J. Math. Fluid Mech. 20 (2018), 1541–1565

1422-6928/18/041541-25
https://doi.org/10.1007/s00021-018-0376-3

Journal of Mathematical
Fluid Mechanics

Global Regularity and Time Decay for the 2D Magnetohydrodynamic Equations with Fractional Dissipation and Partial Magnetic Diffusion

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Communicated by Roman Shvydkoy

Abstract. This paper focuses on a system of the 2D magnetohydrodynamic (MHD) equations with the kinematic dissipation given by the fractional operator $(-\Delta)^\alpha$ and the magnetic diffusion by partial Laplacian. We are able to show that this system with any $\alpha > 0$ always possesses a unique global smooth solution when the initial data is sufficiently smooth. In addition, we make a detailed study on the large-time behavior of these smooth solutions and obtain optimal large-time decay rates. Since the magnetic diffusion is only partial here, some classical tools such as the maximal regularity property for the 2D heat operator can no longer be applied. A key observation on the structure of the MHD equations allows us to get around the difficulties due to the lack of full Laplacian magnetic diffusion. The results presented here are the sharpest on the global regularity problem for the 2D MHD equations with only partial magnetic diffusion.

Mathematics Subject Classification. 35Q35, 35B40, 35B65, 76B03.

Keywords. MHD equations, Global regularity, Optimal decay rates, Partial dissipation, Time decay.

1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They consist of a coupled system of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Since their initial derivation by the Nobel Laureate H. Alfvén in 1924, the MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [3,12]). The standard incompressible MHD equations can be written as

$$\begin{aligned} &u_t + u\cdot\nabla u = -\nabla\pi + \mu\Delta u + b\cdot\nabla b,\\ &b_t + u\cdot\nabla b = \eta\Delta b + b\cdot\nabla u,\\ &\nabla\cdot u = 0, \quad \nabla\cdot b = 0, \end{aligned} \tag{1.1}$$

where u denotes the velocity field, b the magnetic field, π the pressure, $\nu \geq 0$ the kinematic viscosity and $\eta \geq 0$ the magnetic diffusivity. Besides their wide physical applicability, the MHD equations are also of great interest in mathematics. As a coupled system, the MHD equations contain much richer structures than the Navier–Stokes equations. They are not merely a combination of two parallel Navier–Stokes type equations but an interactive and integrated system.

One of the fundamental problems concerning the MHD equations is whether physically relevant regular solutions remain smooth for all time or they develop finite time singularities. In recent years the MHD regularity problem has attracted considerable interests and one focus has been on the 2D MHD equations with partial or fractional dissipation. Important progress has been made (see, e.g, [4–10,13–17,19–23, 25,27,28,30,34–48]). The MHD equations with partial or fractional dissipation is physically relevant and mathematically important. One special partial dissipation case is the 2D resistive MHD equations, namely

$$\begin{aligned} &u_t + u \cdot \nabla u = -\nabla \pi + b \cdot \nabla b, \\ &b_t + u \cdot \nabla b = \eta\, \Delta b + b \cdot \nabla u, \\ &\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{aligned} \tag{1.2}$$

where $\eta > 0$ denotes the magnetic diffusivity (resistivity). Equation (1.2) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence. Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system such as solar flares and northern lights. The mathematical study of (1.2) may help understand the Sweet-Parker model arising in magnetic reconnection theory [29]. Although the global regularity problem on (1.2) is not completely solved at this moment, recent efforts on this problem have significantly advanced our understanding.

In certain physical regimes and under suitable scaling, the full Laplacian dissipation is reduced to a partial dissipation. One notable example is the Prandtl boundary layer equation in which only the vertical dissipation is included in the horizontal component (see, e.g., [33]). This paper focuses on a system of the 2D MHD equations that is closely related to (1.2),

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \mu(-\Delta)^\alpha u + \nabla \pi = b \cdot \nabla b, \\ &\partial_t b_1 + u \cdot \nabla b_1 - \kappa \partial_{22} b_1 = b \cdot \nabla u_1, \\ &\partial_t b_2 + u \cdot \nabla b_2 - \kappa \partial_{11} b_2 = b \cdot \nabla u_2, \\ &\nabla \cdot u = \nabla \cdot b = 0. \end{aligned} \tag{1.3}$$

The velocity equation in (1.3) involves a fractional Laplacian dissipation. The Navier–Stokes equations with fractional dissipation can be derived from the Boltzmann equation under suitable scaling [18]. The equation of the magnetic field contains only partial dissipation. We attempt to achieve two main goals: first, to prove the global existence and regularity of solutions of (1.3) by fully exploiting the special structure of this system, and second, to develop a systematic approach for systems with partial dissipation to extract large-time decay rates for solutions of (1.3).

When the partial magnetic diffusion in (1.3) is replaced by full Laplacian dissipation, the global regularity has been obtained in [14] by using the maximal regularity property of the 2D heat operator. However, when there is only partial dissipation, some of the classical analysis tools such as the maximal regularity estimates can no longer be applied. To be more precise, we make a comparison between (1.2) and (1.3). In terms of the full Laplacian operator in (1.2), we can write the equation of b in (1.2) as

$$b(t) = e^{\eta t \Delta} b_0 + \int_0^t e^{\eta(t-s)\Delta} (b \cdot \nabla u - u \cdot \nabla b)(s)\, ds.$$

Applying the maximal regularity of the 2D heat operator and combining with the equation of the vorticity $\omega \equiv \nabla \times u$,

$$\partial_t \omega + u \cdot \nabla \omega + \mu(-\Delta)^\alpha \omega = b \cdot \nabla j$$

with $j = \nabla \times b$ being the current density, we can show that, for any $1 < p, q < \infty$ and $T > 0$,

$$\Delta b \in L^q(0, T; L^p(\mathbb{R}^2))\, and \quad \omega \in L^\infty(0, T; L^p(\mathbb{R}^2)).$$

More details can be found in [14,21]. In contrast, the equation of b in (1.3) involves only partial dissipation and the maximal regularity of the 2D heat operator no longer applies. We need a new approach and a key new observation is the special structure in the equation of b. This observation allows us to obtain similar global *a priori* bounds on solutions to (1.3) as those previously obtained for (1.2), namely

$$\|\Delta b\|_{L^q(0,T;L^p)} \le C(T, u_0, b_0) \quad \text{and} \quad \|\omega\|_{L^\infty(0,T;L^p)} \le C(T, u_0, b_0)$$

for any $1 < p, q < \infty$ and $T > 0$. We explain this more explicitly. By using the 1D heat operator, we can write the equations of b_1 and b_2 (components of b) as

$$b_1(t) = e^{\kappa t \partial_{22}} b_0 + \int_0^t e^{\kappa(t-s)\partial_{22}} (b \cdot \nabla u_1 - u \cdot \nabla b_1)(s)\, ds,$$
$$b_2(t) = e^{\kappa t \partial_{11}} b_0 + \int_0^t e^{\kappa(t-s)\partial_{11}} (b \cdot \nabla u_2 - u \cdot \nabla b_2)(s)\, ds.$$

The maximal regularity of the 1D heat operator only leads to the global bounds

$$\|\partial_{22} b_1\|_{L^q(0,T;L^p)} \leq C(T, u_0, b_0) \quad \text{and} \quad \|\partial_{11} b_2\|_{L^q(0,T;L^p)} \leq C(T, u_0, b_0).$$

A key observation on the structure of the equation of b allows us to prove global bounds for $\|\partial_{12} b_1\|_{L^q(0,T;L^p)}$ and $\|\partial_{12} b_2\|_{L^q(0,T;L^p)}$. More precisely, combining the representation

$$\partial_{12} b_1(t) = \partial_{12} e^{\kappa t \partial_{22}} b_0 + \int_0^t e^{\kappa(t-s)\partial_{22}} \partial_{12} (b \cdot \nabla u_1 - u \cdot \nabla b_1)(s)\, ds$$

with the special structure

$$\begin{aligned} \partial_{12}\left(b \cdot \nabla u_1 - u \cdot \nabla b_1\right) &= \partial_{12}\left(\partial_1(b_1 u_1) + \partial_2(b_2 u_1) - \partial_1(u_1 b_1) - \partial_2(u_2 b_1)\right) \\ &= \partial_{22}\left(\partial_1(b_2 u_1) - \partial_1(u_2 b_1)\right) \end{aligned}$$

leads to the desired global bound for $\|\partial_{12} b_1\|_{L^q(0,T;L^p)}$. The bound for $\|\partial_{12} b_2\|_{L^q(0,T;L^p)}$ is similarly obtained.

A special consequence of these global bounds is the following global existence and regularity result on (1.3).

Theorem 1.1. *Consider (1.3) with $\mu > 0$, $0 < \alpha \leq 1$ and $\kappa > 0$. Assume the initial data $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s \geq 3$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then (1.3) has a unique global solution (u, b) satisfying, for any $T > 0$,*

$$u \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2)),$$
$$b \in C([0,\infty); H^s(\mathbb{R}^2)), \quad \partial_2 b_1, \partial_1 b_2 \in L^2(0,T; H^s(\mathbb{R}^2)).$$

Our second main result details the large-time behavior of the global solutions obtained in Theorem 1.1 and provides explicit decay rates for various norms of the solutions. We have been aiming at developing effective approaches to understand the large-time behavior of partially dissipated systems. Systematic procedures such as the Fourier splitting method of Schonbek have been developed to deal with various fully dissipative partial differential equations and a very rich array of results have been established (see, e.g, [1,31]). To extend these results to partially dissipated systems, we need to overcome some major difficulties. For example, the Fourier splitting method does not directly apply to partially dissipated systems. Here we fully exploit the special structure of the system in (1.3) and are able to prove the following large-time decay results.

Theorem 1.2. *Consider (1.3) with $\mu > 0$, $\kappa > 0$ and $0 < \alpha < \frac{1}{2}$. Assume $(u_0, b_0) \in H^s$ with $s \geq 3$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.3) as stated in Theorem 1.1. If the initial data (u_0, b_0) satisfies*

$$|\widehat{u}_0(\xi)| \leq C\sqrt{|\xi|}, \quad \|\widehat{b}_{01}(\xi)\|_{L^2_{\xi_1}} \leq C\sqrt{|\xi_2|} \quad \textit{and} \quad \|\widehat{b}_{02}(\xi)\|_{L^2_{\xi_2}} \leq C\sqrt{|\xi_1|}, \tag{1.4}$$

then (u, b) obeys, for any $t > 0$,

$$\|b(t)\|_{L^2} \leq C\,(1+t)^{-\frac{1}{2}}, \quad \|\nabla b(t)\|_{L^2} \leq C\,(1+t)^{-1};$$

and

$$\|u(t)\|_{L^2} \leq C(1+t)^{-(\frac{3}{2}-\frac{\alpha}{2})}, \quad \|\nabla u(t)\|_{L^2} \leq C(1+t)^{-1-\frac{\alpha}{2}}.$$

We remark that the assumptions in (1.4) are typical in the study of large-time behavior of solutions (see, e.g., [31,32]). Functions in suitable Sobolev spaces of negative indices would fulfill these assumptions. Since our main focus here is to understand the large-time behavior of the nonlinear terms under the fractional and partial dissipation, these assumptions are made here to coordinate this main goal. The proof of Theorem 1.2 is divided into several steps. The first step shows that

$$(1+t)\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\right)\to 0 \quad \text{as} \quad t\to\infty,$$

which follows from the uniform global H^1 bounds for (u,b) and the time integrability of $\|\Lambda^\alpha u\|_{L^2}^2$, $\|(\partial_2 b_1,\partial_1 b_2)\|_{L^2}^2$ and $\|\Lambda^\alpha\omega\|_{L^2}^2$. The second step asserts that, for any $\epsilon>0$,

$$\|(u(t),b(t))\|_{L^2}\le C(1+t)^{-\frac{1}{2}+\epsilon} \quad \text{for any } \ t>0.$$

To prove this decay rate, we represent u, b_1 and b_2 in integral forms and make use of the special structures of the equations of b_1 and b_2. By differentiating the integral representations of b_1 and b_2, dividing the time integral into several pieces and taking advantage of the special structure of the equations of b_1 and b_2, we further show that, for any $t>0$,

$$\|\nabla b(t)\|_{L^2}\le C(1+t)^{-1},$$

which in turn allows us to improve the decay rate for $\|b(t)\|_{L^2}$,

$$\|b(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}}.$$

The decay rates for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$ are obtained by applying and generalizing the Fourier splitting method of Schonbek [31].

