COMMUN. MATH. SCI. © 2020 International Press
Vol. 18, No. 1, pp. 259–292

2D TROPICAL CLIMATE MODEL WITH FRACTIONAL DISSIPATION AND WITHOUT THERMAL DIFFUSION*

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Abstract. This paper investigates the global existence and regularity problem on a 2D tropical climate model with fractional dissipation. The inviscid version of this model was derived by Frierson, Majda and Pauluis for large-scale dynamics of precipitation fronts in the tropical atmosphere. The fractionally dissipated system studied here is capable of modeling nonlocal and long-range interactions. Mathematically this system involves two parameters controlling the regularization due to the dissipation and our aim is the global regularity for smallest possible parameters. The model considered here has some very special features. This nonlinear system involves interactions between a divergence-free vector field and a non-divergence-free vector field. We introduce an efficient way to control the gradient of the non-divergence-free vector field and make sharp estimates by controlling the regularity of related quantities simultaneously. The global estimates on the Sobolev norms of the solutions are extremely involved and lengthy. We take advantage of some of the most recent developments and tools on the fractional Laplacian operators and introduce some new techniques.

Keywords. Tropical climate model; fractional dissipation; global regularity.

AMS subject classifications. 35D35; 35B65; 76D03.

1. Introduction

This paper studies the global existence and regularity of solutions to the following two dimensional (2D) tropical climate model

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \Lambda^{2\alpha}u + \nabla p + \nabla\cdot(v\otimes v) = 0, \ x\in\mathbb{R}^2, t>0, \\ \partial_t v + (u\cdot\nabla)v + \Lambda^{2\beta}v + \nabla\theta + (v\cdot\nabla)u = 0, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nabla\cdot v = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u=(u_1(x,t),u_2(x,t))$ is the barotropic mode, $v=(v_1(x,t),v_2(x,t))$ is the first baroclinic mode of the vector velocity, $p=p(x,t)$ is the scalar pressure and $\theta=\theta(x,t)$ is the scalar temperature, respectively. Here $v\otimes v$ denotes the tensor product, namely $v\otimes v=(v_iv_j)$, $\alpha>0$ and $\beta>0$ are real parameters and $\Lambda=(-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator. Λ and more general fractional Laplacian operators Λ^κ are defined through the Fourier transform, namely

$$\widehat{\Lambda^\kappa f}(\xi) = |\xi|^\kappa \widehat{f}(\xi).$$

Great attention has recently been paid to the study of the fractional Laplacian problems, not only for pure mathematical generalization, but also for applications in many different fields. In fact, the fractional Laplacian operator is closely related to many real-world phenomena, including models from geophysics [25], and from plasma physics and

*Received: April 07, 2019; Accepted (in revised form): October 03, 2019. Communicated by Alexander Kiselev.

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flame propagation [5], conservation laws [4], probability and finance [12], and anomalous diffusion problems [1, 20, 30]. When $\alpha=\beta=1$, the fractional Laplacian terms $\Lambda^{2\alpha}u$ and $\Lambda^{2\beta}v$ reduce to the standard $-\Delta u$ and $-\Delta v$. Equation (1.1) without dissipation was derived by Frierson, Majda and Pauluis for large-scale dynamics of precipitation fronts in the tropical atmosphere [18]. Its viscous counterpart with the standard Laplacian can be derived by the same argument from the viscous primitive equations (see [27]). More relevant background on the tropical climate model can be found in [19, 28, 29]. The model studied here, namely (1.1), is appended with fractional dissipation terms, which may be relevant in some physical circumstances. One example is in the study of viscous flows in the thinning of atmosphere. Flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian.

Mathematically (1.1) possesses some special features. The first is that (1.1) involves the coupling of a divergence-free vector field u and a non-divergence-free vector field v. This mix poses mathematical challenges. In order to control the gradient of v, we need to bound both the curl of v, $\nabla\times v$ and the divergence of v, $\nabla\cdot v$. This paper demonstrates how to effectively bound ∇v in terms of $\nabla\cdot v$ and $\nabla\times v$. The second feature is that (1.1) allows us to examine two-parameter families of systems simultaneously and to understand how the regularity of the solutions is affected as the sizes of the parameters vary. Our aim here is to establish the global regularity for (1.1) with the smallest amount of dissipation and provide the the sharpest possible global well-posedness results with respect to α and β.

Our main result can be stated as follows.

THEOREM 1.1. *Consider (1.1) with α and β satisfying*

$$0<\alpha<1, \qquad \beta\geq\frac{3-\alpha}{2}. \tag{1.2}$$

Assume the initial data $(u_0,v_0,\theta_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ with $s>2$, and $\nabla\cdot u_0=0$. In addition, we assume $\theta_0\in\dot{H}^{2-2\beta}(\mathbb{R}^2)$. Then (1.1) admits a unique global solution (u,v,θ) such that for any given $T>0$,

$$\begin{aligned}
u&\in C([0,T];H^s(\mathbb{R}^2))\cap L^2(0,T;H^{s+\alpha}(\mathbb{R}^2)),\\
v&\in C([0,T];H^s(\mathbb{R}^2))\cap L^2(0,T;H^{s+\beta}(\mathbb{R}^2)),\\
\theta&\in C([0,T];H^s(\mathbb{R}^2))\cap L^\infty(0,T;\dot{H}^{2-2\beta}(\mathbb{R}^2)).
\end{aligned}$$

There are several important previous global regularity results for different ranges of the parameters. Li and Titi in [26] dealt with the case when $\alpha=1$ and $\beta=1$. By introducing a combined quantity of v and θ, Li and Titi were able to establish the global (in time) H^1 bound. Ye in [35] obtained the global regularity for (1.1) when $\alpha>0$, $\beta=1$ and the equation of θ also contains $\Delta\theta$. Dong, Wang, Wu and Zhang proved the global regularity for the climate model in the case when there is no thermal diffusion, and when $\alpha\leq\frac{1}{2}$ and the total fractional dissipation in the equations of u and v is at the order of two Laplacians [16]. For more results on the global regularity of the 2D tropical climate model, we refer to [15, 17].

We remark that following the same arguments adopted in proving Theorem 1.1, the global regularity regularity result also holds true for the following tropical climate model with moisture (see [27]) for α and β satisfying (1.2)

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+\Lambda^{2\alpha}u+\nabla p+\nabla\cdot(v\otimes v)=0,\ x\in\mathbb{R}^2,\, t>0,\\ \partial_t v+(u\cdot\nabla)v+\Lambda^{2\beta}v+(v\cdot\nabla)u=\dfrac{1}{1+\gamma}\nabla(T_e-q_e),\\ \partial_t T_e+(u\cdot\nabla)T_e-(1-\bar{Q})\nabla\cdot v=0,\\ \partial_t q_e+(u\cdot\nabla)q_e+(\bar{Q}+\gamma)\nabla\cdot v=-\dfrac{1+\gamma}{\varepsilon}q_{e+},\\ \nabla\cdot u=0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad T_e(x,0)=T_{e0}(x),\quad q_e(x,0)=q_{e0}(x),\end{cases}$$

where $f_+:=\max\{f,0\}$ denotes the positive part of f, ε is a convective adjustment timescale parameter, and the constants γ and $\bar{Q}$ are required to satisfy (see [18])

$$0<\bar{Q}<1,\qquad \gamma+\bar{Q}>0.$$

We also remark that the tropical climate model studied here bears some similarities but is different from the magnetohydrodynamic (MHD) equations. The MHD equations are a coupled system of the Navier-Stokes or Euler equation with a Lorentz forcing and the induction equation for the magnetic field, namely

$$\begin{cases}\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla p-\nabla\cdot(b\otimes b)=0,\ x\in\mathbb{R}^2,\, t>0,\\ \partial_t b+(u\cdot\nabla)b-\Delta b-(b\cdot\nabla)u=0,\\ \nabla\cdot u=0,\quad \nabla\cdot b=0.\end{cases}\tag{1.3}$$

Clearly (1.1) resembles (1.3), but there are some key differences. One obvious one is that b can be assumed divergence-free due to the fact that this property is preserved as time evolves, but v in (1.1) is not divergence-free. A more subtle difference is that $b\cdot\nabla u$ in (1.3) has a negative sign while $v\cdot\nabla u$ has a positive sign. This sign makes a big difference in the study of the global existence and regularity problem on (1.3). The well-posedness problem on the MHD system with partial or fractional dissipation has recently attracted extensive interests and there are considerable developments (see, e.g., [7–9, 14, 34, 36]). It is hoped that this study on the tropical climate model will also help advance the course on the investigation of the MHD equations.

The proof of Theorem 1.1 is not trivial and involves the combination of an array of tools and new techniques. The core of the proof is to establish a global *a priori* bound for (u,v,θ) in H^s. This is obtained by consecutively proving more and more regular global bounds. The global L^2 bound for (u,v,θ), along with the time integrability of $\|\Lambda^\alpha u\|_{L^2}^2$, $\|\Lambda^\beta v\|_{L^2}^2$, is immediate due to the special structure of (1.1) and $\nabla\cdot u=0$.

However, the nonlinear coupling in (1.1) makes it very difficult to obtain global bounds on the Sobolev norms of (u,v,θ). Since there is no thermal diffusion in the equation of θ, bounding any Sobolev norm of θ would require the control of $\|\nabla u\|_{L^\infty}$. However, to bound $\|\nabla u\|_{L^\infty}$, one has to first obtain the global bound on the forcing in the equation of u, namely

$$\|\nabla(\nabla\cdot(v\otimes v))\|_{L^\infty}<\infty.$$

We point out that here and in what follows the $(k,j)_{1\le k,j\le 2}$ component of $\nabla(\nabla\cdot(v\otimes v))$ reads $\partial_k\partial_i(v_iv_j)$, where we use the summation convention over repeated indices. Unfortunately the equation of v involves $\nabla\theta$ and one has to know the regularity of θ

first in order to bound v. This tangling makes the estimates of the Sobolev norms of (u,v,θ) very difficult.

Our observation is the special structure in the equation of v, which contains $\nabla\theta$. We can eliminate θ if we take the curl of this equation. However, since v is not necessarily divergence-free, we also need to control the divergence $\nabla\cdot v$ in order to bound ∇v. We are naturally led to consider the system of (ω,j,h), where

$$\omega=\nabla\times u:=\partial_1 u_2-\partial_2 u_1,\qquad j=\nabla\times v,\qquad h=\nabla\cdot v:=\partial_1 v_1+\partial_2 v_2.$$

It follows from (1.1) that (ω,j,h) satisfies

$$\begin{cases}\partial_t\omega+(u\cdot\nabla)\omega+\Lambda^{2\alpha}\omega+(v\cdot\nabla)j+2hj-(v_1\partial_2 h-v_2\partial_1 h)=0,\\ \partial_t j+(u\cdot\nabla)j+\Lambda^{2\beta}j+(v\cdot\nabla)\omega+h\omega=0,\\ \partial_t h+(u\cdot\nabla)h+\Lambda^{2\beta}h+\Delta\theta+Q(\nabla u,\nabla v)=0,\\ \nabla\cdot u=0,\end{cases}$$

where $Q(\nabla u,\nabla v)$ is given by

$$Q(\nabla u,\nabla v)=2\partial_1 u_1(\partial_1 v_1-\partial_2 v_2)+2\partial_1 u_2\partial_2 v_1+2\partial_2 u_1\partial_1 v_2.$$

The equation of h involves $\Delta\theta$, which would make it impossible to obtain a global bound for h. Our idea is to hide $\Delta\theta$ by introducing a combined quantity. We define H such that

$$\Lambda^{2\beta}H=\Lambda^{2\beta}h+\Delta\theta$$

or

$$H=h-\Lambda^{2-2\beta}\theta.$$

By combining the equations of h and θ, we find that H satisfies

$$\partial_t H+(u\cdot\nabla)H+\Lambda^{2\beta}H=-Q(\nabla u,\nabla v)+\Lambda^{2-2\beta}h+[\Lambda^{2-2\beta},u\cdot\nabla]\theta.$$

Owing to the vector identity

$$\Delta v=\nabla(\nabla\cdot v)+\nabla^\perp(\nabla\times v),\qquad \nabla^\perp=(\partial_2,-\partial_1),$$

we can write

$$\Delta v=\nabla h+\nabla^\perp j.$$

Alternatively,

$$\nabla v=\mathcal{R}_1 h+\mathcal{R}_2 j=\mathcal{R}_1 H+\mathcal{R}_1\Lambda^{2-2\beta}\theta+\mathcal{R}_2 j,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are Riesz transforms given by (see Chapter 3 of [10])

$$\mathcal{R}_1=\nabla\nabla(\Delta)^{-1},\qquad \mathcal{R}_2=\nabla\nabla^\perp(\Delta)^{-1}.$$

By making use of the system of (ω,j,H), we are able to obtain a global-in-time bound for $\|(\omega,j,H)\|_{L^2}$. In this process, we need to bound quite a few triple product terms and various tools including the Littlewood-Paley decomposition are invoked. The worst term (the most regularity demanding term) comes from $v_1\partial_2 h-v_2\partial_1 h$ in the equation of ω. In fact, it is this term that requires (1.2).

The global L^2 bound of (ω, j, H) is the foundation for higher regularity. Making use of the maximal regularity estimates for the fractional Laplacian operators, we can further show that, for any $p \in [2,\infty)$,

$$\|\theta(t)\|_{L^p} \le C(t), \quad \|\nabla v(t)\|_{L^p} \le C(t), \quad \|\nabla H(t)\|_{L^p} \le C(t),$$

where $C(t)$ is an upper bound depending on t and the initial data. Since the equation of θ has no thermal diffusion, the global H^s-bound of (u,v,θ) relies on the Lipschitz property of u, namely $\|\nabla u(t)\|_{L^\infty}$ or $\|\omega(t)\|_{L^\infty}$. We find that the control of these quantities depends crucially on the size of α. We consider two cases: $\alpha > \frac{1}{2}$ and $\alpha \le \frac{1}{2}$. The first case is resolved using the maximal regularity estimate while the second case is much more involved. The difficulty for the case when $\alpha \le \frac{1}{2}$ is that the vorticity equation involves the term

$$\nabla \times (\nabla \cdot (v \otimes v)),$$

but the global bound on v is for $\|\nabla v(t)\|_{L^p}$. Therefore we have to shift one derivative from this term when we estimate $\|\omega\|_{L^\infty}$. When $\alpha \le \frac{1}{2}$, there is not enough dissipation to absorb this derivative. The way we handle this difficulty is to write the second-order derivative $\nabla^2 v$ in terms of j, H and $\Lambda^{2-2\beta}\theta$ and make use of the bounds on j and H. In addition, we prove and use a De Giorgi-Nash regularization estimate involving the fractional Laplacian operators. Once the crucial bound $\|\omega\|_{L^\infty}$ is established, the global H^s-bound for (u,v,θ) is then within reach.

