Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx
pp. **X–XX**

GLOBAL REGULARITY RESULTS FOR THE CLIMATE MODEL WITH FRACTIONAL DISSIPATION

Boqing Dong and Wenjuan Wang

School of Mathematical Sciences
Anhui University
Hefei 230601, China

Jiahong Wu*

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, USA

Hui Zhang

School of Mathematics and computation Sciences
Anqing Normal University
Anqing, 246133, China

(Communicated by Peter E. Kloeden)

Abstract. This paper studies the global well-posedness problem on a tropical climate model with fractional dissipation. This system allows us to simultaneously examine a family of equations characterized by the fractional dissipative terms $(-\Delta)^\alpha u$ in the equation of the barotropic mode u and $(-\Delta)^\beta v$ in the equation of the first baroclinic mode v. We establish the global existence and regularity of the solutions when the total fractional power is 2, namely $\alpha + \beta = 2$.

1. **Introduction.** This paper focuses on the global existence and regularity of solutions to the initial-value problem on a tropical climate model with fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^\alpha u + \nabla p + \nabla \cdot (v \otimes v) = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \partial_t v + u \cdot \nabla v + v \cdot \nabla u + \eta(-\Delta)^\beta v + \nabla\theta = 0, \\ \partial_t \theta + u \cdot \nabla\theta + \nabla \cdot v = 0, \\ \nabla \cdot u = 0, \\ (u, v, \theta)(x, 0) = (u_0(x), v_0(x), \theta_0(x)), \end{cases} \tag{1}$$

where the vector fields $u = (u_1, u_2)$ and $v = (v_1, v_2)$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, and the scalar p denotes

---

2010 *Mathematics Subject Classification.* Primary: 35Q35, 35B65; Secondary: 76B03.

Key words and phrases. Atmospheric dynamics, baroclinic mode, climate model, fractional dissipation, global well-posedness.

B. Dong was partially supported by the NNSFC (No. 11271019, No. 11571240). J. Wu was supported by NSF grant DMS 1614246 and the AT&T Foundation at Oklahoma State University. H. Zhang was as partially supported by the Research Fund of SMS at Anhui University.

* Corresponding author: Jiahong Wu.

the pressure and θ the temperature, and $\nu \geq 0$, $\eta \geq 0$, $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$ are real parameters.

When $\nu = \eta = 0$, (1) reduces to the original tropical climate model derived by Frierson, Majda and Pauluis [5]. When $\nu > 0$, $\eta > 0$, $\alpha = 1$ and $\beta = 1$, (1) reduces to the viscous counterpart of the Frierson-Majda-Pauluis model that was studied by Li and Titi [10]. We also mention a recent work of Ye [14], which includes the standard Laplacian dissipation in the equation for θ. In this paper the equation for θ has no dissipation. (1) with $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$ may be relevant in modeling tropical atmospheric dynamics. These fractional diffusion operators model the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., [1, 6, 11]). Especially, (1) allows us to study long-range diffusive interactions. Our main focus here is on the fundamental mathematical issues such as the global (in time) existence and regularity problem. Mathematically (1) has the advantage of allowing us to examine a family of equations simultaneously.

We are able to show that, for $\alpha + \beta = 2$ and $1 < \beta \leq \frac{3}{2}$, (1) always possesses a unique global solution when the initial data (u_0, v_0, θ_0) is sufficiently smooth. More precisely, we obtain the following theorem.

Theorem 1.1. *Let $s > 2$. Assume the initial data (u_0, v_0, θ_0) satisfies*

$$u_0, v_0 \in H^s(\mathbb{R}^2), \quad \nabla \cdot u_0 = 0, \quad \theta_0 \in \dot{H}^{-1}(\mathbb{R}^2) \cap H^{s+1-\beta}(\mathbb{R}^2).$$

Consider (1) with α and β satisfying

$$\alpha + \beta = 2, \quad 1 < \beta \leq \frac{3}{2}.$$

Then (1) has a unique global solution (u, v, θ) satisfying, for any $t > 0$,

$$u,\, v \in C([0, t); H^s(\mathbb{R}^2)), \qquad \theta \in C([0, t); \dot{H}^{-1}(\mathbb{R}^2) \cap H^{s+1-\beta}(\mathbb{R}^2)).$$

We remark that the case $\alpha + \beta = 2$ and $\frac{3}{2} < \beta < 2$ is no more difficult than the case presented here and will be worked out later. In addition, we also examine a special case of $\alpha + \beta = 2$, namely $\alpha = 2$ and $\beta = 0$. We establish the global existence and uniqueness of the solutions when the initial data is in H^s with $s > 2$.

Theorem 1.2. *Let $s > 2$. Assume $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$. Consider (1) with $\alpha = 2$ and $\eta = 0$. Then (1) has a unique global solution (u, v, θ) satisfying, for any $t > 0$*

$$u, v, \theta \in C(0, t; H^s(\mathbb{R}^2)).$$

Our approach for proving Theorem 1.1 is new and different from those in [10] and [14]. The proof of Theorem 1.1 boils down to proving global *a priori* bounds. The global bound for the L^2-norm of (u, v, θ) follows directly from (1). The proof of the global H^1-bound is more difficult and is at the core of the proof of Theorem 1.1. For notational convenience, we set $\nu = \eta = 1$. Our approach for proving the global H^1-bound is new. We take the structure of (1) into full account and reformulate (1) in terms of the variables

$$\omega = \nabla \times u, \qquad j = \nabla \times v, \qquad h = \nabla \cdot v, \qquad H = h - \Lambda^{2-2\beta}\theta$$

as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + (-\Delta)^{\alpha} \omega + v \cdot \nabla j + 2h\, j + v_2 \partial_1 h - v_1 \partial_2 h = 0, \\ \partial_t j + u \cdot \nabla j + v \cdot \nabla \omega + \omega\, h + \eta (-\Delta)^{\beta} j = 0, \\ \partial_t h + u \cdot \nabla h + (-\Delta)^{\beta} h + \Delta \theta = -2 \nabla u : \nabla v, \\ \partial_t \theta + u \cdot \nabla \theta + h = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{2}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and the notation $A : B$ for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as

$$A : B = \sum_{i,j} a_{ij}\, b_{ij}.$$

The advantage of (2) is that the equations of ω and j do not involve θ. Due to the lack of dissipation in the equation of θ, it is difficult to deal with the term $\Delta\theta$ in the equation for h. This motivates us to consider the following combined quantity

$$H = h - \mathcal{R}_\beta \theta \quad \text{with} \quad \mathcal{R}_\beta = \Lambda^{2-2\beta},$$

which satisfies (by combining the equations of h and θ)

$$\partial_t H + u \cdot \nabla H + (-\Delta)^{\beta} H = -2 \nabla u : \nabla v + \Lambda^{2-2\beta} h + [\Lambda^{2-2\beta}, u \cdot \nabla] \theta. \tag{3}$$

We work with (2) and (3) to prove the global bound for (ω, j, H), which reads

$$\|(\omega, j, H)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{\alpha} \nabla u\|_{L^2}^2 + \|\Lambda^{\beta} (j, H)\|_{L^2}^2 \right) d\tau \le C(\|(u_0, v_0, \theta_0)\|_{H^1}, t) < \infty.$$

This global bound does not immediately translate into a global bound for

$$\int_0^t \|\Lambda^{\beta} \nabla v\|_{L^2}^2 \, d\tau < \infty.$$

As in Lemma 1.3 stated below, the L^2-norm of the gradient can be represented in terms of the L^2-norms of the curl and the divergence, namely

$$\|\Lambda^{\beta} \nabla v\|_{L^2}^2 = \|\Lambda^{\beta} j\|_{L^2}^2 + \|\Lambda^{\beta} h\|_{L^2}^2 = \|\Lambda^{\beta} j\|_{L^2}^2 + \|\Lambda^{\beta} H + \Lambda^{2-\beta} \theta\|_{L^2}^2,$$

but the trouble is that we do not have control of $\|\Lambda^{2-\beta}\theta\|_{L^2}$. Since there is no dissipation in the equation of θ, we need to control $\|\nabla u\|_{L^\infty}$ or its equivalent such as $\|\omega\|_{H^1}$ in order to control any derivative of θ.