The rest of the paper is divided into two sections. Section 2 proves Theorem 1.1 while Sect. 3 supplies the proof of Theorem 1.2.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, the global existence and uniqueness of smooth solution to (1.3). The key component of the proof is the global *a priori* bound for (u,b) in H^s with $s\ge 3$. For the sake of clarity, we divide the estimates into several regularity levels. The first subsection proves the global H^1 bound, the second subsection establishes L^q-bounds for ω and Δb for any $q\in(1,\infty)$ and the third subsection provides the global bound for $\|\nabla u\|_{L^\infty}$ and finishes the proof of Theorem 1.1.

2.1. Global H^1 Estimate

We prove that any classical solution of (1.3) admits a global H^1-bound, as stated in the following proposition.

Proposition 2.1. *Assume $\alpha>0,\mu\ge 0,\kappa>0$ and $(u_0,b_0)\in H^1(\mathbb{R}^2)$ with $\nabla\cdot u_0=\nabla\cdot b_0=0$. Then the corresponding solution (u,b) obeys the following uniform bounds, for any $0<t<\infty$,*

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+2\int_0^t\left(\mu\|\Lambda^\alpha u\|_{L^2}^2 ds+\kappa\|\partial_2 b_1\|_{L^2}^2+\kappa\|\partial_1 b_2\|_{L^2}^2\right)ds\le C, \tag{2.1}$$

$$\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+2\mu\int_0^t\|\Lambda^\alpha\omega(s)\|_{L^2}^2 ds+\int_0^t H(b,s)ds\le C, \tag{2.2}$$

where $\Lambda^s=(-\Delta)^{s/2}$,

$$H(b,t)=\kappa\int_{\mathbb{R}^2}\left((\partial_{11}b_2)^2+(\partial_{11}b_1)^2+(\partial_{22}b_2)^2+(\partial_{22}b_1)^2\right)dx$$

and C's are positive constants depending on κ and $\|(u_0,b_0)\|_{H^1}$ only.

Proof. Taking the L^2 inner product of (1.3) with (u, b) and integrating in time yield

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\mu \int_0^t \|\Lambda^\alpha u(s)\|_{L^2}^2 ds + 2\kappa \int_0^t \left(\|\partial_2 b_1(s)\|_{L^2}^2 + \|\partial_1 b_2(s)\|_{L^2}^2\right) ds \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2,$$

which is (2.1). The vorticity $\omega = \nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and the current density $j = \nabla \times b = \partial_1 b_2 - \partial_2 b_1$ satisfy

$$\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega + \mu(-\Delta)^\alpha \omega = b \cdot \nabla j, \\ &\partial_t j + u \cdot \nabla j - \kappa \partial_{111} b_2 + \kappa \partial_{222} b_1 = b \cdot \nabla \omega + T(\nabla u, \nabla b), \end{aligned} \tag{2.3}$$

where $T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)$. Taking the inner products of the vorticity equation in (2.3) with ω and the current density equation with j, adding the results and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2\right) + \mu\|\Lambda^\alpha \omega\|_{L^2}^2 + I = \int_{\mathbb{R}^2} T\, j\, dx,$$

where

$$I = \kappa \int_{\mathbb{R}^2} (-\partial_{111} b_2 + \partial_{222} b_1) j\, dx.$$

We first show $I = H(b, t)$. In fact,

$$\begin{aligned} I &= \kappa \int_{\mathbb{R}^2} (-\partial_{111} b_2 + \partial_{222} b_1)(\partial_1 b_2 - \partial_2 b_1) dx \\ &= \kappa \int_{\mathbb{R}^2} (-\partial_{111} b_2 \partial_1 b_2 + \partial_{111} b_2 \partial_2 b_1 + \partial_{222} b_1 \partial_1 b_2 - \partial_{222} b_1 \partial_2 b_1) dx \\ &= \kappa \int_{\mathbb{R}^2} \left((\partial_{11} b_2)^2 + (\partial_{11} b_1)^2 + (\partial_{22} b_2)^2 + (\partial_{22} b_1)^2\right) dx \equiv H(b, t) \end{aligned}$$

due to $\partial_1 b_1 + \partial_2 b_2 = 0$. By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned} \int_{\mathbb{R}^2} T j dx &\leq C \,\|\nabla u\|_{L^2} \|\nabla b\|_{L^4} \|j\|_{L^4} \\ &\leq C \,\|\omega\|_{L^2} \|j\|_{L^4}^2 \leq C(\kappa)\, \|\omega\|_{L^2}^2 \|j\|_{L^2}^2 + \frac{\kappa}{8}\|\nabla j\|_{L^2}^2, \end{aligned}$$

where we have used the fact that the Calderon-Zygmund operators are bounded on L^p $(1 < p < \infty)$. It is easy to verify that

$$\frac{\kappa}{4}\|\nabla j\|_{L^2}^2 \leq H(b, t).$$

Indeed,

$$\begin{aligned} \kappa\|\nabla j\|_{L^2}^2 &= \kappa\|(\partial_1 j, \partial_2 j)\|_{L^2}^2 = \kappa\|((\partial_{11} b_2 - \partial_{12} b_1), (\partial_{12} b_2 - \partial_{22} b_1))\|_{L^2}^2 \\ &= \kappa\|((\partial_{11} b_2 + \partial_{22} b_2), (-\partial_{11} b_1 - \partial_{22} b_1))\|_{L^2}^2 \leq 4H(b, t). \end{aligned}$$

Combining the estimates above yields

$$\frac{d}{dt}\left(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2\right) + 2\mu\|\Lambda^\alpha \omega\|_{L^2}^2 + H(b, t) \leq C(\kappa)\|\omega\|_{L^2}^2 \|j\|_{L^2}^2.$$

Gronwall's inequality, together with the fact

$$\|j\|_{L^2}^2 \leq 2\|\partial_1 b_2\|_{L^2}^2 + 2\|\partial_2 b_1\|_{L^2}^2$$

allows us to conclude that

$$\begin{aligned}&\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+2\mu\int_0^t\|\Lambda^\alpha\omega(s)\|_{L^2}^2ds+\int_0^t H(b,s)ds\\&\le\left(\|\omega_0\|_{L^2}^2+\|j_0\|_{L^2}^2\right)\exp\left\{C(\kappa)\int_0^t\|j(s)\|_{L^2}^2ds\right\}\\&\le\left(\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2\right)\exp\left\{C(\kappa)\left(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2\right)\right\}.\end{aligned}$$

This completes the proof of Proposition 2.1. □

2.2. L^q-Bounds for ω and Δb with $q\in(1,\infty)$

This subsection presents the global bound for $\|\Delta b\|_{L_t^2L_x^q}$ and $\|\omega\|_{L_t^\infty L_x^q}$ for $q\in(1,\infty)$. As aforementioned in the introduction, due to the lack of full Laplacian dissipation, the maximal regularity estimate for the 2D heat operator can not be used here. Instead, we make use of a key observation on the special structure of the MHD equations and the maximal regularity estimate for the 1D heat operator. We remark this step does not allow us to obtain the global bounds for $q=\infty$.

Proposition 2.2. *Assume that u_0 and b_0 satisfy the conditions in Theorem 1.1. Let (u,b) be the corresponding solution of the (1.3). Then (u,b) obeys, for any $q\in(1,\infty)$ and any $0<t<\infty$,*

$$\|\Lambda^{1+\alpha}b\|_{L^\infty(0,t;L^2)},\ \|\Delta b\|_{L^2(0,t;L^q)},\quad \|\omega\|_{L^\infty(0,t;L^q)}\le C(t,u_0,b_0). \tag{2.4}$$

To prove this proposition, we recall the maximal regularity property for the heat operator (see, e.g., [2], [26, p.64]).

Lemma 2.3. *Assume $K_d(x,t)$ is the heat kernel of d-dimensional heat equation*

$$K_d(x,t)=(4\pi t)^{-\frac d2}e^{-\frac{|x|^2}{4t}}$$

and define the operator A as

$$Af(x,t)\equiv\int_0^t\int_{\mathbb{R}^d}K_d(y,s)\Delta_x f(x-y,t-s)\,dyds.$$

Then, for any $T\in(0,\infty]$ and $p,q\in(1,\infty)$, the operator A maps $L^p(0,T;L^q(\mathbb{R}^d))$ to $L^p(0,T;L^q(\mathbb{R}^d))$.

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. We first bound $\|\Lambda^{1+\alpha}b\|_{L^\infty(0,t;L^2)}$. Taking the inner product of magnetic equations in (1.3) with $\Lambda^{2+2\alpha}b$ leads to

$$\frac12\frac{d}{dt}\|\Lambda^{1+\alpha}b(t)\|_{L^2}^2+\frac{\kappa}{2}\|\Lambda^{2+\alpha}b\|_{L^2}^2\le\int_{\mathbb{R}^2}(b\cdot\nabla u-u\cdot\nabla b)\Lambda^{2+2\alpha}b\,dx, \tag{2.5}$$

where we have used the following observation

$$\begin{aligned}&\|\Lambda^{1+\alpha}\partial_2 b_1\|_{L^2}^2+\|\Lambda^{1+\alpha}\partial_1 b_2\|_{L^2}^2\\&\ge\frac12\|\Lambda^{1+\alpha}(\partial_1b_2-\partial_2b_1)\|_{L^2}^2=\frac12\|\Lambda^{1+\alpha}j\|_{L^2}^2\ge\frac12\|\Lambda^{2+\alpha}b\|_{L^2}^2.\end{aligned}$$

