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# Influence of a background magnetic field on a 2D magnetohydrodynamic flow

**Wen Feng**[1], **Farzana Hafeez**[2,*] **and Jiahong Wu**[2]

[1] Department of Mathematics, 5795 Lewiston Rd, Niagara University, Lewiston, NY 14109, United States of America

[2] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States of America

E-mail: wfeng@niagara.edu, farzana.hafeez@okstate.edu and jiahong.wu@okstate.edu

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**Abstract**

Physical experiments and numerical simulations have demonstrated that background magnetic fields stabilize electrically conducting fluids. This paper establishes these observations as mathematically rigorous facts on a 2D magnetohydrodynamic (MHD) system. This system is anisotropic with the velocity equation involving only the vertical dissipation. Flows governed by the 2D Navier–Stokes equations with only vertical dissipation are not known to be stable. Under the influence of a background magnetic field, the velocity field is shown here to stabilize and decay in time through the coupling and the interaction. Mathematically we reduce the MHD system concerned here to a system of degenerate and damped wave equations and exploit the smoothing and stabilizing effects of the wave structure. We are able to prove that any perturbation near a background magnetic field remains asymptotically stable. In addition, certain explicit large time behavior is also established.

Keywords: magnetohydrodynamic equations, partial dissipation, stability, wave equations

Mathematics Subject Classification numbers: 35A01, 35B35, 35Q35, 76D09.

## 1. Introduction

The stabilization and smoothing effect of a background magnetic field on electrically conducting fluids has been observed in physical experiments and numerical simulations, and demonstrated in theoretical analysis (see, e.g., [1–3, 15, 16]). In addition, the stabilization effect of a strong magnetic field has been employed in the development of magnetic polymers and paints (see, e.g., [23]). One goal of this paper is to understand the mechanism of the stabilization and

*Author to whom any correspondence should be addressed.

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establish the observations as a mathematically rigorous fact on a system modeling the electrically conducting fluids. We consider the following 2D incompressible magnetohydrodynamic (MHD) system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ \ b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u$ denotes the velocity field, $b$ the magnetic field and $P$ the pressure, and $\nu > 0$ and $\eta$ are the viscosity and the damping coefficient, respectively. Here the velocity $u$ obeys a degenerate Navier–Stokes equation with only vertical dissipation $\nu\partial_{22}u$ and with a Lorentz forcing term. The magnetic field $b$ satisfies the induction equation. The extra two terms $\partial_1 b$ and $\partial_1 u$ are created when we write the original magnetic field as the sum of a background magnetic field and a perturbation, namely $(1,0) + b$. The system focused here governs the motion of the perturbation near a background magnetic field.

The justification for including only one-directional dissipation in (1.1) is two fold. The first is that the Laplacian dissipation in some partial differential equation systems modeling fluids reduces to the degenerate case in certain physical regimes and after suitable scaling. One prominent example is Prandtl's boundary layer equation. The second justification is to demonstrate the smoothing and stabilization effect of the magnetic field. Mathematically only one directional dissipation in the Navier–Stokes equations makes the stability problem much more difficult. Without the coupling with the magnetic field, the velocity of the Navier–Stokes equation with only vertical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u, \quad x \in \mathbb{R}^2, \quad t > 0, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

is not known to be stable near the trivial solution. Some physically relevant infinite energy solutions of (1.2) can grow rather rapidly [8]. One expects the solution of (1.2) in the Sobolev space setting to be unstable, but a proof is currently lacking. When there is no dissipation at all, the 2D Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, \quad x \in \mathbb{R}^2, \quad t > 0, \\ \nabla \cdot u = 0. \end{cases}$$

can generate solutions that grow exponentially or even double exponentially in time (see, e.g., [10, 22, 50]). In contrast, solutions to the 2D Navier–Stokes equations with full dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u, \quad x \in \mathbb{R}^2, \quad t > 0, \\ \nabla \cdot u = 0. \end{cases}$$

in the Sobolev spaces are always asymptotically stable with explicit decay rates (see [30, 32]).