This prompts us to prove global bounds for ω in more regular settings. We first show a global bound for $\|\omega\|_{L^{\frac{2}{\alpha}}}$, which allows us to further prove the global bound for

$$\int_0^t \|\nabla \omega\|_{L^2}^2 \, d\tau < \infty.$$

This global bound is enough for us to control the nonlinear terms in the estimate of the H^s via the logarithmic Sobolev inequality, for $s > 2$,

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \le C \left(1 + \|u\|_{L^2(\mathbb{R}^2)} + \|\nabla \omega\|_{L^2(\mathbb{R}^2)} \log(1 + \|u\|_{H^s(\mathbb{R}^2)}) \right).$$

The proof of Theorem 1.2 is achieved through a two-stage process. The first stage proves a global H^1-bound via energy estimates and an improved version of Gronwall's inequality while the second stage establishes the global H^s bound by making use of the global H^1 bound.

Finally we supply a basic fact that relates ∇F to $\nabla \times F$ and $\nabla \cdot F$ for a vector field F. As we know, for a divergence-free vector $\nabla \cdot F = 0$,

$$\|\nabla F\|_{L^2} = \|\nabla \times F\|_{L^2}.$$

If F is not divergence-free, $\nabla \cdot F \neq 0$, in general $\|\nabla F\|_{L^2} \neq \|\nabla \times F\|_{L^2}$. The following lemma relates the L^2-norms of ∇F, $\nabla \times F$ and $\nabla \cdot F$ and provides a bound for the L^q-norm of ∇F. This lemma will be used repeatedly throughout the rest of this paper.

Lemma 1.3. *For any vector field F,*

$$\|\nabla F\|_{L^2}^2 = \|\nabla \times F\|_{L^2}^2 + \|\nabla \cdot F\|_{L^2}^2 \tag{4}$$

and, for $2 < q < \infty$,

$$\|\nabla F\|_{L^q} \leq C\,(\|\nabla \times F\|_{L^q} + \|\nabla \cdot F\|_{L^q}). \tag{5}$$

(4) follows from the identity

$$\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \Delta F, \tag{6}$$

Plancherel's theorem and a direct calculation. (5) follows from a variant of (6),

$$\nabla F = \nabla(-\Delta)^{-1}\nabla \times (\nabla \times F) - \nabla(-\Delta)^{-1}\nabla(\nabla \cdot F),$$

the Calderon-Zygmund inequality on singular integral operators.

The rest of this paper is divided into two sections followed by an appendix. Section 2 provides the proof of Theorem 1.1 while Section 3 proves Theorem 1.2. The appendix supplies some of the inequalities as well as the definition of Besov spaces.

2. Proof of Theorem 1.1.

This section proves Theorem 1.1. As we know, the proof of Theorem 1.1 boils down to the global *a priori* bounds in H^s. The proof is achieved in several steps, which successively establish the global bounds in more and more regular functional settings.

2.1. Global H^1-bound.

The subsection proves the following global H^1 bound.

Proposition 1. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.1. Assume $\alpha + \beta = 2$ and $1 < \beta \leq \frac{3}{2}$. Let (u, v, θ) be the corresponding solution. Then (u, v, θ) obeys the following global H^1-bound, for any $t > 0$,*

$$\begin{aligned}\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 &+ \int_0^t \left(\|\Lambda^\alpha \nabla u\|_{L^2}^2 + \|\Lambda^\beta (j, H)\|_{L^2}^2\right) d\tau \\ &\leq C(\|(u_0, v_0, \theta_0)\|_{H^1}, t) < \infty,\end{aligned}$$

where $C(\|(u_0, v_0, \theta_0)\|_{H^1}, t)$ depends on t and the initial H^1-norm $\|(u_0, v_0, \theta_0)\|_{H^1}$.

In order to prove Proposition 1, we first state the following global L^2 bound for (u_0, v_0, θ_0) and the global L^q-bound for θ.

Lemma 2.1. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then (u, v, θ) obeys the following global L^2-bound, for any $t > 0$,*

$$\|(u, v, \theta)\|_{L^2}^2 + 2\int_0^t (\|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2)\, d\tau = \|(u_0, v_0, \theta_0)\|_{L^2}^2. \tag{1}$$

In addition, for any q satisfying $2 \le q \le \frac{2}{2-\beta}$ and for any $t > 0$,

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} + \sqrt{t}\|\Lambda^\beta v\|_{L^2_t L^2} < \infty. \tag{2}$$

Proof of Lemma 2.1. The global L^2-bound in (1) follows from a standard energy estimate involving integration by parts and the application of $\nabla \cdot u = 0$. The global bound in (2) follows from the fact that, for $q = \frac{2}{2-\beta} > 2$, by taking the scalar product of $\theta|\theta|^{q-1}$ with the equation for θ in (1.2),

$$\begin{aligned}
\|\theta(t)\|_{L^{\frac{2}{2-\beta}}} &\le \|\theta_0\|_{L^{\frac{2}{2-\beta}}} + \int_0^t \|h\|_{L^q}\, d\tau \\
&\le \|\theta_0\|_{L^{\frac{2}{2-\beta}}} + \int_0^t \|\Lambda^\beta v\|_{L^2}\, d\tau \\
&\le \|\theta_0\|_{L^{\frac{2}{2-\beta}}} + \sqrt{t}\|\Lambda^\beta v\|_{L^2_t L^2}.
\end{aligned}$$

The global bound for $\|\theta\|_{L^q}$ with $2 \le q \le \frac{2}{2-\beta}$ follows from a simple interpolation inequality. $\square$

We now turn to the proof of Proposition 1.

Proof of Proposition 1. Dotting the equation of ω with ω, the equation of j with j and the equation of H with H, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|(\omega, j, H)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2 + \|\Lambda^\beta H\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
&I_1 = -3\int w\, h\, j, \qquad I_2 = -\int (v_2 \partial_1 h - v_1 \partial_2 h)\,\omega, \qquad I_3 = -2\int \nabla u : \nabla v\, H, \\
&I_4 = \int \Lambda^{2-2\beta} h\, H, \qquad I_5 = \int H\,[\Lambda^{2-2\beta}, u\cdot\nabla]\theta.
\end{aligned}$$

We now estimate the terms on the right-hand side. By Hölder's inequality and Sobolev embedding inequality

$$\begin{aligned}
|I_1| &\le 3\,\|j\|_{L^2}\,\|\omega\|_{L^p}\,\|h\|_{L^q} \\
&\le C\,\|j\|_{L^2}\,\|\Lambda^\alpha \omega\|_{L^2}\,\|\Lambda^{\beta-1} h\|_{L^2} \\
&\le C\,\|j\|_{L^2}\,\|\Lambda^\alpha \omega\|_{L^2}\,\|\Lambda^\beta v\|_{L^2} \\
&\le \frac{1}{16}\,\|\Lambda^\alpha \omega\|_{L^2}^2 + C\,\|\Lambda^\beta v\|_{L^2}^2\,\|j\|_{L^2}^2,
\end{aligned}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad \frac{1}{p} = \frac{1}{2} - \frac{\alpha}{2}, \qquad \frac{1}{q} = \frac{1}{2} - \frac{\beta - 1}{2}. \tag{3}$$

To estimate I_2, we recall that $h = H + \Lambda^{2-2\beta}\theta$ and write I_2 as

$$\begin{aligned}
I_2 &= -\int (v_2\partial_1 H - v_1 \partial_2 H)\,\omega - \int (v_2 \partial_1 \Lambda^{2-2\beta}\theta - v_1 \partial_2 \Lambda^{2-2\beta}\theta)\,\omega \\
&:= I_{21} + I_{22}.
\end{aligned}$$