Applying Hölder inequality, Proposition 2.1 and Young's inequality, the right hand side of (2.5) can be bounded by

$$\begin{aligned}&\int_{\mathbb{R}^2}(b\cdot\nabla u)\cdot\Lambda^{2+2\alpha}b\,dx\\&\le\|\Lambda^\alpha(b\cdot\nabla u)\|_{L^2}\,\|\Lambda^{2+\alpha}b\|_{L^2}\\&\le\Big(\|\Lambda^\alpha b\|_\infty\|\nabla u\|_{L^2}+\|b\|_\infty\|\Lambda^\alpha\nabla u\|_{L^2}\Big)\|\Lambda^{2+\alpha}b\|_{L^2}\\&\le\frac{\kappa}{8}\|\Lambda^{2+\alpha}b\|_{L^2}^2+C\Big(\|b\|_{L^2}^2+\|\Lambda^2b\|_{L^2}^2\Big)\|\omega\|_{L^2}^2+C\Big(\|b\|_{L^2}^2+\|\Lambda^{1+\alpha}b\|_{L^2}^2\Big)\|\Lambda^\alpha\omega\|_{L^2}^2\end{aligned}$$

$$\leq \frac{\kappa}{8}\|\Lambda^{2+\alpha}b\|_{L^2}^2 + C\|\Lambda^{1+\alpha}b\|_{L^2}^2\|\Lambda^\alpha\omega\|_{L^2}^2 + C\Big(H(b,t) + \|\Lambda^\alpha\omega\|_{L^2}^2 + 1\Big)$$

and

$$\begin{aligned}\int_{\mathbb{R}^2}(u\cdot\nabla b)\cdot\Lambda^{2+2\alpha}b\,dx &\leq \|\Lambda^\alpha(u\cdot\nabla b)\|_{L^2}\,\|\Lambda^{2+\alpha}b\|_{L^2}\\ &\leq \Big(\|\Lambda^\alpha u\|_{L^{\frac{2}{\alpha}}}\|\nabla b\|_{L^{\frac{2}{1-\alpha}}} + \|u\|_\infty\|\Lambda^\alpha\nabla b\|_{L^2}\Big)\|\Lambda^{2+\alpha}b\|_{L^2}\\ &\leq \frac{\kappa}{8}\|\Lambda^{2+\alpha}b\|_{L^2} + C\|\Lambda^{1+\alpha}b\|_{L^2}^2\|\omega\|_{L^2}^2 + C\Big(\|u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2\Big)\|\Lambda^{1+\alpha}b\|_{L^2}^2\\ &\leq \frac{\kappa}{8}\|\Lambda^{2+\alpha}b\|_{L^2} + C\|\Lambda^{1+\alpha}b\|_{L^2}^2\Big(\|\Lambda^\alpha\omega\|_{L^2}^2 + 1\Big).\end{aligned}$$

Combining the estimates and employing Gronwall's inequality give

$$\|\Lambda^{1+\alpha}b(t)\|_{L^2}^2 \leq \Big(\|\Lambda^{1+\alpha}b_0\|_{L^2}^2 + C + Ct\Big)\exp(C+Ct). \tag{2.6}$$

In addition, (2.6), together with Proposition 2.1, implies

$$\|b(t)\|_{L^\infty} \leq \|b(t)\|_{L^2} + \|\Lambda^{1+\alpha}b(t)\|_{L^2} \leq C.$$

We now prove the global L^q-bounds for ω and Δb. To serve this purpose, we write the second equation of (1.3) in the integral form

$$\begin{aligned}b_1(x_1,x_2,t) &= \int_{\mathbb{R}} K_1(y_2,t)\,b_{01}(x_1,x_2-y_2)\,dy_2\\ &\quad + \int_0^t\int_{\mathbb{R}} K_1(y_2,s)\,(b\cdot\nabla u_1 - u\cdot\nabla b_1)(x_1,x_2-y_2,t-s)\,dy_2ds\\ &= J_1 + J_2.\end{aligned}$$

For any $2 < q < \infty$, we bound $\|\partial_{22}b_1\|_{L^2(0,t;L^q(\mathbb{R}^2))}$. Taking the L^q-norm with respect to x_1 and then the L^q-norm in x_2, we have

$$\begin{aligned}&\int_0^t \|\partial_{22}J_1\|_{L^q(\mathbb{R}^2)}^2 ds\\ &\quad\leq \int_0^t \left\|\int_{\mathbb{R}} K_1(y_2,s)\ \partial_{x_2x_2}b_{01}(x_1,x_2-y_2)dy_2\right\|_{L^q(\mathbb{R}^2)}^2 ds\\ &\quad\leq C\int_0^t \|K_1\|_{L^1(\mathbb{R})}^2\,\|\partial_{x_2x_2}b_{01}\|_{L^q(\mathbb{R}^2)}^2\,ds\\ &\quad\leq C\,t\,\|b_{01}\|_{H^3}^2.\end{aligned} \tag{2.7}$$

where we have used the fact $\|K_1\|_{L^1(\mathbb{R})} = 1$. To estimate J_2, we first take the L^q-norm in x_1 and then the L^q-norm in x_2, we obtain, after applying Lemma 2.3,

$$\begin{aligned}&\int_0^t \|\partial_{22}J_2\|_{L^q(\mathbb{R}^2)}^2 ds\\ &\quad\leq C\int_0^t \|b\cdot\nabla u_1 - u\cdot\nabla b_1\|_{L^q(\mathbb{R}^2)}^2\,ds\\ &\quad\leq C\int_0^t \Big(\|b\|_{L^\infty}^2\|\nabla u_1\|_{L^q}^2 + \|u\|_{L^{2q}}^2\|\nabla b_1\|_{L^{2q}}^2\Big)ds\\ &\quad\leq C\int_0^t \Big(\|\omega\|_{L^q}^2 + \Big(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\Big)\Big(\|b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\Big)ds\\ &\quad\leq C\int_0^t \|\omega\|_{L^q}^2 ds + C\int_0^t (H(b,s)+1)ds\end{aligned}$$

$$\leq C\int_0^t \|\omega\|_{L^q}^2 ds + C(t+1), \tag{2.8}$$

where Hölder's inequality, Sobolev's inequality and Proposition 2.1 have been used. Combining the estimates above, we have

$$\int_0^t \|\partial_{22} b_1(s)\|_{L^q(\mathbb{R}^2)}^2 ds \leq C\int_0^t \|\omega\|_{L^q}^2 ds + Ct + C. \tag{2.9}$$

We now prove that $\|\partial_{12} b_1\|_{L^p(0,t;L^q(\mathbb{R}^2)}$ is bounded globally. $\|\partial_{12} J_1\|_{L^q(\mathbb{R}^2)}^2$ can be bounded as in (2.7),

$$\int_0^t \|\partial_{12} J_1\|_{L^q(\mathbb{R}^2)}^2 ds \leq C\, t\, \|b_{01}\|_{H^3}^2.$$

But $\|\partial_{12} J_2\|_{L^q(\mathbb{R}^2)}^2$ is estimated differently from (2.8) due to the lack of dissipation in x_1-direction. The special structure of the equation of b still allows us to gain the needed derivative. Due to $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we have the following observation

$$\begin{aligned}\partial_{12}\left(b\cdot\nabla u_1 - u\cdot\nabla b_1\right) &= \partial_{12}\left(\partial_1(b_1u_1) + \partial_2(b_2u_1) - \partial_1(u_1b_1) - \partial_2(u_2b_1)\right)\\ &= \partial_{22}\left(\partial_1(b_2u_1) - \partial_1(u_2b_1)\right).\end{aligned}$$

As a consequence, we have, by Lemma 2.3 and Proposition 2.1,

$$\begin{aligned}&\int_0^t \|\partial_{12} J_2\|_{L^q(\mathbb{R}^2)}^2 ds\\ &= \int_0^t \left\|\int_0^s\int_{\mathbb{R}} K_1(y_2,\tau)\partial_{12}(b\cdot\nabla u_1 - u\cdot\nabla b_1)(x_1, x_2-y_2, s-\tau)dy_2 d\tau\right\|_{L^q(\mathbb{R}^2)}^2 ds\\ &= \int_0^t \left\|\partial_{22}\int_0^s\int_{\mathbb{R}} K_1(y_2,\tau)(\partial_1(b_2u_1) - \partial_1(u_2b_1))(x_1, x_2-y_2, s-\tau)dy_2 d\tau\right\|_{L^q(\mathbb{R}^2)}^2 ds\\ &\leq C\int_0^t \|\partial_1(b_2u_1) - \partial_1(u_2b_1)\|_{L^q(\mathbb{R}^2)}^2 ds\\ &\leq C\int_0^t \left(\|b\|_{L^\infty}^2\|\nabla u\|_{L^q}^2 + \|u\|_{L^{2q}}^2\|\nabla b\|_{L^{2q}}^2\right) ds\\ &\leq C\int_0^t \|\omega\|_{L^q}^2 ds + \int_0^t \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)\left(\|b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\right) ds\\ &\leq C\int_0^t \|\omega\|_{L^q}^2 ds + C\int_0^t (H(b,s)+1) ds\\ &\leq C\int_0^t \|\omega\|_{L^q}^2 ds + C(t+1).\end{aligned}$$

Hence we have reached the following bound

$$\int_0^t \|\partial_{12} b_1(s)\|_{L^q(\mathbb{R}^2)}^2 ds \leq C\int_0^t \|\omega\|_{L^q}^2 ds + Ct + C. \tag{2.10}$$

We now consider b_2, the second component of b. Again we write the equation for b_2 in (1.3) in the following integral form

$$\begin{aligned}b_2(x_1,x_2,t) = &\int_{\mathbb{R}} K_1(y_1,t) b_{02}(x_1-y_1, x_2) dy_1\\ &+ \int_0^t\int_{\mathbb{R}} K_1(y_1,s)(b\cdot\nabla u_2 - u\cdot\nabla b_2)(x_1-y_1, x_2, t-s) dy_1 ds.\end{aligned}$$

Likewise, we obtain

$$\int_0^t \|\partial_{11} b_2(s)\|_{L^q(\mathbb{R}^2)}^2 ds + \int_0^t \|\partial_{12} b_2(s)\|_{L^q(\mathbb{R}^2)}^2 ds \leq C \int_0^t \|\omega\|_{L^q}^2 ds + Ct + C, \tag{2.11}$$

where the only difference in the estimation of $\partial_{12} b_2(s)$ lies in the relation

$$\begin{aligned}\partial_{12}\left(b \cdot \nabla u_2 - u \cdot \nabla b_2\right) &= \partial_{12}\left(\partial_1(b_1 u_2) + \partial_2(b_2 u_2) - \partial_1(u_1 b_2) - \partial_2(u_2 b_2)\right) \\ &= \partial_{11}\left(\partial_2(b_1 u_2) - \partial_2(u_1 b_2)\right).\end{aligned}$$

Combining (2.9), (2.10) and (2.11) leads to

$$\begin{aligned}&\int_0^t \|\Delta b(s)\|_{L^q(\mathbb{R}^2)}^2 ds \\ &\leq C \int_0^t \Big(\|\partial_{22} b_1(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{22} b_2(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{11} b_2(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{11} b_1(s)\|_{L^q(\mathbb{R}^2)}^2\Big) ds \\ &= C \int_0^t \Big(\|\partial_{22} b_1(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{12} b_1(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{11} b_2(s)\|_{L^q(\mathbb{R}^2)}^2 + \|\partial_{12} b_2(s)\|_{L^q(\mathbb{R}^2)}^2\Big) ds \\ &\leq C \int_0^t \|\omega\|_{L^q}^2 ds + Ct + C.\end{aligned} \tag{2.12}$$

Due to the Calderon–Zygmund inequality,

$$\begin{aligned}\int_0^t \|\nabla j(s)\|_{L^q(\mathbb{R}^2)}^2 ds &\leq C \int_0^t \|\Delta b(s)\|_{L^q(\mathbb{R}^2)}^2 ds \\ &\leq C \int_0^t \|\omega\|_{L^q}^2 ds + Ct + C.\end{aligned} \tag{2.13}$$

