Stretching Effect.** A second important feature of the 2D Boussinesq system is that it shares a similar vortex stretching effect as that in 3D flows. As a matter of fact, by letting $\mathbf{W} = \nabla \theta = (\partial_x \theta, \partial_y \theta)^T$, one can show that for classical solutions, (1.1) with $\nu_x = \nu_y = \kappa_x = \kappa_y = 0$ is equivalent to

$$\begin{cases} \partial_t \omega + \mathbf{u} \cdot \nabla \omega = \partial_x \theta, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{W} + \mathbf{u} \cdot \nabla \mathbf{W} = -(\nabla \mathbf{u}^T) \mathbf{W}, \end{cases}$$

where $(\nabla \mathbf{u}^T) = (\nabla u \nabla v)$ is a 2×2 matrix. The first equation indicates that qualitatively the growth of the vorticity depends on the temporal accumulation of $\partial_x \theta$ which is the first component of \mathbf{W} . From the third equation we see that \mathbf{W} has the same degree of regularity as $\nabla \mathbf{u}$. Therefore, qualitatively the growth of the vorticity depends on the temporal accumulation of $\nabla \mathbf{u}$ – a scenario that is similar to the vortex stretching effect in the 3D incompressible Euler equations. In fact, in view of the vorticity equation of the 3D incompressible Euler equations:

$$\partial_t \Sigma + \mathbf{U} \cdot \nabla \Sigma = \Sigma \cdot \nabla \mathbf{U},$$

where \mathbf{U} is the 3D velocity and $\Sigma = \nabla \times \mathbf{U}$, we see that the growth of the vorticity depends on the temporal accumulation of the gradient of the velocity field, as well.

1.2. Literature review

Due to its physical background and mathematical features, the 2D Boussinesq system has attracted considerable attention from the community of mathematical fluid mechanics in recent years. The following results concerning the qualitative behavior (global/local well-posedness, blowup criteria, regularity, explicit solutions, finite-time singularities, etc.) of (1.1) have been well documented in the literature:

- global well-posedness, blowup criteria and regularity when $\nu_x, \nu_y, \kappa_x, \kappa_y$ are not all zeros [4–26],
- local well-posedness, blowup criteria, explicit solutions and finite-time singularities when all four parameters are zeros [27–36],
- well-posedness and regularity with critical and supercritical dissipation [37–43].

Comparing with the magnitude of research conducted on the well-posedness of the model, the long-time asymptotic behavior of (1.1) has been investigated relatively little. To the authors' knowledge, the following results have been achieved so far:

- exponential decay of θ to constant states and uniform-in-time boundedness of kinetic energy for initial-boundary value problems on 2D bounded smooth domains for large data when $\nu_x = \nu_y = 0, \kappa_x = \kappa_y = \kappa > 0$ [26],
- uniform-in-time boundedness of kinetic energy for initial-boundary value problems on 2D bounded smooth domains for large data when $\nu_x = \nu_y = \nu > 0, \kappa_x = \kappa_y = 0$ [22],
- algebraic decay of (\mathbf{u}, θ) to constant ground states for the Cauchy problem on \mathbb{R}^3 for small data when $\nu_x = \nu_y = \nu > 0, \kappa_x = \kappa_y = \kappa > 0$ [44].
- long time averaged heat transport sustained by thermal boundary conditions, i.e., bounds for Rayleigh–Bénard convection [45].
- existence of a global attractor containing infinitely many invariant manifolds on 2D periodic domains when $\nu_x = \nu_y = \nu > 0, \kappa_x = \kappa_y = 0$ [46].

1.3. Motivation and goal

The 2D Boussinesq system describes the motion of buoyancy driven flows. The hydrostatic equilibria associated with buoyancy driven flows take the form $[\mathbf{0}, P_{\text{he}}(y), \theta_{\text{he}}(y)]$ where the vertically stratified pressure and temperature satisfy $\partial_y P_{\text{he}}(y) = \theta_{\text{he}}(y)$ obtained from the first equation of (1.1) by setting $\mathbf{u} = \mathbf{0}$. Moreover, when the thermal diffusion coefficients $\kappa_x, \kappa_y > 0$, by formally substituting the ansatz into the second equation of (1.1), we see that $\partial_{yy} \theta_{\text{he}}(y) = 0$ which suggests that the equilibrium temperature is an affine function of the depth. Mathematically, it is then natural to ask whether such kind of ansatz is stable under appropriate initial and boundary conditions.

When the dissipation parameters satisfy $\nu_x = \nu_y = \nu > 0$ and $\kappa_x = \kappa_y = \kappa > 0$, it is not difficult to check that when (1.1) is set on a bounded domain with sufficiently smooth boundary and subject to the boundary conditions: $\theta|_{\partial\Omega} = \alpha y + \bar{\theta}$ & $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ (no-flow), or $\theta|_{\partial\Omega} = \alpha y + \bar{\theta}$ & $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \omega|_{\partial\Omega} = 0$ (stress-free), where $\omega = \partial_x v - \partial_y u$ denotes the 2D vorticity, the ansatz $(\mathbf{u}_\infty, \theta_\infty) = (\mathbf{0}, \alpha y + \bar{\theta})$ is globally asymptotically stable, provided $\alpha > 0$. Indeed, the global asymptotic stability follows from the energy conservation law:

$$\begin{aligned} \frac{d}{dt} (\alpha \|\mathbf{u}\|_{L^2}^2 + \|\theta - \alpha y - \bar{\theta}\|_{L^2}^2) + \alpha \nu \|\omega\|_{L^2}^2 \\ + \kappa \|\nabla(\theta - \alpha y - \bar{\theta})\|_{L^2}^2 = 0, \end{aligned} \tag{1.2}$$

and the Poincaré inequality.

On the other hand, in certain circumstances when the thermal diffusion is insignificant, (1.1) becomes

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu_x \partial_{xx} \mathbf{u} + \nu_y \partial_{yy} \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.3}$$

which arises naturally as a relevant system in geophysics [47–49]. Comparing with (1.1), the system (1.3) is only partially dissipative

with missing dissipation in the second major function of interest. Such a degeneracy, together with the nonlinear coupling through advection and gravity, makes the mathematical analysis of (1.3) considerably more challenging than that for (1.1). Although the global well-posedness of (1.3) has been proved under various initial and/or boundary conditions, the long-time behavior is still elusive. So far the only such results can be found in [22,46], in which it was shown that the kinetic energy associated with (1.3) on bounded domains is uniformly bounded with respect to time, which indicates that the flow is trapped inside of a finite ball in the energy space. Moreover, it was shown that (1.3) has a so-called weak sigma-attractor which contains infinitely many invariant manifolds in which several universal properties of the Batchelor, Kraichnan, Leith theory of turbulence are potentially present. However, we note that the results in [22,46] do not imply the unconditional “slowing down” of the flow, and therefore provide no information about the global stability of hydrostatic equilibria associated with (1.3).

The objective of this paper is to study the global well-posedness of (1.3) on 2D bounded domains with non-smooth boundaries for large data, and investigate the stability/instability of hydrostatic equilibria associated with the partially-dissipative system. The spatial domain $\Omega \subset \mathbb{R}^2$ under consideration is either a rectangle or more generally a connected bounded domain with Lipschitz boundary satisfying some minor constraints that we are specifying now. Since $\partial\Omega$ concerned here is not smooth and may have corners such as in the case of rectangles, we impose additional conditions on Ω (in addition to $\partial\Omega$ being Lipschitz) so that some of the classical analysis tools such as the Calderón–Zygmund inequality are still valid for such domains. Jerison and Kenig in [50] give a striking example in which the domain Ω has C^1 boundary and the Calderón–Zygmund inequality fails. We focus on bounded and simply connected planar domains with $\partial\Omega$ belonging to $C^{2,\gamma}$ ($\gamma \in (0, 1)$) except at a finite number of points, where $\partial\Omega$ is a corner of angle in $(0, \frac{\pi}{2}]$. Planar polygonal domains such as rectangles in \mathbb{R}^2 are special examples of such domains. As shown in [51–53], the Calderón–Zygmund inequality and some other tools still hold for such domains.

This paper attempts to achieve three main goals. The first is to establish the global-in-time existence and uniqueness of classical solutions to the following initial–boundary value problem:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ (\mathbf{u}, \theta)(\mathbf{x}, 0) = (\mathbf{u}_0, \theta_0)(\mathbf{x}), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^2$ is as specified in the previous paragraph, \mathbf{n} is the unit outward normal to $\partial\Omega$, and $\omega = \partial_x v - \partial_y u$ is the 2D vorticity. A particular type of such domains are the rectangles. (1.4) in the whole space \mathbb{R}^2 or in a smooth bounded domain with the Dirichlet boundary condition has been studied previously and important results on the well-posedness have been obtained (see, e.g., [10,19,46,21,22,23]). In our study presented in this paper, we attempt to make weak regularity assumptions on the initial data. In addition, the boundary of the domain here is not smooth.

Motivated by the reasoning for the fully dissipative system (see (1.2) and the paragraph therein), the second goal is to understand the large time behavior and the global stability of the perturbation near the hydrostatic equilibrium $[\mathbf{u}_{\text{he}}, P_{\text{he}}(y), \theta_{\text{he}}(y)]$ given by

$$\mathbf{u}_{\text{he}} = \mathbf{0}, \quad \theta_{\text{he}}(y) = \mathcal{T}(y) \equiv \alpha y + \bar{\theta}, \quad P_{\text{he}} = \int_0^y \mathcal{T}(z) dz.$$

The perturbation

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{\text{he}}, \quad \tilde{P} = P - P_{\text{he}}(y), \quad \tilde{\theta} = \theta - \theta_{\text{he}}(y) \quad (1.5)$$

satisfies

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \nabla \tilde{P} = \nu \Delta \tilde{\mathbf{u}} + \tilde{\theta} \mathbf{e}_2, \\ \partial_t \tilde{\theta} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\theta} + \tilde{v} \mathcal{T}'(y) = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0; \\ (\tilde{\mathbf{u}}, \tilde{\theta})(\mathbf{x}, 0) = (\tilde{\mathbf{u}}_0, \tilde{\theta}_0)(\mathbf{x}), \\ \tilde{\mathbf{u}} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \tilde{\omega}|_{\partial\Omega} = 0, \end{cases} \quad (1.6)$$

where $\tilde{\omega} = \nabla \times \tilde{\mathbf{u}}$ denotes the corresponding vorticity.

The third goal is to prove the linear stability of the hydrostatic equilibrium when the vertically stratified temperature $\mathcal{T}(y)$ satisfies $\mathcal{T}'(y) = \alpha > 0$ and the linear instability when $\mathcal{T}'(y) = \alpha < 0$. In order to achieve this goal, we first derive an equivalent system of equations for the perturbation near the hydrostatic equilibrium (see (4.7) below) and the corresponding linearized system (4.8).

1.4. Results

We now state our main results. Before doing so, we introduce some notations for convenience.

Notation 1.1. Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and the usual Sobolev space $W^{s,p}(\Omega)$, respectively. For $p = 2$, we denote the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|$ and $\|\cdot\|_{H^s}$, respectively. Unless specified, c_i denote generic constants which are independent of the unknown functions and t , but may depend on ν , Ω and initial data. The values of the constants may vary line by line according to the context.

We present three main results. The first one assesses the global-in-time existence and uniqueness of solutions to the initial–boundary value problem (1.4). The second one describes the large time behavior of the nonlinear perturbation while the third establishes the linear stability when the vertically stratified temperature $\mathcal{T}(y)$ satisfies $\mathcal{T}'(y) = \alpha > 0$ and the linear instability for periodic perturbations when $\mathcal{T}'(y) = \alpha < 0$.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ be a rectangle or a more general bounded and simply connected planar domain with $\partial\Omega$ belonging to $C^{2,\gamma}$ ($\gamma \in (0, 1)$) except at a finite number of points, where $\partial\Omega$ is a corner of angle in $(0, \frac{\pi}{2}]$. Consider the initial–boundary value problem (1.4). Suppose that the initial data $\mathbf{u}_0 \in H^2(\Omega)$ and $\theta_0 \in H^1(\Omega)$, $\nabla \cdot \mathbf{u}_0 = 0$ and are compatible with the boundary conditions. Then (1.4) has a unique global-in-time solution (\mathbf{u}, θ) satisfying

$$\begin{aligned} \mathbf{u} &\in L^\infty((0, t); H^2(\Omega)) \cap L^2((0, t); H^3(\Omega)) \quad \text{and} \\ \theta &\in L^\infty((0, t); H^1(\Omega)) \end{aligned} \quad (1.7)$$

for any $t > 0$. In addition, if $(\mathbf{u}_0, \theta_0) \in H^3(\Omega)$, then $\mathbf{u} \in L^\infty((0, t); H^3(\Omega)) \cap L^2((0, t); H^4(\Omega))$ and $\theta \in L^\infty((0, t); H^3(\Omega))$ for any $t > 0$.

Our second main result focuses on the large time behavior of solutions to (1.6), the full nonlinear equations for the perturbation $[\tilde{\mathbf{u}}, \tilde{P}, \tilde{\theta}]$.

Theorem 1.2. Assume $\Omega \subset \mathbb{R}^2$ is either a rectangle or a more general Lipschitz domain as specified previously. Consider the initial–boundary value problem (1.6) for the perturbation $[\tilde{\mathbf{u}}, \tilde{P}, \tilde{\theta}]$. Assume the initial perturbations $\tilde{\mathbf{u}}_0 \in H^2(\Omega)$, $\tilde{\theta}_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $\nabla \cdot \tilde{\mathbf{u}}_0 = 0$, and are compatible with the boundary conditions. Then (1.6) possesses a unique and global-in-time solution $(\tilde{\mathbf{u}}, \tilde{\theta})$ satisfying

$$\begin{aligned} \tilde{\mathbf{u}} &\in L^\infty((0, t); H^2(\Omega)) \cap L^2((0, t); H^3(\Omega)), \\ \tilde{\theta} &\in L^\infty((0, t); H^1(\Omega) \cap L^\infty(\Omega)). \end{aligned}$$

Furthermore, if $\mathcal{T}'(y) = \alpha > 0$, the solution $(\tilde{\mathbf{u}}, \tilde{\theta})$ obeys the following large-time behavior, as $t \rightarrow \infty$,

$$\begin{aligned} & \bullet \|\tilde{\mathbf{u}}(t)\|_{L^2} \rightarrow 0, \quad \|\nabla \tilde{\mathbf{u}}(t)\|_{L^2} \rightarrow 0, \quad \|\partial_t \tilde{\mathbf{u}}(t)\| \rightarrow 0, \\ & \bullet \|\tilde{\theta}(t)\| \rightarrow c_0, \quad 0 < c_0 < \sqrt{\alpha \|\tilde{\mathbf{u}}_0\|^2 + \|\tilde{\theta}_0\|^2}, \\ & \bullet \|\nabla \tilde{P} - \tilde{\theta} \mathbf{e}_2\|_{H^{-1}} \rightarrow 0, \end{aligned} \tag{1.8}$$

and there exists a positive constant C , which is independent of $t > 0$, such that

$$\|\tilde{\mathbf{u}}(t)\|_{H^2} \leq C, \quad \forall t > 0. \tag{1.9}$$

Remark 1.1. We remark that by combining the uniform estimate (1.9) with the arguments in [46], one can establish the existence of global attractors in stronger topological spaces, as was pointed out in [46].