By Hölder's inequality and Sobolev's embedding inequality,

$$\begin{aligned}
|I_{21}| &\le \|\omega\|_{L^2}\|v\|_{L^{\frac{2}{\beta-1}}}\,\|\nabla H\|_{L^{\frac{2}{2-\beta}}} \\
&\le \|\omega\|_{L^2}\,\|v\|_{H^\beta}\,\|\Lambda^\beta H\|_{L^2}.
\end{aligned}$$

To bound I_{22}, we shift the derivatives away from θ to obtain

$$I_{22} = \int \theta \Lambda^{2-2\beta}(\partial_1(v_2\omega) - \partial_2(v_1\omega)).$$

By Hölder's inequality and Sobolev's embedding inequality,

$$\begin{aligned}
|I_{22}| &\le \|\theta\|_{L^{\frac{2}{2-\beta}}} \|\Lambda^{3-2\beta}(v\omega)\|_{L^{\frac{2}{\beta}}} \\
&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}} \left(\|\Lambda^{3-2\beta} v\|_{L^2}\|\omega\|_{L^{\frac{2}{1-\alpha}}} + \|\Lambda^{3-2\beta}\omega\|_{L^{\frac{2}{2-\beta}}} \|v\|_{L^{\frac{1}{\beta-1}}}\right) \\
&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}} \|v\|_{H^\beta} \|\Lambda^\alpha\omega\|_{L^2} \\
&\le \frac{1}{16}\|\Lambda^\alpha\omega\|_{L^2}^2 + C\|\theta\|^2_{L^{\frac{2}{2-\beta}}} \|v\|^2_{H^\beta}.
\end{aligned}$$

Again, by Hölder's inequality and Sobolev embedding inequality,

$$\begin{aligned}
|I_3| &\le 2\|\nabla u\|_{L^p}\|\nabla v\|_{L^q}\|H\|_{L^2} \\
&\le C\|\Lambda^\alpha\omega\|_{L^2}\|\Lambda^{\beta-1}\nabla v\|_{L^2}\|H\|_{L^2} \\
&\le \frac{1}{16}\|\Lambda^\alpha\omega\|_{L^2}^2 + C\|\Lambda^\beta v\|^2_{L^2}\|H\|^2_{L^2},
\end{aligned}$$

where p and q are defined in (3). Thanks to $1 < \beta \le \frac{3}{2}$, by an interpolation inequality,

$$\begin{aligned}
|I_4| &\le \|\Lambda^{3-2\beta}v\|_{L^2}\|H\|_{L^2} \\
&\le C\|v\|_{L^2}^{3(1-\frac{1}{\beta})}\|\Lambda^\beta v\|_{L^2}^{\frac{3}{\beta}-2}\|H\|_{L^2}.
\end{aligned}$$

Due to $\nabla\cdot u = 0$,

$$[\Lambda^{2-2\beta}, u\cdot\nabla]\theta = \Lambda^{2-2\beta}\nabla\cdot(u\theta) - u\cdot\nabla\Lambda^{2-2\beta}\theta.$$

By Lemma A.1,

$$\begin{aligned}
|I_5| &\le \|\Lambda^{2-2\beta}\nabla\cdot(u\theta) - u\cdot\nabla\Lambda^{2-2\beta}\theta\|_{L^2}\|H\|_{L^2} \\
&\le C\|H\|_{L^2}\left(\|\Lambda^{3-2\beta}u\|_{L^{\frac{2}{\beta-1}}}\|\theta\|_{L^{\frac{2}{2-\beta}}} + \|\nabla u\|_{L^{\frac{1}{\beta-1}}}\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{3-2\beta}}}\right) \\
&\le C\|H\|_{L^2}\left(\|\Lambda^{5-3\beta}u\|_{L^2}\|\theta\|_{L^{\frac{2}{2-\beta}}} + \|\Lambda^{3-2\beta}\nabla u\|_{L^2}\|\theta\|_{L^2}\right) \\
&\le C\|H\|_{L^2}(\|\omega\|_{L^2} + \|\Lambda^\alpha\omega\|_{L^2})(\|\theta\|_{L^2} + \|\theta\|_{L^{\frac{2}{2-\beta}}}),
\end{aligned}$$

where, due to $\beta > 1$, we have $5-3\beta \le 1+\alpha$ and $4-2\beta \le 1+\alpha$, and

$$\|\Lambda^{5-3\beta}u\|_{L^2},\ \|\Lambda^{3-2\beta}\nabla u\|_{L^2} \le C(\|\omega\|_{L^2} + \|\Lambda^\alpha\omega\|_{L^2}).$$

Further, by Young's inequality and Lemma 2.1,

$$|I_5| \le \frac{1}{16}\|\Lambda^\alpha\omega\|^2_{L^2} + C(\|H\|^2_{L^2} + \|\omega\|^2_{L^2}).$$

Combining all the estimates above, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\omega, j, H)\|^2_{L^2} + \|\Lambda^\alpha\omega\|^2_{L^2} + \|\Lambda^\beta j\|^2_{L^2} + \|\Lambda^\beta H\|^2_{L^2} \\
&\qquad \le C(1+\|v\|^2_{H^\beta})(\|\omega\|^2_{L^2} + \|j\|^2_{L^2} + \|H\|^2_{L^2} + \|H\|_{L^2}) + C\|\theta\|^2_{L^{\frac{2}{2-\beta}}}\|v\|^2_{H^\beta}.
\end{aligned}$$

Gronwall's inequality then implies the global H^1-bound in Proposition 1 $\square$

2.2. **Global $W^{1,p}$ bound for v.** This subsection proves a global bound for $\|\theta\|_{L^q}$ with any $q \in [2,\infty]$ and for $\|\nabla v\|_{L^q}$ for any $2 \le q < \infty$.

Proposition 2. *Assume (u, v, θ) solves (1). Then, for any $t > 0$,*

$$\begin{aligned} &\|\theta(t)\|_{L^q} \le C(t, \|(u_0, v_0, \theta_0)\|_{H^1}), \quad 2 \le q \le \infty, \\ &\|\nabla v(t)\|_{L^q} \le C(t, \|\theta_0\|_{\dot{H}^{-1}}, \|(u_0, v_0, \theta_0)\|_{H^1}), \quad 2 \le q < \infty. \end{aligned} \tag{4}$$

Especially,

$$\|v(t)\|_{L^\infty} \le C(t, \|\theta_0\|_{\dot{H}^{-1}}, \|(u_0, v_0, \theta_0)\|_{H^1}).$$

To prove Proposition 2, we need to consider $\|\theta\|_{\dot{H}^{-1}}$. It appears reasonable to consider the $\dot{H}^{-1}$-norm of θ. The equation of θ

$$\partial_t \theta + u \cdot \nabla \theta + \nabla \cdot v = 0$$

indicates that $\|\theta\|_{L^2} \sim \|\nabla \cdot v\|_{L^2}$. Consequently, the counterpart of $\|v\|_{L^2}$ is $\|\theta\|_{\dot{H}^{-1}}$. The following asserts that $\|\theta\|_{\dot{H}^{-1}}$ remains bounded for all time.

Lemma 2.2. *Assume $\theta_0 \in \dot{H}^{-1} \cap \dot{H}^1$. Then, for all time $t > 0$,*

$$\|\theta(t)\|_{\dot{H}^{-1}} \le C(t, \|\theta_0\|_{\dot{H}^{-1}}, \|(u_0, v_0, \theta_0)\|_{H^1}).$$

Proof of Lemma 2.2. We write $\theta = \Delta\widetilde{\theta}$. Inserting this in the equation for θ and dotting the equation by $\widetilde{\theta}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\theta}\|_{L^2}^2 &\le \|u\theta\|_{L^2}\|\nabla\widetilde{\theta}\|_{L^2} + \|v\|_{L^2}\|\nabla\widetilde{\theta}\|_{L^2} \\ &\le (\|u\|_{L^\infty}\|\theta\|_{L^2} + \|v\|_{L^2})\|\nabla\widetilde{\theta}\|_{L^2} \end{aligned}$$

The embedding inequality

$$\|u\|_{L^\infty} \le C\,\|u\|_{L^2}^{\frac{1}{1+\alpha}}\,\|\Lambda^\alpha \omega\|_{L^2}^{\frac{\alpha}{1+\alpha}}$$

and the global bound in Proposition 1 then implies that

$$\|\theta(t)\|_{\dot{H}^{-1}} = \|\nabla\widetilde{\theta}\|_{L^2} \le \|\theta_0\|_{\dot{H}^{-1}} + \int_0^t (\|u\|_{L^\infty}\|\theta\|_{L^2} + \|v\|_{L^2})\,d\tau < \infty.$$

This completes the proof of Lemma 2.2. □

We now prove Proposition 2.