Multiplying the vorticity equation in (2.3), namely

$$\partial_t \omega + u \cdot \nabla \omega + \mu(-\Delta)^\alpha \omega = b \cdot \nabla j$$

by $|\omega|^{q-2}\omega$ and integrating on $\mathbb{R}^2$, we have

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_{L^q}^q + C(q)\,\mu\|\Lambda^\alpha(|\omega|^{\frac{q}{2}})\|_{L^2}^2 \\ &\leq \int_{\mathbb{R}^2} b \cdot \nabla j |\omega|^{q-2}\omega dx \leq \|b\|_{L^\infty}\|\nabla j\|_{L^q}\|\omega\|_{L^q}^{q-1} \\ &\leq C\|\nabla j\|_{L^q}\|\omega\|_{L^q}^{q-1} \leq C\Big(\|\nabla j\|_{L^q}^2 + \|\omega\|_{L^q}^2\Big)\|\omega\|_{L^q}^{q-2},\end{aligned} \tag{2.14}$$

where we have invoked the lower bound, for any $\alpha \in (0, 1]$,

$$\int_{\mathbb{R}^2} \omega|\omega|^{q-2}(-\Delta)^\alpha \omega\, dx \geq C(q)\,\|\Lambda^\alpha(|\omega|^{\frac{q}{2}})\|_{L^2}^2.$$

Integrating (2.14) in time and combining with (2.13) lead to

$$\begin{aligned}\|\omega(t)\|_{L^q}^2 &\leq \|\omega_0\|_{L^q}^2 + \int_0^t \Big(\|\nabla j\|_{L^q}^2 + \|\omega\|_{L^q}^2\Big) ds \\ &\leq C \int_0^t \|\omega\|_{L^q}^2 ds + Ct + C.\end{aligned}$$

Applying Gronwall's inequality yields the desired bound

$$\|\omega(t)\|_{L^q}^2 \leq C(t+1)\,(1 + C\,t\,\exp(Ct))$$

and, due to (2.12),

$$\int_0^t \|\Delta b(s)\|_{L^q(\mathbb{R}^2)}^2 ds \leq Ct(t+1)(1 + C\,t\exp(Ct)).$$

Sobolev's inequality then implies

$$\int_0^t \|\nabla b(s)\|_{L^\infty(\mathbb{R}^2)} ds \le C(t, u_0, b_0).$$

This completes the proof of Proposition 2.2. □

2.3. L^∞-Bound for the Gradient ∇u

Making use of the global bounds obtained in two previous subsections, we obtain a global bound for the L^∞-norm of ∇u. This crucial global bound then ensures a global bound for $\|(u, b)\|_{H^s}$ for any $s \ge 3$.

Proposition 2.4. *Assume* (u_0, b_0) *satisfies the conditions stated in Theorem 1.1. Let* (u, b) *be the corresponding solution of* (1.3)*. Then,* (u, b) *admits the following global bounds, for any* $0 < t < \infty$*,*

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau \le C(t, u_0, b_0), \quad \|(u(t), b(t))\|_{H^s} \le C(t, u_0, b_0).$$

Proof. Taking the inner product of (2.3) with $(-\Delta\omega, -\Delta j)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\omega(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2\Big) + \mu\|\Lambda^{1+\alpha}\omega\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta j\|_{L^2}^2 \\
&\quad \le \int_{\mathbb{R}^2} (u\cdot\nabla\omega - b\cdot\nabla j)\Delta\omega dx + \int_{\mathbb{R}^2} (u\cdot\nabla j - b\cdot\nabla\omega)\Delta j dx \\
&\qquad - \int_{\mathbb{R}^2} (T(\nabla u, \nabla b))\Delta j dx,
\end{aligned}$$

where we have used the following observation

$$\begin{aligned}
&\int_{\mathbb{R}^2} (-\partial_{111}b_2 + \partial_{222}b_1)(-\Delta\partial_1 b_2 + \Delta\partial_2 b_1)dx \\
&\quad = \int_{\mathbb{R}^2} (-\partial_{111}\Lambda b_2\partial_1\Lambda b_2 + \partial_{111}\Lambda b_2\partial_2\Lambda b_1 + \partial_{222}\Lambda b_1\partial_1\Lambda b_2 - \partial_{222}\Lambda b_1\partial_2\Lambda b_1)dx \\
&\quad = \int_{\mathbb{R}^2} \Big((\partial_{11}\Lambda b_2)^2 + (\partial_{11}\Lambda b_1)^2 + (\partial_{22}\Lambda b_2)^2 + (\partial_{22}\Lambda b_1)^2\Big)dx \\
&\quad \ge \frac{1}{2}\int_{\mathbb{R}^2} |\Delta\Lambda b|^2 dx = \frac{1}{2}\|\Delta j\|_{L^2}^2.
\end{aligned}$$

Applying Propositions 2.1 and 2.2, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (u\cdot\nabla\omega - b\cdot\nabla j)\Delta\omega dx \\
&\quad \le C\,\|\nabla u\|_{L^{\frac{2}{\alpha}}}\|\nabla\omega\|_{L^2}\|\nabla\omega\|_{L^{\frac{2}{1-\alpha}}} + \Big(\|b\|_{L^\infty}\|\Delta j\|_{L^2} + \|\nabla b\|_{L^{\frac{2}{1-\alpha}}}\|\nabla j\|_{L^{\frac{2}{\alpha}}}\Big)\|\nabla\omega\|_{L^2} \\
&\quad \le C\,\|\omega\|_{L^{\frac{2}{\alpha}}}\|\nabla\omega\|_{L^2}\|\Lambda^{1+\alpha}\omega\|_{L^2} + \Big(\|b\|_{L^\infty}\|\Delta j\|_{L^2} + \|\Lambda^{1+\alpha}b\|_{L^2}\|\nabla j\|_{L^{\frac{2}{\alpha}}}\Big)\|\nabla\omega\|_{L^2} \\
&\quad \le \frac{\mu}{8}\|\Lambda^{1+\alpha}\omega\|_{L^2}^2 + \frac{\kappa}{8}\|\Lambda^2 j\|_{L^2}^2 + C\Big(1 + \|\nabla j\|_{L^{\frac{2}{\alpha}}}^2\Big)\|\nabla\omega\|_{L^2}^2, \\
&\int_{\mathbb{R}^2} (u\cdot\nabla j - b\cdot\nabla\omega)\Delta j dx \\
&\quad \le C\,(\|u\|_{L^4}\|\nabla j\|_{L^4} + \|b\|_{L^\infty}\|\nabla\omega\|_{L^2})\|\Delta j\|_{L^2} \\
&\quad \le \frac{\kappa}{8}\|\Lambda^2 j\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2 + C\|\nabla j\|_{L^4}^2
\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2} T(\nabla u, \nabla b)\, \Delta j\, dx \\ &\quad \le C\, \|\nabla u\|_{L^{\frac{2}{\alpha}}} \|\nabla b\|_{L^{\frac{2}{1-\alpha}}} \|\Delta j\|_{L^2} \\ &\quad \le \frac{\kappa}{8}\|\Lambda^2 j\|_{L^2}^2 + C\|\omega\|_{L^{\frac{2}{\alpha}}}^2 \|\Lambda^{1+\alpha} b\|_{L^2}^2 \le \frac{\kappa}{8}\|\Lambda^2 j\|_{L^2}^2 + C.\end{aligned}$$

Combining these estimates and applying Gronwall's inequality yield

$$\begin{aligned}&(\|\nabla\omega(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2) + \frac{\mu}{2}\int_0^t \|\Lambda^{1+\alpha}\omega\|_{L^2}^2 ds + \frac{\kappa}{2}\int_0^t \|\Lambda^2 j\|_{L^2}^2 ds \\ &\quad \le \Big(\|\nabla\omega_0\|_{L^2}^2 + \|\nabla j_0\|_{L^2}^2\Big)(Ct + \exp(C+Ct)),\end{aligned}$$

which implies

$$\int_0^t \|\nabla u(s)\|_{L^\infty} ds \le \int_0^t \Big(\|\nabla u\|_{L^2} + \|\Lambda^{2+\alpha} u\|_{L^2}\Big) ds \le C(t, u_0, b_0).$$

To show the global bound for (u, b) in H^s, we start with the energy inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\Big) + \mu\|\Lambda^\alpha u\|_{H^s}^2 + \frac{\kappa}{2}\|\Lambda b\|_{H^s}^2 \\ &\quad = \int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla]u \cdot \Lambda^s u dx + \int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla] b\cdot \Lambda^s b dx \\ &\qquad + \int_{\mathbb{R}^2} [\Lambda^s, b\cdot\nabla] b \cdot \Lambda^s u dx + \int_{\mathbb{R}^2} [\Lambda^s, b\cdot\nabla] u\cdot \Lambda^s b dx,\end{aligned} \tag{2.15}$$

where $[a, b]$ is the standard commutator notation, namely $[a, b] = ab - ba$. Invoking the commutator estimate (see, e.g., [24])

$$\|[\Lambda^s, f]g\|_{L^p} \le C\|\nabla f\|_{L^q}\, \|\Lambda^{s-1} g\|_{L^r} + C\, \|\nabla^s f\|_{L^{q_1}}\, \|g\|_{L^{r_1}}$$

where $s > 0$, $p, r, q_1 \in (1, \infty)$, $q, r_1 \in [1, \infty]$ and $1/p = 1/q + 1/r = 1/q_1 + 1/r_1$, the right hand side of (2.15) can be bounded by

$$C\left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}\right)\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right).$$

Gronwall's inequality then leads to

$$\|u\|_{H^s}^2 + \|b\|_{H^s}^2 \le C(t, u_0, b_0).$$

This completes the proof of Proposition 2.4. □

With the global bounds in the previous propositions at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is achieved via a standard procedure. First we seek the solution of a regularized system. We begin by introducing a few notation. For $\varepsilon > 0$, we denote by ϕ_ε the standard mollifier, namely

$$\phi_\varepsilon(x) = \varepsilon^{-2}\phi(\varepsilon^{-1}|x|)$$

with

$$\phi \in C_0^\infty(\mathbb{R}^2), \quad \phi(x) = \phi(|x|), \quad \text{supp}\phi \subset \{x \,|\, |x| < 1\}, \quad \int_{\mathbb{R}^2} \phi(x)\, dx = 1.$$

For any locally integrable function v, define the mollification $\mathcal{J}_\varepsilon v$ by

$$\mathcal{J}_\varepsilon v = \phi_\varepsilon * v.$$

Let $\mathbb{P}$ denote the Leray projection operator (onto divergence-free vector fields). We seek a solution $(u^\varepsilon, b^\varepsilon)$ of the system

$$\begin{aligned} &\partial_t u^\varepsilon + \mathbb{P}\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)) + \mu\mathcal{J}_\varepsilon^2(-\Delta)^\alpha u^\varepsilon = \mathbb{P}\mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon b^\varepsilon)),\\ &\partial_t b_1^\varepsilon + \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon b_1^\varepsilon)) - \kappa\mathcal{J}_\varepsilon^2\partial_{22} b_1^\varepsilon = \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u_1^\varepsilon)),\\ &\partial_t b_2^\varepsilon + \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon b_2^\varepsilon)) - \kappa\mathcal{J}_\varepsilon^2\partial_{11} b_2^\varepsilon = \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u_2^\varepsilon)),\\ &\nabla\cdot u^\varepsilon = \nabla\cdot b^\varepsilon = 0\\ &(u^\varepsilon, w^\varepsilon)(x,0) = (u_0 * \phi_\varepsilon, \quad b_0 * \phi_\varepsilon) = (u_0^\varepsilon, b_0^\varepsilon). \end{aligned} \tag{2.16}$$