The rest of this paper is devoted to the proof of Theorem 1.1. The proof is extremely technical and long. For the sake of clarity, we divide it into two sections together with an appendix. The first section contains the proof of the global bounds on (ω, j, H) in L^2 and θ, ∇v and ∇H in L^p with $p \ge 2$. The second section proves a global bound for the H^s-norm of (u,v,θ). Each section is further divided into several subsections to make our presentation easily accessible. The appendix presents the Littlewood-Paley decomposition, Besov spaces and related tools used in the proof of Theorem 1.1.

2. The proof of Theorem 1.1, Part I

The proof of Theorem 1.1 is lengthy and we split it into two main parts. The first part contains the proofs of the global bounds on (ω, j, H) in L^2 and θ, ∇v and ∇H in L^p with $p \ge 2$, which is presented in this section. The second part obtains a global bound on $\|\nabla u(t)\|_{L^\infty}$ and then a global bound for the H^s-norm of (u,v,θ), which is given in Section 3. More precisely, this section proves the two propositions stated below.

The first proposition provides the global bound on $\|(\omega, j, H)\|_{L^2}$, where ω, j and H are as defined in the introduction. As explained in the introduction, these quantities are employed to untangle the coupling in (1.1).

PROPOSITION 2.1. *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. If $\beta \ge \frac{3-\alpha}{2}$ for $0 < \alpha < 1$, then for any corresponding smooth solution (u,v,θ) of (1.1), we have, for any given $T > 0$ and any $t \in [0,T]$,*

$$\begin{aligned} &\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \\ &\quad + \int_0^t (\|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 + \|\Lambda^\beta j(\tau)\|_{L^2}^2 + \|\Lambda^\beta H(\tau)\|_{L^2}^2)\, d\tau \le C_0(t), \end{aligned} \tag{2.1}$$

where $C_0(t)$ is a finite quantity depending only on t and initial data.

The second proposition establishes global bounds on $\|\theta(t)\|_{L^p}$, $\|\nabla v(t)\|_{L^p}$ and $\|\nabla H(t)\|_{L^p}$ for any $p \ge 2$.

Proposition 2.2. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. If $\beta\geq\frac{3-\alpha}{2}$ for $0<\alpha<1$, then for any corresponding smooth solution (u,v,θ) of (1.1), we have, for $p\in[2,\infty]$, any given $T>0$ and any $t\in[0,T]$,*

$$\|\theta(t)\|_{L^p}\leq C_0(t), \tag{2.2}$$

$$\|\nabla v(t)\|_{L^p}\leq C_0(t), \tag{2.3}$$

$$\|\nabla H(t)\|_{L^p}\leq C_0(t), \tag{2.4}$$

where $C_0(t)$ depends only on t and the initial data. Especially,

$$\|v(t)\|_{L^\infty}\leq C_0(t). \tag{2.5}$$

For the sake of clarity, we divide the rest of this section into three subsections. The first subsection provides a useful lemma and two global upper bounds. The second subsection proves Proposition 2.1 while the third subsection proves Proposition 2.2.

2.1. A tool and two upper bounds. The useful lemma relates the regularity (in the space-time norm) of the solution of a generalized heat equation to that of the right-hand side.

Lemma 2.1. *Consider the following linear equation with $\sigma>0$,*

$$\partial_t f+\Lambda^{2\sigma}f=g, \qquad f(x,0)=f_0(x), \quad x\in\mathbb{R}^d, \tag{2.6}$$

then for any $0<\widetilde{\varepsilon}\leq 2\sigma$ and for any $1\leq p,q\leq\infty$, we have

$$\|\Lambda^{2\sigma-\widetilde{\varepsilon}}f\|_{L^q_tL^p_x}\leq C(t,f_0)+C(t)\,\|g\|_{L^q_tL^p_x},$$

where $C(t,f_0)=\|e^{-\Lambda^{2\sigma}t}\Lambda^{2\sigma-\widetilde{\varepsilon}}f_0\|_{L^q_tL^p_x}$ and $C(t)$ depends on t only.

We remark that the classical maximal regularity estimate for generalized heat operators allows $\widetilde{\varepsilon}=0$, but requires that $1<p,q<\infty$. This tool lemma allows $1\leq p,q\leq\infty$ but asks that $\widetilde{\varepsilon}>0$ (a loss of a little spatial regularity). This lemma serves our purpose very well.

Proof. **(Proof of Lemma 2.1.)** We first remark that this lemma is used in the *a priori* estimates with the assumption that the solution is smooth. Of course, we can also make sense of this lemma in the given functional spaces here. Then, the equation in (2.6) of Lemma 2.1 is really understood in the integral form

$$f=e^{-t\Lambda^{2\alpha}}f_0+\int_0^t e^{-(t-\tau)\Lambda^{2\sigma}}g\,d\tau, \tag{2.7}$$

which follows from (2.6) via the Fourier transform. We apply $\Lambda^{2\sigma-\widetilde{\varepsilon}}$ to (2.7) to get

$$\partial_t\Lambda^{2\sigma-\widetilde{\varepsilon}}f+\Lambda^{2\sigma}\Lambda^{2\sigma-\widetilde{\varepsilon}}f=\Lambda^{2\sigma-\widetilde{\varepsilon}}g. \tag{2.8}$$

Using the standard Duhamel formula, we write (2.8) as

$$\Lambda^{2\sigma-\widetilde{\varepsilon}}f=e^{-t\Lambda^{2\alpha}}\Lambda^{2\sigma-\widetilde{\varepsilon}}f_0+\int_0^t e^{-(t-\tau)\Lambda^{2\sigma}}\Lambda^{2\sigma-\widetilde{\varepsilon}}g\,d\tau.$$

Invoking the estimate (see [32, Lemma 3.1] for example)

$$\|\Lambda^\rho e^{-\Lambda^{2\sigma}t}f\|_{L^k(\mathbb{R}^d)}\leq Ct^{-\frac{\rho}{2\sigma}-\frac{d}{2\sigma}(\frac{1}{m}-\frac{1}{k})}\|f\|_{L^m(\mathbb{R}^d)}$$

with $\rho\geq 0$ and $k\geq m$, we obtain

$$\begin{aligned}\|\Lambda^{2\sigma-\widetilde{\varepsilon}}f\|_{L^p(\mathbb{R}^d)}&\leq\|e^{-t\Lambda^{2\sigma}}\Lambda^{2\sigma-\widetilde{\varepsilon}}f_0\|_{L^p(\mathbb{R}^d)}+\int_0^t\|e^{-(t-\tau)\Lambda^{2\sigma}}\Lambda^{2\sigma-\widetilde{\varepsilon}}g\|_{L^p(\mathbb{R}^d)}\,d\tau\\&\leq\|e^{-t\Lambda^{2\sigma}}\Lambda^{2\sigma-\widetilde{\varepsilon}}f_0\|_{L^p(\mathbb{R}^d)}+C\int_0^t(t-\tau)^{\frac{2\sigma-\widetilde{\varepsilon}}{2\sigma}}\|g(\tau)\|_{L^p(\mathbb{R}^d)}\,d\tau.\end{aligned}$$

By taking L_t^q and applying the Young convolution inequality, we have

$$\|\Lambda^{2\sigma-\widetilde{\varepsilon}}f\|_{L_t^qL_x^p}\leq C(t,f_0)+C(t)\,\|g\|_{L_t^qL_x^p}.$$

This conclude the proof of Lemma 2.1. ◻

Now we present two upper bounds to be used in the proofs of Proposition 2.1 and Proposition 2.2. We start with the global L^2-bound.

LEMMA 2.2. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. Then for any corresponding smooth solution (u,v,θ) of (1.1), we have, for any $t>0$,*

$$\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2+\int_0^t(\|\Lambda^\alpha u(\tau)\|_{L^2}^2+\|\Lambda^\beta v(\tau)\|_{L^2}^2)\,d\tau\leq C(\|(u_0,v_0,\theta_0)\|_{L^2}). \tag{2.9}$$

As a special consequence,

$$\|\theta(t)\|_{L^{\frac{2}{2-\beta}}}\leq C(t,\|(u_0,v_0,\theta_0)\|_{L^2}). \tag{2.10}$$

Proof. Taking the inner product of (1.1) with (u,v,θ), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2)+\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2\\&=-\int_{\mathbb{R}^2}\nabla\cdot(v\otimes v)\cdot u\,dx-\int_{\mathbb{R}^2}(v\cdot\nabla)u\cdot v\,dx-\int_{\mathbb{R}^2}\nabla\theta\cdot v\,dx-\int_{\mathbb{R}^2}(\nabla\cdot v)\theta\,dx\\&\quad-\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot u\,dx-\int_{\mathbb{R}^2}(u\cdot\nabla)v\cdot v\,dx-\int_{\mathbb{R}^2}(u\cdot\nabla)\theta\theta\,dx-\int_{\mathbb{R}^2}\nabla p\cdot u\,dx.\end{aligned}$$

Due to the fact that (u,v,θ) are smooth, we can get by integrating by parts that

$$\begin{aligned}-\int_{\mathbb{R}^2}\nabla\cdot(v\otimes v)\cdot u\,dx-\int_{\mathbb{R}^2}(v\cdot\nabla)u\cdot v\,dx&=-\int_{\mathbb{R}^2}\partial_i(v_iv_j)u_j\,dx-\int_{\mathbb{R}^2}v_i\partial_iu_jv_j\,dx\\&=\int_{\mathbb{R}^2}(v_iv_j)\partial_iu_j\,dx-\int_{\mathbb{R}^2}v_i\partial_iu_jv_j\,dx\\&=0,\end{aligned}$$

$$\begin{aligned}-\int_{\mathbb{R}^2}\nabla\theta\cdot v\,dx-\int_{\mathbb{R}^2}(\nabla\cdot v)\theta\,dx&=-\int_{\mathbb{R}^2}\partial_i\theta v_i\,dx-\int_{\mathbb{R}^2}\partial_iv_i\theta\,dx\\&=\int_{\mathbb{R}^2}\theta\partial_iv_i\,dx-\int_{\mathbb{R}^2}\partial_iv_i\theta\,dx\\&=0.\end{aligned}$$

As (u,v,θ) are smooth, one derives by integrating by parts and using the incompressibility condition $\nabla\cdot u=0$ that

$$-\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot u\,dx=-\int_{\mathbb{R}^2}u_i\partial_iu_ju_j\,dx$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{\mathbb{R}^2} u_i \partial_i u_j^2\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \partial_i u_i u_j^2\,dx \\
&= 0.
\end{aligned}
$$

Similarly, we obtain

$$
\int_{\mathbb{R}^2} (u\cdot\nabla)v\cdot v\,dx = \int_{\mathbb{R}^2} (u\cdot\nabla)\theta\theta\,dx = \int_{\mathbb{R}^2} \nabla p\cdot u\,dx = 0.
$$

Combining all the above estimates yields

$$
\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2 = 0. \tag{2.11}
$$

Integrating (2.11) in the time variable over $(0,t)$ implies (2.9). Multiplying the equation of θ in (1.1) by $|\theta|^{\frac{2}{2-\beta}-2}\theta$ and integrating with respect to x, we have

$$
\frac{d}{dt}\|\theta(t)\|_{L^{\frac{2}{2-\beta}}} \le C\|\nabla v\|_{L^{\frac{2}{2-\beta}}} \le C\|\Lambda^\beta v\|_{L^2}.
$$

Integrating in time over $(0,t)$ and using (2.9), we obtain (2.10). This completes the proof of (2.10). ∎

The second upper bound controls the lower mode of $\Lambda^{2-2\beta}\theta$, namely $\Delta_{-1}\Lambda^{2-2\beta}\theta$, where Δ_{-1} is the Fourier restriction operator as defined in the appendix. This upper bound will be used in the proofs of Proposition 2.1 and Proposition 2.2.

LEMMA 2.3. *Assume that (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. Then for any corresponding smooth solution (u,v,θ) of (1.1), we have, for any $2\le q<\frac{2}{3-2\beta}$ and for any $t>0$,*

$$
\|\Delta_{-1}\Lambda^{2-2\beta}\theta(t)\|_{L^2} \le C_0(t), \qquad \|\Lambda^{2-2\beta}\theta(t)\|_{L^q} \le C_0(t), \tag{2.12}
$$

where $C_0(t)$ depends only on t and the initial data.