Proof of Proposition 2. First we show that, for any $2 \le q \le \infty$,

$$\|\theta(t)\|_{L^q} \le C < \infty.$$

In fact, for any $2 \le \widetilde{q} < q \le \infty$ satisfying

$$2 \le \widetilde{q} < \frac{2}{2-\beta}, \qquad \frac{1}{q} = \frac{1}{\widetilde{q}} - \frac{2\beta - 2}{2}$$

we have, from the equation of θ and the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} \|\theta(t)\|_{L^q} &\le \|\theta_0\|_{L^q} + \int_0^t \|h\|_{L^q}\,d\tau \\ &\le \|\theta_0\|_{L^q} + \int_0^t \|H + \Lambda^{2-2\beta}\theta\|_{L^q}\,d\tau \\ &\le \|\theta_0\|_{L^q} + \int_0^t \|H\|_{L^q}\,d\tau + \int_0^t \|\theta\|_{L^{\widetilde{q}}}\,d\tau. \end{aligned} \tag{5}$$

Because of the embedding inequality

$$\|H\|_{L^q} \le C\,\|H\|_{L^2}^{1-\frac{q-2}{\beta q}}\,\|\Lambda^\beta H\|_{L^2}^{\frac{q-2}{\beta q}},$$

the global bounds in Proposition 1 and Lemma 2.1, we use (5) to control $\|\theta(t)\|_{L^q}$ by $\|\theta\|_{L^{\tilde q}}$ and an iterative process leads to a global bound on $\|\theta(t)\|_{L^q}$ for all q.

We now establish a global bound on $\|\nabla v\|_{L^q}$. We rewrite the equation for v in the integral form

$$v = g * v_0 - \int_0^t g(t-\tau) * (u\cdot\nabla v + v\cdot\nabla u + \nabla\theta)(\tau)\,d\tau,$$

where g is the kernel function associated with the operator $e^{-t\,(-\Delta)^\beta}$, or

$$\widehat{g}(\xi,t) = e^{-t\,|\xi|^{2\beta}}.$$

By Young's inequality, for $q_1, q_2 \in [1,\infty]$ and $1+\frac{1}{q} = \frac{1}{q_1}+\frac{1}{q_2}$,

$$\|\nabla v\|_{L^q} \le \|g * \nabla v_0\|_{L^q} + \int_0^t \|\nabla\nabla g(t-\tau)\|_{L^{q_1}}\,\|(uv+\theta)(\tau)\|_{L^{q_2}}\,d\tau.$$

Noticing that

$$g(x,t) = t^{-\frac{1}{\beta}}\, g\left(\frac{x}{t^{\frac{1}{2\beta}}},1\right) =: t^{-\frac{1}{\beta}}\, g_0\left(\frac{x}{t^{\frac{1}{2\beta}}}\right),$$

we have

$$\|\nabla\nabla g(t)\|_{L^{q_1}} \le C\,t^{-\frac{2}{\beta}+\frac{1}{q_1\beta}}\,\|g_0\|_{L^{q_1}}.$$

In addition,

$$\|uv+\theta\|_{L^{q_2}} \le \|u\|_{L^{2q_2}}\|v\|_{L^{2q_2}} + \|\theta\|_{L^{q_2}} \le C\,\|\omega\|_{L^2}\|\nabla v\|_{L^2} + \|\theta\|_{L^{q_2}}$$

and, due to $\beta \le \frac{3}{2}$, by an interpolation inequality and Lemma 2.2

$$\begin{aligned}
\|\nabla v\|_{L^2} &\le C\,(\|j\|_{L^2} + \|h\|_{L^2})\\
&= C\,(\|j\|_{L^2} + \|H\|_{L^2} + \|\Lambda^{2-2\beta}\theta\|_{L^2})\\
&\le C\,(\|j\|_{L^2} + \|H\|_{L^2} + \|\theta\|_{\dot H^{-1}} + \|\theta\|_{L^2}) < \infty.
\end{aligned}$$

By Proposition 1, for any $t>0$,

$$\|uv+\theta\|_{L^{q_2}} < \infty.$$

Therefore, if $1 < q_1 < \frac{1}{2-\beta}$, then

$$-1 < -\frac{2}{\beta} + \frac{1}{q_1\beta} < 0$$

and thus

$$\begin{aligned}
\|\nabla v\|_{L^q} &\le \|g\|_{L^1}\|\nabla v_0\|_{L^q} + C\int_0^t (t-\tau)^{-\frac{2}{\beta}+\frac{1}{q_1\beta}}\,d\tau\\
&\le \|\nabla v_0\|_{L^q} + C < \infty.
\end{aligned}$$

This completes the proof of Proposition 2. $\square$

2.3. **Global bound for $\|\omega\|_{L^{\frac{2}{\alpha}}}$ and $\|\nabla\omega\|_{L^2_tL^2}$.** This subsection proves a global bound for $\|\nabla\omega\|_{L^2_tL^2}$, an important step in proving the global H^s bound for the solution. To do so, we need a global bound for $\|\omega\|_{L^{\frac{2}{\alpha}}}$.

Proposition 3. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.1 with $\alpha+\beta=2$ and $1<\beta\le\frac{3}{2}$. Let (u,v,θ) be the corresponding solution. Then $\omega=\nabla\times u$ satisfies, for any $t>0$,*

$$\|\omega(t)\|_{L^{\frac{2}{\alpha}}}\le C(t,u_0,v_0,\theta_0), \qquad \int_0^t\|\nabla\omega(\tau)\|_{L^2}^2\,d\tau\le C(t,u_0,v_0,\theta_0). \tag{6}$$

Proof of Proposition 3. For notational convenience, we write $q=\frac{2}{\alpha}$. Recalling the equation of ω,

$$\partial_t\omega+u\cdot\nabla\omega+(-\Delta)^\alpha\omega+\nabla\times\nabla\cdot(v\otimes v)=0, \tag{7}$$

we obtain

$$\frac{1}{q}\frac{d}{dt}\|\omega\|_{L^q}^q+\int|\omega|^{q-2}\omega\,(-\Delta)^\alpha\omega\,dx=-\int_{\mathbb{R}^2}\nabla\times\nabla\cdot(v\otimes v)|\omega|^{q-2}\omega\,dx. \tag{8}$$

The dissipative part admits the lower bound,

$$\int|\omega|^{q-2}\omega\,(-\Delta)^\alpha\omega\,dx\ge C_0\,\|\omega\|^q_{L^{\frac{q}{1-\alpha}}}. \tag{9}$$

We write v in terms of j and h. According to (6),

$$\begin{aligned} v &= (-\Delta)^{-1}\nabla\times(\nabla\times v)-(-\Delta)^{-1}\nabla(\nabla\cdot v)\\ &= (-\Delta)^{-1}\nabla\times j-(-\Delta)^{-1}\nabla h\\ &= (-\Delta)^{-1}\nabla\times j-(-\Delta)^{-1}\nabla H-(-\Delta)^{-1}\nabla\Lambda^{2-2\beta}\theta. \end{aligned} \tag{10}$$