Following the lines as those in the proofs of Propositions 2.1, 2.2 and 2.4, we can establish the global bound, for any $t \in (0,\infty)$,

$$\|u^\varepsilon(t)\|_{H^s}^2 + \|b^\varepsilon(t)\|_{H^s}^2 \le C(t, u_0, b_0). \tag{2.17}$$

A standard compactness argument allows us to obtain the global existence of the classical solution (u,b) to (1.3). The uniqueness can also be easily established. We omit further details. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

This section proves Theorem 1.2. For the sake of clarity, we divide the estimates into several decay levels. Correspondingly, this section is divided into three subsections. The first subsection shows that

$$(1+t)\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right) \to 0 \quad \text{as} \quad t\to\infty,$$

which follows from the uniform global H^1 bounds for (u,b) and the time integrability of $\|\Lambda^\alpha u\|_{L^2}^2$, $\|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2$ and $\|\Lambda^\alpha \omega\|_{L^2}^2$. The second subsection proves the global optimal bounds for b,

$$\|\nabla b(t)\|_{L^2} \le C(1+t)^{-1}, \qquad \|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.$$

To achieve this goal, we first obtain the intermediate decay rate, for any $\epsilon > 0$,

$$\|(u(t), b(t))\|_{L^2} \le C(1+t)^{-\frac{1}{2}+\epsilon} \quad \text{for any } t > 0. \tag{3.1}$$

To prove (3.1), we represent u, b_1 and b_2 in integral forms and make use of the special structures of the equations of b_1 and b_2. Once (3.1) is proven, we further differentiate the integral representations of b_1 and b_2, divide the time integral into several pieces and take advantage of the special structure of the equations of b_1 and b_2 to show that, for any $t > 0$,

$$\|\nabla b(t)\|_{L^2} \le C(1+t)^{-1},$$

which in turn allows us to improve the decay rate for $\|b(t)\|_{L^2}$,

$$\|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.$$

The third subsection provides faster decay rates for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$, which are obtained by applying and generalizing the Fourier splitting method of Schonbek [31].

3.1. Decay Estimates for $\|(\nabla u, \nabla b)\|_{L^2}$

This subsection shows that the L^2-norm of ∇u and ∇b decays faster than $(1+t)^{-\frac{1}{2}}$ as $t \to \infty$. More precisely, the following proposition holds.

Proposition 3.1. *Suppose (u,b) is a solution of (1.3) with the corresponding initial data $(u_0, b_0) \in H^1$. Then*

$$t\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right) \to 0 \quad \textit{as} \quad t \to \infty. \tag{3.2}$$

Proof. As in the proof of Proposition 2.1, we have for $0 \le s < t \le \infty$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\mu \int_s^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau + 2\kappa \int_s^t \Big(\|\partial_2 b_1(\tau)\|_{L^2}^2 + \|\partial_1 b_2(\tau)\|_{L^2}^2\Big) d\tau \le \|u(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2 \tag{3.3}$$

and

$$\begin{aligned}\|\omega(t)\|_{L^2}^2 &+ \|j(t)\|_{L^2}^2 + 2\mu \int_s^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 d\tau \\ &\le \Big(\|\omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2\Big) \exp\Big\{C(\kappa) \int_s^t \|j(\tau)\|_{L^2}^2 d\tau\Big\} \\ &\le \Big(\|\omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2\Big) \exp\Big\{C(\kappa) \int_0^t \|j(\tau)\|_{L^2}^2 d\tau\Big\} \\ &\le \Big(\|\omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2\Big) \exp\Big\{C(\kappa)(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)\Big\}.\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}\int_0^\infty \|\nabla b(s)\|_{L^2}^2 ds &= \int_0^\infty \|j(s)\|_{L^2}^2 ds \\ &\le 2\int_0^\infty \Big(\|\partial_2 b_1(\tau)\|_{L^2}^2 + \|\partial_1 b_2(\tau)\|_{L^2}^2\Big) d\tau \le C(\|u_0\|_{L^2} + \|b_0\|_{L^2}).\end{aligned}$$

By Sobolev's inequality,

$$\int_0^\infty \|\nabla u(s)\|_{L^2}^2 ds \le \int_0^\infty \Big(\|\Lambda^\alpha u(s)\|_{L^2}^2 + \|\Lambda^\alpha \omega(s)\|_{L^2}^2) ds \le C(\|u_0\|_{H^1} + \|b_0\|_{H^1}\Big).$$

A special consequence is that

$$\int_{\frac{t}{2}}^t \Big(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2\Big) d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

(3.4) then implies

$$\begin{aligned}&\frac{t}{2} \exp\{-C(\kappa)(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)\}\Big(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\Big) \\ &\quad \le \int_{\frac{t}{2}}^t \Big(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2\Big) d\tau,\end{aligned}$$

which yields the desired decay rate

$$(1+t)\Big(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\Big) \to 0 \quad \text{as} \quad t \to \infty.$$

This completes the proof of Proposition 3.1. □

3.2. Optimal Decay Rates for b and ∇b

This subsection derives the optimal decay rates for $\|b(t)\|_{L^2}$ and $\|\nabla b(t)\|_{L^2}$. To do so, we need to overcome the difficulty due to the lack of full Laplacian dissipation. We make use of a key observation on the structure of the equation of b. First we recall the $L^p - L^q$ decay estimates of the heat operator associated with the fractional Laplacian.

Lemma 3.2. (Schonbek [32]) *Let $\alpha > 0$, $\mu > 0$, $1 \le p \le q \le \infty$ and $m \ge 0$. The following $L^p - L^q$ estimate on the semigroup $e^{-\kappa(-\Delta)^\alpha t}$ is valid for any $t > 0$,*

$$\|\nabla^m e^{-\mu(-\Delta)^\alpha t} f\|_{L^q(\mathbb{R}^2)} \le C\, t^{-\frac{m}{2\alpha} - \frac{1}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^2)}. \tag{3.5}$$

Proposition 3.3. *Assume the same conditions as in Theorem 1.2. Then the corresponding solution* (u, b) *of (1.3) admits the following decay rates*

$$\|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}, \quad \|\nabla b(t)\|_{L^2} \le C(1+t)^{-1}. \tag{3.6}$$

Proof. We write the momentum equation of (1.3) in the integral form,

$$\begin{aligned} u(t) &= e^{-\mu(-\Delta)^\alpha t}u_0 + \int_0^t e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(s)\,ds \\ &= e^{-\mu(-\Delta)^\alpha t}u_0 + \int_0^{\frac{t}{2}} e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(s)\,ds \\ &\quad + \int_{\frac{t}{2}}^t e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(s)\,ds, \end{aligned} \tag{3.7}$$

where $\mathbb{P}$ denotes the Leray projection onto divergence-free vector fields. This projection allows us to eliminate the pressure term. We start with the estimates of the first term. For $t \ge 1$, we use Plancherel's Theorem and (1.4) to obtain

$$\|e^{-\mu(-\Delta)^\alpha t}u_0\|_{L^2(\mathbb{R}^2_x)} = \|e^{-\mu|\xi|^{2\alpha}t}\widehat{u}_0\|_{L^2(\mathbb{R}^2_\xi)} \le C\|e^{-\mu|\xi|^{2\alpha}t}\sqrt{|\xi|}\|_{L^2(\mathbb{R}^2_\xi)} \le Ct^{-\frac{3}{4\alpha}}.$$

For $t < 1$,

$$\|e^{-\mu(-\Delta)^\alpha t}u_0\|_{L^2(\mathbb{R}^2_x)} \le \|u_0\|_{L^2(\mathbb{R}^2_x)}.$$

Therefore,

$$\|e^{-\mu(-\Delta)^\alpha t}u_0\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{3}{4\alpha}},$$

where $C = C(u_0)$ is a constant independent of t. By Lemma 3.2,

$$\begin{aligned} &\left\|\int_0^{\frac{t}{2}} e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(s)\,ds\right\|_{L^2} \\ &= \left\|\int_0^{\frac{t}{2}} \nabla e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\otimes b - u\otimes u)(s)\,ds\right\|_{L^2} \\ &\le C\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{\alpha}}\left(\|b(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^2\right)ds. \end{aligned}$$

Also by Lemma 3.2, for any $2 < \frac{1}{\alpha} < r < \frac{2}{\alpha}$,

$$\begin{aligned} &\left\|\int_{\frac{t}{2}}^t e^{-\mu(-\Delta)^\alpha (t-s)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(s)\,ds\right\|_{L^2} \\ &= \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{\alpha}(\frac{2+r}{2r}-\frac{1}{2})}\|b\cdot\nabla b - u\cdot\nabla u\|_{L^{\frac{2r}{2+r}}}\,ds \\ &\le C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{\alpha r}}\left(\|b(s)\|_{L^r}\|\nabla b(s)\|_{L^2} + \|u(s)\|_{L^r}\|\nabla u(s)\|_{L^2}\right)ds \\ &\le C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{\alpha r}}\left(\|b(s)\|_{L^2}^{\frac{2}{r}}\|\nabla b(s)\|_{L^2}^{2-\frac{2}{r}} + \|u(s)\|_{L^2}^{\frac{2}{r}}\|\nabla u(s)\|_{L^2}^{2-\frac{2}{r}}\right)ds, \end{aligned}$$

where we have used Hölder's inequality and the Gagliardo-Nirenberg inequality. Inserting these estimates in (3.7), we have

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4\alpha}} + C\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{\alpha}} (\|b(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^2)\, ds$$
$$+ C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{\alpha r}} \left(\|b(s)\|_{L^2}^{\frac{2}{r}} \|\nabla b(s)\|_{L^2}^{2-\frac{2}{r}} + \|u(s)\|_{L^2}^{\frac{2}{r}} \|\nabla u(s)\|_{L^2}^{2-\frac{2}{r}}\right) ds. \tag{3.8}$$

We recall the integral form of b_1 in(1.3),

$$b_1(x_1,x_2,t) = K_1(x_2,t) * b_{01} + \int_0^t K_1(x_2,t-s) * (b\cdot\nabla u_1 - u\cdot\nabla b_1)(s)ds \tag{3.9}$$

where K_1 denotes the 1D heat kernel. For $t < 1$,

$$\|K_1(x_2,t) * b_{01}\|_{L^2} \le \|b_{01}\|_{L^2}.$$

For $t \ge 1$, by Plancherel's Theorem and (1.4),

$$\begin{aligned}\|K_1(x_2,t) * b_{01}(x_1,x_2)\|_{L^2(\mathbb{R}^2)} &= \|\widehat{K}_1(\xi_2,t)\widehat{b}_{01}(\xi_1,\xi_2)\|_{L^2_\xi} \\ &\le C\, \|\widehat{K}_1(\xi_2,t)\, \|\widehat{b}_{01}(\xi_1,\xi_2)\|_{L^2_{\xi_1}}\|_{L^2_{\xi_2}} \\ &\le C\, \left\|e^{-\kappa|\xi_2|^2 t}\sqrt{|\xi_2|}\,\right\|_{L^2_{\xi_2}} \le C\, t^{-\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\|K_1(x_2,t) * b_{01}(x_1,x_2)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2}}. \tag{3.10}$$