The third main result assesses the linear stability when the vertically stratified temperature $\mathcal{T}(y)$ satisfies $\mathcal{T}'(y) = \alpha > 0$ and the linear instability for periodic perturbations when $\mathcal{T}'(y) = \alpha < 0$. As a preparation, we derive an equivalent system of equations for the perturbation near the hydrostatic equilibrium and the corresponding linearized system. In contrast to the periodic case or the whole space case, this process is more involved due to the nonsmoothness of the domain. We invoke a result asserting the existence and boundedness of the operator projecting onto divergence-free vector fields for bounded convex domains (see [51,54]).

Denote by $\mathcal{D}(\Omega)$ the space of smooth vector fields compactly supported on Ω . Let

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega) : \nabla \cdot \mathbf{u} = 0\}.$$

Let $L^p(\Omega)$ with $1 < p < \infty$ denote the standard Lebesgue space and set

$$L^p_\sigma(\Omega) = \{\mathbf{u} \in L^p(\Omega), \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Alternatively, $L^p_\sigma(\Omega)$ is the closure of \mathcal{V} in $L^p(\Omega)$. Let $P_\Omega : L^2(\Omega) \rightarrow L^2_\sigma(\Omega)$ be the orthogonal projector. According to Theorem 1.1.4 of [55], for any Lipschitz domain Ω , P_Ω is well defined. The following lemma [54] states that P_Ω can be extended to general $L^p(\Omega)$.

Lemma 1.3. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let $1 < p < \infty$. Then

- (1) P_Ω can be extended to a bounded linear operator from $L^p(\Omega)$ to $L^p_\sigma(\Omega)$ and

$$\|P_\Omega \mathbf{u}\|_{L^p} \leq C(\Omega, p) \|\mathbf{u}\|_{L^p}.$$

In addition,

$$P_\Omega \mathbf{u} = \mathbf{u} \text{ for any } \mathbf{u} \in L^p_\sigma(\Omega). \tag{1.10}$$

- (2) For any $\mathbf{u} \in L^p(\Omega)$, there exists a unique $\pi \in W^{1,p}(\Omega)$ (up to an additive constant) such that

$$\mathbf{u} = P_\Omega \mathbf{u} + \nabla \pi.$$

Additionally, in the weak sense, $\Delta \pi = \nabla \cdot \mathbf{u}$, which allows us to write

$$\pi = \Delta^{-1} \nabla \cdot \mathbf{u}, \tag{1.11}$$

where π 's different by an additive constant are regarded the same. In addition,

$$\|\nabla \pi\|_{L^p} \leq C(\Omega, p) \|\mathbf{u}\|_{L^p}.$$

With this lemma at our disposal, we can now state the linear stability and instability results.

Theorem 1.4. Assume the spatial domain $\Omega \subset \mathbb{R}^2$ is as specified in Theorem 1.2. Let $(\tilde{\mathbf{u}}, \tilde{P}, \tilde{\theta})$ denote the perturbation as stated in Theorem 1.2 with the initial perturbation $(\tilde{\mathbf{u}}_0, \tilde{\theta}_0)$ satisfying $\tilde{\mathbf{u}}_0 \in H^2(\Omega)$, $\tilde{\theta}_0 \in H^1(\Omega)$, $\nabla \cdot \tilde{\mathbf{u}}_0 = 0$ and the compatibility conditions. Let

$$P_\Omega : L^2(\Omega) \rightarrow L^2_\sigma(\Omega)$$

denote the projection operator as in Lemma 1.3.

- (1) (1.6) is equivalent to the following system without the pressure term,

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + P_\Omega(\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) = \nu \Delta \tilde{\mathbf{u}} + \tilde{\theta} \mathbf{e}_2 - \nabla \Delta^{-1} \nabla \cdot (\tilde{\theta} \mathbf{e}_2), \\ \partial_t \tilde{\theta} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\theta} + \tilde{v} \mathcal{T}'(y) = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ (\tilde{\mathbf{u}}, \tilde{\theta})(\mathbf{x}, 0) = (\tilde{\mathbf{u}}_0, \tilde{\theta}_0)(\mathbf{x}), \\ \tilde{\mathbf{u}} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \tilde{\omega}|_{\partial\Omega} = 0, \end{cases} \tag{1.12}$$

with the corresponding linearized system of equations given by

$$\begin{cases} \partial_t \mathbf{U} = \nu \Delta \mathbf{U} + \Theta \mathbf{e}_2 - \nabla \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2), \\ \partial_t \Theta + V \mathcal{T}'(y) = 0, \\ \nabla \cdot \mathbf{U} = 0, \\ (U, \Theta)(\mathbf{x}, 0) = (U_0(\mathbf{x}), \Theta_0(\mathbf{x})), \\ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \end{cases} \tag{1.13}$$

where we have used capital letters to distinguish the solutions of the linearized system from those of the full system, and $\mathbf{U} = (U, V)$ with V being the vertical component and $\omega = \nabla \times \mathbf{U}$.

- (2) Let $\mathcal{T}'(y) = \alpha > 0$. Assume the initial data (U_0, Θ_0) satisfying $U_0 \in H^2$, $\nabla \cdot U_0 = 0$ and $\Theta_0 \in H^1$. When Ω is a rectangle, we also make the natural assumption that Θ_0 is zero on the top and the bottom. When Ω is a general domain, we impose that Θ is zero on the boundary $\partial\Omega$. Then (1.13) has a unique global solution (U, Θ) that is also stable in the sense, as $t \rightarrow \infty$,

$$\begin{aligned} & \bullet \|\mathbf{U}(t)\|, \quad \|\nabla \mathbf{U}(t)\|, \quad \|\Delta \mathbf{U}(t)\| \rightarrow 0, \\ & \bullet \|\Theta(t)\| \rightarrow C_0, \quad 0 < C_0 < \sqrt{\alpha \|\mathbf{U}_0\|^2 + \|\Theta_0\|^2}, \\ & \bullet \|\nabla \Theta(t)\| \rightarrow C_1, \quad 0 < C_1 < \sqrt{\alpha \|\nabla \mathbf{U}_0\|^2 + \|\nabla \Theta_0\|^2}. \end{aligned} \tag{1.14}$$

- (3) If $\mathcal{T}'(y) = \alpha < 0$, then any spatially periodic solution (U, Θ) is unstable.

1.5. Idea of proof

The core of the proof of Theorem 1.1 is the global-in-time a priori bound for (\mathbf{u}, θ) in (1.7). We use L^2 -based energy method. The main difficulties are due to the lack of dissipation in the temperature equation and the weak regularity setup. We begin with the Lyapunov functional (free energy formulation) associated with (1.4):

$$\frac{d}{dt} (\alpha \|\mathbf{u}\|^2 + \|\theta - (\alpha y + \bar{\theta})\|^2) + 2\alpha \nu \|\omega\|^2 = 0, \tag{1.15}$$

where $\alpha > 0$ and $\bar{\theta}$ are arbitrary constants. It follows from (1.15) that the kinetic energy is uniformly bounded with respect to time, and $\|\omega\|^2$ is uniformly integrable with respect to time. Since the spatial domain is bounded, the uniform integrability of $\|\omega\|^2$, together with the Poincaré inequality, implies that the kinetic energy is also uniformly integrable with respect to time. The global bound for $\|\nabla \mathbf{u}\|$ and the time integrability of $\|\mathbf{u}\|_{H^2}$ are obtained via the vorticity equation. To further the estimates, we seek the global bounds for $\|\nabla \omega\|$ and $\|\nabla \theta\|$ simultaneously. Due to the lack of dissipation in the temperature equation, we need to control

$\|\nabla \mathbf{u}\|_{L^\infty}$. This is achieved via a logarithmic Sobolev inequality for domains that may not necessarily have smooth boundaries (see Lemma 2.2). In addition, due to the regularity limitation, we also invoke an improved Gronwall inequality (see Lemma 2.3). We make no effort in this proof to optimize the bounds in terms of time.

To prove the large-time asymptotics for the perturbation $(\tilde{\mathbf{u}}, \tilde{\theta})$ in Theorem 1.2, we start with a lemma stating that any nonnegative, integrable and uniformly continuous function $f = f(t)$ on $(0, \infty)$ must decay to zero as $t \rightarrow \infty$ (see Lemma 3.1). When the vertically stratified temperature $\mathcal{T}(y)$ satisfies $\mathcal{T}'(y) = \alpha > 0$, we first show $\alpha \|\tilde{\mathbf{u}}(t)\|^2 + \|\tilde{\theta}(t)\|^2$ decreases in time t , and $\|\nabla \tilde{\mathbf{u}}(t)\|^2$ is integrable on $(0, \infty)$ and $\|\tilde{\mathbf{u}}(t)\|^2$ is uniformly continuous in $t \in [0, \infty)$. Poincaré’s inequality implies that $\|\tilde{\mathbf{u}}(t)\|^2 \in L^1(0, \infty)$. Lemma 3.1 then leads to $\|\tilde{\mathbf{u}}(t)\|^2 \rightarrow 0$ as $t \rightarrow \infty$. To show $\|\nabla \tilde{\mathbf{u}}(t)\| \rightarrow 0$, we resort to the vorticity ω and show $\|\omega(t)\|^2$ is uniformly continuous in $t \in (0, \infty)$. An application of Lemma 3.1 then yields the desired property for $\|\nabla \tilde{\mathbf{u}}(t)\|$. The proof of $\|\partial_t \tilde{\mathbf{u}}(t)\| \rightarrow 0$ as $t \rightarrow \infty$ is obtained by applying Lemma 3.1 to the quantity $A(t) \equiv e^{-\int_0^t \|\omega(\tau)\|^2 d\tau} \|\partial_t \tilde{\mathbf{u}}(t)\|^2$. The large-time behavior $\|\tilde{\mathcal{P}} - \tilde{\theta} \mathbf{e}_2\|_{H^{-1}} \rightarrow 0$ and the uniform boundedness of $\|\tilde{\mathbf{u}}(t)\|_{H^2}$ are consequences of the decay properties of the velocity $\tilde{\mathbf{u}}$.

The proof of Theorem 1.4 starts with a derivation of an equivalent system of equations for the perturbation near the hydrostatic equilibrium and the corresponding linearized system. To derive the equivalent systems, we need to eliminate the pressure term. Since the domain Ω may not be smooth, we make use of a recent work on the existence of the projection operator P_Ω (see [54]) and show that, for the stress-free boundary conditions, $P_\Omega \Delta = \Delta$, where Δ denotes the Laplacian operator. With the suitable form of the linearized system at our disposal, the linear stability when $\mathcal{T}'(y) = \alpha > 0$ is obtained by careful energy estimates and the application of Lemma 3.1. The linear instability of periodic perturbations when $\mathcal{T}'(y) = \alpha < 0$ follows from the fact that one of the eigenvalues of the matrix (in Fourier space) associated with the linear system is always positive.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.1. Section 3 proves Theorem 1.2 while Section 4 proves Theorem 1.4. Section 5 is devoted to the numerical studies of (1.4), in which we try to understand some of the unresolved issues concerning the long-time behavior of the problem. The paper ends with concluding remarks.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We first state the Calderón–Zygmund inequality which is frequently used in the subsequent sections. As aforementioned, we assume that Ω is a bounded and simply connected planar domains with $\partial\Omega$ belonging to $C^{2,\gamma}$ ($\gamma \in (0, 1)$) except at a finite number of points, where $\partial\Omega$ is a corner of angle in $(0, \frac{\pi}{2}]$. Planar polygonal domains such as rectangles in \mathbb{R}^2 are special examples of such domains. As shown in [51–53], the Calderón–Zygmund inequality holds for such domains.

Lemma 2.1. *Assume that Ω is a bounded and simply connected planar domains with $\partial\Omega$ belonging to $C^{2,\gamma}$ ($\gamma \in (0, 1)$) except at a finite number of points, where $\partial\Omega$ is a corner of angle in $(0, \frac{\pi}{2}]$. Let $\mathbf{v} \in W^{s,p}(\Omega)$ be a vector-valued function satisfying $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$ where \mathbf{n} is the unit outward normal to $\partial\Omega$. Then there exists a constant $c = c(s, p, \Omega)$, such that*

$$\|\mathbf{v}\|_{W^{s,p}} \leq c \|\nabla \times \mathbf{v}\|_{W^{s-1,p}}$$

for any $s \geq 1$ and $p \in (1, \infty)$.