Proof. Applying $\Delta_{-1}\Lambda^{2-2\beta}$ to the third equation of (1.1), dotting with $\Delta_{-1}\Lambda^{2-2\beta}\theta$ and using the Bernstein inequality (see the appendix), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}^2 &\le \|\Delta_{-1}\Lambda^{2-2\beta}(u\cdot\nabla\theta)\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\quad + \|\Delta_{-1}\Lambda^{2-2\beta}\nabla\cdot v\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\le C\|\Delta_{-1}\Lambda^{2-2\beta}\nabla\cdot(u\theta)\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\quad + \|\Delta_{-1}\Lambda^{2-2\beta}\nabla\cdot v\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\le C\|\Delta_{-1}\Lambda^{3-2\beta}(u\theta)\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\quad + C\|\Delta_{-1}\Lambda^{3-2\beta}v\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\le C\|\Delta_{-1}(u\theta)\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\quad + C\|\Delta_{-1}v\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2} \\
&\le C\|\Delta_{-1}(u\theta)\|_{L^1}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|v\|_{L^2}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}\\
&\le C(\|u\|_{L^2}\|\theta\|_{L^2}+\|v\|_{L^2})\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}.
\end{aligned}
$$

This leads to

$$\frac{d}{dt}\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}\le C(\|u\|_{L^2}\|\theta\|_{L^2}+\|v\|_{L^2}).$$

By (2.9),

$$\|\Delta_{-1}\Lambda^{2-2\beta}\theta(t)\|_{L^2}\le C_0(t).$$

For any $2\le q<\frac{2}{3-2\beta}$, by Bernstein's inequality,

$$
\begin{aligned}
\|\Lambda^{2-2\beta}\theta\|_{L^q}&\le\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^q}+\sum_{j\ge0}\|\Delta_j\Lambda^{2-2\beta}\theta\|_{L^q}\\
&\le C\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}+C\sum_{j\ge0}2^{2(\frac12-\frac1q)j+2(1-\beta)j}\|\theta\|_{L^2}\\
&\le C+C\|\theta\|_{L^2}
\end{aligned}
$$

due to $(\frac12-\frac1q)+1-\beta<0$ and (2.9). This proves (2.12). The lemma is proven. ∎

2.2. Proof of Proposition 2.1. This subsection is devoted to the proof of Proposition 2.1. As described in the introduction, we make use of the equations of (ω,j,H) with

$$\omega=\nabla\times u,\qquad j=\nabla\times v,\qquad h=\nabla\cdot v,\qquad H=h-\Lambda^{2-2\beta}\theta.$$

(ω,j,h) satisfies

$$
\begin{cases}
\partial_t\omega+(u\cdot\nabla)\omega+\Lambda^{2\alpha}\omega+(v\cdot\nabla)j+2hj-(v_1\partial_2h-v_2\partial_1h)=0,\\
\partial_tj+(u\cdot\nabla)j+\Lambda^{2\beta}j+(v\cdot\nabla)\omega+h\omega=0,\\
\partial_th+(u\cdot\nabla)h+\Lambda^{2\beta}h+\Delta\theta+Q(\nabla u,\nabla v)=0,\\
\nabla\cdot u=0,
\end{cases}
\tag{2.13}
$$

where $Q(\nabla u,\nabla v)$ is given by

$$Q(\nabla u,\nabla v)=2\partial_1u_1(\partial_1v_1-\partial_2v_2)+2\partial_1u_2\partial_2v_1+2\partial_2u_1\partial_1v_2.$$

By combining the equations of h and θ, we find that H satisfies

$$\partial_tH+(u\cdot\nabla)H+\Lambda^{2\beta}H=-Q(\nabla u,\nabla v)+\Lambda^{2-2\beta}h+[\Lambda^{2-2\beta},u\cdot\nabla]\theta.\tag{2.14}$$

The main reason for introducing H is to hide $\Delta\theta$ in the equation of h. Owing to the following identity

$$\Delta v=\nabla(\nabla\cdot v)+\nabla^\perp(\nabla\times v),\qquad \nabla^\perp=(\partial_2,-\partial_1),$$

we can recover v from h and j via

$$\Delta v=\nabla h+\nabla^\perp j.$$

Alternatively, we can compute ∇v via H, θ and j,

$$\nabla v = \mathcal{R}_1 h + \mathcal{R}_2 j = \mathcal{R}_1 H + \mathcal{R}_1 \Lambda^{2-2\beta}\theta + \mathcal{R}_2 j, \tag{2.15}$$

where

$$\mathcal{R}_1 = \nabla\nabla(\Delta)^{-1}, \quad \mathcal{R}_2 = \nabla\nabla^{\perp}(\Delta)^{-1}.$$

We are now ready to prove Proposition 2.1.

Proof. **(Proof of Proposition 2.1.)** Multiplying the Equations $(2.13)_1$, $(2.13)_2$ and (2.14) by ω, j and H, respectively, integrating over $\mathbb{R}^2$ and summing them up lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\Lambda^{\alpha}\omega\|_{L^2}^2 + \|\Lambda^{\beta} j\|_{L^2}^2 + \|\Lambda^{\beta} H\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^2} \Big((v\cdot\nabla)j\omega + 2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,dx - \int_{\mathbb{R}^2} Q(\nabla u, \nabla v) H\,dx \\
&\quad + \int_{\mathbb{R}^2} \Lambda^{2-2\beta} h H\,dx + \int_{\mathbb{R}^2} (v_1\partial_2 h - v_2\partial_1 h)\omega\,dx + \int_{\mathbb{R}^2} [\Lambda^{2-2\beta}, u\cdot\nabla]\theta H\,dx \\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{2.16}$$

Integrating by parts yields that

$$\begin{aligned}
J_1 &= -2\int_{\mathbb{R}^2} hj\omega\,dx - \int_{\mathbb{R}^2} \Big((v\cdot\nabla)j\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,dx \\
&= -2\int_{\mathbb{R}^2} hj\omega\,dx - \int_{\mathbb{R}^2} \nabla\cdot(v\omega j)\,dx \\
&= -2\int_{\mathbb{R}^2} hj\omega\,dx \\
&= -2\int_{\mathbb{R}^2} Hj\omega\,dx - 2\int_{\mathbb{R}^2} \Lambda^{2-2\beta}\theta j\omega\,dx.
\end{aligned}$$

By the Gagliardo-Nirenberg inequality, we have, for $1 < \beta \le \frac{3}{2}$

$$\begin{aligned}
-2\int_{\mathbb{R}^2} \Lambda^{2-2\beta}\theta j\omega\,dx &\le C\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{3-2\beta}}} \|j\|_{L^{\frac{1}{\beta-1}}} \|\omega\|_{L^2} \\
&\le C\|\theta\|_{L^2}\|j\|_{L^2}^{1-\frac{3-2\beta}{\beta}} \|\Lambda^{\beta} j\|_{L^2}^{\frac{3-2\beta}{\beta}} \|\omega\|_{L^2} \\
&\le \epsilon\|\Lambda^{\beta} j\|_{L^2}^2 + C_{\epsilon}\|\theta_0\|_{L^2}^{\frac{2\beta}{4\beta-3}} (\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2),
\end{aligned}$$

where we have used the Hardy-Littlewood-Sobolev inequality,

$$\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{3-2\beta}}} \le C\|\theta\|_{L^2}.$$

Applying the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4} \le C\|f\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1} \|\Lambda^{\beta} f\|_{L^2}^{\lambda_1}, \quad \lambda_1 = \frac{3-2\beta}{2},$$

we have

$$-2\int_{\mathbb{R}^2} Hj\omega\,dx \le C\|\omega\|_{L^2}\|H\|_{L^4}\|j\|_{L^4}$$

$$
\begin{aligned}
&\le C\|\omega\|_{L^2}\|H\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1}\|\Lambda^\beta H\|_{L^2}^{\lambda_1}\|j\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1}\|\Lambda^\beta j\|_{L^2}^{\lambda_1}\\
&\le C\|\omega\|_{L^2}(\|h\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1}+\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1})\|\Lambda^\beta H\|_{L^2}^{\lambda_1}\|j\|_{L^{\frac{2}{2-\beta}}}^{1-\lambda_1}\|\Lambda^\beta j\|_{L^2}^{\lambda_1}\\
&\le C\|\omega\|_{L^2}(1+\|\Lambda^\beta v\|_{L^2}^{1-\lambda_1}+\|\theta\|_{L^2}^{1-\lambda_1})\|\Lambda^\beta H\|_{L^2}^{\lambda_1}\|\Lambda^\beta v\|_{L^2}^{1-\lambda_1}\|\Lambda^\beta j\|_{L^2}^{\lambda_1}\\
&\le \epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\beta j\|_{L^2}^2+C_\epsilon(1+\|\theta\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2)\|\omega\|_{L^2}^{\frac{1}{1-\lambda_1}}\\
&\le \epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\beta j\|_{L^2}^2+C_\epsilon(1+\|\theta_0\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2)(1+\|\omega\|_{L^2}^2),
\end{aligned}
$$

where we have used Lemma 2.3 to bound $\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{2-\beta}}}$. Therefore,

$$
J_1\le \epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\beta j\|_{L^2}^2+C_\epsilon(1+\|\theta\|_{L^2}^{\frac{2\beta}{4\beta-3}}+\|\theta\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2)(1+\|\omega\|_{L^2}^2+\|j\|_{L^2}^2).
$$

Next J_2 can be bounded as follows,

$$
\begin{aligned}
J_2\le& C\int_{\mathbb{R}^2}|\nabla u|\;|\nabla v|\;|H|\,dx\\
\le& C\int_{\mathbb{R}^2}|\nabla u|\;(|\mathcal{R}_1 H|+|\mathcal{R}_1\Lambda^{2-2\beta}\theta|+|\mathcal{R}_2 j|)\;|H|\,dx\\
\le& C\|\nabla u\|_{L^2}\|H\|_{L^4}^2+C\|\Lambda^{2-2\beta}\theta\|_{L^{\frac{2}{3-2\beta}}}\|H\|_{L^{\frac{1}{\beta-1}}}\|\nabla u\|_{L^2}\\
&+C\|\nabla u\|_{L^2}\|H\|_{L^4}\|j\|_{L^4}\\
\le& \epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\beta j\|_{L^2}^2+C_\epsilon(1+\|\theta_0\|_{L^2}^{\frac{2\beta}{4\beta-3}}+\|\theta_0\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2)\\
&\times(1+\|\omega\|_{L^2}^2+\|j\|_{L^2}^2+\|H\|_{L^2}^2).
\end{aligned}
$$

J_3 can be easily bounded as

$$
\begin{aligned}
J_3\le& C\|\Lambda^{3-2\beta}v\|_{L^2}\|H\|_{L^2}\\
\le& C\|v_0\|_{L^2}^{\frac{3(\beta-1)}{\beta}}\|\Lambda^\beta v\|_{L^2}^{\frac{3-2\beta}{\beta}}(1+\|H\|_{L^2}^2).
\end{aligned}
$$

The estimate of J_4 is much more involved.

$$
J_4\le C\Big|\int_{\mathbb{R}^2}v\nabla H\omega\,dx\Big|+C\Big|\int_{\mathbb{R}^2}v\nabla\Lambda^{2-2\beta}\theta\omega\,dx\Big|.
$$

Thanks to $\beta>1$ and by Sobolev's inequality,

$$
\begin{aligned}
C\Big|\int_{\mathbb{R}^2}v\nabla H\omega\,dx\Big|\le& C\|v\|_{L^{\frac{2}{\beta-1}}}\|\nabla H\|_{L^{\frac{2}{2-\beta}}}\|\omega\|_{L^2}\\
\le& C\|v\|_{H^\beta}\|\Lambda^\beta H\|_{L^2}\|\omega\|_{L^2}\\
\le& \epsilon\|\Lambda^\beta H\|_{L^2}^2+C_\epsilon\|v\|_{H^\beta}^2\|\omega\|_{L^2}^2.
\end{aligned}
$$

By the Kato-Ponce inequality, we have, for $\beta\ge\frac{3-\alpha}{2}$ and $\alpha<1$,

$$
\begin{aligned}
C\Big|\int_{\mathbb{R}^2}v\nabla\Lambda^{2-2\beta}\theta\omega\,dx\Big|\le& C\|\theta\|_{L^{\frac{2}{2-\beta}}}\|\Lambda^{3-2\beta}(v\omega)\|_{L^{\frac{2}{\beta}}}\\
\le& C\|\theta\|_{L^{\frac{2}{2-\beta}}}(\|\Lambda^{3-2\beta}v\|_{L^{\frac{2}{\alpha+\beta-1}}}\|\omega\|_{L^{\frac{2}{1-\alpha}}}\\
&+\|\Lambda^{3-2\beta}\omega\|_{L^{\frac{2}{4-\alpha-2\beta}}}\|v\|_{L^{\frac{2}{\alpha+3\beta-4}}})
\end{aligned}
$$

$$\begin{aligned}&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}}(\|v\|_{H^\beta}\|\Lambda^\alpha\omega\|_{L^2}+\|\Lambda^\alpha\omega\|_{L^2}\|v\|_{H^\beta})\\&\le\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2+C_\epsilon\|v\|_{H^\beta}^2\|\theta\|_{L^{\frac{2}{2-\beta}}}^2.\end{aligned}$$

where we need the condition $\beta\ge\frac{3-\alpha}{2}$. Therefore,

$$J_4\le\epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2+C_\epsilon\|v\|_{H^\beta}^2\|\omega\|_{L^2}^2+C_\epsilon\|v\|_{H^\beta}^2\|\theta\|_{L^{\frac{2}{2-\beta}}}^2.$$

We rewrite J_5 as

$$\begin{aligned}J_5&=\int_{\mathbb{R}^2}\Lambda^{2-2\beta}(u\cdot\nabla\theta)H\,dx+\int_{\mathbb{R}^2}u\cdot\Lambda^{2-2\beta}\nabla\theta H\,dx\\&=\int_{\mathbb{R}^2}\Lambda^{2-2\beta}\nabla\cdot(u\theta)H\,dx+\int_{\mathbb{R}^2}u\cdot\Lambda^{2-2\beta}\nabla\theta H\,dx.\end{aligned}$$

The first term admits the bound

$$\begin{aligned}\int_{\mathbb{R}^2}\Lambda^{2-2\beta}\nabla\cdot(u\theta)H\,dx&\le C\|u\theta\|_{L^2}\|\Lambda^{3-2\beta}H\|_{L^2}\\&\le C\|u\|_{L^\infty}\|\theta\|_{L^2}\|H\|_{L^2}^{1-\frac{3-2\beta}{\beta}}\|\Lambda^\beta H\|_{L^2}^{\frac{3-2\beta}{\beta}}\\&\le C\|u\|_{L^2}^{1-\frac{1}{\alpha+1}}\|\Lambda^\alpha\omega\|_{L^2}^{\frac{1}{\alpha+1}}\|\theta\|_{L^2}\|H\|_{L^2}^{1-\frac{3-2\beta}{\beta}}\|\Lambda^\beta H\|_{L^2}^{\frac{3-2\beta}{\beta}}\\&\le\epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2+C_\epsilon(\|u_0\|_{L^2},\|\theta_0\|_{L^2})(1+\|H\|_{L^2}^2).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}^2}u\cdot\Lambda^{2-2\beta}\nabla\theta H\,dx&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}}\|\Lambda^{3-2\beta}(uH)\|_{L^{\frac{2}{\beta}}}\\&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}}(\|u\|_{L^{\frac{2}{\beta-1}}}\|\Lambda^{3-2\beta}H\|_{L^2}+\|\Lambda^{3-2\beta}u\|_{L^2}\|H\|_{L^{\frac{2}{\beta-1}}})\\&\le C\|\theta\|_{L^{\frac{2}{2-\beta}}}(\|u\|_{L^2}^{1-\frac{2-\beta}{1+\alpha}}\|\Lambda^\alpha\omega\|_{L^2}^{\frac{2-\beta}{1+\alpha}}\|H\|_{L^2}^{1-\frac{3-2\beta}{\beta}}\|\Lambda^\beta H\|_{L^2}^{\frac{3-2\beta}{\beta}}\\&\quad+\|u\|_{L^2}^{1-\frac{3-2\beta}{1+\alpha}}\|\Lambda^\alpha\omega\|_{L^2}^{\frac{3-2\beta}{1+\alpha}}\|H\|_{L^2}^{1-\frac{2-\beta}{\beta}}\|\Lambda^\beta H\|_{L^2}^{\frac{2-\beta}{\beta}})\\&\le\epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2+C_\epsilon(t,\|u_0\|_{L^2},\|\theta_0\|_{L^2})(1+\|H\|_{L^2}^2).\end{aligned}$$