Thanks to

$$b\cdot\nabla u_1 - u\cdot\nabla b_1 = \partial_1(b_1u_1) + \partial_2(b_2u_1) - \partial_1(u_1b_1) - \partial_2(u_2b_1) = \partial_2(b_2u_1 - u_2b_1)$$

and Lemma 3.2, the second term in (3.9) is bounded by

$$\begin{aligned}&\left\|\int_0^t K_1(x_2,t-s) * (b\cdot\nabla u_1 - u\cdot\nabla b_1)(x_1,x_2,s)ds\right\|_{L^2(\mathbb{R}^2)} \\ &\le \int_0^t \left\|\partial_2 K_1(x_2,t-s) * \|(b_2u_1 - u_2b_1)(x_1,x_2,s)\|_{L^2_{x_1}}\right\|_{L^2_{x_2}} ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|(b_2u_1 - u_2b_1)(s)\|_{L^2(\mathbb{R}^2)}ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|u\|_{L^{\frac{2}{1-\alpha}}}\|b\|_{L^{\frac{2}{\alpha}}}ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\|b_1(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-s)^{-\frac{1}{2}}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds. \tag{3.12}$$

b_2 can be similarly estimated. In fact, we begin with the integral form of b_2,

$$b_2(x_1,x_2,t) = K_1(x_1,t) * b_{02} + \int_0^t K_1(x_1,t-s) * (b\cdot\nabla u_2 - u\cdot\nabla b_2)(x_1,x_2,s)ds,$$

As in (3.10),

$$\|K_1(x_1,t) * b_{02}\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.$$

As in (3.11), by revoking the identity

$$b\cdot\nabla u_2 - u\cdot\nabla b_2 = \partial_1(b_1u_2) + \partial_2(b_2u_2) - \partial_1(u_1b_2) - \partial_2(u_2b_2) = \partial_1(b_1u_2 - u_1b_2),$$

we have

$$\begin{aligned}&\left\|\int_0^t K_1(x_1,t-s)*(b\cdot\nabla u_2-u\cdot\nabla b_2)(x_1,x_2,s)ds\right\|_{L^2(\mathbb{R}^2)}\\&\quad\le\int_0^t\left\|\partial_1K_1(x_1,t-s)*\|(b_1u_2-u_1b_2)(x_1,x_2,s)\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}ds\\&\quad\le C\int_0^t(t-s)^{-\frac12}\|(b_1u_2-u_1b_2)(s)\|_{L^2}ds\\&\quad\le C\int_0^t(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds.\end{aligned}$$

Therefore b_2 obeys the same bound. Therefore, after splitting the time integral into two parts and applying an interpolation inequality, we have

$$\begin{aligned}\|b(t)\|_{L^2}&\le C(1+t)^{-\frac12}+C\int_0^t(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds\\&\le C(1+t)^{-\frac12}+C\int_0^{\frac t2}(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds\\&\quad+C\int_{\frac t2}^t(t-s)^{-\frac12}\|u\|_{L^2}^{1-\alpha}\,\|\nabla u\|_{L^2}^\alpha\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds.\end{aligned}\tag{3.13}$$

Adding (3.8) and (3.13) yields

$$\begin{aligned}&\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\\&\le\ C(1+t)^{-\frac{3}{4\alpha}}+C(1+t)^{-\frac12}+C\int_0^{\frac t2}(t-s)^{-\frac1\alpha}(\|b(s)\|_{L^2}^2+\|u(s)\|_{L^2}^2)\,ds\\&\quad+C\int_{\frac t2}^t(t-s)^{-\frac{1}{\alpha r}}\left(\|b(s)\|_{L^2}^{\frac2r}\|\nabla b(s)\|_{L^2}^{2-\frac2r}+\|u(s)\|_{L^2}^{\frac2r}\|\nabla u(s)\|_{L^2}^{2-\frac2r}\right)ds\\&\quad+C\int_0^{\frac t2}(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds\\&\quad+C\int_{\frac t2}^t(t-s)^{-\frac12}\|u\|_{L^2}^{1-\alpha}\,\|\nabla u\|_{L^2}^\alpha\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds.\end{aligned}\tag{3.14}$$

First we show that, under the condition that $0<\alpha<\frac12$ and for any small $\epsilon>0$,

$$\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\le C\,(1+t)^{-\frac12+\epsilon}.\tag{3.15}$$

This is achieved via an iterative procedure. It is the term

$$C\int_0^{\frac t2}(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^\alpha\|\nabla b\|_{L^2}^{1-\alpha}ds$$

in (3.14) that forces us to go through such a procedure. The first step is to show, for any $t\ge0$,

$$\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\le C\,(1+t)^{-\left(\frac12-\frac\alpha2\right)}.\tag{3.16}$$

For notational convenience, writing

$$\mathcal{M}_1(t)=\sup_{0\le s\le t}\left\{(1+s)^{\frac12-\frac\alpha2}(\|u(s)\|_{L^2}+\|b(s)\|_{L^2})\right\}$$

and

$$\varphi(t)=t^{\frac12}\Big(\|\nabla u(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}\Big),$$

we have from (3.14)

$$\begin{aligned}\mathcal{M}_1(t) &\le C(1+t)^{\frac12-\frac{3}{4\alpha}-\frac{\alpha}{2}} + C(1+t)^{-\frac{\alpha}{2}}\\ &\quad + C(1+t)^{\frac12-\frac{\alpha}{2}}\int_0^{\frac t2}(t-s)^{-\frac1\alpha}\left(\|b(s)\|_{L^2}^2+\|u(s)\|_{L^2}^2\right)ds\\ &\quad + C\mathcal{M}_1^{\frac2r}(t)(1+t)^{\frac12-\frac\alpha2}\int_{\frac t2}^t (t-s)^{-\frac{1}{\alpha r}}s^{-1+\frac1r}\varphi^{2-\frac2r}(s)s^{-\frac2r(\frac12-\frac\alpha2)}\,ds\\ &\quad + C(1+t)^{\frac12-\frac\alpha2}\int_0^{\frac t2}(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{1-\alpha}ds\\ &\quad + C\mathcal{M}_1(t)(1+t)^{\frac12-\frac\alpha2}\int_{\frac t2}^t (t-s)^{-\frac12}s^{-1+\frac\alpha2}\varphi(s)ds. \end{aligned} \tag{3.17}$$

The terms on the right-hand side can be further bounded as follows. For $0<\alpha<\frac12$,

$$\begin{aligned}&(1+t)^{\frac12-\frac\alpha2}\int_0^{\frac t2}(t-s)^{-\frac1\alpha}\left(\|b(s)\|_{L^2}^2+\|u(s)\|_{L^2}^2\right)ds\\ &\le C\left(\|b_0\|_{L^2}^2+\|u_0\|_{L^2}^2\right)(1+t)^{\frac32-\frac1\alpha-\frac\alpha2} \quad \to 0 \quad \text{as } t\to\infty. \end{aligned}\tag{3.18}$$

Due to $\frac{1}{\alpha r}>\frac12$ and $\varphi(t)\to 0$ as $t\to\infty$,

$$(1+t)^{\frac12-\frac\alpha2}\int_{\frac t2}^t (t-s)^{-\frac{1}{\alpha r}}s^{-1+\frac1r}\varphi^{2-\frac2r}(s)s^{-\frac2r(\frac12-\frac\alpha2)}\,ds \quad \to 0 \quad \text{as } t\to\infty. \tag{3.19}$$

By (3.3) and Hölder's inequality,

$$\begin{aligned}&(1+t)^{\frac12-\frac\alpha2}\int_0^{\frac t2}(t-s)^{-\frac12}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{1-\alpha}ds\\ &\le (\|b_0\|_{L^2}^2+\|u_0\|_{L^2}^2)^{\frac\alpha2}\left(\int_0^{\frac t2}\|\Lambda^\alpha u\|_{L^2}^2\,ds\right)^{\frac12}\left(\int_0^{\frac t2}\|\nabla b\|_{L^2}^2\,ds\right)^{\frac{1-\alpha}{2}}\\ &\le C. \end{aligned}\tag{3.20}$$

We remark that this is the term that prevents us from getting higher-order decay than the one in (3.16). Due to $\varphi(t)\to 0$ as $t\to\infty$, By (3.3) and Hölder's inequality,

$$(1+t)^{\frac12-\frac\alpha2}\int_{\frac t2}^t (t-s)^{-\frac12}s^{-1+\frac\alpha2}\varphi(s)ds \quad \to 0 \quad \text{as } t\to\infty. \tag{3.21}$$

Inserting the estimates (3.18), (3.19), (3.20) and (3.16) in (3.17), we obtain

$$\mathcal{M}_1(t)\le \ C + C\mathcal{M}_1^{\frac2r}(t)+\frac12\mathcal{M}_1(t)\le C+\frac12\mathcal{M}_1(t)$$

which implies $\mathcal{M}_1(t)\le C$ or the desired bound in (3.16). The second step makes use of (3.16) to show the higher-order decay, for any $t\ge 0$,

$$\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\le C\,(1+t)^{-\left(\frac12-\frac\alpha2\right)(1+\alpha)}. \tag{3.22}$$

The proof of (3.22) is similar to the that of (3.16). For notational convenience, we write $\rho_2=\left(\frac12-\frac\alpha2\right)(1+\alpha)$. We begin by setting

$$\mathcal{M}_2(t)=\sup_{0\le s\le t}\left\{(1+s)^{\rho_2}(\|u(s)\|_{L^2}+\|b(s)\|_{L^2})\right\}$$

and then proceed as in (3.17),

$$\begin{aligned}\mathcal{M}_2(t) &\le C(1+t)^{\rho_2-\frac{3}{4\alpha}} + C(1+t)^{-\frac{1}{2}+\rho_2}\\ &\quad + C(1+t)^{\rho_2}\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{\alpha}}\left(\|b(s)\|_{L^2}^2+\|u(s)\|_{L^2}^2\right)ds\\ &\quad + C\mathcal{M}_2^{\frac{2}{r}}(t)(1+t)^{\rho_2}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{\alpha r}}s^{-1+\frac{1}{r}}\varphi^{2-\frac{2}{r}}(s)s^{-\frac{2}{r}\rho_2}\,ds\\ &\quad + C(1+t)^{\rho_2}\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2}}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{1-\alpha}ds\\ &\quad + C\mathcal{M}_2(t)(1+t)^{\rho_2}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}}s^{-1+\frac{\alpha}{2}}\varphi(s)ds. \end{aligned} \tag{3.23}$$

The terms on the right of (3.23) can be similarly estimated as before except the term

$$(1+t)^{\rho_2}\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2}}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{1-\alpha}ds,$$

which can be bounded, due to (3.16),

$$\begin{aligned}&(1+t)^{\rho_2}\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2}}\|\Lambda^\alpha u\|_{L^2}\,\|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{1-\alpha}ds\\ &\le (1+t)^{\rho_2}(1+t)^{-\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha u\|_{L^2}\,dx\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla b\|_{L^2}^2\,ds\right)^{\frac{1-\alpha}{2}}\left(\int_0^{\frac{t}{2}}\|b(s)\|_{L^2}^2ds\right)^{\frac{\alpha}{2}}\\ &\le C\,(1+t)^{-\frac{\alpha^2}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha u\|_{L^2}\,dx\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla b\|_{L^2}^2\,ds\right)^{\frac{1-\alpha}{2}}\left(\int_0^{\frac{t}{2}}(1+s)^{-1+\alpha}\,ds\right)^{\frac{\alpha}{2}}\\ &\le C\,(1+t)^{-\frac{\alpha^2}{2}}\left(\int_0^{\frac{t}{2}}\|\Lambda^\alpha u\|_{L^2}\,dx\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|\nabla b\|_{L^2}^2\,ds\right)^{\frac{1-\alpha}{2}}(1+t)^{\frac{\alpha^2}{2}}\\ &\le C.\end{aligned}$$

Inserting this estimate and the estimates for other terms in (3.23) yields the global bound

$$\mathcal{M}_2(t) < \infty.$$

Proceeding in this fashion, we can further show that

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \le C\,(1+t)^{-\left(\frac{1}{2}-\frac{\alpha}{2}\right)(1+\alpha+\alpha^2)}$$

and more generally, for any natural number N,

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \le C\,(1+t)^{-\left(\frac{1}{2}-\frac{\alpha}{2}\right)(1+\alpha+\alpha^2+\cdots+\alpha^N)}.$$

Since, as $N\to\infty$,

$$1+\alpha+\alpha^2+\cdots+\alpha^N \quad \to \quad \frac{1}{1-\alpha},$$

for any given $\epsilon > 0$, we can take N sufficiently large such that

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \le C\,(1+t)^{-\frac{1}{2}+\epsilon},$$

which is the desired bound in (3.15).