The proof of Theorem 1.1 also involves a logarithmic Sobolev inequality for domains that may not necessarily have smooth boundaries. The following lemma provides a logarithmic Sobolev interpolation inequality for a bounded convex domain with a finite number of corners. Planar polygonal domains such as rectangles in \mathbb{R}^2 are special examples of such domains.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain verifying that there exist constants $A > 1, \delta_0 \in (0, \text{diam}(\Omega))$ such that for any $r \in (0, \delta_0)$ and $x \in \Omega$,*

$$A|\Omega(x, r)| \geq |B(x, r)| \geq |\Omega(x, r)|,$$

where $\Omega(x, r) = \Omega \cap B(x, r)$ with $B(x, r)$ being the ball with center x and radius r . Then there exists a constant C depending on δ_0 such that, for any smooth function f and $0 < \alpha, \beta \leq 1$,

$$\|f\|_{L^\infty(\Omega)} \leq C \beta^{-\alpha} \left(\sup_{p \geq 2} \frac{\|f\|_{L^p(\Omega)}}{p^\alpha} + 1 \right) \log^\alpha(e + \|f\|_{C^{0,\beta}(\Omega)}).$$

A special consequence is the following logarithmic Sobolev interpolation inequality. Assume Ω is a bounded and simply connected domain of \mathbb{R}^2 , with $\partial\Omega$ belonging to $C^{2,\gamma}$ ($\gamma \in (0, 1)$) except at a finite number of points, where $\partial\Omega$ is a corner of angle between 0 and $\frac{\pi}{2}$. Assume \mathbf{u} is smooth vector. Then, for some constant C ,

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq C(1 + \|\nabla \mathbf{u}\|_{H^1(\Omega)}) \log(1 + \|\Delta \nabla \mathbf{u}\|_{L^2(\Omega)}).$$

The first part of Lemma 2.2 can be found in [56]. The second part is a special consequence of the first part. In fact, if we take $\alpha = 1$ and any $\beta < 1$, we obtain the second part by recalling the Sobolev inequalities,

$$\|f\|_{L^p(\Omega)} \leq C p \|\nabla f\|_{L^2(\Omega)}, \quad \|f\|_{C^{0,\beta}(\Omega)} \leq C \|f\|_{H^2(\Omega)},$$

which hold for Lipschitz domains.

We also need an improved Gronwall type inequality (see, e.g., [57]).

Lemma 2.3. *Assume that Y, Z, A and B are non-negative functions satisfying*

$$\frac{d}{dt} Y(t) + Z(t) \leq A(t) Y(t) + B(t) Y(t) \ln(1 + Z(t)). \tag{2.1}$$

Let $T > 0$. Assume $A \in L^1(0, T)$ and $B \in L^2(0, T)$. Then, for any $t \in [0, T]$,

$$Y(t) \leq (1 + Y(0)) e^{\int_0^t B(\tau) d\tau} e^{\int_0^t e^{\int_s^t B(\tau) d\tau} (A(s) + B^2(s)) ds} \tag{2.2}$$

and

$$\int_0^t Z(\tau) d\tau \leq Y(t) \int_0^t A(\tau) d\tau + Y^2(t) \int_0^t B^2(\tau) d\tau < \infty. \tag{2.3}$$

For the convenience of the readers, we provide a proof of this lemma.

Proof of Lemma 2.3. Setting

$$Y_1(t) = \ln(1 + Y(t)), \quad Z_1(t) = Z(t)/(1 + Y(t)),$$

we have

$$\begin{aligned} \frac{d}{dt} Y_1(t) + Z_1(t) &\leq A(t) + B(t) \ln(1 + Z(t)) \\ &\leq A(t) + B(t) \ln(1 + (1 + Y(t)) Z_1(t)) \\ &\leq A(t) + B(t) \ln(1 + Y(t))(1 + Z_1(t)) \\ &\leq A(t) + B(t) Y_1(t) + B(t) \ln(1 + Z_1(t)). \end{aligned}$$

Using the simple fact that, for $f \geq 0$,

$$\ln(1 + f(t)) \leq f^{\frac{1}{2}}(t), \tag{2.4}$$

we obtain

$$\frac{d}{dt} Y_1(t) + Z_1(t) \leq A(t) + B(t) Y_1(t) + B^2(t) + \frac{1}{4} Z_1(t).$$

Gronwall's inequality then implies

$$Y_1(t) \leq Y_1(0) e^{\int_0^t B(\tau) d\tau} + \int_0^t e^{\int_s^t B(\tau) d\tau} (A(s) + B^2(s)) ds,$$

which yields (2.2). In addition, (2.2) allows us to obtain (2.3) by using the inequality (2.4) in (2.1) and integrating in time. This completes the proof of Lemma 2.3. \square

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. The key component of the proof is the global-in-time *a priori* bound for (\mathbf{u}, θ) in the Sobolev space

$$\mathbf{u} \in L^\infty((0, t); H^2(\Omega)) \cap L^2((0, t); H^3(\Omega)), \quad \theta \in L^\infty((0, t); H^1(\Omega))$$

for any $t > 0$. Once the global bound is established, the existence of solutions then follows from some of the standard procedures such as various approximation schemes. The uniqueness of the solutions is a simple consequence of the fact

$$\int_0^t \|\nabla \mathbf{u}(\cdot, \tau)\|_{L^\infty(\Omega)} d\tau < \infty$$

due to $\mathbf{u} \in L^2((0, t); H^3(\Omega))$. This remark explains why it suffices to prove the global *a priori* bounds. In general the bounds depend on the time t and we shall not make efforts to obtain the optimal bounds.

First of all, since θ is purely transported by a divergence-free vector field, it holds that

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad \forall t > 0, \quad \forall p \in [2, \infty]. \tag{2.5}$$

For any fixed constants $a > 0$ and $b, c \in \mathbb{R}$, we write $\tilde{\theta} = \theta - ay - b$ and $\tilde{P} = P - \frac{a}{2}y^2 - by - c$ to convert (1.4) to

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \tilde{P} = \nu \Delta \mathbf{u} + \tilde{\theta} \mathbf{e}_2, \\ \partial_t \tilde{\theta} + \mathbf{u} \cdot \nabla \tilde{\theta} = -a\mathbf{u} \cdot \mathbf{e}_2, \\ \nabla \cdot \mathbf{u} = 0; \\ (\mathbf{u}, \tilde{\theta})(\mathbf{x}, 0) = (\mathbf{u}_0, \theta_0 - ay - b)(\mathbf{x}), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0. \end{cases} \tag{2.6}$$

Taking the L^2 -inner products of the first equation of (2.6) with \mathbf{u} and the second equation with $\tilde{\theta}$, and using the boundary conditions, we have

$$\frac{1}{2} \frac{d}{dt} (a\|\mathbf{u}\|^2 + \|\tilde{\theta}\|^2) + a\nu\|\omega\|^2 = 0, \tag{2.7}$$

which implies, in terms of the original functions, that

$$\frac{1}{2} \frac{d}{dt} (a\|\mathbf{u}\|^2 + \|\theta - ay - b\|^2) + a\nu\|\omega\|^2 = 0. \tag{2.8}$$

This provides a Lyapunov functional associated with (1.4). Moreover, since

$$\frac{d}{dt} \|\theta\|^2 = 0 \quad \text{and} \quad \frac{d}{dt} \int_\Omega \theta \, d\mathbf{x} = 0,$$

the Lyapunov functional is equivalent to

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}\|^2 - \int_\Omega y\theta \, d\mathbf{x} \right) + \nu\|\omega\|^2 = 0,$$

where the quantity inside the parenthesis is the total mechanical energy associated with (1.4). Our numerical simulations, see Section 5, illustrate that the total mechanical energy decreases as time evolves.

Upon integrating (2.8) with respect to time, we obtain

$$\begin{aligned} a\|\mathbf{u}(t)\|^2 + \|\theta(t) - ay - b\|^2 + 2a\nu \int_0^t \|\omega(\tau)\|^2 d\tau \\ = a\|\mathbf{u}_0\|^2 + \|\theta_0 - ay - b\|^2, \quad \forall t \geq 0, \end{aligned} \tag{2.9}$$

which implies

$$\begin{aligned} \|\mathbf{u}(t)\|^2 + \nu \int_0^t \|\omega(\tau)\|^2 d\tau + \nu \int_0^t \|\nabla \mathbf{u}(\tau)\|^2 d\tau \\ \leq c_1, \quad \forall t \geq 0. \end{aligned} \tag{2.10}$$

To prove the global H^1 -bound, we resort to the vorticity equation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega + \partial_x \theta. \tag{2.11}$$

Multiplying (2.11) by ω , integrating over Ω and applying the boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\omega|_{\partial\Omega} = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 = - \int_\Omega \theta \partial_x \omega \, d\mathbf{x}. \tag{2.12}$$

Inserting the inequality

$$\left| \int_\Omega \theta \partial_x \omega \, d\mathbf{x} \right| \leq \frac{\nu}{2} \|\nabla \omega\|^2 + c_2 \|\theta\|^2$$

in (2.12) and integrating in time yields

$$\|\omega(t)\|^2 + \nu \int_0^t \|\nabla \omega(\tau)\|^2 d\tau \leq 2c_2 \|\theta_0\|^2 t. \tag{2.13}$$

According to Lemma 2.1, (2.10) and (2.13) especially imply

$$\int_0^t \|\mathbf{u}(\tau)\|_{H^2} d\tau \leq c_3 t. \tag{2.14}$$

Next we prove a global bound for $\|\mathbf{u}\|_{H^2}$ and $\|\nabla \theta\|$. This is achieved by combining the energy inequalities for $\|\nabla \omega\|$ and $\|\nabla \theta\|$ and invoking Lemmas 2.2 and 2.3. Multiplying the vorticity equation (2.11) by $\Delta \omega$, integrating over Ω and applying the boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\omega|_{\partial\Omega} = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla \omega\|^2 + \nu \|\Delta \omega\|^2 = - \int_\Omega \partial_x \theta \Delta \omega \, d\mathbf{x} - \int_\Omega (\nabla \omega)^T \nabla \mathbf{u} \nabla \omega \, d\mathbf{x}.$$

By the Ladyzhenskaya inequality,

$$\begin{aligned} \left| \int_\Omega (\nabla \omega)^T \nabla \mathbf{u} \nabla \omega \, d\mathbf{x} \right| &\leq \|\nabla \mathbf{u}\| \|\nabla \omega\|_{L^4}^2 \\ &\leq c_4 \|\nabla \mathbf{u}\| \|\nabla \omega\| \|\Delta \omega\| \\ &\leq \frac{\nu}{4} \|\Delta \omega\|^2 + c_5 \|\nabla \mathbf{u}\|^2 \|\nabla \omega\|^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \|\nabla \omega\|^2 + \nu \|\Delta \omega\|^2 \leq c_6 \|\nabla \theta\|^2 + c_7 \|\nabla \mathbf{u}\|^2 \|\nabla \omega\|^2. \tag{2.15}$$

It follows from the temperature equation that

$$\frac{d}{dt} \|\nabla \theta\|^2 \leq 2\|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \theta\|^2.$$

Invoking the logarithmic Sobolev inequality in Lemma 2.2, we have

$$\frac{d}{dt} \|\nabla \theta\|^2 \leq c_8 (1 + \|\omega\|_{H^1} \log(1 + \|\Delta \omega\|)) \|\nabla \theta\|^2. \tag{2.16}$$

Combining (2.15) and (2.16) and setting $Y(t) = \|\nabla \omega\|^2 + \|\nabla \theta\|^2$ and $Z(t) = \|\Delta \omega\|^2$, we obtain

$$\begin{aligned} \frac{d}{dt} Y(t) + \nu Z(t) &\leq c_9 (1 + \|\nabla \mathbf{u}\|^2) Y(t) \\ &\quad + c_{10} \|\mathbf{u}\|_{H^2} Y(t) \log(1 + Z(t)). \end{aligned} \tag{2.17}$$

Applying Lemma 2.3 to (2.17) and recalling the bound for $\|\mathbf{u}\|_{H^2}$ in (2.14), we obtain the desired global bound for $\|\nabla\omega\|$ and $\|\nabla\theta\|$,

$$\|\nabla\omega(t)\|, \quad \|\nabla\theta(t)\|, \quad \int_0^t \|\Delta\omega(\tau)\|^2 d\tau \leq c_{11} e^{c_{12}e^{c_{13}t}}.$$

By Lemma 2.1 and the Sobolev embedding, we have, for $2 < p < \infty$,

$$\int_0^t \|\mathbf{u}(\tau)\|_{W^{2,p}}^2 d\tau \leq c_{14} e^{c_{12}e^{c_{13}t}}. \quad (2.18)$$

We have made no effort here to optimize the time growth rate.

We derive further energy estimates for the solution when $(\mathbf{u}_0, \theta_0) \in H^3$, and complete the proof of Theorem 1.1. We start with the L^p -estimate of $\nabla\theta$. By taking the gradient ∇ of the temperature equation, then testing the result with $p|\nabla\theta|^{p-2}\nabla\theta$, we have

$$\frac{d}{dt} \|\nabla\theta\|_{L^p}^p \leq p \|\nabla\mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^p}^p, \quad 2 < p < \infty,$$

which, together with the Sobolev embedding, $L^\infty \hookrightarrow W^{1,p}$ ($2 < p < \infty$), implies

$$\frac{d}{dt} \|\nabla\theta\|_{L^p} \leq c_{15} \|\mathbf{u}\|_{W^{2,p}} \|\nabla\theta\|_{L^p}, \quad 2 < p < \infty. \quad (2.19)$$

We remark that no boundary condition for θ is necessary in the derivation of (2.19). In a similar fashion, we can show that

$$\begin{aligned} \frac{d}{dt} \|D^2\theta\|_{L^p} &\leq c_{16} \|\nabla\theta\|_{W^{1,p}} \|\mathbf{u}\|_{W^{2,p}} + c_{17} \|\mathbf{u}\|_{W^{2,p}} \|D^2\theta\|_{L^p}, \\ 2 < p < \infty. \end{aligned} \quad (2.20)$$

Combining (2.19) and (2.20), we obtain

$$\frac{d}{dt} \|\nabla\theta\|_{W^{1,p}} \leq c_{18} \|\mathbf{u}\|_{W^{2,p}} \|\nabla\theta\|_{W^{1,p}}, \quad 2 < p < \infty.$$

Gronwall's inequality and (2.18) then imply

$$\|\nabla\theta\|_{W^{1,p}} \leq e^{c_{19}e^{c_{12}e^{c_{13}t}}} \|\nabla\theta_0\|_{W^{1,p}}, \quad 2 < p < \infty. \quad (2.21)$$

Moreover, direct calculations show that

$$\begin{aligned} \|\theta_t\|_{L^p} &\leq \|\mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^p} \\ &\leq c_{20} \|\mathbf{u}\|_{H^2} \|\nabla\theta\|_{L^p} \\ &\leq c_{21} e^{c_{12}e^{c_{13}t}} \|\nabla\theta\|_{L^p}, \\ \|\nabla\theta_t\|_{L^p} &\leq c_{22} \|\nabla\mathbf{u}\|_{H^1} \|\nabla\theta\|_{L^{2p}} + c_{23} \|\mathbf{u}\|_{H^2} \|D^2\theta\|_{L^p} \\ &\leq c_{24} e^{c_{12}e^{c_{13}t}} (\|\nabla\theta\|_{L^{2p}} + \|D^2\theta\|_{L^p}), \quad 2 < p < \infty, \end{aligned}$$

which, together with (2.21), shows that

$$\|\theta_t\|_{W^{1,p}} \leq c_{25} e^{c_{19}e^{c_{12}e^{c_{13}t}}}, \quad 2 < p < \infty. \quad (2.22)$$