Combining the above estimates yields

$$J_5\le\epsilon\|\Lambda^\beta H\|_{L^2}^2+\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2+C_\epsilon(t,\|u_0\|_{L^2},\|\theta_0\|_{L^2})(1+\|H\|_{L^2}^2).$$

Inserting the estimates of J_1 through J_5 in (2.16) and taking ϵ small enough, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\|H(t)\|_{L^2}^2)+\|\Lambda^\alpha\omega\|_{L^2}^2+\|\Lambda^\beta j\|_{L^2}^2+\|\Lambda^\beta H\|_{L^2}^2\\&\le C(t,\|u_0\|_{L^2},\|\theta_0\|_{L^2})(1+\|v\|_{H^\beta}^2)(1+\|\omega\|_{L^2}^2+\|j\|_{L^2}^2+\|H\|_{L^2}^2).\end{aligned}$$

By Grönwall's inequality,

$$\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\|H(t)\|_{L^2}^2+\int_0^t(\|\Lambda^\alpha\omega(\tau)\|_{L^2}^2+\|\Lambda^\beta j(\tau)\|_{L^2}^2+\|\Lambda^\beta H(\tau)\|_{L^2}^2)\,d\tau\le C_0(t).$$

Thanks to (2.15), we have

$$\|\nabla v\|_{L^2}\le\|\mathcal{R}_1H\|_{L^2}+\|\mathcal{R}_1\Lambda^{2-2\beta}\theta\|_{L^2}+\|\mathcal{R}_2 j\|_{L^2}\le C_0(t).$$

This ends the proof of Proposition 2.1. ∎

2.3. Proof of Proposition 2.2. This subsection presents the proof of Proposition 2.2, which establishes global bounds for $\|\theta(t)\|_{L^p}$, $\|\nabla v(t)\|_{L^p}$ and $\|\nabla H(t)\|_{L^p}$ for any $2\le p\le\infty$. The proof makes use of the previous global bounds and Lemma 2.1.

Proof. **(Proof of Proposition 2.2).** Multiplying both sides of the θ-equation by $|\theta|^{p-2}\theta$ and integrating by parts, we have

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p \le \|h\|_{L^p}\|\theta\|_{L^p}^{p-1}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\|\theta(t)\|_{L^p} &\le C\|h\|_{L^p}\\ &\le C\|H\|_{L^p}+C\|\Lambda^{2-2\beta}\theta\|_{L^p}\\ &\le C\|H\|_{H^\beta}+C\|\Lambda^{2-2\beta}\theta\|_{L^2}^{\varrho}\|\theta\|_{L^p}^{1-\varrho}, \quad \varrho=\frac{\beta-1}{\beta-\frac{1}{2}-\frac{1}{p}}.\end{aligned}$$

For $2\le p$, we have $\varrho\le 1$. The global bounds on $\int_0^t\|H\|_{H^\beta}^2\,d\tau$ and $\|\Lambda^{2-2\beta}\theta\|_{L^2}$ obtained previously imply the global bound

$$\|\theta(t)\|_{L^p}\le C(t),$$

where $C(t)$ depends on t and the initial data. Since $C(t)$ is independent of p, this global bound also holds for $p=\infty$.

Recall the second equation of (1.1),

$$\partial_t v+(u\cdot\nabla)v+\Lambda^{2\beta}v+\nabla\theta+(v\cdot\nabla)u=0.$$

Applying Λ^{-1} to both sides of the above equation yields

$$\partial_t\Lambda^{-1}v+\Lambda^{2\beta}\Lambda^{-1}v=-\Lambda^{-1}(v\cdot\nabla u)-\Lambda^{-1}\nabla\cdot(uv)-\Lambda^{-1}\nabla\theta. \tag{2.17}$$

Applying Lemma 2.1 to (2.17) yields that, for any $\varepsilon_1\in(0,2\beta-1)$ and for any $p\in[2,\infty)$,

$$\begin{aligned}\|\Lambda^{2\beta-1-\varepsilon}v\|_{L^p} &\le C(t,v_0)+C\|\Lambda^{-1}(v\cdot\nabla u)\|_{L^p}+C\|\Lambda^{-1}\nabla\cdot(uv)\|_{L^p}+C\|\Lambda^{-1}\nabla\cdot\theta\|_{L^p}\\ &\le C(t,v_0)+C\|\Lambda^{-1}(\nabla vu)\|_{L^p}+C\|uv\|_{L^p}+C\|\theta\|_{L^p},\end{aligned}$$

where we have used the fact that the Calderon-Zygmund operator $\Lambda^{-1}\nabla$ is bounded on L^p $(1<p<\infty)$ (see [6]). The terms on the right can be further bounded as follows. By the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}C\|\Lambda^{-1}(\nabla vu)\|_{L^p} &\le C\|\nabla vu\|_{L^{\frac{2p}{p+2}}}\\ &\le C\|\nabla v\|_{L^2}\|u\|_{L^p}\\ &\le C(\|h\|_{L^2}+\|j\|_{L^2})\|u\|_{H^1}\\ &\le C(\|H\|_{L^2}+\|\Lambda^{2-2\beta}\theta\|_{L^2}+\|j\|_{L^2})\|u\|_{H^1}\\ &\le C(t).\end{aligned}$$

For $p>\frac{2}{2\beta-1-\varepsilon_1}$, we have $\|v\|_{L^\infty}\le C\|v\|_{L^2}^{1-\gamma}\|\Lambda^{2\beta-1-\varepsilon_1}v\|_{L^p}^{\gamma}$ and thus

$$C\|uv\|_{L^p}\le C\|u\|_{L^p}\|v\|_{L^\infty}$$

$$\begin{aligned}&\le C\|u\|_{H^1}\|v\|_{L^2}^{1-\gamma}\|\Lambda^{2\beta-1-\varepsilon_1}v\|_{L^p}^{\gamma}\\&\le\frac{1}{2}\|\Lambda^{2\beta-1-\varepsilon_1}v\|_{L^p}+C\|u\|_{H^1}^{\frac{1}{1-\gamma}}\|v\|_{L^2}\\&\le\frac{1}{2}\|\Lambda^{2\beta-1-\varepsilon_1}v\|_{L^p}+C(t).\end{aligned}$$

Therefore, for any $\varepsilon_1\in(0,2\beta-1)$ and for any $p\in(2,\infty)$,

$$\|\Lambda^{2\beta-1-\varepsilon_1}v\|_{L^p}\le C(t). \tag{2.18}$$

In particular, taking $\varepsilon_1=2\beta-2$ in (2.18) yields

$$\|\nabla v\|_{L^p}\le C(t).$$

Taking $p>\frac{2}{2\beta-2-\varepsilon_1}$ in (2.18) leads to

$$\|\nabla v\|_{L^\infty}\le C(t).$$

By the simple interpolation inequality,

$$\|v\|_{L^\infty}\le C(\|v\|_{L^2}+\|\nabla v\|_{L^p})\le C(t).$$

We now show that

$$\|\nabla H(t)\|_{L^p}\le C(t).$$

Applying Λ^{-1} to (2.14) yields

$$\begin{aligned}\partial_t\Lambda^{-1}H+\Lambda^{2\beta}\Lambda^{-1}H=&-\Lambda^{-1}Q(\nabla u,\nabla v)+\Lambda^{2-2\beta}\Lambda^{-1}h\\&+\Lambda^{-1}([\Lambda^{2-2\beta},u\cdot\nabla]\theta)-\Lambda^{-1}\nabla\cdot(uH).\end{aligned} \tag{2.19}$$

Applying Lemma 2.1 to (2.19) with any $\varepsilon_3\in(0,2\beta-1)$ gives

$$\begin{aligned}\|\Lambda^{2\beta-1-\varepsilon_3}H\|_{L^p}=&\|\Lambda^{2\beta-\varepsilon_3}\Lambda^{-1}H\|_{L^p}\\&\le C(t,H_0)+C\|\Lambda^{-1}Q(\nabla u,\nabla v)\|_{L^p}+C\|\Lambda^{2-2\beta}\Lambda^{-1}h\|_{L^p}\\&\quad+C\|\Lambda^{-1}([\Lambda^{2-2\beta},u\cdot\nabla]\theta)\|_{L^p}+C\|\Lambda^{-1}\nabla\cdot(uH)\|_{L^p}.\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality, for any $p\in(2,\infty)$,

$$\begin{aligned}C\|\Lambda^{-1}Q(\nabla u,\nabla v)\|_{L^p}&\le C\|\nabla u\nabla v\|_{L^{\frac{2p}{p+2}}}\\&\le C\|\nabla u\|_{L^2}\|\nabla v\|_{L^p}\\&\le C(t).\end{aligned}$$

It is easy to check that for any $p\in(\frac{2}{3-2\beta},\infty)$

$$\begin{aligned}C\|\Lambda^{2-2\beta}\Lambda^{-1}h\|_{L^p}&\le C\|\Lambda^{2-2\beta}v\|_{L^p}\\&\le C\|v\|_{L^{\frac{p}{(\beta-1)p+1}}}\\&\le C(\|v\|_{L^2}+\|v\|_{L^\infty})\\&\le C(t).\end{aligned}$$

Similarly, we have for any $p\in(\frac{2}{3-2\beta},\infty)$

$$\begin{aligned}\|\Lambda^{-1}([\Lambda^{2-2\beta},u\cdot\nabla]\theta)\|_{L^p}\leq&\|\Lambda^{-1}\Lambda^{2-2\beta}(u\cdot\nabla\theta)\|_{L^p}+\|\Lambda^{-1}(u\cdot\nabla\Lambda^{2-2\beta}\theta)\|_{L^p}\\ \leq&\|\Lambda^{-1}\Lambda^{2-2\beta}\nabla(u\theta)\|_{L^p}+\|\Lambda^{-1}\nabla(u\Lambda^{2-2\beta}\theta)\|_{L^p}\\ \leq&C\|\Lambda^{2-2\beta}(u\theta)\|_{L^p}+C\|u\Lambda^{2-2\beta}\theta\|_{L^p}\\ \leq&C\|u\theta\|_{L^{\frac{p}{(\beta-1)p+1}}}+C\|u\|_{L^{2p}}\|\Lambda^{2-2\beta}\theta\|_{L^{2p}}\\ \leq&C\|u\|_{L^{\frac{2p}{(\beta-1)p+1}}}\|\theta\|_{L^{\frac{2p}{(\beta-1)p+1}}}+C\|u\|_{L^{2p}}\|\theta\|_{L^{\frac{2p}{2(\beta-1)p+1}}}\\ \leq&C(t).\end{aligned}$$

The last term admits the bound for any $p\in(\max\{2,\frac{2}{2\beta-1-\varepsilon_2}\},\infty)$

$$\begin{aligned}C\|\Lambda^{-1}\nabla\cdot(uH)\|_{L^p}\leq&C\|uH\|_{L^p}\\ \leq&C\|u\|_{L^p}\|H\|_{L^\infty}\\ \leq&C\|u\|_{L^p}\|H\|_{L^2}^{1-\eta}\|\Lambda^{2\beta-1-\varepsilon_3}H\|_{L^p}^{\eta}\\ \leq&\frac{1}{2}\|\Lambda^{2\beta-1-\varepsilon_3}H\|_{L^p}+C(t).\end{aligned}$$

Putting the above estimates together yields that, for any $p>\max\{\frac{2}{3-2\beta},\frac{2}{2\beta-1-\varepsilon_3}\}$,

$$\|\Lambda^{2\beta-1-\varepsilon_3}H\|_{L^p}\leq C(t). \tag{2.20}$$

In particular, we have by selecting $\varepsilon_3=2\beta-2>0$ in (2.20),

$$\|\nabla H\|_{L^p}\leq C(t)$$

for any $p\in(\frac{2}{3-2\beta},\infty)$. Of course, if one takes $p>\frac{2}{2\beta-2-\varepsilon_3}$, then

$$\|\nabla H\|_{L^\infty}\leq C(t).$$

This completes the proof. □

3. Proof of Theorem 1.1, Part II

This section continues and finishes the proof of Theorem 1.1. The proof depends crucially on the size of α. The proof for the case when $\alpha>\frac{1}{2}$ takes a different path from that for the case when $\alpha\leq\frac{1}{2}$. The case when $\alpha\leq\frac{1}{2}$ is much more involved.

When $\alpha>\frac{1}{2}$, the dissipation in the equation of u, together with the global bounds obtained in the previous section, allows us to show that

$$\|\nabla u(t)\|_{L^\infty}\leq C(t),$$

where $C(t)$ is a finite quantity depending on t and the initial data only. We further show that, for any $2\leq p<\infty$,

$$\|\nabla\theta(t)\|_{L^p}\leq C(t).$$

With these preparations at our disposal, the global H^s-bound then follows as a consequence.

When $\alpha\leq\frac{1}{2}$, the proof of Theorem 1.1 is quite involved. The difficulty is that the vorticity equation involves the term

$$\nabla\times(\nabla\cdot(v\otimes v)),$$

but the global bound on v is for $\|\nabla v(t)\|_{L^p}$. Therefore we have to shift one derivative from this term when we estimate $\|\omega\|_{L^\infty}$. When $\alpha\le\frac{1}{2}$, there is not enough dissipation to absorb this derivative. The way we handle this difficulty is to write the second-order derivative $\nabla^2 v$ in terms of j, H and $\Lambda^{2-2\beta}\theta$ and make use of the bounds on j and H. In order to prove the global bound

$$\|\omega(t)\|_{L^\infty}\le C(t), \tag{3.1}$$

we first prove a De Giorgi-Nash regularization estimate (see Lemma 3.4). In addition, we also need to bound $\|\nabla j(t)\|_{L^p}$ in terms of $\|\omega(t)\|_{L^p}$. Once (3.1) is shown, the global H^s-bound on (u,v,θ) is then within our reach.