We now turn to the decay estimate of $\|\nabla b\|_{L^2}$ and the goal is the improved decay rate in (3.6). As in the proof of (3.12), we have

$$
\begin{aligned}
&\|\partial_2 b_1(x_1,x_2,t)\|_{L^2(\mathbb{R}^2)}\\
&\quad\le \|\partial_2 K_1(x_2,t)*b_{01}\|_{L^2(\mathbb{R}^2)}\\
&\qquad+\int_0^t \|\partial_2 K_1(x_2,t-s)*(b\cdot\nabla u_1-u\cdot\nabla b_1)(s)\|_{L^2(\mathbb{R}^2)}ds\\
&\quad\le \|\partial_2 K_1(x_2,t)*b_{01}\|_{L^2(\mathbb{R}^2)}\\
&\qquad+\int_0^{\frac t2}\Big\|\partial_{22}K_1(x_2,t-s)*\|(b_2u_1-u_2b_1)(x_1,x_2,s)\|_{L^2_{x_1}}\Big\|_{L^2_{x_2}}\,ds\\
&\qquad+\int_{\frac t2}^t\Big\|\partial_{22}|\partial_2|^{-\frac14}K_1(x_2,t-s)*\||\partial_2|^{\frac14}(b_2u_1-u_2b_1)(x_1,x_2,s)\|_{L^2_{x_1}}\Big\|_{L^2_{x_2}}\,ds\\
&\quad=: I_1+I_2+I_3.
\end{aligned}
$$

By Plancherel's theorem and (1.4),

$$
I_1=\|\partial_2K_1(x_2,t)*b_{01}(x_1,x_2)\|_{L^2(\mathbb{R}^2)}\le C(1+t)^{-1}.
$$

By Lemma 3.2, for any small $\epsilon>0$,

$$
\begin{aligned}
I_2&\le\int_0^{\frac t2}(t-s)^{-1}\|b_2u_1-u_2b_1\|_{L^2}ds\\
&\le\int_0^{\frac t2}(t-s)^{-1}\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\,ds\\
&\le C\int_0^{\frac t2}(t-s)^{-1}(1+s)^{-\frac34+\frac{\epsilon}{2}}\varphi^{\frac12}(s)\|\nabla b\|_{L^2}^{\frac12}\,ds,
\end{aligned}
$$

where $\varphi(t)$ is defined as

$$
\varphi(t)=(1+t)^{\frac12}\Big(\|\nabla u(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}\Big).
$$

Similarly,

$$
\begin{aligned}
I_3&\le\int_{\frac t2}^t(t-s)^{-1+\frac18}\Big\||\partial_2|^{\frac14}(b_2u_1-u_2b_1)\Big\|_{L^2(\mathbb{R}^2)}ds\\
&\le\int_{\frac t2}^t(t-s)^{-\frac78}\Big(\||\partial_2|^{\frac14}b\|_{L^4}\|u\|_{L^4}+\||\partial_2|^{\frac14}u\|_{L^4}\|b\|_{L^4}\Big)ds\\
&\le\int_{\frac t2}^t(t-s)^{-\frac78}\Big(\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|b\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}^{\frac34}+\|u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}^{\frac34}\|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\Big)ds\\
&\le C\int_{\frac t2}^t(t-s)^{-\frac78}\Big((1+s)^{-\frac58+\frac{\epsilon}{4}}\varphi^{\frac12}(s)\|\nabla b\|_{L^2}^{\frac34}+(1+s)^{-\frac34+\frac{\epsilon}{2}}\varphi^{\frac34}(s)\|\nabla b\|_{L^2}^{\frac12}\Big)ds.
\end{aligned}
$$

Putting these estimates together yields

$$
\begin{aligned}
\|\partial_2b_1\|_{L^2}&\le C(1+t)^{-1}+C\int_0^{\frac t2}(t-s)^{-1}(1+s)^{-\frac34+\frac{\epsilon}{2}}\varphi^{\frac12}(s)\|\nabla b\|_{L^2}^{\frac12}ds\\
&\quad+C\int_{\frac t2}^t(t-s)^{-\frac78}\Big((1+s)^{-\frac58+\frac{\epsilon}{4}}\varphi^{\frac12}(s)\|\nabla b\|_{L^2}^{\frac34}+(1+s)^{-\frac34+\frac{\epsilon}{2}}\varphi^{\frac34}(s)\|\nabla b\|_{L^2}^{\frac12}\Big)ds.
\end{aligned}
$$

$\|\partial_1b_2\|_{L^2}$ admits the same bound. Thanks to $\nabla\cdot b=0$,

$$
\|\nabla b\|_{L^2}=\|\nabla\times b\|_{L^2}\le2\|\partial_2b_1\|_{L^2}+2\|\partial_1b_2\|_{L^2}
$$

and thus $\|\nabla b\|_{L^2}$ obeys the same bound. Therefore,

$$
\mathcal{M}_3(t)=\sup_{0\le s\le t}\{(1+s)\|\nabla b(s)\|_{L^2}\}
$$

satisfies

$$\begin{aligned}\mathcal{M}_3(t) &\le C + C\mathcal{M}_3^{\frac12}(t)(1+t)\int_0^{\frac t2}(t-s)^{-1}(1+s)^{-\frac54+\frac{\epsilon}{2}}\varphi^{\frac12}(s)ds\\&\quad + C\mathcal{M}_3^{\frac34}(t)(1+t)\int_{\frac t2}^t (t-s)^{-\frac78}(1+s)^{-\frac{11}{8}+\frac{\epsilon}{4}}\varphi^{\frac12}(s)ds\\&\quad + C\mathcal{M}_3^{\frac12}(t)(1+t)\int_{\frac t2}^t (t-s)^{-\frac78}(1+s)^{-\frac54+\frac{\epsilon}{2}}\varphi^{\frac12}(s)ds\\&\le C + C\mathcal{M}_3^{\frac12}(t) + C\mathcal{M}_3^{\frac34}(t) \le C + \frac12\mathcal{M}_3(t),\end{aligned}$$

which implies,

$$\mathcal{M}_3(t) \le C \quad \text{or} \quad \|\nabla b(t)\|_{L^2} \le C(1+t)^{-1}.$$

To obtain the optimal bound for $\|b(t)\|_{L^2}$. We insert the new decay rate for $\|\nabla b\|_{L^2}$ in (3.14) to obtain

$$\begin{aligned}&\|u(t)\|_{L^2} + \|b(t)\|_{L^2}\\&\le\ C(1+t)^{-\frac{3}{4\alpha}} + C(1+t)^{-\frac12} + C\int_0^{\frac t2}(t-s)^{-\frac1\alpha}\Big(\|b(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^2\Big)\,ds\\&\quad + C\int_{\frac t2}^t (t-s)^{-\frac{1}{\alpha r}}\Big(\|b(s)\|_{L^2}^{\frac2r}\|\nabla b(s)\|_{L^2}^{2-\frac2r} + \|u(s)\|_{L^2}^{\frac2r}\|\nabla u(s)\|_{L^2}^{2-\frac2r}\Big)\,ds\\&\quad + C\int_0^t (t-s)^{-\frac12}\Big(\|u\|_{L^2} + \|b\|_{L^2}\Big)\|\nabla u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}ds\\&\le\ C(1+t)^{-\frac{1}{2\alpha}} + C(1+t)^{-\frac12} + C\int_0^{\frac t2}(t-s)^{-\frac1\alpha}(1+s)^{-1+2\epsilon}\,ds\\&\quad + C\int_{\frac t2}^t (t-s)^{-\frac{1}{\alpha r}}(1+s)^{-1+\frac{\epsilon}{r}}\,ds + C\int_0^t (t-s)^{-\frac12}(1+s)^{-\frac54+\epsilon}ds\\&\le\ C(1+t)^{-\frac12}. \qquad (3.24)\end{aligned}$$

The proof of Proposition 3.3 is now completed. □

3.3. Faster Decay Rates for $\|u\|_{L^2}$ and $\|\nabla u\|_{L^2}$

This subsection improves the decay rates for $\|u\|_{L^2}$ and $\|\nabla u\|_{L^2}$ by applying and generalizing the Fourier splitting method [31].