In addition, by taking the temporal derivative to the equation of vorticity, then calculating the L^2 inner product of the resulting equation with $\partial_t\omega$, we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t\omega\|^2 + \nu \|\nabla\partial_t\omega\|^2 \\ = - \int_{\Omega} (\partial_t\mathbf{u} \cdot \nabla\omega) \partial_t\omega \, d\mathbf{x} + \int_{\Omega} (\partial_t\partial_x\theta) \partial_t\omega \, d\mathbf{x} \\ \leq \|\partial_t\mathbf{u}\|_{L^4} \|\nabla\omega\|_{L^4} \|\partial_t\omega\| + \frac{1}{2} \|\partial_t\omega\|^2 + \frac{1}{2} \|\partial_t\partial_x\theta\|^2 \\ \leq c_{26} \|\mathbf{u}\|_{W^{2,4}} \|\partial_t\omega\|^2 + \frac{1}{2} \|\partial_t\omega\|^2 + \frac{1}{2} \|\partial_t\partial_x\theta\|^2, \end{aligned} \quad (2.23)$$

where we have applied Gagliardo–Nirenberg interpolation inequality and Lemma 2.1. Applying the Gronwall inequality to (2.23) and using (2.18) and (2.22), we find

$$\partial_t\omega \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \forall 0 < T < \infty. \quad (2.24)$$

Applying the classic elliptic regularity theory to the equation of vorticity and using (2.21) and (2.24), we can easily show that

$$\omega \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \quad \forall 0 < T < \infty,$$

which, together with Lemma 2.1, implies

$$\mathbf{u} \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \quad \forall 0 < T < \infty. \quad (2.25)$$

Finally, applying (2.25) to the equation of temperature, we can show that

$$\theta \in L^\infty(0, T; H^3(\Omega)), \quad \forall 0 < T < \infty.$$

This completes the proof of Theorem 1.1. \square

3. Proof of Theorem 1.2

This section proves Theorem 1.2. The following decay lemma will be used several times.

Lemma 3.1. *Assume $f \in L^1(0, \infty)$ is a nonnegative and uniformly continuous function. Then,*

$$f(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In particular, if $f \in L^1(0, \infty)$ is nonnegative and satisfies, for a constant C and any $0 \leq s < t < \infty$,

$$|f(t) - f(s)| \leq C |t - s|,$$

$$\text{then } f(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The proof of this lemma is not difficult.

Proof. We prove by the definition of limit. For any $\epsilon > 0$, the uniform continuity of f implies that there exists $\delta > 0$ such that, for any $0 \leq s < t < \infty$,

$$|f(t) - f(s)| < \frac{\epsilon}{2} \quad \text{whenever } |t - s| < \delta.$$

Since $f \in L^1(0, \infty)$, there exists $T > 0$ such that

$$\int_T^\infty f(t) dt < \frac{\epsilon\delta}{2}.$$

Especially, for any $k = 0, 1, 2, \dots$,

$$\int_{T+k\delta}^{T+(k+1)\delta} f(t) dt < \frac{\epsilon\delta}{2},$$

which implies that there exists $t_k \in (T+k\delta, T+(k+1)\delta)$ such that

$$f(t_k) < \frac{\epsilon}{2}.$$

Then, for any $t > T$, there exists k_0 satisfying $t \in (T+k_0\delta, T+(k_0+1)\delta]$ and thus

$$f(t) \leq f(t_{k_0}) + |f(t) - f(t_{k_0})| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This concludes the proof of Lemma 3.1. \square

Although primarily we focus on the case when Ω is a rectangle, our results hold for more general domains as specified in Theorem 1.1. When Ω is a rectangle, the curvature of the boundary κ is zero and no boundary term gets into play. For a more general domain, a boundary term related to κ plays a role. We will use \mathbf{n} to denote the unit outward normal vector to $\partial\Omega$. If s is arc length and we parameterize the curve by arc length, then $\frac{d\mathbf{n}}{ds} \cdot \mathbf{n} = 0$; i.e., $\frac{d\mathbf{n}}{ds}$ is perpendicular to \mathbf{n} . The magnitude of $\frac{d\mathbf{n}}{ds}$ is called the curvature of the curve and is denoted by κ . That is to say, $\kappa = |\frac{d\mathbf{n}}{ds}|$. The unit vector in the direction of $\frac{d\mathbf{n}}{ds}$ is defined as the unit tangent vector to the curve and is denoted as $\boldsymbol{\tau}$. The following lemma facilitates the process of energy estimates in the proof of Theorem 1.2.

Lemma 3.2. Let Ω be a rectangle or a more general domain as specified in [Theorem 1.1](#). Assume the vector field \mathbf{u} and its corresponding $\omega = \nabla \times \mathbf{u}$ satisfy the boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0.$$

Then the following identity holds

$$(\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\tau} + \kappa \mathbf{u} \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\Omega,$$

where \mathbf{n} and $\boldsymbol{\tau}$ denote the unit outward normal and tangent vector to $\partial\Omega$, respectively.

We provide a proof of [Lemma 3.2](#).

Proof. Since $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, the directional derivative of $\mathbf{u} \cdot \mathbf{n}$ along $\partial\Omega$ should also be zero, namely

$$\frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{n}) = 0 \quad \text{on } \partial\Omega.$$

The product rule then yields

$$\begin{aligned} \left(\frac{d}{d\tau} \mathbf{u}\right) \cdot \mathbf{n} + \mathbf{u} \cdot \left(\frac{d}{d\tau} \mathbf{n}\right) &= 0 \quad \text{or} \\ \boldsymbol{\tau} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + \mathbf{u} \cdot (\boldsymbol{\tau} \cdot \nabla \mathbf{n}) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Due to $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$,

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{n})\mathbf{n} + (\mathbf{u} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = (\mathbf{u} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \quad \text{on } \partial\Omega.$$

Therefore, due to $\kappa = \frac{dn}{ds} \cdot \boldsymbol{\tau} = \left(\frac{\partial n}{\partial x_1} \frac{dx_1}{ds} + \frac{\partial n}{\partial x_2} \frac{dx_2}{ds}\right) \cdot \boldsymbol{\tau} = \left(\frac{\partial n}{\partial x_1} \tau_1 + \frac{\partial n}{\partial x_2} \tau_2\right) \cdot \boldsymbol{\tau} = \boldsymbol{\tau} \cdot \nabla \mathbf{n} \cdot \boldsymbol{\tau}$,

$$\begin{aligned} \boldsymbol{\tau} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + (\boldsymbol{\tau} \cdot \nabla \mathbf{n} \cdot \boldsymbol{\tau})(\mathbf{u} \cdot \boldsymbol{\tau}) &= 0 \quad \text{or} \\ \boldsymbol{\tau} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + \kappa \mathbf{u} \cdot \boldsymbol{\tau} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since $\omega = 0$ on $\partial\Omega$,

$$\nabla \mathbf{u} - (\nabla \mathbf{u})^T = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = 0 \quad \text{on } \partial\Omega.$$

That is, $\nabla \mathbf{u} = (\nabla \mathbf{u})^T$. Therefore,

$$(\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\tau} = \boldsymbol{\tau} \cdot (\nabla \mathbf{u})^T \cdot \mathbf{n} = \boldsymbol{\tau} \cdot \nabla \mathbf{u} \cdot \mathbf{n} = -\kappa \mathbf{u} \cdot \boldsymbol{\tau},$$

which is the desired equality. \square

We now prove [Theorem 1.2](#).

Proof of Theorem 1.2. Since the global-in-time existence and uniqueness can be similarly obtained as in the proof of [Theorem 1.1](#), we focus on the asymptotic behavior. Assume $\mathcal{T}'(y) = \alpha > 0$. Taking the inner product of the first two equations in [\(1.6\)](#) with $(\tilde{\mathbf{u}}, \tilde{\theta})$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\alpha \|\tilde{\mathbf{u}}\|^2 + \|\tilde{\theta}\|^2) &= v \alpha \int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} - \alpha \int_{\Omega} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}} \, d\mathbf{x} \\ &\quad - \alpha \int_{\Omega} \nabla \tilde{P} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} - \int_{\Omega} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\theta}) \cdot \tilde{\mathbf{u}} \, d\mathbf{x}. \end{aligned} \tag{3.1}$$

The boundary conditions allow us to eliminate the last three terms,

$$\int_{\Omega} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = \frac{1}{2} \int_{\partial\Omega} \mathbf{n} \cdot \tilde{\mathbf{u}} |\tilde{\mathbf{u}}|^2 \, dS(\mathbf{x}) = 0, \tag{3.2}$$

$$\int_{\Omega} \nabla \tilde{P} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{n} \cdot \tilde{\mathbf{u}} \tilde{P} \, dS(\mathbf{x}) = 0, \tag{3.3}$$

$$\int_{\Omega} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\theta}) \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = \frac{1}{2} \int_{\partial\Omega} \mathbf{n} \cdot \tilde{\mathbf{u}} \tilde{\theta}^2 \, dS(\mathbf{x}) = 0. \tag{3.4}$$

For the dissipative term, we first apply the divergence theorem to obtain

$$\int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = \int_{\partial\Omega} (\mathbf{n} \cdot \nabla) \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, dS(\mathbf{x}) - \int_{\Omega} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x}.$$

Since $\tilde{\mathbf{u}} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we can write

$$\tilde{\mathbf{u}} = (\tilde{\mathbf{u}} \cdot \mathbf{n})\mathbf{n} + (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \quad \text{on } \partial\Omega.$$

By [Lemma 3.2](#),

$$\begin{aligned} \int_{\partial\Omega} (\mathbf{n} \cdot \nabla) \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, dS(\mathbf{x}) &= \int_{\partial\Omega} (\mathbf{n} \cdot \nabla) \tilde{\mathbf{u}} \cdot \boldsymbol{\tau} (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau}) \, dS(\mathbf{x}) \\ &= - \int_{\partial\Omega} \kappa (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})^2 \, dS(\mathbf{x}). \end{aligned}$$

Therefore,

$$\int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = - \int_{\partial\Omega} \kappa (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})^2 \, dS(\mathbf{x}) - \int_{\Omega} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x}. \tag{3.5}$$

In the special case when Ω is a rectangle, the boundary term vanishes and

$$\int_{\Omega} \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} = - \int_{\Omega} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x}.$$

Inserting [\(3.2\)](#), [\(3.3\)](#), [\(3.4\)](#) and [\(3.5\)](#) in [\(3.1\)](#) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\alpha \|\tilde{\mathbf{u}}\|^2 + \|\tilde{\theta}\|^2) + v \alpha \int_{\Omega} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x} \\ + v \alpha \int_{\partial\Omega} \kappa (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})^2 \, dS(\mathbf{x}) &= 0. \end{aligned} \tag{3.6}$$

Integrating in time yields, for any $0 \leq s < t < \infty$,

$$\begin{aligned} \alpha \|\tilde{\mathbf{u}}(t)\|^2 + \|\tilde{\theta}(t)\|^2 + v \alpha \int_s^t \|\nabla \tilde{\mathbf{u}}(\rho)\|^2 \, d\rho \\ + v \alpha \int_s^t \|\sqrt{\kappa} (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau})(\rho)\|_{L^2(\partial\Omega)}^2 \, d\rho \\ \leq \alpha \|\tilde{\mathbf{u}}(s)\|^2 + \|\tilde{\theta}(s)\|^2. \end{aligned} \tag{3.7}$$

Especially, [\(3.7\)](#) implies, $Y(t) \equiv \alpha \|\tilde{\mathbf{u}}(t)\|^2 + \|\tilde{\theta}(t)\|^2$ is a decreasing function of $t \in [0, \infty)$ and

$$0 < Y(t) \leq Y_0 \equiv \alpha \|\tilde{\mathbf{u}}_0\|^2 + \|\tilde{\theta}_0\|^2.$$

As a consequence, there exists $c_0 > 0$ such that, as $t \rightarrow \infty$,

$$Y(t) \equiv \alpha \|\tilde{\mathbf{u}}(t)\|^2 + \|\tilde{\theta}(t)\|^2 \rightarrow c_0^2 < Y_0^2. \tag{3.8}$$

Next we show that, as $t \rightarrow \infty$,

$$\|\tilde{\mathbf{u}}(t)\| \rightarrow 0 \quad \text{and} \quad \|\nabla \tilde{\mathbf{u}}(t)\| \rightarrow 0.$$

We recall the Poincaré inequality, for any $\mathbf{u} \in H^1(\Omega)$ with $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$,

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \mathbf{u}\|_{L^2(\Omega)}.$$

Invoking the Poincaré inequality in [\(3.7\)](#) yields, for $C > 0$,

$$\alpha \|\tilde{\mathbf{u}}(t)\|^2 + C \int_0^\infty \|\tilde{\mathbf{u}}(\tau)\|^2 \, d\tau \leq \alpha \|\tilde{\mathbf{u}}_0\|^2 + \|\tilde{\theta}_0\|^2,$$

which in particular implies

$$\|\tilde{\mathbf{u}}(t)\|^2 \in L^1(0, \infty). \tag{3.9}$$

We also need another ingredient, the uniform continuity of $\|\tilde{\mathbf{u}}(t)\|^2$ as a function of $t \in [0, \infty)$. Taking the inner product of the equation of $\tilde{\mathbf{u}}$ in [\(1.6\)](#) with $\tilde{\mathbf{u}}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{u}}\|^2 + v \alpha \|\nabla \tilde{\mathbf{u}}\|^2 \leq \int \tilde{\theta} \tilde{\mathbf{u}} \, d\mathbf{x} \leq \|\tilde{\mathbf{u}}\| \|\tilde{\theta}\| \leq \frac{Y_0}{2\sqrt{\alpha}}.$$

Therefore, for any $0 \leq s \leq t < \infty$ and $C = \frac{Y_0}{\sqrt{\alpha}}$,

$$\|\tilde{\mathbf{u}}(t)\|_{L^2}^2 - \|\tilde{\mathbf{u}}(s)\|_{L^2}^2 \leq C |t - s|. \tag{3.10}$$