Inserting this representation in (8) and applying Hölder's inequality yield

$$\begin{aligned} &\left|\int_{\mathbb{R}^2}\nabla\times\nabla\cdot(v\otimes v)|\omega|^{q-2}\omega\,dx\right|\\ \le\;& C\,\|v\|_{L^\infty}\,(\|\nabla\times j\|_{L^q}+\|\nabla H\|_{L^q})\,\|\omega\|_{L^q}^{q-1}\\ &+C\,\|v\|_{L^\infty}\,\|\theta\|_{L^{q_1}}\,\|\Lambda^{3-2\beta}(|\omega|^{q-2}\omega)\|_{L^{q_2}}\\ \le\;& C\,\|v\|_{L^\infty}\,\left(\|\Lambda^\beta j\|_{L^2}+\|\Lambda^\beta H\|_{L^2}\right)\,\|\omega\|_{L^q}^{q-1}\\ &+C\,\|v\|_{L^\infty}\,\|\theta\|_{L^{q_1}}\,\|\Lambda^{3-2\beta}(|\omega|^{q-2}\omega)\|_{L^{\frac{1}{\frac{\alpha}{2}+2(1-\alpha)^2}}}\\ &+C\,\|\omega\|_{L^q}^{q-1}\,\|\nabla v\|_{L^{q_2}}^2, \end{aligned}$$

where $\frac{1}{q_1}+\frac{\alpha}{2}+2(1-\alpha)^2=1$ and $q_2=2q$. Due to $3-2\beta<\alpha$, we can choose σ satisfying

$$3-2\beta<\sigma<\alpha.$$

By Lemma A.5,

$$\begin{aligned} \|\Lambda^{3-2\beta}(|\omega|^{q-2}\omega)\|_{L^{\frac{1}{\frac{\alpha}{2}+2(1-\alpha)^2}}} &\le C\,\|\omega\|_{B^\sigma_{\frac{1}{\frac{\alpha}{2}+(1-\alpha)^2},\frac{1}{\frac{\alpha}{2}+2(1-\alpha)^2}}}\,\|\omega\|^{q-2}_{L^{\frac{q-2}{(1-\alpha)^2}}}\\ &\le C\,\|\omega\|_{B^\alpha_{2,2}}\,\|\omega\|^{q-2}_{L^{\frac{q}{1-\alpha}}}\\ &= C\,\|\omega\|_{H^\alpha}\,\|\omega\|^{q-2}_{L^{\frac{q}{1-\alpha}}}, \end{aligned}$$

where we have used a Besov embedding inequality and a simple identity

$$\|\omega\|_{B^{\sigma}_{\frac{1}{\frac{\alpha}{2}+(1-\alpha)^2},\frac{1}{\frac{\alpha}{2}+2(1-\alpha)^2}}} \le C\,\|\omega\|_{B^{\alpha}_{2,2}}, \qquad \frac{q-2}{(1-\alpha)^2}=\frac{q}{1-\alpha} \quad \text{for} \quad q=\frac{2}{\alpha}.$$

Therefore,

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \nabla\times\nabla\cdot(v\otimes v)|\omega|^{q-2}\omega\,dx\\
&\quad\le \frac{C_0}{16}\,\|\omega\|^q_{L^{\frac{q}{1-\alpha}}} + C\,\|v\|_{L^\infty}\,\left(\|\Lambda^\beta j\|_{L^2}+\|\Lambda^\beta H\|_{L^2}\right)\,\|\omega\|^{q-1}_{L^q}\\
&\quad\quad + C\,\|\omega\|^{q-1}_{L^q}\,\|\nabla v\|^2_{L^{q_2}} + C\,\|v\|^{\frac{1}{\alpha}}_{L^\infty}\,\|\theta\|^{\frac{1}{\alpha}}_{L^{q_1}}\,\|\omega\|^{\frac{1}{\alpha}}_{H^\alpha}.
\end{aligned} \tag{11}$$

Inserting (9) and (11) in (8), we obtain, due to $\alpha\ge\frac12$ because $\alpha+\beta=2$ and $\beta\le\frac32$,

$$\|\omega\|_{L^{\frac{2}{\alpha}}} \le C(t,u_0,v_0,\theta_0)<\infty.$$

Let $\gamma=\alpha+2\beta-3$. Due to $\alpha+\beta=2$ and $\beta>1$, we have

$$\gamma>0, \qquad \gamma+\alpha=1.$$

Applying Λ^γ to (7) and then dotting with $\Lambda^\gamma\omega$, we find

$$\frac12\frac{d}{dt}\|\Lambda^\gamma\omega\|^2_{L^2}+\|\Lambda^{\alpha+\gamma}\omega\|^2_{L^2}=J_1+J_2, \tag{12}$$

where

$$\widetilde{J}_1=-\int \Lambda^\gamma(u\cdot\nabla\omega)\,\Lambda^\gamma\omega\,dx,$$
$$\widetilde{J}_2=-\int \Lambda^\gamma\omega\,\Lambda^\gamma\nabla\times\nabla\cdot(v\otimes v)\,dx.$$

To bound $\widetilde{J}_1$, we write it as, due to $\nabla\cdot u=0$,

$$\widetilde{J}_1=-\int(\Lambda^\gamma(u\cdot\nabla\omega)-u\cdot\nabla\Lambda^\gamma\omega)\,\Lambda^\gamma\omega\,dx.$$

By Lemma A.1,

$$\begin{aligned}
|\widetilde{J}_1| &\le C\,\|\Lambda^\gamma\omega\|_{L^2}\,\left(\|\Lambda^\gamma\nabla u\|_{L^{\frac{2}{1-\alpha}}}\,\|\omega\|_{L^{\frac{2}{\alpha}}}+\|\nabla u\|_{L^{\frac{2}{\alpha}}}\,\|\Lambda^\gamma\omega\|_{L^{\frac{2}{1-\alpha}}}\right)\\
&\le C\,\|\Lambda^\gamma\omega\|_{L^2}\,\|\omega\|_{L^{\frac{2}{\alpha}}}\,\|\Lambda^{\alpha+\gamma}\omega\|_{L^2}\\
&\le \frac{1}{16}\|\Lambda^{\alpha+\gamma}\omega\|^2_{L^2}+C\,\|\omega\|^2_{L^{\frac{2}{\alpha}}}\,\|\Lambda^\gamma\omega\|^2_{L^2}.
\end{aligned}$$

To bound $\widetilde{J}_2$, we invoke (10) and apply Hölder's inequality

$$\begin{aligned}
|\widetilde{J}_2| &\le C\,\|v\|_{L^\infty}\,\|(j,H)\|_{H^1}\,\|\Lambda^{\alpha+\gamma}\omega\|_{L^2}+C\,\|v\|_{L^\infty}\,\|\theta\|_{L^2}\,\|\Lambda^{3-2\beta}\Lambda^{2\gamma}\omega\|_{L^2}\\
&\le \frac{1}{16}\|\Lambda^{\alpha+\gamma}\omega\|^2_{L^2}+C\,\|v\|^2_{L^\infty}\,\|(j,H)\|^2_{H^1}+C\,\|v\|^2_{L^\infty}\,\|\theta\|^2_{L^2},
\end{aligned}$$

where we have written $3-2\beta+\gamma=\alpha$. Inserting the bounds for $\widetilde{J}_1$ and $\widetilde{J}_2$ in (12) and invoking the global bound for $\|\omega\|_{L^{\frac{2}{\alpha}}}$, we obtain

$$\|\Lambda^\gamma\omega(t)\|^2_{L^2}+\int_0^t\|\Lambda^{\alpha+\gamma}\omega(\tau)\|^2_{L^2}\,d\tau\le C(t,u_0,v_0,\theta_0)<\infty.$$

Noticing $\alpha+\gamma=1$ finishes the proof of (6) and the proof of Proposition 3. $\square$

2.4. **Global H^s bound.** This subsection establishes the global H^s bound for the solution. More precisely, we prove the following proposition.

Proposition 4. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then (u, v, θ) obeys, for any $t > 0$,*

$$\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^{s+1-\beta}}^2 + \int_0^t \left(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2\right) d\tau$$
$$\leq C(u_0, v_0, \theta_0, t),$$

where $C(u_0, v_0, \theta_0, t)$ depends on t and the initial data.