Naturally the rest of this section is divided into two subsections. The first subsection deals with $\alpha>\frac{1}{2}$ while the second subsection handles the case $\alpha\le\frac{1}{2}$.

3.1. Proof of Theorem 1.1 for $\alpha>\frac{1}{2}$. Before we can prove Theorem 1.1, we prove two lemmas first as a preparation.

Lemma 3.1. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. If α and β satisfy*

$$\frac{1}{2}<\alpha<1, \qquad \beta\ge\frac{3-\alpha}{2}, \tag{3.2}$$

then any corresponding smooth solution (u,v,θ) of (1.1) obeys

$$\|\nabla u(t)\|_{L^\infty}\le C_0(t), \tag{3.3}$$

where $C_0(t)$ is a finite quantity depending on t and the initial data only.

Proof. The proof makes use of Lemma 2.1. Applying Λ^{-1} to both sides of the vorticity equation yields

$$\partial_t\Lambda^{-1}\omega+\Lambda^{2\alpha}\Lambda^{-1}\omega=-\Lambda^{-1}\nabla\times(\nabla\cdot(v\otimes v))-\Lambda^{-1}\nabla\cdot(u\omega). \tag{3.4}$$

Assume $\alpha>\frac{1}{2}$. Applying Lemma 2.1 to (3.4) yields, for any $\varepsilon_2\in(0,2\alpha-1)$ and for any $p\in(\frac{2}{2\alpha-1-\varepsilon_2},\infty)$,

$$\begin{aligned}\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p}&=\|\Lambda^{2\alpha-\varepsilon_2}\Lambda^{-1}\omega\|_{L^p}\\&\le C(t,u_0)+C\|\Lambda^{-1}\nabla\times(\nabla\cdot(v\otimes v))\|_{L^p}+C\|\Lambda^{-1}\nabla\cdot(u\omega)\|_{L^p}\\&\le C(t,u_0)+C\|v\nabla v\|_{L^p}+C\|u\omega\|_{L^p}\\&\le C(t,u_0)+C\|v\|_{L^\infty}\|\nabla v\|_{L^p}+C\|u\|_{L^p}\|\omega\|_{L^\infty}\\&\le C(t,u_0)+C\|v\|_{L^\infty}\|\nabla v\|_{L^p}+C\|u\|_{H^1}\|\omega\|_{L^2}^{1-\varrho}\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p}^{\varrho}\\&\le\frac{1}{2}\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p}+C(t),\end{aligned} \tag{3.5}$$

where we have used the embedding, for $p\in(\frac{2}{2\alpha-1-\varepsilon_2},\infty)$,

$$\|\omega\|_{L^\infty}\le C\|\omega\|_{L^2}^{1-\varrho}\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p}^{\varrho}, \qquad \frac{1}{2}(1-\varrho)+\varrho\left(\frac{1}{p}-\frac{2\alpha-1-\varepsilon_2}{2}\right)=0.$$

Equation (3.5) implies that, for any $\varepsilon_2\in(0,2\alpha-1)$ and for any $p\in(\frac{2}{2\alpha-1-\varepsilon_2},\infty)$

$$\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p}\le C(t).$$

As a special consequence, for $p\in(\frac{2}{2\alpha-1-\varepsilon_2},\infty)$,

$$\|\nabla u\|_{L^\infty}\leq C(\|\nabla u\|_{L^2}+\|\Lambda^{2\alpha-1-\varepsilon_2}\omega\|_{L^p})\leq C(t).$$

This ends the proof of Lemma 3.1. □

To facilitate the proof of Theorem 1.1, we need one more global bound, namely a global bound on $\|\nabla\theta\|_{L^p}$ for any $q\in[2,\infty)$. Once this global bound is at our disposal, we are ready to prove Theorem 1.1.

Lemma 3.2. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. If α and β satisfy (3.2), then any corresponding smooth solution (u,v,θ) of (1.1) obeys, for any $q\in[2,\infty]$ and any given $T>0$ and any $t\in[0,T]$,*

$$\|\nabla\theta(t)\|_{L^q}\leq C_0(t),$$

where $C_0(t)$ is a finite quantity depending only on t and the initial data.

Proof. We multiply the equation of θ by $|\nabla\theta|^{q-2}\nabla\theta$ to obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\nabla\theta(t)\|_{L^q}^q\leq& C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q+C\|\nabla h\|_{L^q}\|\nabla\theta\|_{L^q}^{q-1}\\ \leq& C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q+C(\|\nabla H\|_{L^q}+\|\Lambda^{2-2\beta}\nabla\theta\|_{L^q})\|\nabla\theta\|_{L^q}^{q-1}\end{aligned}$$

or

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^q}\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}+C(\|\nabla H\|_{L^q}+\|\Lambda^{2-2\beta}\nabla\theta\|_{L^q}).$$

For $\beta>1$, as in the proof of Lemma 2.3,

$$\begin{aligned}\|\Lambda^{2-2\beta}\nabla\theta\|_{L^q}\leq&\|\Delta_{-1}\Lambda^{2-2\beta}\nabla\theta\|_{L^q}+\sum_{j\geq0}\|\Delta_j\Lambda^{2-2\beta}\nabla\theta\|_{L^q}\\ \leq& C\|\Delta_{-1}\Lambda^{2-2\beta}\theta\|_{L^2}+C\sum_{j\geq0}2^{2(1-\beta)j}\|\nabla\theta\|_{L^q}\\ \leq& C+C\|\nabla\theta\|_{L^q}.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^q}\leq C(1+\|\nabla u\|_{L^\infty})\|\nabla\theta\|_{L^q}+C\|\nabla H\|_{L^q}.$$

Grönwall's inequality, (2.4) and (3.3) imply that, for any $q\in[2,\infty]$,

$$\|\nabla\theta(t)\|_{L^q}\leq C.$$

This completes the proof of Lemma 3.2. □

We are ready to finish the proof of Theorem 1.1 for the case when $\alpha>\frac{1}{2}$.

Proof. **(Proof of Theorem 1.1.)** Applying Λ^s to (1.1) and then taking the L^2-inner product with $(\Lambda^s u,\Lambda^s v,\Lambda^s\theta)$ yield

$$\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+\beta}v\|_{L^2}^2$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^2}\Big(\Lambda^s\nabla\cdot(v\otimes v)\cdot\Lambda^s u+\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\Big)\,dx-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,dx\\
&\quad -\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,dx-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&:= H_1+H_2+H_3+H_4. \qquad (3.6)
\end{aligned}
$$

Using the commutator and bilinear estimates (see, e.g., [22–24]),

$$
\begin{aligned}
&\|[\Lambda^s,f]g\|_{L^p}\le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}}+\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),\\
&\|\Lambda^s(fg)\|_{L^p}\le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}}+\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}})
\end{aligned}
$$

with $s>0$ and $p_2,p_3\in(1,\infty)$, $p_1,p_4\in[1,\infty]$ satisfying $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$, we have

$$
\begin{aligned}
H_1 &\le C\|\Lambda^s\nabla\cdot(v\otimes v)\|_{L^2}\|\Lambda^s u\|_{L^2}+C\|\Lambda^{s-1}(v\cdot\nabla u)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C\|v\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}\|\Lambda^s u\|_{L^2}+C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{s-1}v\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C\|v\|_{L^\infty}(\|v\|_{L^2}+\|\Lambda^{s+\beta}v\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad +C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}(\|v\|_{L^2}+\|\Lambda^s v\|_{L^2})\|\Lambda^{s+1}v\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\beta}v\|_{L^2}^2+C(\|v\|_{L^2}^2+\|v\|_{L^\infty}^2)\|\Lambda^s u\|_{L^2}^2+C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2(\|v\|_{L^2}^2\|\Lambda^s v\|_{L^2}^2),
\end{aligned}
$$

$$
\begin{aligned}
H_2 &\le C\|[\Lambda^s,u\cdot\nabla]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^s u\|_{L^{\frac{2}{1-\alpha}}})\|\Lambda^s\theta\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+\alpha}u\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}^2)\|\Lambda^s\theta\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
H_3 &\le C\|\Lambda^s(u\cdot\nabla v)\|_{L^2}\|\Lambda^s v\|_{L^2}\\
&\le C(\|u\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}+\|\nabla v\|_{L^{\frac{2}{\alpha}}}\|\Lambda^s u\|_{L^{\frac{2}{1-\alpha}}})\|\Lambda^s v\|_{L^2}\\
&\le C(\|u\|_{L^\infty}\|\Lambda^{s+\beta}v\|_{L^2}+\|u\|_{L^\infty}\|v\|_{L^2}+\|\nabla v\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+\alpha}u\|_{L^2})\|\Lambda^s v\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2+\frac{1}{8}\|\Lambda^{s+\beta}v\|_{L^2}^2+C(\|u\|_{L^\infty}^2+\|v\|_{L^2}^2+\|\nabla v\|_{L^{\frac{2}{\alpha}}}^2)(1+\|\Lambda^s v\|_{L^2}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
H_4 &\le C\int_{\mathbb{R}^2}|[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u|\,dx\le C\|\Lambda^s u\|_{L^2}\|[\Lambda^s,u\cdot\nabla]u\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}
$$

Substituting all the preceding estimates into (3.6), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+\beta}v\|_{L^2}^2\\
&\le CG(t)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2),
\end{aligned}
$$

where

$$
G(t):=1+\|v(t)\|_{L^2}^2+\|v(t)\|_{L^\infty}^2+\|\nabla u(t)\|_{L^{\frac{2}{1-\alpha}}}^2+\|\nabla u(t)\|_{L^\infty}+\|\nabla\theta(t)\|_{L^{\frac{2}{\alpha}}}+\|\nabla v(t)\|_{L^{\frac{2}{\alpha}}}^2.
$$

The global bounds we have obtained previously imply that

$$\int_0^t G(\tau)\,d\tau<\infty.$$

Grönwall's inequality then yields

$$\begin{aligned}&\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2\\&\quad+\int_0^t(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2)\,d\tau<\infty,\end{aligned}$$

which provides the desired global bound in Theorem 1.1. This completes the proof of Theorem 1.1 for the case when $\alpha>\frac{1}{2}$. ∎

3.2. Proof of Theorem 1.1 for the case when $\alpha\le\frac{1}{2}$. As aforementioned, the proof for this case is not straightforward. The core is to prove the bound

$$\|\omega(t)\|_{L^\infty}\le C(t).$$

We need two lemmas. The first lemma bounds $\|\nabla j\|_{L^q}$ in terms of $\|\omega\|_{L^q}$ for any $2\le q\le\infty$. The second lemma is a De Giorgi-Nash regularization estimate.

LEMMA 3.3. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. If α and β satisfy*

$$0<\alpha<1,\quad \beta\ge\frac{3-\alpha}{2}.$$

then any corresponding smooth solution (u,v,θ) of (1.1) obeys, for $q\in[2,\infty]$ and for any given $T>0$ and any $t\in[0,T]$,

$$\|\nabla j(t)\|_{L^p}\le C(t)+C(t)\|\omega\|_{L^p}, \tag{3.7}$$

where $C(t)$ is a finite quantity depending only on t and the initial data.

Proof. Applying Λ^{-1} to the second equation of (2.13) gives

$$\partial_t\Lambda^{-1}j+\Lambda^{2\beta}\Lambda^{-1}j=-\Lambda^{-1}(u\cdot\nabla j)-\Lambda^{-1}(v\cdot\nabla\omega)-\Lambda^{-1}(h\omega). \tag{3.8}$$

Applying Lemma 2.1 to (3.8) yields, for any $\varepsilon_4\in(0,2\beta-1)$,

$$\begin{aligned}\|\Lambda^{2\beta-1-\varepsilon_4}j\|_{L^p}&=\|\Lambda^{2\beta-\varepsilon_4}\Lambda^{-1}j\|_{L^p}\\&\le C(t,j_0)+C\|\Lambda^{-1}(u\cdot\nabla j)\|_{L^p}+C\|\Lambda^{-1}(h\omega)\|_{L^p}+C\|\Lambda^{-1}(v\cdot\nabla\omega)\|_{L^p}.\end{aligned}$$

By the boundedness of Riesz transforms on L^p with $1<p<\infty$,

$$\begin{aligned}C\|\Lambda^{-1}(u\cdot\nabla j)\|_{L^p}&\le C\|\Lambda^{-1}\nabla(uj)\|_{L^p}\\&\le C\|uj\|_{L^p}\\&\le C\|u\|_{L^p}\|j\|_{L^\infty}\\&\le C\|u\|_{L^p}\|j\|_{L^2}^{1-\lambda}\|\Lambda^{2\beta-1-\varepsilon_4}j\|_{L^p}^{\lambda}\\&\le\frac{1}{2}\|\Lambda^{2\beta-1-\varepsilon_4}j\|_{L^p}+C(t),\end{aligned}$$

where we have used Sobolev's inequality with some $\lambda\in(0,1)$.

$$\begin{aligned}C\|\Lambda^{-1}(h\omega)\|_{L^p}\le&C\|h\omega\|_{L^{\frac{2p}{p+2}}}\\ \le&C\|\omega\|_{L^2}\|\nabla v\|_{L^p}\\ \le&C(t).\end{aligned}$$

The last term can be bounded by

$$\begin{aligned}C\|\Lambda^{-1}(v\cdot\nabla\omega)\|_{L^p}\le&C\|\Lambda^{-1}\nabla(v\omega)\|_{L^p}+C\|\Lambda^{-1}(\nabla v\omega)\|_{L^p}\\ \le&C\|v\omega\|_{L^p}+C\|\nabla v\omega\|_{L^{\frac{2p}{p+2}}}\\ \le&C\|v\|_{L^\infty}\|\omega\|_{L^p}+\|\nabla v\|_{L^p}\|\omega\|_{L^2}\\ \le&C(t)+C(t)\|\omega\|_{L^p}.\end{aligned}$$

Putting all the above estimates together yields that for any $p\in(\frac{2}{2\beta-1-\varepsilon_4},\infty)$

$$\|\Lambda^{2\beta-1-\varepsilon_4}j\|_{L^p}\le C(t)+C(t)\|\omega\|_{L^p},$$

which yields (3.7) if $\varepsilon_4=2\beta-2$. This completes the proof of Lemma 3.3. □

The following De Giorgi-Nash estimate plays a very important role in showing that the vorticity is bounded.