(3.9) and (3.10), together with Lemma 3.1, lead to the desired decay

$$\|\tilde{\mathbf{u}}(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{3.11}$$

In addition, (3.8) and (3.11) yield

$$\|\tilde{\theta}(t)\| \rightarrow c_0.$$

We now prove

$$\|\nabla \tilde{\mathbf{u}}(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{3.12}$$

According to (3.7), $\|\nabla \tilde{\mathbf{u}}(t)\|^2 \in L^1(0, \infty)$, which, in particular, implies

$$\|\tilde{\omega}(t)\|^2 \in L^1(0, \infty). \tag{3.13}$$

Taking the inner product of the vorticity equation

$$\partial_t \tilde{\omega} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\omega} = \nu \Delta \tilde{\omega} + \partial_1 \tilde{\theta}.$$

with ω and invoking the boundary conditions yield

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\omega}\|^2 + \nu \|\nabla \tilde{\omega}\|^2 = - \int \partial_1 \omega \tilde{\theta} \, d\mathbf{x} \leq \frac{\nu}{2} \|\nabla \tilde{\omega}\|^2 + \frac{1}{2\nu} \|\tilde{\theta}_0\|^2$$

or

$$\frac{d}{dt} \|\tilde{\omega}\|^2 + \nu \|\nabla \tilde{\omega}\|^2 \leq \frac{1}{\nu} \|\tilde{\theta}_0\|^2.$$

Integrating in time yields, for any $0 \leq s < t < \infty$,

$$\|\tilde{\omega}(t)\|^2 - \|\tilde{\omega}(s)\|^2 \leq \frac{1}{\nu} |t - s| \|\tilde{\theta}_0\|^2. \tag{3.14}$$

(3.13) and (3.14), and Lemma 3.1 allow us to conclude that

$$\|\tilde{\omega}(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{3.15}$$

By Lemma 2.1, $\|\nabla \tilde{\mathbf{u}}(t)\| \leq C \|\tilde{\omega}(t)\|$ and (3.15) implies (3.12). Next we show that, as $t \rightarrow \infty$,

$$\|\partial_t \tilde{\mathbf{u}}(t)\| \rightarrow 0. \tag{3.16}$$

Applying ∂_t to the equation of $\tilde{\mathbf{u}}$ in (1.6) yields

$$\partial_{tt} \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}}_t \cdot \nabla \tilde{\mathbf{u}} + \nabla \partial_t \tilde{P} = \nu \Delta \partial_t \tilde{\mathbf{u}} + \partial_t \tilde{\theta} \mathbf{e}_2. \tag{3.17}$$

Taking the L^2 -inner product of (3.17) with $\partial_t \tilde{\mathbf{u}}$ and invoking the equation of $\tilde{\theta}$ in (1.6), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t \tilde{\mathbf{u}}(t)\|^2 + \nu \|\partial_t \tilde{\omega}\|^2 \\ &= \int_{\Omega} (\partial_t \tilde{\theta}) (\mathbf{e}_2 \cdot \partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} - \int_{\Omega} (\partial_t \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) \cdot (\partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} \\ &= \int_{\Omega} \tilde{\theta} \tilde{\mathbf{u}} \cdot \nabla (\mathbf{e}_2 \cdot \partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} - a \int_{\Omega} \tilde{v} (\mathbf{e}_2 \cdot \partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} \\ & \quad - \int_{\Omega} (\partial_t \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) \cdot (\partial_t \tilde{\mathbf{u}}) \, d\mathbf{x}. \end{aligned} \tag{3.18}$$

By the Hölder inequality, Poincaré inequality and Sobolev inequality,

$$\begin{aligned} \left| \int_{\Omega} \tilde{\theta} \tilde{\mathbf{u}} \cdot \nabla (\mathbf{e}_2 \cdot \partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} \right| &\leq \|\tilde{\theta}\|_{L^\infty} \|\tilde{\mathbf{u}}\| \|\nabla \partial_t \tilde{\mathbf{u}}\| \\ &\leq \frac{\nu}{8} \|\partial_t \tilde{\omega}\|^2 + C \|\tilde{\mathbf{u}}\|^2, \\ \left| \int_{\Omega} \tilde{v} (\mathbf{e}_2 \cdot \partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} \right| &\leq \|\tilde{v}\| \|\partial_t \tilde{\mathbf{u}}\| \leq C \|\tilde{v}\| \|\nabla \partial_t \tilde{\mathbf{u}}\| \\ &\leq \frac{\nu}{8} \|\partial_t \tilde{\omega}\|^2 + C \|\tilde{v}\|^2 \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Omega} (\partial_t \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}) \cdot (\partial_t \tilde{\mathbf{u}}) \, d\mathbf{x} \right| &\leq \|\nabla \tilde{\mathbf{u}}\| \|\partial_t \tilde{\mathbf{u}}\|_{L^4}^2 \leq C \|\omega\| \|\partial_t \tilde{\mathbf{u}}\| \|\nabla \partial_t \tilde{\mathbf{u}}\| \\ &\leq \frac{\nu}{4} \|\partial_t \tilde{\omega}\|^2 + C \|\omega\|^2 \|\partial_t \tilde{\mathbf{u}}\|^2. \end{aligned}$$

Inserting the estimates above in (3.18) yields

$$\frac{d}{dt} \|\partial_t \tilde{\mathbf{u}}(t)\|^2 + \nu \|\partial_t \tilde{\omega}\|^2 \leq C \|\omega\|^2 \|\partial_t \tilde{\mathbf{u}}\|^2 + C \|\tilde{\mathbf{u}}\|^2. \tag{3.19}$$

Gronwall's inequality, together with (3.9) and (3.13), implies

$$\|\partial_t \tilde{\mathbf{u}}(t)\|^2 + \nu \int_0^t \|\partial_\tau \tilde{\omega}(\tau)\|^2 \, d\tau \leq C.$$

Poincaré's inequality then implies

$$\int_0^\infty \|\partial_t \tilde{\mathbf{u}}(t)\|^2 \, dt \leq C \int_0^\infty \|\partial_t \tilde{\omega}(t)\|^2 \, dt \leq C. \tag{3.20}$$

Writing (3.19) as

$$\frac{d}{dt} \left(e^{-C \int_0^t \|\omega\|^2 \, d\tau} \|\partial_t \tilde{\mathbf{u}}\|^2 \right) \leq C e^{-C \int_0^t \|\omega\|^2 \, d\tau} \|\tilde{\mathbf{u}}\|^2,$$

integrating in time and using the fact the right-hand side is bounded uniformly in time, we obtain

$$|A(t) - A(s)| \leq C |t - s|, \tag{3.21}$$

where A is given by

$$A(t) = e^{-C \int_0^t \|\omega(\tau)\|^2 \, d\tau} \|\partial_t \tilde{\mathbf{u}}(t)\|^2.$$

In addition, (3.20) implies

$$A(t) \in L^1(0, \infty).$$

Lemma 3.1 then asserts $A(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$\|\partial_t \tilde{\mathbf{u}}(t)\|^2 = e^{C \int_0^t \|\omega(\tau)\|^2 \, d\tau} A(t) \leq C A(t) \rightarrow 0. \tag{3.22}$$

It then follows from the equation of $\tilde{\mathbf{u}}$ in (1.6) that

$$\begin{aligned} \|\nabla \tilde{P} - \tilde{\theta} \mathbf{e}_2\|_{H^{-1}} &\leq \|\partial_t \tilde{\mathbf{u}}\| + \|\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}\| + \|\nabla \tilde{\mathbf{u}}\| \\ &\leq \|\partial_t \tilde{\mathbf{u}}\| + C \|\tilde{\mathbf{u}}\| \|\nabla \tilde{\mathbf{u}}\| + \|\nabla \tilde{\mathbf{u}}\|. \end{aligned}$$

(3.11), (3.12) and (3.22) then lead to

$$\|\nabla \tilde{P} - \tilde{\theta} \mathbf{e}_2\|_{H^{-1}} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Finally we prove (1.9). We note that the equation for $\tilde{\mathbf{u}}$ in (1.6) can be written in the component form as

$$\begin{aligned} \nu \partial_y \tilde{\omega} + \partial_x \tilde{P} &= -\partial_t \tilde{u} - \tilde{\mathbf{u}} \cdot \nabla \tilde{u}, \\ -\nu \partial_x \tilde{\omega} + \partial_y \tilde{P} &= -\partial_t \tilde{v} - \tilde{\mathbf{u}} \cdot \nabla \tilde{v} + \tilde{\theta}. \end{aligned}$$

Due to the boundary condition $\tilde{\omega}|_{\partial\Omega} = 0$, it holds that

$$\begin{aligned} & \|\nu \partial_y \tilde{\omega} + \partial_x \tilde{P}\|^2 + \|\nu \partial_x \tilde{\omega} + \partial_y \tilde{P}\|^2 \\ &= \nu^2 \|\nabla \tilde{\omega}\|^2 + \|\nabla \tilde{P}\|^2 + 2\nu \int_{\Omega} (\partial_y \tilde{\omega} \partial_x \tilde{P} - \partial_x \tilde{\omega} \partial_y \tilde{P}) \, d\mathbf{x} \\ &= \nu^2 \|\nabla \tilde{\omega}\|^2 + \|\nabla \tilde{P}\|^2 + 2\nu \int_{\Omega} \nabla \cdot (-\tilde{\omega} \partial_y \tilde{P}, \tilde{\omega} \partial_x \tilde{P}) \, d\mathbf{x} \\ &= \nu^2 \|\nabla \tilde{\omega}\|^2 + \|\nabla \tilde{P}\|^2, \end{aligned}$$

which, together with the preceding equations, implies

$$\|\nabla \tilde{\omega}\|^2 \lesssim \|\partial_t \tilde{\mathbf{u}}\|^2 + \|\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}\|^2 + \|\tilde{\theta}\|^2. \tag{3.23}$$

According to the Ladyzhenskaya inequality, Lemma 2.1 and Poincaré inequality, we have

$$\begin{aligned} \|\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}\|^2 &\lesssim \|\tilde{\mathbf{u}}\|_{L^4}^2 \|\nabla \tilde{\mathbf{u}}\|_{L^4}^2 \\ &\lesssim \|\tilde{\mathbf{u}}\| \|\nabla \tilde{\mathbf{u}}\|^2 \|\nabla \tilde{\omega}\| \\ &\lesssim \frac{1}{2} \|\nabla \tilde{\omega}\|^2 + \|\tilde{\mathbf{u}}\|^2 \|\nabla \tilde{\mathbf{u}}\|^4, \end{aligned}$$

which, together with (3.23), implies

$$\|\nabla\tilde{\omega}\|^2 \lesssim \|\partial_t\tilde{\mathbf{u}}\|^2 + \|\tilde{\mathbf{u}}\|^2\|\nabla\tilde{\mathbf{u}}\|^4 + \|\tilde{\theta}\|^2. \quad (3.24)$$

Since $\|\tilde{\theta}\|^2 \leq 2(\|\theta\|^2 + \|\alpha\gamma + \bar{\theta}\|^2)$, the uniform boundedness of $\|\nabla\omega(t)\|$ then follows from (3.24), (3.11), (3.12), (3.22) and (2.5). As a consequence of Lemma 2.1 and the Poincaré inequality, we conclude that $\|\mathbf{u}(t)\|_{H^2}$ is uniformly bounded in time. This completes the proof of Theorem 1.2. \square

4. Proof of Theorem 1.4

This section proves Theorem 1.4. For the sake of clarity, it is divided into three subsections. The first subsection derives the system of Eqs. (1.13) while the second proves the linear stability when $\mathcal{T}'(y) = \alpha > 0$ and the third proves the linear instability of periodic perturbations when $\mathcal{T}'(y) = \alpha < 0$.

4.1. Derivation of (1.13)

Although our functional setting is $\tilde{\mathbf{u}} \in H^2(\Omega)$ and $\tilde{\theta} \in H^1(\Omega)$, we present a derivation that is valid for more general Sobolev spaces

$$\tilde{\mathbf{u}} \in W^{2,p}(\Omega), \quad \tilde{\theta} \in W^{1,p}(\Omega), \quad 1 < p < \infty.$$

Let P_Ω denote the projection operator from $L^p(\Omega)$ to $L^p_\sigma(\Omega)$ with $1 < p < \infty$ as stated in Lemma 1.3. Applying P_Ω to the velocity equation in (1.6) yields

$$\partial_t\tilde{\mathbf{u}} + P_\Omega(\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}) = \nu P_\Omega\Delta\tilde{\mathbf{u}} + P_\Omega(\tilde{\theta}\mathbf{e}_2). \quad (4.1)$$

We first show that

$$P_\Omega\Delta\tilde{\mathbf{u}} = \Delta\tilde{\mathbf{u}}. \quad (4.2)$$

In general the operators P_Ω and Δ do not commute and (4.2) may not be true, but the stress-free boundary conditions in (1.6) allow us to prove (4.2). For $\tilde{\mathbf{u}} \in W^{2,p}(\Omega)$ with $\nabla \cdot \tilde{\mathbf{u}} = 0$, we have

$$\Delta\tilde{\mathbf{u}} \in L^p(\Omega) \quad \text{and} \quad \nabla \cdot \Delta\tilde{\mathbf{u}} = 0 \quad \text{in } \Omega. \quad (4.3)$$

In addition, we show that

$$\Delta\tilde{\mathbf{u}} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (4.4)$$

In fact, $\tilde{\omega} = 0$ on $\partial\Omega$ implies the directional derivative of $\tilde{\omega}$ along the tangential direction τ on $\partial\Omega$ is zero, namely

$$\frac{d\tilde{\omega}}{d\tau} = 0 \quad \text{on } \partial\Omega,$$

where τ denotes the unit tangent vector to $\partial\Omega$. That is,

$$\tau \cdot \nabla\tilde{\omega} = 0 \quad \text{or} \quad \tau_1\partial_1\tilde{\omega} + \tau_2\partial_2\tilde{\omega} = 0 \quad \text{on } \partial\Omega. \quad (4.5)$$

Since $\tau \cdot \mathbf{n} = 0$, we have $\tau = \mathbf{n}^\perp$ or $(\tau_1, \tau_2) = (-n_2, n_1)$. Thus, (4.5) becomes

$$-n_2\partial_1\tilde{\omega} + n_1\partial_2\tilde{\omega} = 0 \quad \text{on } \partial\Omega.$$