Proof of Proposition 4. Let $J = (I - \Delta)^{\frac{1}{2}}$ denote the inhomogeneous differentiation operator. Taking the inner product of (1) with $(J^{2s}u, J^{2s}v, J^{2s+2-2\beta}\theta)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^{s+1-\beta}}^2\right) + \|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2 \\
\leq\;& -\int J^s(u\cdot\nabla u)\cdot J^s u\,dx - \int J^s\nabla\cdot(v\otimes v)\cdot J^s u\,dx - \int J^s(v\cdot\nabla u)\cdot J^s v\,dx \\
& - \int J^s(u\cdot\nabla v)\cdot J^s v\,dx - \int J^s\nabla\theta\cdot J^s v\,dx \\
& - \int J^{s+1-\beta}(u\cdot\nabla\theta)\; J^{s+1-\beta}\theta\,dx - \int J^{s+1-\beta}h\, J^{s+1-\beta}\theta\,dx \\
=:\;& J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \qquad (13)
\end{aligned}$$

To bound J_1, we apply $\nabla\cdot u = 0$ to write it as

$$J_1 = -\int \left(J^s(u\cdot\nabla u) - u\cdot\nabla J^s u\right)\cdot J^s u\,dx.$$

By Lemma A.1 and Sobolev's inequality,

$$\begin{aligned}
|J_1| &\leq C\,\|J^s u\|_{L^2}\,\|J^s u\|_{L^{\frac{2}{1-\alpha}}}\,\|\nabla u\|_{L^{\frac{2}{\alpha}}} \leq C\,\|J^s u\|_{L^2}\,\|\Lambda^\alpha u\|_{H^s}\,\|\omega\|_{L^{\frac{2}{\alpha}}} \\
&\leq \frac{1}{64}\|\Lambda^\alpha u\|_{H^s}^2 + C\,\|\omega\|_{L^{\frac{2}{\alpha}}}^2\,\|u\|_{H^s}^2.
\end{aligned}$$

We estimate J_2 and J_3 together. Since $\nabla\cdot(v\otimes v) = v(\nabla\cdot v) + v\cdot\nabla v$,

$$\begin{aligned}
J_2 + J_3 &= -\int J^s(v\cdot\nabla v)\cdot J^s u\,dx - \int J^s(v\cdot\nabla u)\cdot J^s v\,dx \\
&\quad - \int J^s(v\nabla\cdot v)\cdot J^s u\,dx \\
&= -\int \left(J^s(v\cdot\nabla v) - v\cdot\nabla J^s v\right)\cdot J^s u\,dx \\
&\quad - \int \left(J^s(v\cdot\nabla u) - v\cdot\nabla J^s u\right)\cdot J^s v\,dx \\
&\quad - \int (\nabla\cdot v)\,J^s u\cdot J^s v\,dx - \int J^s(v\nabla\cdot v)\cdot J^s u\,dx.
\end{aligned}$$

By Lemma A.1, Sobolev's inequality and $\alpha+\beta=2$,

$$\begin{aligned}
|J_2+J_3| \quad &\le \quad C\,\|J^s u\|_{L^{\frac{2}{1-\alpha}}}\,\|\nabla v\|_{L^{\frac{2}{\alpha}}}\,\|J^s v\|_{L^2}\\
&\quad + C\,\|J^s v\|_{L^2}\,\left(\|J^s v\|_{L^{\frac{2}{1-\alpha}}}\,\|\nabla u\|_{L^{\frac{2}{\alpha}}} + \|\nabla v\|_{L^{\frac{2}{\alpha}}}\,\|J^s u\|_{L^{\frac{2}{1-\alpha}}}\right)\\
&\quad + \|J^s v\|_{L^2}\,\|\nabla\cdot v\|_{L^{\frac{2}{\alpha}}}\,\|J^s u\|_{L^{\frac{2}{1-\alpha}}} + \|v\|_{L^\infty}\,\|J^s u\|_{L^2}\,\|\nabla J^s v\|_{L^2}\\
&\le \quad C\,\left(\|\Lambda^\beta v\|_{L^2} + \|\omega\|_{L^{\frac{2}{\alpha}}}\right)\|\Lambda^\alpha u\|_{H^s}\,\|v\|_{H^s}\\
&\quad + C\,\|v\|_{L^\infty}\,\|u\|_{H^s}\,\|\Lambda^\beta v\|_{H^s}\\
&\le \quad \frac{1}{64}\,\left(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2\right) + C\,\left(\|\Lambda^\beta v\|_{L^2}^2 + \|\omega\|_{L^{\frac{2}{\alpha}}}^2\right)\|v\|_{H^s}^2\\
&\quad + C\,\|v\|_{L^\infty}^2\,\|J^s u\|_{L^2}^2.
\end{aligned}$$

J_4 can be estimated similarly,

$$|J_4| \le \frac{1}{64}\,\left(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2\right) + C\,\left(\|\Lambda^\beta v\|_{L^2}^2 + \|\omega\|_{L^{\frac{2}{\alpha}}}^2\right)\|(u,v)\|_{H^s}^2.$$

By Hölder's inequality,

$$\begin{aligned}
|J_5| &\le \|J^{s+1-\beta}\theta\|_{L^2}\|\Lambda^\beta v\|_{H^s} \le \frac{1}{64}\,\|\Lambda^\beta v\|_{H^s}^2 + C\,\|J^{s+1-\beta}\theta\|_{L^2}^2,\\
|J_7| &\le \|J^{s+1-\beta}h\|_{L^2}\,\|J^{s+1-\beta}\theta\|_{L^2} \le \frac{1}{64}\,\|\Lambda^\beta v\|_{H^s}^2 + C\,\|J^{s+1-\beta}\theta\|_{L^2}^2.
\end{aligned}$$

To estimate J_6, we write, due to $\nabla\cdot u=0$,

$$J_6 = -\int \left(J^{s+1-\beta}(u\cdot\nabla\theta) - u\cdot\nabla J^{s+1-\beta}\theta\right)\; J^{s+1-\beta}\theta\,dx.$$

By Lemma A.1,

$$|J_6| \le \|J^{s+1-\beta}\theta\|_{L^2}\,\left(\|\theta\|_{L^\infty}\,\|J^{s+2-\beta}u\|_{L^2} + \|\nabla u\|_{L^\infty}\,\|J^{s+1-\beta}\theta\|_{L^2}\right).$$

We invoke the logarithmic Sobolev inequality,

$$\begin{aligned}
\|\nabla u\|_{L^\infty} \quad &\le \quad C\,(1+\|u\|_{L^2} + \|\nabla u\|_{\dot H^1}\,\log(1+\|u\|_{H^s}))\\
&\le \quad C\,(1+\|u\|_{L^2} + \|\nabla \omega\|_{L^2}\,\log(1+\|u\|_{H^s})).
\end{aligned}$$

Due to $\alpha+\beta=2$,

$$\begin{aligned}
|J_6| \quad &\le \quad \frac{1}{64}\,\|\Lambda^\beta u\|_{H^s}^2 + C\,\|\theta\|_{L^\infty}^2\|\theta\|_{H^{s+1-\beta}}^2\\
&\quad + C\,(1+\|u\|_{L^2} + \|\nabla\omega\|_{L^2}\,\log(1+\|u\|_{H^s}))\|\theta\|_{H^{s+1-\beta}}^2.
\end{aligned}$$

Inserting the bounds in (13), applying Osgood's inequality and taking into account of the bounds in Propositions 2 and 3, we obtain the desired global bound in Proposition 4. □

3. **Proof of Theorem 1.2.** This section proves Theorem 1.2. Again the proof boils down to establishing the global *a priori* bounds. First we prove the global H^1 bound.

Proposition 5. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.2. Consider (1) with $\alpha = 2$ and $\eta = 0$. Let (u, v, θ) be the corresponding solution. Then, for any $t > 0$,*

$$\|(u,v,\theta)\|_{L^2}^2 + 2\int_0^t \|\Delta u\|_{L^2}^2\, d\tau \le \|(u_0,v_0,\theta_0)\|_{L^2}^2, \tag{1}$$

$$\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2 + \int_0^t \|\nabla\Delta u\|_{L^2}^2\, d\tau \le C(t, \|(u_0,v_0,\theta_0)\|_{H^1}^2), \tag{2}$$

where C depends on t and $\|(u_0,v_0,\theta_0)\|_{H^1}^2$.