Since the partially dissipated Navier–Stokes equation itself alone is not known to be stable, we must seek the stabilizing effect from the magnetic field in order to achieve any stability. The two terms in (1.1) related to the magnetic field, namely $b \cdot \nabla b$ and $\partial_1 b$, do not appear to be helpful at first glance, but the smoothing and damping effect would emerge when we convert the MHD system in (1.1) into an equivalent form. To do so, we first apply the Helmholtz–Leray projection operator

$$\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$$

to eliminate the pressure term to obtain

$$\partial_t u = \nu\partial_{22}u + \partial_1 b + N_1, \qquad N_1 = \mathbb{P}(-u\cdot\nabla u + b\cdot\nabla b). \tag{1.3}$$

By separating the linear terms from the nonlinear ones in (1.1), the equation of $b$ can be written as

$$\partial_t b = -\eta b + \partial_1 u + N_2, \qquad N_2 = -u\cdot\nabla b + b\cdot\nabla u. \tag{1.4}$$

Differentiating (1.3) and (1.4) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt}u - (\nu\partial_{22} - \eta)\partial_t u - (\partial_{11}u + \eta\nu\partial_{22}u) = N_3, \\ \partial_{tt}b - (\nu\partial_{22} - \eta)\partial_t b - (\partial_{11}b + \eta\nu\partial_{22}b) = N_4, \end{cases} \tag{1.5}$$

where $N_3$ and $N_4$ are given by

$$N_3 = (\partial_t + \eta)N_1 + \partial_1 N_2, \qquad N_4 = (\partial_t - \nu\partial_{22})N_2 + \partial_1 N_1.$$

Surprisingly, both $u$ and $b$ are found to satisfy nonhomogeneous wave equations with exactly the same linear parts. Clearly, (1.5) exhibits much more regularization than its original counterpart in (1.1). Similarly, the equations of the vorticity $\omega = \nabla\times u$ and the current density $j = \nabla\times b$ given by

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j, \\ \partial_t j + u\cdot\nabla j + \eta j = b\cdot\nabla\omega + Q + \partial_1\omega, \end{cases} \tag{1.6}$$

with

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$$

can also be converted into the following system of wave equations

$$\begin{cases} \partial_{tt}\omega - (\nu\partial_{22} - \eta)\partial_t\omega - (\partial_{11}\omega + \eta\nu\partial_{22}\omega) = N_5, \\ \partial_{tt}j - (\nu\partial_{22} - \eta)\partial_t j - (\partial_{11}j + \eta\nu\partial_{22}j) = N_6, \end{cases} \tag{1.7}$$

where $N_5$ and $N_6$ are given by

$$N_5 = (\partial_t + \eta)(-u\cdot\nabla\omega + b\cdot\nabla j) + \partial_1(b\cdot\nabla\omega - u\cdot\nabla j + Q),$$
$$N_6 = (\partial_t - \nu\partial_{22})(b\cdot\nabla\omega - u\cdot\nabla j + Q) + \partial_1(-u\cdot\nabla\omega + b\cdot\nabla j).$$

Again $\omega$ and $j$ share the same wave structure as that for $u$ and $b$. In particular, (1.5) and (1.7) brings in the much-need horizontal regularization even though it is lacking in the original system (1.1).

Our first effort is devoted to understanding how the wave structure affects the regularity and large-time behavior. For simplicity, we consider the linearized portion of (1.5), namely

$$\begin{cases} \partial_{tt}u - (\nu\partial_{22} - \eta)\partial_t u - (\partial_{11}u + \eta\nu\partial_{22}u) = 0, \\ \partial_{tt}b - (\nu\partial_{22} - \eta)\partial_t b - (\partial_{11}b + \eta\nu\partial_{22}b) = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \tag{1.8}$$

or equivalently, the linearization of the original system

$$\begin{cases} \partial_t u = \nu\partial_{22}u + \partial_1 b, \\ \partial_t b = -\eta b + \partial_1 u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x). \end{cases} \tag{1.9}$$