Invoking $\tilde{\omega} = \partial_1\tilde{v} - \partial_2\tilde{u}$ yields

$$\begin{aligned} -n_2\partial_1(\partial_1\tilde{v} - \partial_2\tilde{u}) + n_1\partial_2(\partial_1\tilde{v} - \partial_2\tilde{u}) &= 0 \quad \text{or} \\ -n_2\Delta\tilde{v} - n_1\Delta\tilde{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which is exactly (4.4). Now (4.3) and (4.4) imply that

$$\Delta\tilde{\mathbf{u}} \in L^p_\sigma(\Omega).$$

(1.10) in Lemma 1.3 allows us to conclude (4.2),

$$P_\Omega\Delta\tilde{\mathbf{u}} = \Delta\tilde{\mathbf{u}}.$$

Next we derive an explicit formula for $P_\Omega(\tilde{\theta}\mathbf{e}_2)$. According to Lemma 1.3, we can write, for some $h \in W^{1,p}$,

$$\tilde{\theta}\mathbf{e}_2 = P_\Omega(\tilde{\theta}\mathbf{e}_2) + \nabla h, \quad h = \Delta^{-1}\nabla \cdot (\tilde{\theta}\mathbf{e}_2),$$

where the operator Δ^{-1} is defined in (1.11). Therefore,

$$P_\Omega(\tilde{\theta}\mathbf{e}_2) = \tilde{\theta}\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (\tilde{\theta}\mathbf{e}_2). \quad (4.6)$$

Inserting (4.2) and (4.6) in (4.1) yields

$$\partial_t\tilde{\mathbf{u}} + P_\Omega(\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}) = \nu\Delta\tilde{\mathbf{u}} + \tilde{\theta}\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (\tilde{\theta}\mathbf{e}_2).$$

We thus have obtained the following equivalent form of (1.6)

$$\begin{cases} \partial_t\tilde{\mathbf{u}} + P_\Omega(\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}) = \nu\Delta\tilde{\mathbf{u}} + \tilde{\theta}\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (\tilde{\theta}\mathbf{e}_2), \\ \partial_t\tilde{\theta} + \tilde{\mathbf{u}} \cdot \nabla\tilde{\theta} + \tilde{v}\mathcal{T}'(y) = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0. \end{cases} \quad (4.7)$$

Ignoring the nonlinear terms in (4.7), we obtain the associated linearized system

$$\begin{cases} \partial_t\mathbf{U} = \nu\Delta\mathbf{U} + \Theta\mathbf{e}_2 - \nabla\Delta^{-1}\nabla \cdot (\Theta\mathbf{e}_2), \\ \partial_t\Theta + V\mathcal{T}'(y) = 0, \\ \nabla \cdot \mathbf{U} = 0, \\ (U, \Theta)(\mathbf{x}, 0) = (U_0(\mathbf{x}), \Theta_0(\mathbf{x})), \\ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \end{cases} \quad (4.8)$$

where we have used capital letters to distinguish the solutions of the linearized system from those of the full system (4.7), and $\mathbf{U} = (U, V)$ with V being the vertical component and $\omega = \nabla \times \mathbf{U}$.

4.2. Linear stability for $\mathcal{T}'(y) = \alpha > 0$

This subsection proves the second part of Theorem 1.4. Assume $\mathcal{T}'(y) = \alpha > 0$ and the conditions in the second part of Theorem 1.4. The existence and uniqueness of the corresponding solution can be proven similarly as in the proof of Theorem 1.1. It suffices to prove (1.14). As in the proof of (3.6) in Theorem 1.2, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\alpha\|\mathbf{U}(t)\|^2 + \|\Theta(t)\|^2) \\ + \nu\alpha\|\nabla\mathbf{U}(t)\|^2 + \nu\alpha\|\sqrt{\kappa}(\mathbf{U} \cdot \tau)(t)\|_{L^2(\partial\Omega)}^2 = 0, \end{aligned}$$

where κ denotes the curvature of $\partial\Omega$ and $\kappa = 0$ in the case of a rectangular domain, and we have invoked the fact that

$$\int_\Omega \mathbf{U} \cdot \nabla\Delta^{-1}\nabla \cdot (\Theta\mathbf{e}_2) \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{U} \cdot \mathbf{n} \Delta^{-1}\nabla \cdot (\Theta\mathbf{e}_2) \, dS(\mathbf{x}) = 0.$$

Integrating in time yields

$$\begin{aligned} \alpha\|\mathbf{U}(t)\|^2 + \|\Theta(t)\|^2 + \nu\alpha \int_0^t \|\nabla\mathbf{U}(s)\|^2 \, ds \\ + \nu\alpha \int_0^t \|\sqrt{\kappa}(\mathbf{U} \cdot \tau)(s)\|_{L^2(\partial\Omega)}^2 \, ds \\ \leq \alpha\|\mathbf{U}_0\|^2 + \|\Theta_0\|^2, \end{aligned} \quad (4.9)$$

which yields, as in the proof of Theorem 1.2,

$$\|\mathbf{U}(t)\| \rightarrow 0, \quad \|\Theta(t)\| \rightarrow C_0 < \sqrt{\alpha\|\mathbf{U}_0\|^2 + \|\Theta_0\|^2}.$$

In addition, (4.9) also implies

$$\begin{aligned} \|\nabla\mathbf{U}(t)\|^2 &\in L^1(0, \infty) \quad \text{and} \\ \|\sqrt{\kappa}(\mathbf{U} \cdot \tau)(t)\|_{L^2(\partial\Omega)}^2 &\in L^1(0, \infty). \end{aligned} \quad (4.10)$$

It remains to prove the limits

$$\begin{aligned} \|\nabla\mathbf{U}(t)\| &\rightarrow 0, \quad \|\nabla\Theta(t)\| \rightarrow C_1, \\ \|\Delta\mathbf{U}(t)\| &\rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Dotting the equation of \mathbf{U} in (1.13) with $\Delta\mathbf{U}$ yields

$$\begin{aligned} \int_{\Omega} \Delta\mathbf{U} \cdot \partial_t \mathbf{U} \, d\mathbf{x} &= \nu \|\Delta\mathbf{U}\|^2 + \int_{\Omega} \Theta \Delta V \, d\mathbf{x} \\ &\quad - \int_{\Omega} \Delta\mathbf{U} \cdot \nabla \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2) \, d\mathbf{x}. \end{aligned} \quad (4.11)$$

We integrate by parts in the first term. To be more precise, we write the term in the form of components and apply the divergence theorem,

$$\begin{aligned} \int_{\Omega} \Delta\mathbf{U} \cdot \partial_t \mathbf{U} \, d\mathbf{x} &= \int_{\Omega} \partial_k \partial_k \mathbf{U} \cdot \partial_t \mathbf{U} \, d\mathbf{x} \\ &= \int_{\Omega} \partial_k (\partial_k \mathbf{U} \cdot \partial_t \mathbf{U}) - \int_{\Omega} \partial_k \mathbf{U} \cdot \partial_t \partial_k \mathbf{U} \, d\mathbf{x} \\ &= \int_{\partial\Omega} n_k \partial_k \mathbf{U} \cdot \partial_t \mathbf{U} \, dS(\mathbf{x}) - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{U}|^2 \, d\mathbf{x}, \end{aligned}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$. To further simplify it, we write, due to $\mathbf{U} \cdot \mathbf{n} = 0$ on $\partial\Omega$,

$$\mathbf{U} = (\mathbf{U} \cdot \boldsymbol{\tau})\boldsymbol{\tau} + (\mathbf{U} \cdot \mathbf{n})\mathbf{n} = (\mathbf{U} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \quad \text{on } \partial\Omega$$

and invoke Lemma 3.2 to obtain

$$\begin{aligned} \int_{\Omega} \Delta\mathbf{U} \cdot \partial_t \mathbf{U} \, d\mathbf{x} &= \int_{\partial\Omega} (\mathbf{n} \cdot \nabla) \mathbf{U} \cdot \boldsymbol{\tau} \, \partial_t (\mathbf{U} \cdot \boldsymbol{\tau}) \, dS(\mathbf{x}) - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{U}|^2 \, d\mathbf{x} \\ &= - \int_{\partial\Omega} \kappa (\mathbf{U} \cdot \boldsymbol{\tau}) \, \partial_t (\mathbf{U} \cdot \boldsymbol{\tau}) \, dS(\mathbf{x}) - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{U}|^2 \, d\mathbf{x} \\ &= - \frac{1}{2} \frac{d}{dt} \left(\int_{\partial\Omega} \kappa (\mathbf{U} \cdot \boldsymbol{\tau})^2 \, dS(\mathbf{x}) + \int_{\Omega} |\nabla \mathbf{U}|^2 \, d\mathbf{x} \right). \end{aligned} \quad (4.12)$$

The last term in (4.11) is actually zero. In fact, by $\nabla \cdot \mathbf{U} = 0$ and the divergence theorem

$$\begin{aligned} \int_{\Omega} \Delta\mathbf{U} \cdot \nabla \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2) \, d\mathbf{x} &= \int_{\Omega} \Delta\mathbf{U} \partial_k \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2) \, d\mathbf{x} \\ &= \int_{\Omega} \partial_k (\Delta\mathbf{U} \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2)) \, d\mathbf{x} \\ &= \int_{\partial\Omega} \Delta\mathbf{U} \cdot \mathbf{n} \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2) \, d\mathbf{x}. \end{aligned}$$

Going through a similar process as in the derivation of (4.4), we have

$$\Delta\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

and thus

$$\int_{\Omega} \Delta\mathbf{U} \cdot \nabla \Delta^{-1} \nabla \cdot (\Theta \mathbf{e}_2) \, d\mathbf{x} = 0. \quad (4.13)$$

By the divergence theorem,

$$\begin{aligned} \int_{\Omega} \Theta \Delta V \, d\mathbf{x} &= \int_{\Omega} \nabla \cdot (\Theta \nabla V) \, d\mathbf{x} - \int_{\Omega} \nabla \Theta \cdot \nabla V \, d\mathbf{x} \\ &= \int_{\partial\Omega} \Theta \mathbf{n} \cdot \nabla V \, dS(\mathbf{x}) - \int_{\Omega} \nabla \Theta \cdot \nabla V \, d\mathbf{x}. \end{aligned}$$

In the case when Ω is a rectangle, $\mathbf{n} \cdot \nabla V = n_1 \partial_1 V + n_2 \partial_2 V = 0$ on the two sides of Ω , and on the top and the bottom of Ω ,

$$\Theta(\mathbf{x}, t) = \Theta_0(\mathbf{x}) + \int_0^t V(\mathbf{x}, \tau) \, d\tau = 0$$

due to $V(\mathbf{x}, \tau) = 0$ and $\Theta_0(\mathbf{x}) = 0$ on the top and the bottom. When Ω is a general domain, we need the condition that $\Theta(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \partial\Omega$. Therefore,

$$\int_{\Omega} \Theta \Delta V \, d\mathbf{x} = - \int_{\Omega} \nabla \Theta \cdot \nabla V \, d\mathbf{x}. \quad (4.14)$$

Inserting (4.12), (4.13) and (4.14) in (4.11), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\partial\Omega} \kappa (\mathbf{U} \cdot \boldsymbol{\tau})^2 \, dS(\mathbf{x}) + \int_{\Omega} |\nabla \mathbf{U}|^2 \, d\mathbf{x} \right) &+ \nu \|\Delta\mathbf{U}\|^2 \\ &= \int_{\Omega} \nabla \Theta \cdot \nabla V \, d\mathbf{x}. \end{aligned} \quad (4.15)$$

Taking the gradient of the equation of Θ in (1.13) and then dotting the resulting equation with $\nabla\Theta$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla\Theta\|^2 + \alpha \int_{\Omega} \nabla\Theta \cdot \nabla V \, d\mathbf{x} = 0. \quad (4.16)$$

Multiplying (4.15) by α and then adding to (4.16) lead to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\alpha \|\nabla\mathbf{U}\|^2 + \alpha \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})\|_{L^2(\partial\Omega)}^2 + \|\nabla\Theta\|_{L^2(\Omega)}^2 \right) &+ \nu \|\Delta\mathbf{U}\|^2 = 0. \end{aligned} \quad (4.17)$$

Integrating in time yields

$$\begin{aligned} \alpha \|\nabla\mathbf{U}\|^2 + \alpha \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})\|_{L^2(\partial\Omega)}^2 + \|\nabla\Theta\|_{L^2(\Omega)}^2 + 2\nu \int_0^t \|\Delta\mathbf{U}\|^2 \, d\tau \\ = \alpha \|\nabla\mathbf{U}_0\|^2 + \alpha \|\sqrt{\kappa} (\mathbf{U}_0 \cdot \boldsymbol{\tau})\|_{L^2(\partial\Omega)}^2 + \|\nabla\Theta_0\|_{L^2(\Omega)}^2 \equiv Z_0^2. \end{aligned} \quad (4.18)$$

As a consequence, (4.16) implies

$$\frac{d}{dt} \|\nabla\Theta\|^2 \leq \alpha \|\nabla\Theta\| \|\nabla\mathbf{U}\| \leq \sqrt{\alpha} Z_0^2$$

and thus, for any $0 \leq s < t < \infty$,

$$\left| \|\nabla\Theta(t)\|^2 - \|\nabla\Theta(s)\|^2 \right| \leq \sqrt{\alpha} Z_0^2 |t - s|.$$

In addition, integrating (4.17) in time from s to t yields

$$\begin{aligned} \alpha \|\nabla\mathbf{U}(t)\|^2 + \alpha \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})(t)\|_{L^2(\partial\Omega)}^2 + \|\nabla\Theta(t)\|_{L^2(\Omega)}^2 \\ \leq \alpha \|\nabla\mathbf{U}(s)\|^2 + \alpha \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})(s)\|_{L^2(\partial\Omega)}^2 + \|\nabla\Theta(s)\|_{L^2(\Omega)}^2, \end{aligned}$$

which especially implies

$$\begin{aligned} \alpha \left| \left(\|\nabla\mathbf{U}(t)\|^2 + \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})(t)\|_{L^2(\partial\Omega)}^2 \right) \right. \\ \left. - \left(\|\nabla\mathbf{U}(s)\|^2 + \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})(s)\|_{L^2(\partial\Omega)}^2 \right) \right| \\ \leq \left| \|\nabla\Theta(t)\|^2 - \|\nabla\Theta(s)\|^2 \right| \leq \sqrt{\alpha} Z_0^2 |t - s|. \end{aligned} \quad (4.19)$$