Proof. Taking the L^2 inner product of equations (1) with (u, v, θ) and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \|\Delta u\|_{L^2}^2 \\
= \;& \int_{\mathbb{R}^2} (-u\cdot\nabla u)\cdot u\, dx + \int_{\mathbb{R}^2} (-u\cdot\nabla v)\cdot v\, dx + \int_{\mathbb{R}^2} (-u\cdot\nabla\theta)\,\theta\, dx \\
& - \int_{\mathbb{R}^2} [(v\cdot\nabla u)\cdot v\, dx + \nabla\cdot(v\otimes v)\cdot u]\, dx + \int_{\mathbb{R}^2} (\nabla\theta\cdot v + \nabla\cdot v\theta)\, dx \\
= \;& 0.
\end{aligned}$$

Integrating in time yields the assertion (1). To prove (2), we take the inner product of (1) with $(\Delta u, \Delta v, \Delta\theta)$ and integrate by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + \|\nabla\Delta u\|_{L^2}^2 \qquad (3)\\
\le\;& \int_{\mathbb{R}^2} (u\cdot\nabla u)\Delta u dx + \int_{\mathbb{R}^2} \nabla\cdot(v\otimes v)\Delta u dx + \int_{\mathbb{R}^2} (u\cdot\nabla v)\Delta v dx \\
&+ \int_{\mathbb{R}^2} (v\cdot\nabla u)\Delta v dx + \int_{\mathbb{R}^2} (u\cdot\nabla\theta)\;\Delta\theta dx + \int_{\mathbb{R}^2} (\nabla\theta\Delta v + \nabla\cdot v\Delta\theta) dx \\
\le\;& \int_{\mathbb{R}^2} (|\nabla u||\nabla u||\nabla u| + |\nabla u||\nabla v||\nabla v| + |\nabla u||\nabla\theta||\nabla\theta|)\, dx + 2\int_{\mathbb{R}^2} |v||\nabla v||\Delta u| dx \\
\le\;& C\|\nabla u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + C\|v\|_{L^4}\|\nabla v\|_{L^2}\|\Delta u\|_{L^4} \\
\le\;& C\|\nabla u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \\
&+ C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{3}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta u\|_{L^2}^{\frac{1}{2}} \\
\le\;& C\|\nabla u\|_{L^\infty}\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2 + C\|v\|_{L^2}^{\frac{2}{3}}(1+\|\Delta u\|_{L^2}^2)\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|\nabla\Delta u\|_{L^2}^2.
\end{aligned}$$

Invoking the logarithmic Sobolev inequality,

$$\|\nabla u\|_{L^\infty} \le C(1 + \|u\|_{L^2} + \|\Delta u\|_{L^2}\log(e + \|u\|_{H^3})),$$

(3) can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 \\
\le\;& C(1 + \|u\|_{L^2} + \|\Delta u\|_{L^2}\log(e + \|u\|_{H^3}))\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2 \\
&+ C\|v\|_{L^2}^{\frac{2}{3}}(1+\|\Delta u\|_{L^2}^2)\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2.
\end{aligned}$$

Lemma A.2 then implies

$$\|(\nabla u,\nabla v,\nabla\theta)\|_{L^2}^2 + \int_0^t \|\nabla\Delta u\|_{L^2}^2 ds \le C(t, \|(u_0,v_0,\theta_0)\|_{H^1}^2).$$

This completes the proof of Proposition 5. □

Proposition 6. *Assume (u_0, v_0, θ_0) obeys the assumptions stated in Theorem 1.2. Consider (1) with $\alpha = 2$ and $\eta = 0$. Let (u, v, θ) be the corresponding solution. Then, for any $t > 0$,*

$$\|(u,v,\theta)\|_{H^s}^2 + \int_0^t \|\Delta u\|_{H^s}^2 d\tau \le C(\|(u_0,v_0,\theta_0)\|_{H^s}, t) < \infty, \tag{4}$$

Proof. Taking the inner product of (1) with $(J^{2s}u, J^{2s}v, J^{2s}\theta)$ and integrating by parts, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^s}^2\right) + \|\Delta u\|_{H^s}^2 \\
\le\ & -\int_{\mathbb{R}^2} (u\cdot\nabla u) J^{2s} u dx + \int_{\mathbb{R}^2} (-u\cdot\nabla v) J^{2s} v dx + \int_{\mathbb{R}^2} (-u\cdot\nabla\theta)\ J^{2s}\theta dx \\
& -\int_{\mathbb{R}^2} \nabla\cdot(v\otimes v) J^{2s} u dx - \int_{\mathbb{R}^2} (v\cdot\nabla u) J^s v dx \\
=:\ & L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned} \tag{5}$$

As in the proof of Proposition 4, by Lemma A.1, we have

$$\begin{aligned}
L_1 + L_2 + L_3 &= -\int_{\mathbb{R}^2} \left([J^s\nabla, u]uJ^s u + [J^s\nabla, u]vJ^s v + [J^s\nabla, u]\theta J^s\theta\right) dx \\
&\le C\|\nabla u\|_{L^\infty}\|(u,v,\theta)\|_{H^s}^2 + \|J^s\Lambda u\|_{L^4}\|(u,v,\theta)\|_{L^4}\|(u,v,\theta)\|_{H^s} \\
&\le C\|\nabla u\|_{L^\infty}\|(u,v,\theta)\|_{H^s}^2 \\
&\quad + (\|u\|_{L^2} + \|\Delta u\|_{H^s})\|(u,v,\theta)\|_{H^1}\|(u,v,\theta)\|_{H^s} \\
&\le C(1 + \|\nabla\Delta u\|_{L^2})\|(u,v,\theta)\|_{H^s}^2 + \frac{1}{4}\|\Delta u\|_{H^s}^2 + C.
\end{aligned}$$

L_4 and L_5 can be bounded by

$$\begin{aligned}
L_4 &= -\int_{\mathbb{R}^2} J^s\nabla\cdot(v\otimes v) J^s u dx \\
&= \int_{\mathbb{R}^2} J^{s-1}(v\otimes v) J^{s+1}\nabla u dx \\
&\le C\|J^{s-1}v\|_{L^4}\|v\|_{L^4}\|J^{s+1}\nabla v\|_{L^2} \\
&\le C\|v\|_{H^s}^2\|\nabla v\|_{L^2}\|v\|_{L^2} + \frac{1}{8}\|\Delta u\|_{H^s} \le C\|u\|_{H^s}^2 + \frac{1}{8}\|\Delta u\|_{H^s}
\end{aligned}$$

and

$$\begin{aligned}
L_5 &= -\int_{\mathbb{R}^2} J^s(v\cdot\nabla u) J^s v dx \\
&\le C\left(\|J^s v\|_{L^2}\|\nabla u\|_{L^\infty} + \|v\|_{L^4}\|J^{\frac{s}{2}}\nabla u\|_{L^4}\right)\|v\|_{H^s} \\
&\le C\|\nabla u\|_{L^\infty}\|v\|_{H^s}^2 + \|\nabla v\|_{L^2}\|v\|_{L^2}\|v\|_{H^s}^2 + \frac{1}{8}\|\Delta u\|_{H^s} \\
&\le C\left(\|\nabla\Delta u\|_{L^2} + 1\right)\|v\|_{H^s}^2 + \frac{1}{2}\|\Delta u\|_{H^s}^2.
\end{aligned}$$

Inserting the estimates above in the right hand side of (5) and applying Gronwall's inequality, we have

$$\|(u,v,\theta)\|_{H^s}^2 + \int_0^t \|\Delta u\|_{H^s}^2 d\tau \le C(\|(u_0,v_0,\theta_0)\|_{H^s}, t) < \infty,$$

This completes the proof of 6. □

Appendix A. **Inequalities and Besov spaces.** This appendix supplies several inequalities and some facts on the Besov spaces used in the previous sections. First we recall two calculus inequalities involving fractional derivatives. Second we provide an improved Gronwall type inequality. Third we describe the definition of the Littlewood-Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. The material presented in this appendix can be found in several books and many papers (see, e.g., [2, 3, 12, 13]).