LEMMA 3.4. *Assume u is smooth. Consider the linear transport-diffusion equation with fractional dissipation*

$$\begin{cases}\partial_t b+(u\cdot\nabla)b+\Lambda^{\delta_1}b=\Lambda^{\delta_2}f,\\ \nabla\cdot u=0,\\ b(x,0)=b_0(x),\end{cases}\tag{3.9}$$

where $0\le 2\delta_2\le\delta_1\le 2$, $b_0\in L^\infty$ and $f\in L^p_TL^q\cap L^2_TL^2$ with

$$T>0,\quad 2<p<\infty,\quad \frac{2}{\delta_1-\delta_2}<q<\infty,\quad \frac{\delta_1}{p}+\frac{2}{q}<\delta_1-\delta_2.\tag{3.10}$$

Then there exists a constant $C=C(\delta_1,\delta_2,p,q,\|f\|_{L^2_TL^2})$ such that for any $0\le t\le T$

$$\|b(t)\|_{L^\infty}\le C\|b_0\|_{L^\infty}+C(1+T^{\frac{\sigma}{\delta_1}(\delta_1-\delta_2-\frac{\delta_1}{p}-\frac{2}{q})})\|f\|^{\sigma}_{L^p_TL^q},$$

where $\sigma=\sigma(\delta_1,\delta_2,q)\in(0,1)$.

Proof. Since the equation in (3.9) is linear, we write b into two parts with each one of them satisfying

$$\begin{cases}\partial_t b+(u\cdot\nabla)b+\Lambda^{\delta_1}b=0,\\ \nabla\cdot u=0,\\ b(x,0)=b_0(x)\end{cases}\tag{3.11}$$

and

$$\begin{cases}\partial_t b+(u\cdot\nabla)b+\Lambda^{\delta_1}b=\Lambda^{\delta_2}f,\\ \nabla\cdot u=0,\\ b(x,0)=0.\end{cases}\tag{3.12}$$

Thanks to the maximum principle, the solution of (3.11) satisfies (see [13, Corollary 2.6])

$$\|b(t)\|_{L^\infty} \le \|b_0\|_{L^\infty}.$$

Therefore it suffices to prove the following bound for the solution of (3.12),

$$\|b\|_{L^\infty_T L^\infty} \le C(1+T^{\frac{\sigma}{\delta_1}(\delta_1-\delta_2-\frac{\delta_1}{p}-\frac{2}{q})})\|f\|^\sigma_{L^p_T L^q}.$$

To obtain this bound, we make use of the De Giorgi-Nash argument. We first prove that, for $T\in(0,1]$,

$$\|b\|_{L^\infty_T L^\infty} \le C\|f\|^\sigma_{L^p_T L^q}, \tag{3.13}$$

where $\sigma=\sigma(\delta_1,\delta_2,q)\in(0,1)$. Let M be a positive number to be fixed later. Set $M_k:=M(1-2^{-k-1})$ for $k\in\mathbb{N}$. It follows from the pointwise-positive property for fractional operators that for any convex function φ (see [11, 21])

$$\varphi'(b)\Lambda^{\delta_1}b \ge \Lambda^{\delta_1}\varphi(b).$$

As a result, we have

$$1_{\{b\ge M_k\}}\Lambda^{\delta_1}b \ge \Lambda^{\delta_1}(b-M_k)_+,$$

where $x_+=\max\{x,0\}$ and $1_{\{b\ge M_k\}}$ is the characteristic function. It then follows from (3.12) that

$$\partial_t(b-M_k)+(u\cdot\nabla)(b-M_k)+\Lambda^{\delta_1}(b-M_k)=\Lambda^{\delta_2}f.$$

We obtain by multiplying it by $(b-M_k)_+$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(b-M_k)_+\|^2_{L^2}+\|(b-M_k)_+\|^2_{\dot H^{\frac{\delta_1}{2}}} &\le \Big|\int_{\mathbb{R}^2}\Lambda^{\delta_2}f(b-M_k)_+\,dx\Big| \\ &\le \|f(t)1_{\{b\ge M_k\}}\|_{\dot H^{-\widetilde{s}}}\|(b-M_k)_+\|_{\dot H^{\widetilde{s}+\delta_2}}.\end{aligned}$$

Denoting

$$A_k=\|(b-M_k)_+\|^2_{L^\infty_T L^2}+\|(b-M_k)_+\|^2_{L^2_T\dot H^{\frac{\delta_1}{2}}},$$

we have

$$A_k\le 2\int_0^T \|f(t)1_{\{b\ge M_k\}}\|_{\dot H^{-\widetilde{s}}}\|(b-M_k)_+\|_{\dot H^{\widetilde{s}+\delta_2}}\,dt.$$

The aim is to prove that the right-hand side of the above inequality can be bounded A^γ_{k-1} with $\gamma>1$. By an interpolation inequality, for any $0\le\widetilde{s}\le\frac{\delta_1-2\delta_2}{2}$,

$$\|(b-M_k)_+\|^2_{L_T^{\frac{\delta_1}{\delta_2+\widetilde{s}}}\dot H^{\widetilde{s}+\delta_2}}\le CA_k.$$

By Hölder's inequality and the Hardy-Littlewood-Sobolev inequality,

$$A_k \le C\|f(t)1_{\{b\ge M_k\}}\|_{L_T^{\frac{\delta_1}{\delta_1-\delta_2-\widetilde{s}}}\dot H^{-\widetilde{s}}}\|(b-M_k)_+\|_{L_T^{\frac{\delta_1}{\delta_2+\widetilde{s}}}\dot H^{\widetilde{s}+\delta_2}}$$

$$\leq C\|f(t)1_{\{b\geq M_k\}}\|_{L_T^{\frac{\delta_1}{\delta_1-\delta_2-\widetilde{s}}}L^{\frac{2}{1+\widetilde{s}}}} A_k^{\frac{1}{2}}$$

or

$$A_k \leq C\|f(t)1_{\{b\geq M_k\}}\|^2_{L_T^{\frac{\delta_1}{\delta_1-\delta_2-\widetilde{s}}}L^{\frac{2}{1+\widetilde{s}}}}.$$

From now on, we choose $\widetilde{s}=\frac{\delta_1-2\delta_2}{2}$. By (3.10),

$$\frac{\delta_1}{\delta_1-\delta_2-\widetilde{s}}=2<p, \qquad \frac{2}{1+\widetilde{s}}<\frac{2}{\delta_1-\delta_2}<q.$$

The Hölder inequality allows us to show

$$\begin{aligned}
A_k \leq & C\|f(t)\|^2_{L_T^pL^q}\|1_{\{b\geq M_k\}}\|^2_{L_T^{\frac{2p}{p-2}}L^{\frac{2q}{(1+\widetilde{s})q-2}}}\\
\leq & C\|f(t)\|^2_{L_T^pL^q}\left(\int_0^T\Big(\int_{\{b\geq M_k\}}dx\Big)^{\frac{(1+\widetilde{s})q-2}{q}\frac{p}{p-2}}dt\right)^{\frac{p-2}{p}}\\
\leq & C\|f(t)\|^2_{L_T^pL^q}\left(\int_0^T|\{b\geq M_k\}|^{\frac{(1+\widetilde{s})q-2}{q}\frac{p}{p-2}}dt\right)^{\frac{p-2}{p}}.
\end{aligned}\tag{3.14}$$

We note that if $b\geq M_k$, then we have

$$b-M_{k-1}\geq M_k-M_{k-1}=2^{-k-1}M.$$

This implies that, for any $\delta\geq 1$,

$$\begin{aligned}
|\{b\geq M_k\}|\leq & \Big|\int_{\{b\geq M_k\}}\Big(\frac{(b-M_{k-1})_+}{2^{-k-1}M}\Big)^{\delta}dx\Big|\\
= & \Big(\frac{2^{k+1}}{M}\Big)^{\delta}\|(b-M_{k-1})_+\|^{\delta}_{L^\delta}.
\end{aligned}$$

Therefore,

$$A_k\leq C\|f(t)\|^2_{L_T^pL^q}\Big(\frac{2^{k+1}}{M}\Big)^{\frac{(1+\widetilde{s})q-2}{q}\delta}\left(\int_0^T\|(b-M_{k-1})_+\|_{L^\delta}^{\frac{(1+\widetilde{s})q-2}{q}\frac{p}{p-2}\delta}dt\right)^{\frac{p-2}{p}}.\tag{3.15}$$

By an interpolation inequality, for any (m,l) satisfying

$$2\leq m\leq\frac{4}{2-\delta_1}, \qquad \frac{\delta_1}{l}+\frac{2}{m}\geq 1,$$

we have

$$\|(b-M_{k-1})_+\|^2_{L_T^lL^m}\leq CA_{k-1}.$$

Now we take $\delta\in[2,\frac{4}{2-\delta_1}]$ satisfying

$$\frac{(1+\widetilde{s})q-2}{q}\delta>2, \qquad \frac{\delta_1}{\frac{(1+\widetilde{s})q-2}{q}\frac{p}{p-2}\delta}+\frac{2}{\delta}\geq 1.$$

That is, δ should obey

$$\max\left\{\frac{2-\delta_1}{4}, \frac{p[(1+\widetilde{s})q-2]}{2p[(1+\widetilde{s})q-2]+\delta_1(p-2)q}\right\}\le\frac{1}{\delta}\le\frac{1}{2}, \qquad \frac{1}{\delta}<\frac{(1+\widetilde{s})q-2}{2q}.$$

It can be checked that such δ exists if

$$\frac{\delta_1}{p}+\frac{2}{q}<\delta_1-\delta_2.$$

For such δ, we deduce from (3.15) that

$$A_k\le C_1\|f(t)\|^2_{L^p_TL^q}\left(\frac{2^{k+1}}{M}\right)^{2\gamma}A^{\gamma}_{k-1}, \tag{3.16}$$

where $\gamma=\frac{(1+\widetilde{s})q-2}{2q}\delta>1$. Multiplying (3.12) by b and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b(t)\|^2_{L^2}+\|b\|^2_{\dot{H}^{\frac{\delta_1}{2}}} &\le\|\Lambda^{\delta_2}b\|_{L^2}\|f\|_{L^2}\\ &\le C\|b\|^{1-\frac{2\delta_2}{\delta_1}}_{L^2}\|b\|^{\frac{2\delta_2}{\delta_1}}_{\dot{H}^{\frac{\delta_1}{2}}}\|f\|_{L^2}\\ &\le\frac{1}{2}\|b\|^2_{\dot{H}^{\frac{\delta_1}{2}}}+C\|b\|^2_{L^2}+C\|f\|^2_{L^2}.\end{aligned}$$

Thanks to $2\delta_2\le\delta_1$ and $\|f\|_{L^2_TL^2}<\infty$, we have

$$\|b(t)\|_{L^\infty_TL^2}\le C(T,\|f\|_{L^2_TL^2})\le C_2.$$

Noticing the fact

$$\left|\left\{b\ge M_0=\frac{M}{2}\right\}\right|\le\left(\frac{2}{M}\right)^2\|b(t)\|^2_{L^\infty_TL^2}\le C_2^2\left(\frac{2}{M}\right)^2$$

and making use of (3.14), we obtain

$$A_0\le C_3\|f(t)\|^2_{L^p_TL^q}\left(\frac{2}{M}\right)^{\frac{4\gamma}{\delta}}. \tag{3.17}$$

We take M satisfying

$$C_3\|f(t)\|^2_{L^p_TL^q}\left(\frac{2}{M}\right)^{\frac{4\gamma}{\delta}}=\left(C_1\|f(t)\|^2_{L^p_TL^q}\left(\frac{2}{M}\right)^{2\gamma}\right)^{-\frac{1}{\gamma-1}}4^{-\frac{\gamma}{(\gamma-1)^2}}$$

or

$$M=C_4\|f(t)\|^{\sigma}_{L^p_TL^q},$$

where $C_4=C_4(\delta_1,\delta_2,p,q,\|f\|_{L^2_TL^2})$ and $\sigma=\sigma(\delta_1,\delta_2,q)\in(0,1)$. Applying Lemma 4.2 to (3.16) and (3.17) leads to

$$\lim_{k\to\infty}A_k=0,$$

or

$$\|(b-M)_+\|_{L^\infty_TL^2}=0.$$

Consequently,

$$b(t,x)\le M,\quad (t,x)\in[0,T]\times\mathbb{R}^2.$$

Applying the same argument to $-b$, we also deduce

$$b(t,x)\ge -M,\quad (t,x)\in[0,T]\times\mathbb{R}^2.$$

Thus we have proven (3.13). Next we shall use a scaling argument to prove the case when $T\ge 1$. To this end, for all $(t,x)\in[0,T]\times\mathbb{R}^2$ and $\lambda>0$, we denote $\tau=\lambda^{-\delta_1}t, y=\lambda^{-1}x$ and define $\widetilde{b}(\tau,y)=b(\lambda^{\delta_1}\tau,\lambda y)=b(t,x)$. One may show that $\widetilde{b}(\tau,y)$ also satisfies the transport-diffusion equation with the scaled divergence-free vector field $\widetilde{u}$

$$\partial_\tau\widetilde{b}+(\widetilde{u}\cdot\nabla)\widetilde{b}+\Lambda^{\delta_1}\widetilde{b}=\Lambda^{\delta_2}\widetilde{f},$$

where $\widetilde{f}(\tau,y)=\lambda^{\delta_1-\delta_2}f(\lambda^{\delta_1}\tau,\lambda y)$. Now if we take $\lambda=T^{\frac{1}{\delta_1}}$, then $(\tau,y)\in[0,T]\times\mathbb{R}^2$. Using (3.13), we have for all $T\ge 1$

$$\|b(t)\|_{L^\infty_T L^\infty}=\|\widetilde{b}(\tau)\|_{L^\infty_1 L^\infty}\le C\|\widetilde{f}\|^\sigma_{L^p_1 L^q}\le C(1+T^{\frac{\sigma}{\delta_1}(\delta_1-\delta_2-\frac{\delta_1}{p}-\frac{2}{q})})\|f\|^\sigma_{L^p_T L^q}.$$

This concludes the proof of Lemma 3.4. □

With the preparations of the above two lemmas, we are ready to prove the global bound for the vorticity.