(4.10) and (4.19), together with Lemma 3.1, allow us to conclude that

$$\|\nabla\mathbf{U}(t)\| \rightarrow 0 \quad \text{and} \quad \|\sqrt{\kappa} (\mathbf{U} \cdot \boldsymbol{\tau})(t)\|_{L^2(\partial\Omega)}^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

and

$$\|\nabla\Theta(t)\| \rightarrow C_1 < Z_0.$$

Finally we show

$$\|\nabla\omega(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (4.20)$$

The vorticity $\omega = \nabla \times \mathbf{U}$ satisfies

$$\partial_t \omega = \nu \Delta \omega + \partial_1 \Theta. \quad (4.21)$$

Taking the inner product of (4.21) with $\Delta\omega$ yields

$$\int_{\Omega} \Delta\omega \, \partial_t \omega \, d\mathbf{x} = \nu \|\Delta\omega\|^2 + \int_{\Omega} \partial_1 \Theta \Delta\omega \, d\mathbf{x}.$$

Due to $\omega = 0$ on $\partial\Omega$,

$$\frac{1}{2} \frac{d}{dt} \|\nabla\omega\|^2 + \nu \|\Delta\omega\|^2 \leq \|\nabla\Theta\| \|\Delta\omega\|.$$

By Young's inequality,

$$\frac{1}{2} \frac{d}{dt} \|\nabla\omega\|^2 + \nu \|\Delta\omega\|^2 \leq \|\nabla\Theta\| \|\Delta\omega\| \leq \frac{\nu}{2} \|\Delta\omega\|^2 + \frac{1}{2\nu} Z_0.$$

Integrating in time yields, for any $0 \leq s < t < \infty$,

$$\left| \|\nabla\omega(t)\|^2 - \|\nabla\omega(s)\|^2 \right| \leq \frac{1}{\nu} |t - s| Z_0. \quad (4.22)$$

Due to the divergence-free condition $\nabla \cdot \mathbf{U} = 0$, $\partial_1 \omega = \Delta U_2$ and $\partial_2 \omega = -\Delta U_1$. Thus, according to (4.18),

$$\int_0^\infty \|\nabla\omega(t)\|^2 dt = \int_0^\infty \|\Delta \mathbf{U}\|^2 dt < \infty$$

or

$$\|\nabla\omega(t)\|^2 \in L^1(0, \infty).$$

Lemma 3.1 then implies (4.20). This completes the proof of linear stability.

4.3. Linear instability for $\mathcal{T}'(y) = \alpha < 0$

Finally we assume $\mathcal{T}'(y) = \alpha < 0$ and prove the instability of any periodic perturbation. If the perturbation is periodic, namely (U, Θ) is spatially periodic, then the instability can be easily seen from the Fourier transform of (1.13). In fact, taking the Fourier transform of (1.13), we obtain

$$\begin{bmatrix} \widehat{U}(k, t) \\ \widehat{V}(k, t) \\ \widehat{\Theta}(k, t) \end{bmatrix} = \begin{bmatrix} -\nu|k|^2 & 0 & -\frac{k_1 k_2}{|k|^2} \\ 0 & -\nu|k|^2 & \frac{k_1^2}{|k|^2} \\ 0 & -\alpha & 0 \end{bmatrix} \begin{bmatrix} \widehat{U}(k, t) \\ \widehat{V}(k, t) \\ \widehat{\Theta}(k, t) \end{bmatrix}.$$

When $\alpha < 0$, the coefficient matrix has a positive eigenvalue. In fact, the characteristic polynomial is given by

$$p(\lambda) = (\lambda + \nu|k|^2) \left(\lambda^2 + \nu|k|^2 \lambda + \alpha \frac{k_1^2}{|k|^2} \right)$$

and the eigenvalues of the coefficient matrix

$$\begin{aligned} \lambda_1 &= -\nu|k|^2, \\ \lambda_2 &= -\frac{1}{2}\nu|k|^2 - \frac{1}{2}\sqrt{\nu^2|k|^4 - 4\alpha \frac{k_1^2}{|k|^2}}, \\ \lambda_3 &= -\frac{1}{2}\nu|k|^2 + \frac{1}{2}\sqrt{\nu^2|k|^4 - 4\alpha \frac{k_1^2}{|k|^2}}. \end{aligned}$$

Clearly, for $\alpha < 0$, $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 > 0$. This implies any periodic perturbation is linearly unstable. This completes the proof of Theorem 1.4.

5. Numerical simulations

This section is devoted to the numerical illustration of the analytical results recorded in Theorems 1.1 through Theorem 1.4, and further investigation of some unresolved problems associated with (1.4). For numerical simplicity, we simulate the partially dissipative system subject to the stress free boundary conditions on $\Omega = [-1, 1] \times [-1, 1]$. The main purpose of this section is try to understand the explicit decay rate of the velocity field and long-time dynamics of the temperature function associated with (1.4).

5.1. Numerical method

We use a projection finite element method for \mathbf{u} and P (see [58] for details). The projection method in the strong form is

$$\begin{aligned} \frac{D\mathbf{u}^{*,k+1}}{\Delta t} - \nu \Delta \mathbf{u}^{*,k+1} &= -\nabla P^{Ex,k+1} + (0, \theta)^{Ex,k+1} - (\mathbf{u} \cdot \nabla \mathbf{u})^{Ex,k+1}, \\ \mathbf{u}^{*,k+1} &= \mathbf{u}^{k+1} + \Delta t \nabla \phi, \\ P^{k+1} &= P^{Ex,k+1} + 1.5\phi - \nu \nabla \cdot \mathbf{u}^{*,k+1}, \end{aligned}$$

where

$$\frac{D\mathbf{u}^{*,k+1}}{\Delta t} = \frac{1.5\mathbf{u}^{*,k+1} - 2\mathbf{u}^k + 0.5\mathbf{u}^{k-1}}{\Delta t}$$

is the BDF2 temporal discretization of $\frac{\partial \mathbf{u}}{\partial t}$ at time t^{k+1} , and

$$(\mathbf{u} \cdot \nabla \mathbf{u})^{Ex,k+1} = 2(\mathbf{u} \cdot \nabla \mathbf{u})^k - (\mathbf{u} \cdot \nabla \mathbf{u})^{k-1}$$

is a second order extrapolation of $(\mathbf{u} \cdot \nabla \mathbf{u})$ at time t^{k+1} . Similarly, $P^{Ex,k+1} = 2P^k - P^{k-1}$ and $\theta^{Ex,k+1} = 2\theta^k - \theta^{k-1}$.

We use the P2/P1 finite element method to solve for \mathbf{u} and P . The weak form is as follows. At time step t^{k+1} , we seek $\mathbf{u}^{k+1} \in (H^1(\Omega))^2$ with $\mathbf{u}^{k+1} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $P^{k+1} \in L^2(\Omega)$, such that for any $\mathbf{v} \in (H^1(\Omega))^2$ with $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\psi \in H^1(\Omega)$ and $Q \in L^2(\Omega)$, it holds that

$$\begin{aligned} \left(\frac{D\mathbf{u}^{*,k+1}}{\Delta t}, \mathbf{v} \right) + \nu (\nabla \mathbf{u}^{*,k+1}, \nabla \mathbf{v}) &= (P^{Ex,k+1}, \nabla \cdot \mathbf{v}) \\ &\quad + ((0, \theta)^{Ex,k+1}, \mathbf{v}) \\ &\quad - ((\mathbf{u} \cdot \nabla \mathbf{u})^{Ex,k+1}, \mathbf{v}), \\ (\nabla \phi, \nabla \psi) &= - \left(\frac{\nabla \cdot \mathbf{u}^{*,k+1}}{\Delta t}, \psi \right), \\ (P^{k+1}, Q) &= (P^{Ex,k+1} + 1.5\phi \\ &\quad - \nu \nabla \cdot \mathbf{u}^{*,k+1}, Q), \\ (\mathbf{u}^{k+1}, \mathbf{v}) &= (\mathbf{u}^{*,k+1} - \Delta t \nabla \phi, \mathbf{v}). \end{aligned}$$

Finally, the transport equation of θ is solved by a third order accurate WENO scheme [59,60]. The numerical tests show this scheme is second order accurate for velocity, pressure, and temperature in L^∞ norm (data not shown). For the long-time numerical simulation for system (1.4), we have used the resolutions 50×50 , 100×100 , and 200×200 . When the time exceeds 100 730, all these simulations exhibit some instabilities whose results might be not reliable anymore. But before this time, we observe convergence when the mesh is refined. What we present here are the results from 200×200 resolution before time = 100 730.

5.2. Decay of velocity field

In this subsection, we numerically illustrate the analytical results obtained in Theorem 1.2, regarding the decay of the velocity field associated with (1.4). Numerical results are presented in Fig. 1.

In the simulation of (1.4), we choose $u_0(x, y) = \sin(2\pi x) \cos(2\pi y)$, $v_0(x, y) = -\cos(2\pi x) \sin(2\pi y)$, $\theta_0 = \sin(3x) \cos(2y) + (x - 0.5)^3 + 1/(y + 10)$, and $\nu = 1$. By Theorem 1.2, the kinetic energy and the first order derivatives of the velocity field converge to zero. Fig. 1 plots the evolution of various norms of the solution for time up to $t = 100\,730$, which illustrates that $\|\mathbf{u}(t)\|_{H^1}$, $\|\partial_t \mathbf{u}(t)\|$ and $\|\mathbf{u}(t)\|_{L^\infty}$ asymptotically approach zero. More interestingly, from Fig. 1 we see that although the decay of the velocity field exhibits a slightly oscillatory fashion, the bulk of the curves presents algebraic decay rates by comparing with polynomial functions, which are slower than exponential decay rates.

5.3. Evolution of pressure and temperature

In this subsection, we numerically illustrate the time evolution of the pressure and temperature functions. Numerical results are presented in Figs. 2 and 3.

Figs. 2 and 3 show that the pressure and temperature become horizontally homogeneous and stratify in the vertical direction as time evolves – a scenario that is consistent with the analytical results recorded in Theorem 1.2, especially the third point of (1.8). The numerical results in Fig. 3 show three stages for the heat to get to the stratification. The first is the interior bulk convection as shown from time = 0 to time = 20, where the bulk red region of

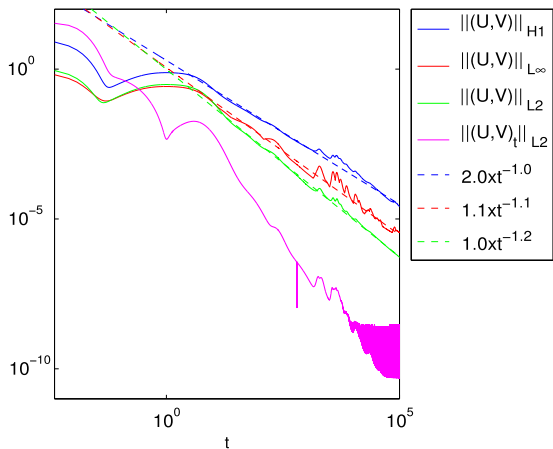


Fig. 1. Decay of velocity field of numerical solution of (1.4) on $[-1, 1] \times [-1, 1]$ with boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\omega|_{\partial\Omega} = 0$, initial data $u_0(x, y) = \sin(2\pi x) \cos(2\pi y)$, $v_0(x, y) = -\cos(2\pi x) \sin(2\pi y)$, $\theta_0(x, y) = \sin(3x) \cos(2y) + (x - 0.5)^3 + 1/(y + 10)$, and $\nu = 1$. This figure plots the evolution of various norms of the velocity for time up to $t = 100\,730$. The dashed lines are the least-squares fittings to the corresponding curves of the same color. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

high temperature and blue region of low temperature are swept to the upper and lower parts of the domain, respectively. The second is boundary layer eruption. One relatively low temperature layer is held on the top wall (Fig. 3, time = 20) and the slip boundary condition ($\mathbf{u} \cdot \mathbf{n} = 0$) is unable to move this layer down to the corresponding positions in the desired stratification. However, driven by gravity, the heat erupts from some separate boundary spots into the interior domain whose pattern exhibits the Rayleigh–Taylor instability (Fig. 3, from time = 2230 to 10730). This stage actually proceeds from time = 1600 to time = 40 000 in the numerical simulation. Finally, after all the eruptions are emitted, the whole system enters a relaxation stage. On account of the complication produced by this instability, we remark that the identification of the exact ansatz of the pressure and temperature functions is still elusive.

To look into the stratification in more detail, we define $T(y, t) = \int_{\Omega} \theta(x, y, t) dx / 2$, the average value of temperature at a y -layer. We

plotted the convergence behavior of $\theta(x, y, t)$ to $T(y, t)$ in Fig. 4[a, b, c]. The L^2 norm shows almost monotone decreasing but the L^∞ norm shows overall very slow decreasing with oscillations. The profile of $T(y, t)$ shows monotonic increasing in y -direction at time = 10^5 (Fig. 4[b, c]). To study the convergence of the mechanical energy, we take the field $T(y, t)$ at $t = 120\,830$ as the reference state. Fig. 4[d] shows the difference of the total mechanical energy (kinetic + potential) at time $\leq 10^5$ and that of the reference state. This indicates a power-rule decreasing of the energy.

6. Conclusion and looking ahead

We have studied the global well-posedness and long-time asymptotic behavior of large-amplitude classical solutions to the 2D Boussinesq equations without thermal diffusion on bounded domains with non-smooth boundaries and subject to the stress-free velocity boundary conditions. Utilizing energy methods we showed that for initial data with low regularity, there exist unique global-in-time solutions to the initial–boundary value problems (IBVPs) of the model, and the L^2 norms of the velocity field and its first order spatial and temporal derivatives converge to zero as $t \rightarrow \infty$. Consequently we found that the pressure and temperature functions stratify in the vertical direction in a weak topology. Moreover, we established the linear stability of the hydrostatic equilibrium $\mathcal{T}(y) = \alpha y + \bar{\theta}$ when $\alpha > 0$, and the instability of periodic perturbations of the ansatz when $\alpha < 0$. Our results indicate how buoyancy plays a dominant role when thermal diffusion is negligible, and partially demonstrate the effectiveness of the 2D Boussinesq equations for modeling stratification-dominated situations in the applied sciences. In addition, the analytical approach developed for proving the long-time behavior results is of independent interest and may be adopted to study other PDE systems with similar partially dissipative structures.