Proposition 3.4. *Assume the initial data (u_0, b_0) satisfies the conditions in Theorem 1.2. Assume $0 < \alpha < \frac12$. Let (u, b) be the corresponding solution of (1.3). Then u admits the following decay rates, for any $t > 0$,*

$$\|u(t)\|_{L^2} \le C(1+t)^{-(\frac32-\frac\alpha2)}, \qquad \|\nabla u(t)\|_{L^2} \le C(1+t)^{-1-\frac\alpha2}. \qquad (3.25)$$

Proof. Taking the inner product of u with the momentum equation in (1.3) yields

$$\frac12\frac{d}{dt}\|u(t)\|_{L^2}^2 + \mu\|\Lambda^\alpha u\|_{L^2}^2 = \int_{\mathbb{R}^2}(b\cdot\nabla)b\cdot u\,dx.$$

By Sobolev's inequality and the decay rates in the previous subsections,

$$\begin{aligned}\int_{\mathbb{R}^2}(b\cdot\nabla)b\cdot u\,dx &\le \int_{\mathbb{R}^2}\Lambda^{-\alpha}\nabla\cdot(b\otimes b)\cdot\Lambda^{\alpha}u\,dx\\ &\le \frac{\mu}{2}\|\Lambda^{\alpha}u\|_{L^2}^2+C\|\Lambda^{1-\alpha}(b\otimes b)\|_{L^2}^2\\ &\le \frac{\mu}{2}\|\Lambda^{\alpha}u\|_{L^2}^2+C\|b\|_{L^{\frac{2}{\alpha}}}^2\,\|\Lambda^{1-\alpha}b\|_{L^{\frac{2}{1-\alpha}}}^2\\ &\le \frac{\mu}{2}\|\Lambda^{\alpha}u\|_{L^2}^2+C\,\|b\|_{L^2}^{2\alpha}\,\|\nabla b\|_{L^2}^{4-2\alpha}\\ &\le \frac{\mu}{2}\|\Lambda^{\alpha}u\|_{L^2}^2+\,C(1+t)^{-4+\alpha},\end{aligned}$$

where we have invoked Sobolev's inequalities. For a large constant $k>0$, we set

$$B(t)=\left\{\xi\in\mathbb{R}^2:\ \mu|\xi|^{2\alpha}<\frac{k-1}{1+t}\right\}.$$

It is then clear that

$$\mu\|\Lambda^{\alpha}u\|_{L^2}^2\ge\mu\int_{B^c(t)}|\xi|^{2\alpha}|\widehat{u}(\xi)|^2d\xi\ge\frac{k-1}{1+t}\int_{\mathbb{R}^2}|\widehat{u}(\xi)|^2d\xi-\frac{k-1}{1+t}\int_{B(t)}|\widehat{u}(\xi)|^2d\xi.$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_{L^2}^2+\frac{k-1}{1+t}\|u\|_{L^2}^2\le\frac{k-1}{1+t}\int_{B(t)}|\widehat{u}(\xi)|^2d\xi+C(1+t)^{-4+\alpha}$$

or

$$\frac{d}{dt}\left((1+t)^{k-1}\|u(t)\|_{L^2}^2\right)\le C(1+t)^{k-2}\int_{B(t)}|\widehat{u}(\xi)|^2d\xi+C(1+t)^{k-5+\alpha}. \tag{3.26}$$

Taking the Fourier transformation of the momentum equation in (1.3) yields

$$|\widehat{u}(\xi)|\le|e^{-\mu|\xi|^{2\alpha}t}\widehat{u}_0|+\int_0^t e^{-\mu|\xi|^{2\alpha}(t-s)}|\widehat{G}(\xi,s)|ds,$$

where G represents the nonlinear terms including the pressure term. Clearly,

$$\begin{aligned}|\widehat{G}(\xi,s)| &\le|\widehat{u\cdot\nabla u}|+|\widehat{b\cdot\nabla b}|+|\widehat{\nabla\pi}|\\ &\le|\xi|\|u\|_{L^2}^2+\|b\|_{L^2}\|\nabla b\|_{L^2}+\left|\sum_{i,j}\frac{\xi_i\xi_j}{|\xi|^2}(\widehat{u\cdot\nabla u}-\widehat{b\cdot\nabla b})\right|\\ &\le|\xi|\|u\|_{L^2}^2+C(1+s)^{-\frac{3}{2}}.\end{aligned}$$

Therefore,

$$\int_0^t e^{-\mu|\xi|^{2\alpha}(t-s)}|\widehat{G}(\xi,s)|ds\le|\xi|\int_0^t\|u\|_{L^2}^2ds+C.$$

Inserting the inequalities above in (3.26), together with the decay of u, we have

$$\begin{aligned}\frac{d}{dt}\left((1+t)^{k-1}\|u(t)\|_{L^2}^2\right) &\le(1+t)^{k-2}\|e^{-\mu(-\Delta)^{\alpha}t}u_0\|_{L^2}^2+(1+t)^{k-2-\frac{1}{\alpha}}\\ &\quad+(1+t)^{k-1}\int_{B(t)}\left||\xi|\int_0^t\|u(\tau)\|_{L^2}^2d\tau\right|^2d\xi\\ &\quad+C(1+t)^{k-5+\alpha}.\end{aligned}$$

Applying Lemma 3.2 and the decay of u in (3.24), we have

$$\frac{d}{dt}\Big((1+t)^{k-1}\|u(t)\|_{L^2}^2\Big)\le(1+t)^{k-2-\frac{3}{2\alpha}}+(1+t)^{k-2-\frac{2}{\alpha}}\ln^2(1+t)+C(1+t)^{k-5+\alpha}.$$

Integrating in time implies

$$\|u(t)\|_{L^2} \le C(1+t)^{-(\frac{3}{2}-\frac{\alpha}{2})}, \quad t>0.$$

We now prove the improved decay rate for $\|\nabla u\|_{L^2}$. Taking the L^2-inner product of ω with the vorticity equation in (2.3) yields

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \mu\|\Lambda^\alpha \omega\|_{L^2}^2 = \int_{\mathbb{R}^2} (b\cdot\nabla) j\, \omega\, dx.$$

The term on the right-hand side can be bounded as, thanks to the decay rates of $\|b\|_{L^2}$ and $\|\nabla b\|_{L^2}$,

$$\begin{aligned}
\int_{\mathbb{R}^2} (b\cdot\nabla) j\, \omega dx &\le \int_{\mathbb{R}^2} \Lambda^{-\alpha}\nabla\cdot(bj)\,\Lambda^\alpha\omega\, dx\\
&\le \frac{\mu}{2}\|\Lambda^\alpha\omega\|_{L^2}^2 + C\|\Lambda^{1-\alpha} j\|^2_{L^{\frac{4}{2-\alpha}}}\|b\|^2_{L^{\frac{4}{\alpha}}} + C\|\Lambda^{1-\alpha} b\|^2_{L^{\frac{2}{1-\alpha}}}\|j\|^2_{L^{\frac{2}{\alpha}}}\\
&\le \frac{\mu}{2}\|\Lambda^\alpha\omega\|_{L^2}^2 + \|b\|_{L^2}^{\alpha}\|\nabla b\|_{L^2}^{2-\alpha}\|j\|_{L^2}^{\alpha}\|\nabla j\|_{L^2}^{2-\alpha}\\
&\quad + C\|\nabla b\|_{L^2}^2\,\|j\|_{L^2}^{2\alpha}\,\|\nabla j\|_{L^2}^{2-2\alpha}\\
&\le \frac{\mu}{2}\|\Lambda^\alpha\omega\|_{L^2}^2 + C(1+t)^{-2-\frac{5\alpha}{2}}\|\nabla j\|_{L^2}^{2-\alpha}\\
&\quad + C(1+t)^{-2-2\alpha}\|\nabla j\|_{L^2}^{2-2\alpha}.
\end{aligned}$$

To further the estimates, we invoke the lower bound

$$\mu\|\Lambda^\alpha\omega\|_{L^2}^2 \ge \mu\int_{B^c(t)} |\xi|^{2\alpha}|\widehat{\omega}(\xi)|^2 d\xi \ge \frac{k-1}{1+t}\int_{\mathbb{R}^2}|\widehat{\omega}(\xi)|^2 d\xi - \frac{k-1}{1+t}\int_{B(t)}|\widehat{\omega}(\xi)|^2 d\xi$$

and the estimate

$$\begin{aligned}
|\widehat{\omega}(\xi)| &\le |e^{-\mu|\xi|^{2\alpha}t}\widehat{\omega}_0| + \int_0^t |e^{-\mu|\xi|^{2\alpha}(t-s)}(\widehat{b\cdot\nabla j - u}\cdot\nabla\omega)|ds\\
&\le ||\xi| e^{-\mu|\xi|^{2\alpha}t}\widehat{u}_0| + C|\xi|\int_0^t \|b\|_{L^2}\|j\|_{L^2}ds + C|\xi|\int_0^t \|u\|_{L^2}\|\omega\|_{L^2}ds\\
&\le |\nabla \widehat{e^{-\mu(-\Delta)^\alpha t}u_0}| + C|\xi|\int_0^t (1+t)^{-\frac{3}{2}}ds + C|\xi|\int_0^t (1+t)^{-\frac{1}{2}-\min\left\{\frac{3}{4\alpha},\ 2-\frac{\alpha}{2}\right\}}ds\\
&\le |\nabla \widehat{e^{-\mu(-\Delta)^\alpha t}u_0}| + C|\xi|.
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}
\frac{d}{dt}\left((1+t)^k\|\omega(t)\|_{L^2}^2\right) &\le C(1+t)^{k-1}\int_{B(t)}|\widehat{\omega}(\xi)|^2 d\xi\\
&\quad + C(1+t)^{k-2-\frac{5\alpha}{2}}\|\nabla j\|_{L^2}^{2-\alpha} + C(1+t)^{k-2-2\alpha}\|\nabla j\|_{L^2}^{2-2\alpha}\\
&\le C(1+t)^{k-1}\|\nabla e^{-\mu|(-\Delta)^\alpha t}u_0\|_{L^2}^2 + C(1+t)^{k-1}\int_{B(t)}|\xi|^2 d\xi\\
&\quad + C(1+t)^{k-2-\frac{5\alpha}{2}}\|\nabla j\|_{L^2}^{2-\alpha} + C(1+t)^{k-2-2\alpha}\|\nabla j\|_{L^2}^{2-2\alpha}\\
&\le C(1+t)^{k-1-\frac{5}{2\alpha}} + C(1+t)^{k-1-\frac{2}{\alpha}}\\
&\quad + C(1+t)^{k-2-\frac{5\alpha}{2}}\|\nabla j\|_{L^2}^{2-\alpha} + C(1+t)^{k-2-2\alpha}\|\nabla j\|_{L^2}^{2-2\alpha}.
\end{aligned}$$

Integrating in time and noting that $\int_0^\infty \|\nabla j\|^2 ds < C$, we obtain, for $t>0$,

$$\begin{aligned}
\|\omega(t)\|_{L^2}^2 &\le C(1+t)^{-k} + C(1+t)^{-\frac{5}{2\alpha}} + C(1+t)^{-\frac{2}{\alpha}}\\
&\quad + C(1+t)^{-k}\int_0^t (1+s)^{k-2-\frac{5\alpha}{2}}\|\nabla j\|_{L^2}^{2-\alpha} ds\\
&\quad + C(1+t)^{-k}\int_0^t (1+s)^{k-2-2\alpha}\|\nabla j\|_{L^2}^{2-2\alpha} ds\\
&\le C(1+t)^{-\frac{2}{\alpha}} + C(1+t)^{-k}\left(\int_0^t (1+s)^{\frac{2}{\alpha}(k-2-\frac{5\alpha}{2})}\right)^{\frac{\alpha}{2}}\left(\int_0^t \|\nabla j\|_{L^2}^2 ds\right)^{\frac{2-\alpha}{2}}\\
&\quad + C(1+t)^{-k}\left(\int_0^t (1+s)^{\frac{1}{\alpha}(k-2-2\alpha)}\right)^{\alpha}\left(\int_0^t \|\nabla j\|_{L^2}^2 ds\right)^{1-\alpha}\\
&\le C(1+t)^{-2-\alpha}.
\end{aligned}$$

This completes the proof of Proposition 3.4. □

Acknowledgements. B. Dong was supported by by the NNSF of China (Nos. 11571240, No.11271019) and Research Fund of Shenzhen University (No. 2017056). Y. Jia is also financed by the Doctor Research Fund of Anhui University (J10117700054) and the NSF of Anhui Education Bureau (KJ2015A045). J. Li was partially supported by the NNSFC Grant No. 11201181. J. Wu was partially supported by the NSF grant DMS 1614246, by the AT&T Foundation at Oklahoma State University, and by NNSFC Grant No. 11471103 (a grant awarded to B. Yuan).

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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(accepted: April 28, 2018; published online: May 15, 2018)