Proposition 3.1. *Assume (u_0,v_0,θ_0) satisfies the conditions stated in Theorem 1.1. If $\beta\ge\frac{3-\alpha}{2}$ with $0<\alpha\le\frac{1}{2}$, then any corresponding smooth solution (u,v,θ) of (1.1) obeys*

$$\|\omega\|_{L^\infty_T L^\infty}\le C_0(T),\tag{3.18}$$

where C_0 is a constant depending only on T and the initial data.

Proof. Recall the vorticity equation

$$\partial_t\omega+(u\cdot\nabla)\omega+\Lambda^{2\alpha}\omega+(v\cdot\nabla)j+2hj-(v_1\partial_2 h-v_2\partial_1 h)=0,\quad \omega(0)=\omega_0.$$

Using

$$H=h-\Lambda^{2-2\beta}\theta,$$

we have

$$\partial_t\omega+(u\cdot\nabla)\omega+\Lambda^{2\alpha}\omega=\sum_{j=1}^{5}f_j,$$

where

$$f_1=-(v\cdot\nabla)j,\qquad f_2=-2(H+\Lambda^{2-2\beta}\theta)j,\qquad f_3=-(v_1\partial_2 H-v_2\partial_1 H),$$

$$f_4=[\partial_1\Lambda^{2-2\beta},v_2]\theta-[\partial_2\Lambda^{2-2\beta},v_1]\theta,\qquad f_5=\partial_2\Lambda^{2-2\beta}(v_1\theta)-\partial_1\Lambda^{2-2\beta}(v_2\theta).$$

We write ω as

$$\omega=\widetilde{\omega}+\sum_{j=1}^{5}\omega_j,$$

where $\widetilde{\omega}$ and ω_j satisfy the equations

$$\partial_t\widetilde{\omega}+(u\cdot\nabla)\widetilde{\omega}+\Lambda^{2\alpha}\widetilde{\omega}=0,\quad \widetilde{\omega}(0)=\omega_0,$$
$$\partial_t\omega_j+(u\cdot\nabla)\omega_j+\Lambda^{2\alpha}\omega_j=f_j,\quad \omega_j(0)=0.$$

By the maximum principle (see [13, Corollary 2.6]),

$$\|\widetilde{\omega}(t)\|_{L^\infty}\leq\|\omega_0\|_{L^\infty}.$$

By (2.5) and (3.7), we have

$$\|f_1\|_{L_t^pL^q}\leq C_0(t)+C(t)\|\omega\|_{L_t^pL^q},\quad \forall 2\leq p,q\leq\infty.$$

By Lemma 3.4,

$$\begin{aligned}\|\omega_1(t)\|_{L^\infty}&\leq C_0(t)+C(T)\|f_1\|_{L_T^pL^q}\\&\leq C_0(T)+C(T)\|\omega\|_{L_T^pL^q}\\&\leq C_0(T)+C(T)\|\omega\|_{L_T^\infty L^\infty}^{1-\frac{2}{q}}\|\omega\|_{L_T^{\frac{2p}{q}}L^2}^{\frac{2}{q}}\\&\leq C_0(T)+\frac{1}{2}\|\omega\|_{L_T^\infty L^\infty}.\end{aligned}$$

The global bounds in Propositions 2.1 and 2.2 yield

$$\|f_2\|_{L_T^pL^q}<\infty,\quad \|f_3\|_{L_T^pL^q}<\infty,\quad \forall 2\leq p,q\leq\infty.$$

Applying Lemma 3.4 yields

$$\|\omega_2(t)\|_{L^\infty}\leq C_0(t),\qquad \|\omega_3(t)\|_{L^\infty}\leq C_0(t).$$

Thanks to [37, Lemma 2.1],

$$\begin{aligned}\|f_4\|_{L^p}&\leq C\|[\nabla\Lambda^{2-2\beta},v]\theta\|_{L^p}\\&\leq C\|[\nabla\Lambda^{2-2\beta},v]\theta\|_{B_{p,\infty}^{\nu}}\\&\leq C(\|\nabla v\|_{L^p}\|\theta\|_{B_{\infty,\infty}^{\nu+2-2\beta}}+\|v\|_{L^2}\|\theta\|_{L^2})\\&\leq C,\end{aligned}$$

where $0<\nu<2\beta-2$. Consequently, we obtain

$$\|f_4\|_{L_T^pL^q}<\infty,\quad \forall 2\leq p,q\leq\infty,$$

which along with Lemma 3.4 yields

$$\|\omega_4(t)\|_{L^\infty}\leq C_0(t).$$

We rewrite f_5 as

$$f_5=\Lambda^{3-2\beta}\mathcal{R}_2(v_1\theta)-\Lambda^{3-2\beta}\mathcal{R}_1(v_2\theta):=\Lambda^{3-2\beta}\widetilde{f}_5.$$

It is obvious that

$$\|\widetilde{f}_5\|_{L_T^2L^2}<\infty,\qquad \|\widetilde{f}_5\|_{L_T^pL^q}<\infty,\quad \forall 2<p,q<\infty.$$

Lemma 3.4 leads to

$$\|\omega_5(t)\|_{L^\infty} \le C_0(T).$$

Summing up all the estimates, we have

$$\begin{aligned}\|\omega\|_{L^\infty_T L^\infty} \le& \|\omega_0\|_{L^\infty} + \sum_{j=1}^{5} \|\omega_j\|_{L^\infty_T L^\infty}\\ \le& C_0(T) + \frac{1}{2}\|\omega\|_{L^\infty_T L^\infty},\end{aligned}$$

or

$$\|\omega\|_{L^\infty_T L^\infty} \le C_0(T).$$

This ends the proof of Proposition 3.1. ∎

With the global bound in Proposition 3.1, we are now ready to prove Theorem 1.1 for the case $\alpha \le \frac{1}{2}$.

Proof. **(Proof of Theorem 1.1 for the case $\alpha \le \frac{1}{2}$.)** First, it follows from the θ-equation that

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta(t)\|_{L^\infty} \le& C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty} + C\|\nabla h\|_{L^\infty}\\ \le& C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty} + C\|\nabla H\|_{L^\infty} + C\|\nabla\Lambda^{2-2\beta}\theta\|_{L^\infty}\\ \le& C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty} + C + C\|\nabla\theta\|_{L^\infty}\\ \le& C(1+\|\nabla u\|_{L^\infty})(1+\|\nabla\theta\|_{L^\infty}).\end{aligned} \tag{3.19}$$

We now recall the following logarithmic-type Sobolev inequality

$$\|\nabla f\|_{L^\infty} \le C\Big(1 + \|f\|_{L^2} + \|\nabla\times f\|_{L^\infty}\ln(e+\|\Lambda^s f\|_{L^2})\Big) \tag{3.20}$$

where $\nabla\cdot f = 0$ and $s > 2$. By applying (3.20) to (3.19), we have

$$\begin{aligned}&1+\|\nabla\theta(t)\|_{L^\infty}\\ &\le (1+\|\nabla\theta(T_0)\|_{L^\infty})\exp\Big[C\int_{T_0}^{t}(1+\|\nabla u(\tau)\|_{L^\infty})\,d\tau\Big]\\ &\le C\exp\Big[C\int_{T_0}^{t}\Big(1+\|u(\tau)\|_{L^2}+\|\omega(\tau)\|_{L^\infty}\ln(e+\|\Lambda^s u(\tau)\|_{L^2}\Big)\,d\tau\Big]\\ &\le C\exp\Big[\int_{T_0}^{t}(1+\|u\|_{L^2})\,ds\Big]\exp\Big[C_0\Big(\int_{T_0}^{t}\|\omega(\tau)\|_{L^\infty}\,d\tau\Big)\ln\big(e+X(t)\big)\Big]\\ &\le C\exp\Big[C_0\Big(\int_{T_0}^{t}\|\omega(\tau)\|_{L^\infty}\,d\tau\Big)\ln\big(e+X(t)\big)\Big],\quad \forall T_0\le t<T,\end{aligned} \tag{3.21}$$

where $C_0 > 0$ is an absolute constant whose value is independent of T or T_0. According to (3.18), one concludes that for any small constant $\epsilon > 0$ to be fixed hereafter, there exists $T_0 = T_0(\epsilon) \in (0,T)$ such that

$$C_0\int_{T_0}^{T}\|\omega(\tau)\|_{L^\infty}\,d\tau \le \epsilon.$$

For any $T_0 \le t \le T$, we denote

$$X(t) := \max_{\tau\in[T_0,t]} (\|\Lambda^s u(\tau)\|_{L^2}^2 + \|\Lambda^s v(\tau)\|_{L^2}^2 + \|\Lambda^s \theta(\tau)\|_{L^2}^2),$$

where $s>2$. Clearly, $X(t)$ is a monotonically nondecreasing function. Our next target is to show

$$\lim_{t\to T^-} X(t) \le C(u_0, v_0, \theta_0, T, X(T_0)) < \infty.$$

As a result, it follows from (3.21) that

$$\|\nabla\theta(t)\|_{L^\infty} \le C(e+X(t))^\epsilon \quad \text{for any } T_0 \le t < T. \tag{3.22}$$

In order to form a closed inequality in (3.22), we estimate $\dot{H}^s$-bound of (u,v,θ). To this end, applying Λ^s with $s>2$ to (1.1) and taking the L^2 inner product with $(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2) + \|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+\beta}v\|_{L^2}^2\\
&= -\int_{\mathbb{R}^2} \Big(\Lambda^s\nabla\cdot(v\otimes v)\cdot\Lambda^s u + \Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\Big)\,dx - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,dx\\
&\quad -\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&:= H_1 + H_2 + H_3 + H_4.
\end{aligned} \tag{3.23}$$

The terms on the right can be bounded as follows.

$$\begin{aligned}
H_1 &\le C\|\Lambda^s\nabla\cdot(v\otimes v)\|_{L^2}\|\Lambda^s u\|_{L^2} + C\|\Lambda^{s-1}(v\cdot\nabla u)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C\|v\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}\|\Lambda^s u\|_{L^2} + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{s-1}v\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C\|v\|_{L^\infty}(\|v\|_{L^2} + \|\Lambda^{s+\beta}v\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}(\|v\|_{L^2} + \|\Lambda^s v\|_{L^2})\|\Lambda^{s+1}v\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C(\|v\|_{L^2}^2 + \|v\|_{L^\infty}^2)\|\Lambda^s u\|_{L^2}^2 + C\|\Lambda^\alpha\omega\|_{L^2}^2(\|v\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2)\\
&\le \frac{1}{8}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C(1+\|\Lambda^\alpha\omega\|_{L^2}^2)(1+\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
H_2 &\le C\|[\Lambda^s, u\cdot\nabla]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|u\|_{L^2}^{\frac{\alpha}{s+\alpha}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{s}{s+\alpha}})\|\Lambda^s\theta\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{2(s+\alpha)}{s+2\alpha}}\|\Lambda^s\theta\|_{L^2}^{\frac{2(s+\alpha)}{s+2\alpha}},
\end{aligned}$$

$$\begin{aligned}
H_3 &\le C\int_{\mathbb{R}^2}|[\Lambda^s, u\cdot\nabla]v\cdot\Lambda^s v|\,dx\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s v\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2)\\
&\le C(1+\|\nabla u\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2)
\end{aligned}$$

and

$$
\begin{aligned}
H_4 \leq & C \int_{\mathbb{R}^2} |[\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u|\,dx \\
\leq & C\|\Lambda^s u\|_{L^2}\|[\Lambda^s, u\cdot\nabla]u\|_{L^2} \\
\leq & C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}
$$

Substituting all the preceding estimates into (3.23) yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+\beta}v\|_{L^2}^2 \\
&\leq C(1+\|\Lambda^\alpha\omega(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^\infty})(1+\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2) \\
&\quad +C\|\nabla\theta\|_{L^\infty}^{\frac{2(s+\alpha)}{s+2\alpha}}\|\Lambda^s\theta\|_{L^2}^{\frac{2(s+\alpha)}{s+2\alpha}} \\
&\leq C(1+\|\Lambda^\alpha\omega(t)\|_{L^2}^2+\|\omega(t)\|_{L^\infty})\ln(e+\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2) \\
&\quad \times(1+\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2)+C\|\nabla\theta\|_{L^\infty}^{\frac{2(s+\alpha)}{s+2\alpha}}\|\Lambda^s\theta\|_{L^2}^{\frac{2(s+\alpha)}{s+2\alpha}}.
\end{aligned} \tag{3.24}
$$

Now we integrate (3.24) over the interval (T_0,t) and use the monotonicity of $X(t)$ as well as (3.22) to conclude

$$
\begin{aligned}
&e+X(t)-\big(e+X(T_0)\big)+\int_{T_0}^t(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2)\,d\tau \\
\leq & C\int_{T_0}^t(1+\|\Lambda^\alpha\omega(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^\infty}) \\
&\quad\times\ln(e+\|\Lambda^s u(\tau)\|_{L^2}^2+\|\Lambda^s v(\tau)\|_{L^2}^2+\|\Lambda^s\theta(\tau)\|_{L^2}^2) \\
&\quad\times(1+\|\Lambda^s u(\tau)\|_{L^2}^2+\|\Lambda^s v(\tau)\|_{L^2}^2+\|\Lambda^s\theta(\tau)\|_{L^2}^2)\,d\tau \\
&\quad +C\int_{T_0}^t\|\nabla\theta(\tau)\|_{L^\infty}^{\frac{2(s+\alpha)}{s+2\alpha}}\|\Lambda^s\theta(\tau)\|_{L^2}^{\frac{2(s+\alpha)}{s+2\alpha}}\,d\tau \\
\leq & C\int_{T_0}^t(1+\|\Lambda^\alpha\omega(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^\infty})\ln\big(e+X(\tau)\big)\big(e+X(\tau)\big)\,d\tau \\
&\quad +C\int_{T_0}^t\big(e+X(\tau)\big)^{\frac{2\epsilon(s+\alpha)}{s+2\alpha}}\big(e+X(\tau)\big)^{\frac{s+\alpha}{s+2\alpha}}\,d\tau.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&e+X(t)-\big(e+X(T_0)\big)+\int_{T_0}^t(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2)\,d\tau \\
\leq & C\int_{T_0}^t(1+\|\Lambda^\alpha\omega(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^\infty})\ln\big(e+X(\tau)\big)\big(e+X(\tau)\big)\,d\tau \\
&\quad +C\int_{T_0}^t\big(e+X(\tau)\big)^{\frac{(2\epsilon+1)(s+\alpha)}{s+2\alpha}}\,d\tau.
\end{aligned}
$$

By taking $0<\epsilon\leq\frac{\alpha}{2(s+\alpha)}$, it yields

$$
e+X(t)+\int_{T_0}^t(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2)\,d\tau
$$

$$\leq e+X(T_0)+C\int_{T_0}^{t}(1+\|\Lambda^\alpha\omega(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^\infty})\ln\big(e+X(\tau)\big)\big(e+X(\tau)\big)\,d\tau.$$

Applying the classical differential-type Grönwall inequality leads to the fact that $X(t)$ remains bounded for any $t\in[0,T]$. Consequently,

$$\sup_{0\leq t\leq T}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)$$
$$+\int_0^T(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+\beta}v(\tau)\|_{L^2}^2)\,d\tau<\infty,$$

This completes the proof of Theorem 1.1 for the case when $\alpha\leq\frac{1}{2}$. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her thoughtful comments and profitable remarks on the initial version of this manuscript. Dong was supported by the National Natural Science Foundation of China (NNSFC) (Grant No. 11871346), the Natural Science Foundation of Guangdong Province (No. 2018A030313024), Natural Science Foundation of Shenzhen City (No. JCYJ20180305125554234) and Research Fund of Shenzhen University (No. 2017056). Wu was partially supported by the National Science Foundation (DMS 1614246) and by the AT&T Foundation at Oklahoma State University. Ye was supported by NNSFC (No. 11701232) and the Natural Science Foundation of Jiangsu Province (No. BK20170224).