Finally we mention that several fundamental questions regarding the long-time behavior of large-amplitude classical solutions to the partially dissipative system (1.4) remain open. We list three of them:

- **Explicit decay rates.** An important piece of information that is missing from our analytical results is the explicit decay rate of the velocity field or the total mechanical energy. This is mainly due to the lack of dissipation in the temperature equation. Unlike the fully dissipative free energy

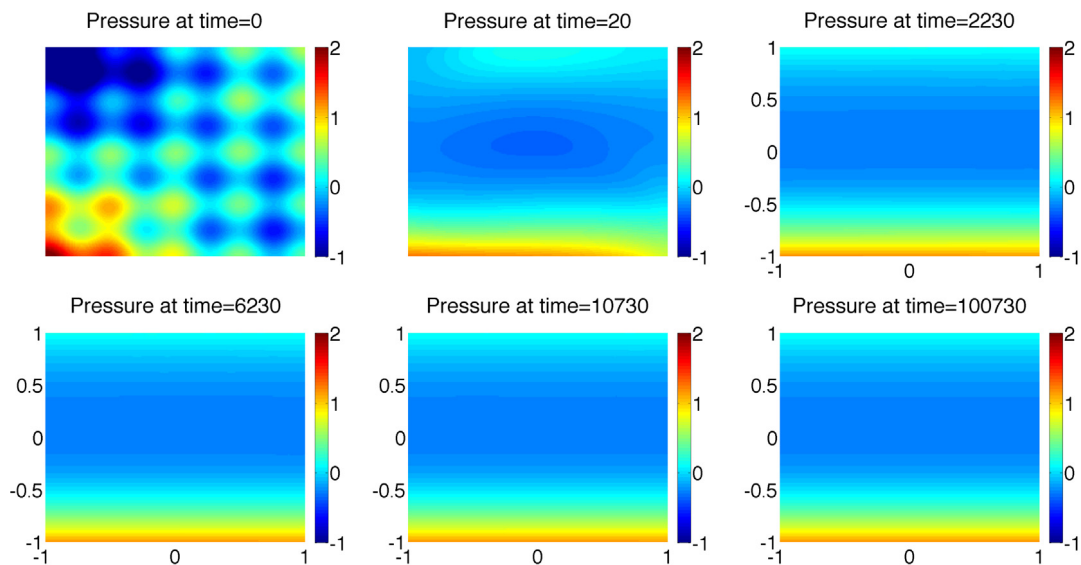


Fig. 2. Numerical solutions of the pressure function of (1.4) subject to the stress free boundary conditions on $[-1, 1] \times [-1, 1]$ at several time steps.

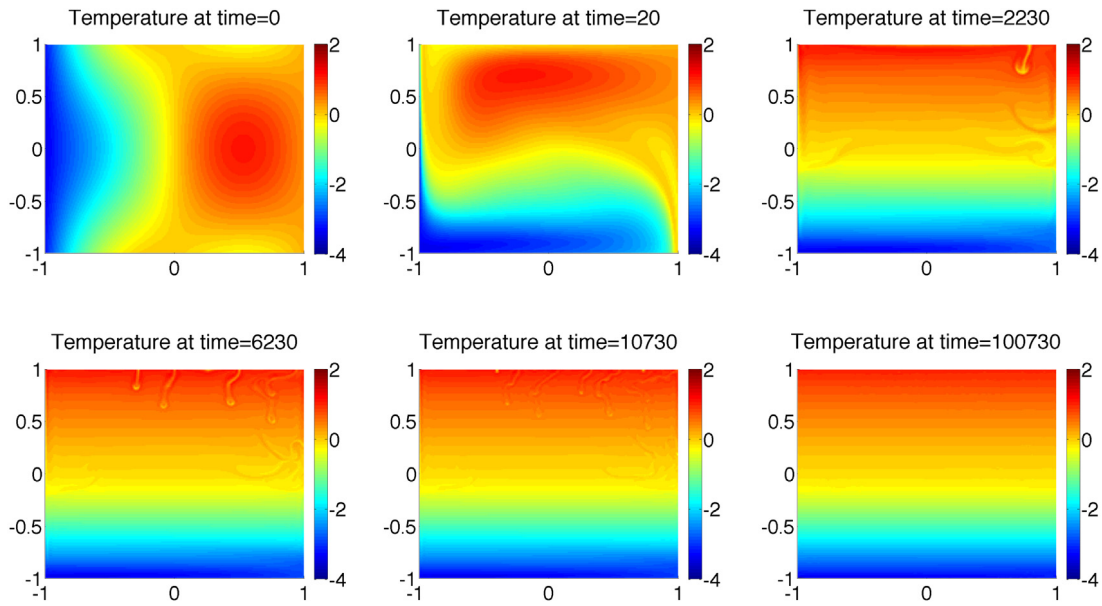


Fig. 3. Numerical solutions of the temperature function of (1.4) subject to the stress free boundary conditions on $[-1, 1] \times [-1, 1]$ at several time steps. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

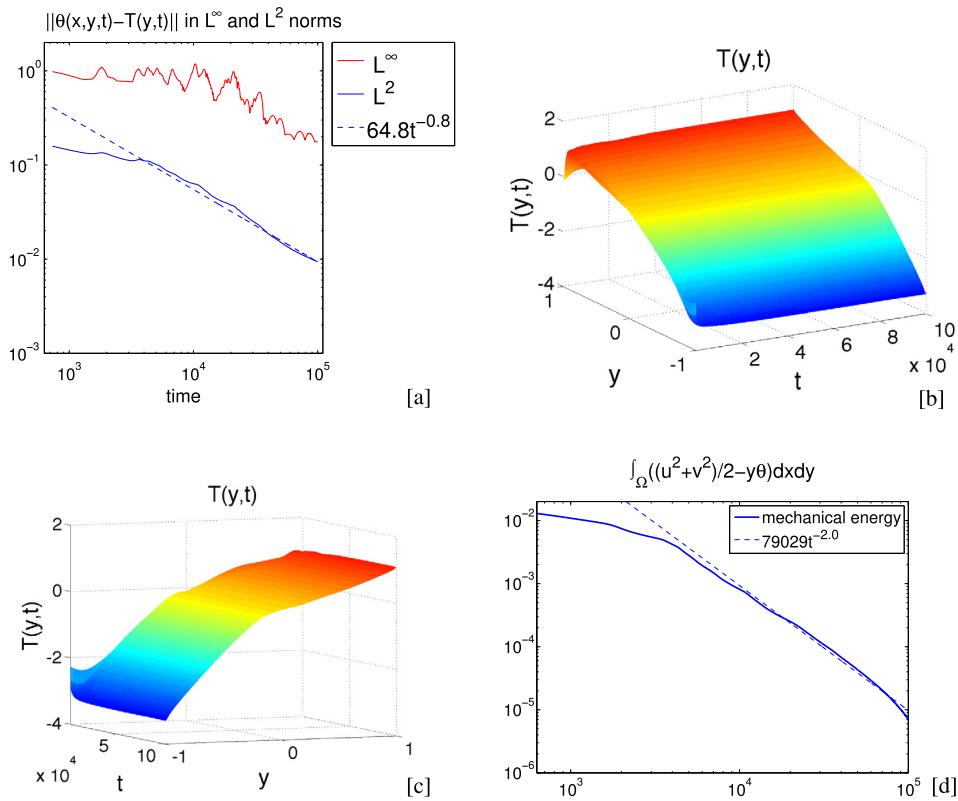


Fig. 4. Numerical results. [a] Convergence of $\theta(x, y, t)$ to $T(y, t)$ over time. [b, c] Plots of the function $T(y, t)$ from two viewpoints. [d] Difference of total mechanical energy before time $\leq 10^5$ with that at time = 120830.

formulation (1.2) which generates exponential decay of the perturbation, the Lyapunov functional associated with the partially dissipative system (1.15) is not capable of bringing about any explicit decay rate of the perturbation. However, our numerical simulation indicates that the velocity field might converge to zero as a power law which provides a

strong hint for a rigorous proof of the phenomenon. We leave this investigation for the future.

- **Thermal structure and stability of the final state.** Another piece of information missing from Theorem 1.2 is a precise description of the final buoyancy distribution in case of general initial conditions. In fact the buoyancy field is just

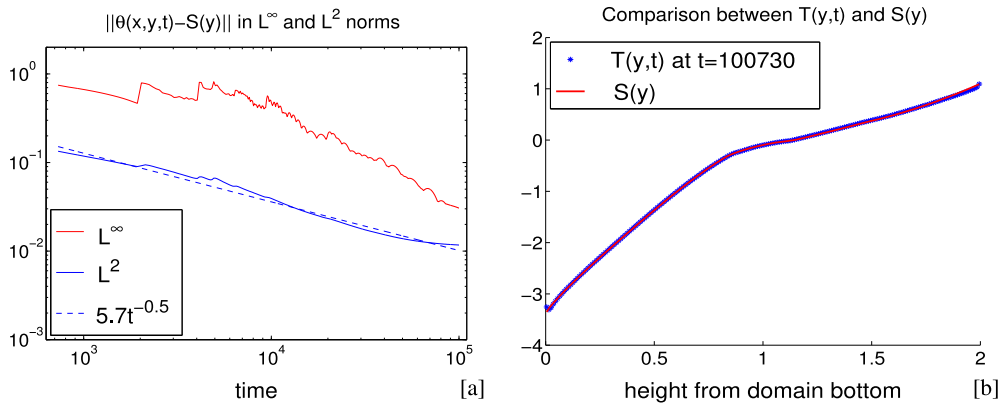


Fig. 5. [a] Convergence of $\theta(x, y, t)$ to $S(y)$. The fitting of L^2 error uses a power function of $5.7t^{-0.5}$ with least squares method. [b] Comparison between $T(y, t)$ at $t = 100730$ and $S(y)$. Here, $T(y, t) = \int_{\Omega} \theta(x, y, t) dx/2$ and $\Omega = [0, 1]^2$.

rearranged by the flow so the area of the domain where $\theta(\mathbf{x}, t)$ ranges between any two values is conserved. This suggests that the final state of the relaxation problem studied here should generically be the unique stably stratified distribution compatible with all these constraints.

We provide a formula of the stably stratified state $\hat{\theta}(y)$: in a rectangular domain $\Omega = [0, W] \times [0, H]$, for any initial value $\theta_0(x, y)$, define a height function of temperature value a as

$$h(a) = \frac{1}{W} \text{area}\{(x, y) \in \Omega : \theta_0(x, y) \leq a\}. \quad (6.1)$$

For a general domain Ω whose horizontal width is given by $w(y)$ at a height y , this formula becomes $\int_0^{h(a)} w(y) dy = \text{area}\{(x, y) \in \Omega : \theta_0(x, y) \leq a\}$. The final stably stratified state stratification function, call it $\hat{\theta}(y)$, is then the inverse function of $h(a)$:

$$\hat{\theta}(y) = S(y) \triangleq h^{-1}(y).$$

The reasoning behind these formulas is as follows. First, because of mass conservation, $h(a)$ must be a constant over time for any a . Second, we speculate that the ultimate final temperature profile must be nondecreasing in the final state. Note the formula (6.1) satisfies this condition. It remains to be shown that any other stratified states compatible with the initial data are linearly unstable, which does not rule out their emergence from properly prepared initial data but suggests a unique final state evolving from generic conditions. For the numerical test presented in the last section, we computed the final state $S(y)$ numerically and tested the convergence of $\theta(x, y, t)$ to it. The results are shown in Fig. 5. Note the L^2 norm of $\theta(x, y, t) - S(y)$ decays as a power rule again and the mean value $T(y, t)$ agrees well with $S(y)$, which is a nonlinear function of y , when time is large. This demonstrates $\hat{\theta}(y) = S(y)$ is indeed the thermal structure of the final state.

- **Infinite Prandtl number convection.** The ratio of a fluid's kinematic viscosity ν to its thermal diffusion coefficient κ is the Prandtl number $Pr = \nu/\kappa$ so the $\kappa = 0$ model studied here corresponds to an infinite Prandtl number limit of the Boussinesq system.

Traditionally (see, e.g., [61]) the time t is rescaled to the dimensionless variable $\kappa t/\ell^2$ where ℓ is an appropriate domain length scale (e.g., the height of a rectangular domain or the thickness of a spherical shell) and velocities are measured in units of κ/ℓ . Then, suitably rescaling buoyancy and pressure, the formal $Pr \rightarrow \infty$ limit in a stress-free thermally

insulating container is

$$\begin{cases} \nabla P = \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \Delta \theta, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \\ \mathbf{n} \cdot \nabla \theta|_{\partial\Omega} = 0. \end{cases}$$

This is a very different system than (1.4), ostensibly describing dynamics on even longer $\mathcal{O}(\ell^2/\kappa)$ time scales than those captured by the strictly non-diffusive model: diffusion is still effective for the temperature on these time scales while the flow is totally viscous dominated. Boundary conditions are still required for the temperature field, but no initial data is required for the velocity field. The dynamics modeled by the system (1.4) studied in this paper, corresponding to measuring time on viscous ($\mathcal{O}(\ell^2/\nu)$) time scales, would be considered an “initial layer” problem for these infinite Prandtl number equations of motion.

Alternatively we could consider dynamics on an intermediate, say $\mathcal{O}(\ell^2/\sqrt{\nu\kappa})$, time scale in which case the formal $Pr \rightarrow \infty$ limit for appropriately rescaled dependent variables is

$$\begin{cases} \nabla P = \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0 \end{cases}$$

requiring neither boundary conditions for the temperature nor initial data for the flow. It is natural to wonder to what extent the long time behavior of the relaxation on viscous time scales might be quantitatively captured by an “over damped” model like this – which keeps the conserved quantities in play for the temperature – keeping in mind that the initial data for the temperature immediately above is *not* simply proportional to the initial temperature distribution in (1.4). Rather, $\theta_0(\mathbf{x})$ for this system would correspond to the sort of rapidly-rearranged temperature distribution seen in the simulations (like that seen around time 20 in Fig. 3). For our purposes the real value of such a reduced model might be to facilitate accurate estimation of the final state and observed kinetic and/or total energy relaxation kinetics.

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