Let $J = (I-\Delta)^{\frac{1}{2}}$ denote the inhomogeneous differentiation operator. We recall following calculus inequalities (see, e.g., [7, p.334]).

Lemma A.1. *Let $s > 0$. Let $p, p_1, p_3 \in (1,\infty)$ and $p_2, p_4 \in [1,\infty]$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then, for two constants C_1 and C_2,

$$\|J^s(f\,g)\|_{L^p} \le C_1\left(\|J^s f\|_{L^{p_1}}\,\|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}}\,\|f\|_{L^{p_4}}\right),$$
$$\|J^s(f\,g) - f\,J^s g\|_{L^p} \le C_2\left(\|J^s f\|_{L^{p_1}}\,\|g\|_{L^{p_2}} + \|J^{s-1} g\|_{L^{p_3}}\,\|\nabla f\|_{L^{p_4}}\right).$$

These estimates still hold if we replace J^s by the homogeneous operator Λ^s.

The second lemma is an improved Gronwall type inequality (see, e.g., [9]).

Lemma A.2. *Assume that Y, Z, A and B are non-negative functions satisfying*

$$\frac{d}{dt}Y(t) + Z(t) \le A(t)\,Y(t) + B(t)\,Y(t)\,\ln(1+Z(t)), \tag{1}$$

Let $T > 0$. Assume $A \in L^1(0,T)$ and $B \in L^2(0,T)$. Then, for any $t \in [0,T]$,

$$Y(t) \le (1+Y(0))^{e^{\int_0^t B(\tau)\,d\tau}}\, e^{\int_0^t e^{\int_s^t B(\tau)\,d\tau}(A(s)+B^2(s))\,ds} \tag{2}$$

and

$$\int_0^t Z(\tau)\,d\tau \le Y(t)\int_0^t A(\tau)\,d\tau + Y^2(t)\int_0^t B^2(\tau)\,d\tau < \infty. \tag{3}$$

Proof. Setting

$$Y_1(t) = \ln(1+Y(t)), \qquad Z_1(t) = Z(t)/(1+Y(t)),$$

we have

$$\begin{aligned}\frac{d}{dt}Y_1(t) + Z_1(t) \le& A(t) + B(t)\,\ln(1+Z(t))\\ \le& A(t) + B(t)\,\ln(1+(1+Y(t))\,Z_1(t))\\ \le& A(t) + B(t)\,\ln(1+Y(t))(1+Z_1(t))\\ \le& A(t) + B(t)\,Y_1(t) + B(t)\,\ln(1+Z_1(t)).\end{aligned}$$

Using the simple fact that, for $f \ge 0$,

$$\ln(1+f(t)) \le f^{\frac{1}{2}}(t), \tag{4}$$

we obtain

$$\frac{d}{dt}Y_1(t) + Z_1(t) \le A(t) + B(t)\,Y_1(t) + B^2(t) + \frac{1}{4}Z_1(t).$$

Gronwall's inequality then implies

$$Y_1(t) \le Y_1(0)\, e^{\int_0^t B(\tau)\,d\tau} + \int_0^t e^{\int_s^t B(\tau)\,d\tau}(A(s)+B^2(s))\,ds,$$

which yields (2). In addition, (2) allows us to obtain (3) by using the inequality (4) in (1) and integrating in time. This completes the proof of Lemma A.2. □

We now describe the Littlewood-Paley decomposition and the Besov spaces. We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma\, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^{\perp} = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ denotes the space of multinomials. We also recall the standard Fourier transform and the inverse Fourier transform,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx, \qquad g^{\vee}(x) = \int_{\mathbb{R}^d} g(\xi)\, e^{2\pi i x\cdot\xi} d\xi.$$

To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d :\ 2^{j-1} \le |\xi| < 2^{j+1} \right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & , \quad \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & , \quad \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\dot{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{5}$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. We set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots . \end{cases} \tag{6}$$

For notational convenience, we write Δ_j for $\dot{\Delta}_j$ when there is no confusion. They are different for $j \le -1$. As provided below, the homogeneous Besov spaces are defined in terms of $\dot{\Delta}_j$ while the inhomogeneous Besov spaces are defined in Δ_j. Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (6). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j and

$$S_j f \rightharpoonup f \qquad \text{in } \mathcal{S}'.$$

In addition, for two tempered distributions u and v, we also recall the notion of paraproducts

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \qquad R(u,v) = \sum_{|i-j| \le 2} \Delta_i u \Delta_j v$$

and Bony's decomposition, see e.g. [2],

$$u\, v = T_u v + T_v u + R(u,v).$$

In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful.

Definition A.3. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ consists of $f \in \mathcal{S}'_0$ satisfying

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js} \|\dot{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

An equivalent norm of the the homogeneous Besov space $\dot{B}^s_{p,q}$ with $s \in (0,1)$ is given by

$$\|f\|_{\dot{B}^s_{p,q}} = \left[\int_{\mathbb{R}^d} \frac{\|f(x+\cdot) - f(\cdot)\|^q_{L^p(\mathbb{R}^d)}}{|x|^{d+sq}}\, dx \right]^{\frac{1}{q}}. \tag{7}$$

Definition A.4. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 7. *For any $s \in \mathbb{R}$,*

$$H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Proposition 8. *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

1) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We have also used the following inequality. It is a generalization of the Kato-Ponce inequality, which requires m to be an integer (see, e.g., [8]). This lemma extends it to any real number $m \ge 2$. A proof for this lemma can be found in [4].

Lemma A.5. *Let $0 < s < \sigma < 1$, $2 \le m < \infty$, and $p, q, r \in (1, \infty)^3$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, there exists $C = C(s, \sigma, m, p, q, r)$ such that*

$$\big\||f|^{m-2} f\big\|_{L^p} + \|\Lambda^s(|f|^{m-2} f)\|_{L^p} \le C\|f\|_{B^\sigma_{q,\,p}}\|f\|^{m-2}_{L^{r(m-2)}}. \tag{8}$$

REFERENCES

[1] S. Abe and S. Thurner, Anomalous diffusion in view of Einsteins 1905 theory of Brownian motion, *Physica A*, **356** (2005), 403–407.

[2] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011.

[3] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[4] B. Dong, J. Wu, X. Xu and Z. Ye, Global regularity for the 2D micropolar equations with fractional dissipation, submitted for publication.

[5] D. Frierson, A. Majda and O. Pauluis, Large scale dynamics of precipitation fronts in the tropical atmosphere: a novel relaxation limit, *Comm. Math. Sci.*, **2** (2004), 591–626.

[6] M. Jara, Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps, *Comm. Pure Appl. Math.*, **62** (2009), 198–214.

[7] C.E. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, *J. American Math. Soc.*, **4** (1991), 323–347.

[8] C.E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, *Comm. Pure Appl. Math.*, **46** (1993), 527–620.

[9] J. Li and E.S. Titi, Global well-posedness of the 2D Boussinesq equations with vertical dissipation, *Arch. Ration. Mech. Anal.*, **220** (2016), 983–1001.

[10] J. Li and E.S. Titi, Global well-posedness of strong solutions to a tropical climate model, *Discrete Contin. Dyn. Syst.*, **36** (2016), 4495–4516.

[11] A. Mellet, S. Mischler and C. Mouhot, Fractional diffusion limit for collisional kinetic equations, *Arch. Rat. Mech. Anal.*, **199** (2011), 493–525.

[12] C. Miao, J. Wu and Z. Zhang, *Littlewood-Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics*, Science Press, Beijing, China, 2012 (in Chinese).

[13] T. Runst and W. Sickel, *Sobolev Spaces of fractional order, Nemytskij operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.

[14] Z. Ye, Global regularity for a class of 2D tropical climate model, *J. Math. Anal. Appl.*, **446** (2017), 307–321.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: bqdong@ahu.edu.cn
E-mail address: wangwenjuan@ahu.edu.cn
E-mail address: jiahong.wu@okstate.edu
E-mail address: zhangaqtc@126.com