Appendix. This appendix recalls the Littlewood-Paley decomposition, Besov spaces and related tools. Materials presented here can be found in many books (see, e.g., [2,3,31,33]). Let (χ,φ) be a pair of smooth functions with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B}:=\{\xi\in\mathbb{R}^n,|\xi|\leq\frac{4}{3}\}$, $\varphi\in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^n,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and (χ,φ) satisfies

$$\chi(\xi)+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\xi)=1,\quad\forall\xi\in\mathbb{R}^n;\qquad\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\ \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

For every $u\in S'$ (tempered distributions), the non-homogeneous Fourier restriction operators are defined as follows,

$$\Delta_j u=0,\ j\leq-2;\quad\Delta_{-1}u=\chi(D)u;\quad\Delta_j u=\varphi(2^{-j}D)u,\ \forall j\in\mathbb{N}.$$

We shall also denote

$$S_j u:=\sum_{-1\leq k\leq j-1}\Delta_k u,\qquad\widetilde{\Delta}_j u:=\Delta_{j-1}u+\Delta_j u+\Delta_{j+1}u.$$

Several simple facts concerning these operators Δ_j have been used frequently. For $j,k,l=-1,0,\cdots$,

$$\Delta_j\Delta_l u\equiv0,\quad|j-l|\geq2\quad\textit{and}\quad\Delta_k(S_l u\Delta_l v)\equiv0\quad|k-l|\geq5.$$

for any u and v. Moreover, it is easy to check that

$$\operatorname{supp}\mathcal{F}\big(S_{j-1}u\Delta_j v\big)=\Big\{\xi\,|\,\frac{1}{12}2^j\leq|\xi|\leq\frac{10}{3}2^j\Big\},$$

$$\operatorname{supp}\mathcal{F}\big(\widetilde{\Delta}_j u\Delta_j v\big)\subset\Big\{\xi\,|\,|\xi|\leq8\times2^j\Big\},$$

where $\mathcal{F}$ denotes the Fourier transform.

We now recall the definition of the inhomogeneous Besov spaces in terms of the Littlewood-Paley decomposition.

DEFINITION 4.1. *Let $s\in\mathbb{R},(p,r)\in[1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is given by*

$$B^s_{p,r}=\{f\in S'(\mathbb{R}^n),\|f\|_{B^s_{p,r}}<\infty\},$$

where

$$\|f\|_{B^s_{p,r}}=\begin{cases}\Big(\sum\limits_{j\geq-1}2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & if r<\infty,\\ \sup\limits_{j\geq-1}2^{js}\|\Delta_j f\|_{L^p}, & if\ r=\infty.\end{cases}$$

Many frequently used function spaces are special cases of Besov spaces. For $s\in\mathbb{R},(p,r)\in[1,+\infty]^2$, we have the following fact

$$\|f\|_{B^s_{2,2}}\approx\|f\|_{H^s}.$$

For any $s\in\mathbb{R}$ and $1<q<\infty$,

$$B^s_{q,\min\{q,2\}}\hookrightarrow W^{s,q}\hookrightarrow B^s_{q,\max\{q,2\}}.$$

Bernstein inequalities are among the most useful tools in dealing with the Fourier restriction operators, which allow us to trade integrability for derivatives.

LEMMA 4.1 (see [2]). *Let $k\in\mathbb{N}\cup\{0\},1\leq a\leq b\leq\infty$. Assume $k=|\alpha|$, then there exist positive constants C_1 and C_2 such that*

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^n:|\xi|\lesssim 2^j\}\Rightarrow\|\partial^\alpha f\|_{L^b}\leq C_1 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a};$$

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^n:|\xi|\approx 2^j\}\Rightarrow C_1 2^{jk}\|f\|_{L^b}\leq\|\partial^\alpha f\|_{L^b}\leq C_2 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}.$$

Here $A\lesssim B$ denotes $A\leq CB$ for some positive constant C.

Finally, we list the following simple lemma concerning an iterative sequence, which has been previously used.

LEMMA 4.2. *Let $a>0$, $b>1$ and $\nu>0$. Assume the nonnegative sequence A_k $(k\in\mathbb{N})$ satisfies the following recurrence relation*

$$A_{k+1}\leq ab^k A_k^{1+\nu}. \tag{4.1}$$

If A_0 satisfies

$$A_0\leq a^{-\frac{1}{\nu}}b^{-\frac{1}{\nu^2}},$$

then

$$\lim_{k\to\infty}A_k=0.$$

Proof. Although the result is well-known to us, we are not able to locate the proof in the literatures. For the sake of completeness, we present an elementary proof. We may assume $A_k>0$ for any given $k\in\mathbb{N}$, otherwise if $A_{k_0}=0$, then by (4.1), we have $A_k=0$ for all $k\geq k_0$. Then, the desired result follows immediately. Therefore, it suffices to consider the case $A_k>0$ for any given $k\in\mathbb{N}$. In this case, it follows from (4.1) that

$$\begin{aligned} \ln A_k &\leq \ln a+(k-1)\ln b+(1+\nu)\ln A_{k-1},\\ \ln A_{k-1} &\leq \ln a+(k-2)\ln b+(1+\nu)\ln A_{k-2},\\ \ln A_{k-2} &\leq \ln a+(k-3)\ln b+(1+\nu)\ln A_{k-3},\\ &\cdots\\ \ln A_1 &\leq \ln a+(k-k)\ln b+(1+\nu)\ln A_0. \end{aligned}$$

This implies

$$\begin{aligned} \ln A_k &\leq \ln a+(k-1)\ln b+(1+\nu)\ln A_{k-1},\\ (1+\nu)\ln A_{k-1} &\leq (1+\nu)\ln a+(k-2)(1+\nu)\ln b+(1+\nu)^2\ln A_{k-2},\\ (1+\nu)^2\ln A_{k-2} &\leq (1+\nu)^2\ln a+(k-3)(1+\nu)^2\ln b+(1+\nu)^3\ln A_{k-3},\\ &\cdots\\ (1+\nu)^{k-1}\ln A_1 &\leq (1+\nu)^{k-1}\ln a+(k-k)(1+\nu)^{k-1}\ln b+(1+\nu)^k\ln A_0. \end{aligned}$$

Iterating the above inequalities yields

$$\ln A_k\leq \sum_{l=0}^{k-1}(1+\nu)^l\ln a+\sum_{l=0}^{k-1}l(1+\nu)^{k-1-l}\ln b+(1+\nu)^k\ln A_0. \tag{4.2}$$

Invoking the sum formula for geometric sequences, we deduce by direct computations

$$\sum_{l=0}^{k-1}(1+\nu)^l\ln a=\frac{\ln a}{\nu}(1+\nu)^k-\frac{\ln a}{\nu}, \tag{4.3}$$

$$\begin{aligned} \sum_{l=0}^{k-1}l(1+\nu)^{k-1-l}\ln b &= (1+\nu)^{k-1}\sum_{l=0}^{k-1}l(1+\nu)^{-l}\ln b\\ &= (1+\nu)^{k-1}\sum_{l=1}^{k-1}l(1+\nu)^{-l}\ln b\\ &= (1+\nu)^{k-1}\left(\frac{(1+\nu)\left((1+\nu)^{k-1}-1\right)}{\nu^2(1+\nu)^{k-1}}-\frac{k-1}{\nu(1+\nu)^{k-1}}\right)\ln b\\ &= \frac{\ln b}{\nu^2}(1+\nu)^k-\frac{(1+\nu)\ln b}{\nu^2}-\frac{(k-1)\ln b}{\nu}, \end{aligned} \tag{4.4}$$

where we have used the following equation

$$\sum_{l=1}^{k-1}l(1+\nu)^{-l}=\frac{(1+\nu)\left((1+\nu)^{k-1}-1\right)}{\nu^2(1+\nu)^{k-1}}-\frac{k-1}{\nu(1+\nu)^{k-1}}. \tag{4.5}$$

The proof of (4.5) is provided at the end. Putting (4.3) and (4.4) into (4.2), we have

$$\ln A_k\leq\left(\ln A_0+\frac{\ln a}{\nu}+\frac{\ln b}{\nu^2}\right)(1+\nu)^k-\frac{\nu\ln a+(1+\nu)\ln b+\nu(k-1)\ln b}{\nu^2},$$

which gives

$$A_k \le \exp\left\{\left(\ln A_0 + \frac{\ln a}{\nu} + \frac{\ln b}{\nu^2}\right)(1+\nu)^k\right\} \times \exp\left\{-\frac{\nu\ln a + (1+\nu)\ln b + \nu(k-1)\ln b}{\nu^2}\right\}. \tag{4.6}$$

Due to $A_0 \le a^{-\frac{1}{\nu}} b^{-\frac{1}{\nu^2}}$, we obtain

$$\ln A_0 + \frac{\ln a}{\nu} + \frac{\ln b}{\nu^2} \le 0.$$

We thus get from (4.6) that

$$0 \le A_k \le \exp\left\{-\frac{\nu\ln a + (1+\nu)\ln b + \nu(k-1)\ln b}{\nu^2}\right\}.$$

According to $\nu > 0$ and $b > 1$, one gets

$$0 \le \lim_{k\to\infty} A_k \le \lim_{k\to\infty} \exp\left\{-\frac{\nu\ln a + (1+\nu)\ln b + \nu(k-1)\ln b}{\nu^2}\right\} = 0.$$

It thus gives the desired estimate, namely,

$$\lim_{k\to\infty} A_k = 0.$$

Finally, we give a direct proof of (4.5). To do so, we denote

$$S_k := \sum_{l=1}^{k-1} l(1+\nu)^{-l} \equiv \sum_{l=1}^{k-1} \frac{l}{(1+\nu)^l}.$$

Then, we conclude that

$$\begin{aligned} S_k - \frac{1}{1+\nu} S_k &= \sum_{l=1}^{k-1} \frac{l}{(1+\nu)^l} - \sum_{l'=1}^{k-1} \frac{l'}{(1+\nu)^{l'+1}} \\ &= \sum_{l=1}^{k-1} \frac{l}{(1+\nu)^l} - \sum_{l=2}^{k} \frac{l-1}{(1+\nu)^l} \\ &= \sum_{l=1}^{k-1} \frac{l}{(1+\nu)^l} - \sum_{l=1}^{k-1} \frac{l-1}{(1+\nu)^l} - \frac{k-1}{(1+\nu)^k} \\ &= \sum_{l=1}^{k-1} \frac{1}{(1+\nu)^l} - \frac{k-1}{(1+\nu)^k} \\ &= \frac{\frac{1}{1+\nu}\left[1 - \frac{1}{(1+\nu)^{k-1}}\right]}{1 - \frac{1}{1+\nu}} - \frac{k-1}{(1+\nu)^k}, \end{aligned} \tag{4.7}$$

where in the last line we have used the sum formula for geometric sequences. We get from (4.7) that

$$\frac{\nu}{1+\nu} S_k = \frac{(1+\nu)^{k-1} - 1}{\nu(1+\nu)^{k-1}} - \frac{k-1}{(1+\nu)^k},$$

which leads to the desired (4.5), namely,

$$S_k = \frac{(1+\nu)\big((1+\nu)^{k-1}-1\big)}{\nu^2(1+\nu)^{k-1}} - \frac{k-1}{\nu(1+\nu)^{k-1}}.$$

We point out that we can prove (4.5) via another approach. More precisely, we define the power series

$$S(z) := \sum_{l=1}^{k-1} l z^l.$$

Then we have

$$S(z) = z\sum_{l=1}^{k-1}(z^l)' = z\left(\sum_{l=1}^{k-1} z^l\right)' = z\left(\frac{z-z^k}{1-z}\right)' = z\frac{1-kz^{k-1}+kz^k-z^k}{(1-z)^2}.$$

Taking $z = \frac{1}{1+\nu}$, one obtains by direct computations

$$\begin{aligned} S_k &= S\left(\frac{1}{1+\nu}\right) \\ &= \frac{1}{1+\nu}\frac{1-\frac{k}{(1+\nu)^{k-1}}+\frac{k}{(1+\nu)^k}-\frac{1}{(1+\nu)^k}}{\frac{\nu^2}{(1-\nu)^2}} \\ &= \frac{(1+\nu)\big((1+\nu)^{k-1}-1\big)}{\nu^2(1+\nu)^{k-1}} - \frac{k-1}{\nu(1+\nu)^{k-1}}. \end{aligned}$$

This ends the proof of the lemma. □

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