The goal here is to obtain all possible regularization due to the dissipation and dispersion effects and to provide a sharp large-time decay rate. To give a precise statement of our result, we define a Fourier multiplier operator $\Phi$,

$$\widehat{\Phi f}(\xi) = \widehat{\Psi}(\xi)\widehat{f}(\xi). \tag{1.10}$$

A very important class of $\Phi$ is the fractional Laplacian operator $(-\Delta)^\gamma$ with $\gamma\in\mathbb{R}$, which can be defined in terms of the Fourier transform,

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma}\widehat{f}(\xi).$$

It is clear that the norm in the standard homogeneous Sobolev space $\mathring{H}^s$ with $s\in\mathbb{R}$ is given by

$$\|f\|_{\mathring{H}^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}.$$

For the sake of conciseness, we shall write $\|(f,g)\|_{L^2}^2$ for $\|f\|_{L^2}^2 + \|g\|_{L^2}^2$.

**Theorem 1.1.** *Consider the linearized system* (1.8)*. Let $\Phi$ be a given Fourier multiplier operator. Assume the initial data $(u_0, b_0)$ satisfies*

$$\Phi u_0,\ \Phi b_0,\ \nabla\Phi u_0,\ \nabla\Phi b_0,\ \partial_{22}\Phi u_0 \in L^2, \qquad \nabla\cdot u_0 = \nabla\cdot b_0 = 0.$$

*Let $(u,b)$ be the corresponding solution of* (1.8)*. Then $(u,b)$ obeys the following regularization and decay estimates.*

*(a) $(u,b)$ is uniformly bounded for all time with the following explicit bounds,*

$$\begin{aligned} &\|\partial_t\Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1\Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2\Phi b(t)\|_{L^2}^2 \\ &\quad + \int_0^t \left(4\nu\|\partial_2\partial_\tau\Phi b\|_{L^2}^2 + 3\eta\|\partial_\tau\Phi b\|_{L^2}^2 + \eta\|\partial_1\Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2\Phi b\|_{L^2}^2\right)\mathrm{d}\tau \\ &\leqslant C(\nu,\eta)\left(\|\Phi b_0\|_{L^2}^2 + \|\partial_1\Phi(u_0,b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right). \end{aligned} \tag{1.11}$$

*and*

$$
\begin{aligned}
&\|\partial_t \Phi u(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi u(t)\|_{L^2}^2 + 2\|\partial_1 \Phi u(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi u(t)\|_{L^2}^2 \\
&\quad + \int_0^t \left(4\nu\|\partial_2\partial_\tau \Phi u\|_{L^2}^2 + 3\eta\|\partial_\tau \Phi u\|_{L^2}^2 + \eta\|\partial_1 \Phi u\|_{L^2}^2 + \nu\eta^2\|\partial_2 \Phi u\|_{L^2}^2\right) \mathrm{d}\tau \\
&\quad \leqslant C(\nu,\eta)\left(\|\Phi u_0\|_{L^2}^2 + \|\partial_1 \Phi(u_0,b_0)\|_{L^2}^2 + \|\partial_2 \Phi u_0\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right). \qquad (1.12)
\end{aligned}
$$

*Especially, for any $s \in \mathbb{R}$ and for $\Phi = (-\Delta)^{\frac{s}{2}}$, we have the uniform bounds in Sobolev (or $L^2$) spaces,*

$$
\begin{aligned}
&\|\partial_t b(t)\|_{\mathring{H}^s}^2 + \frac{\eta^2}{4}\|b(t)\|_{\mathring{H}^s}^2 + 2\|\partial_1 b(t)\|_{\mathring{H}^s}^2 + \frac{5}{2}\eta\nu\|\partial_2 b(t)\|_{\mathring{H}^s}^2 \\
&\quad + \int_0^t \left(4\nu\|\partial_2\partial_\tau b\|_{\mathring{H}^s}^2 + 3\eta\|\partial_\tau b\|_{\mathring{H}^s}^2 + \eta\|\partial_1 b\|_{\mathring{H}^s}^2 + \nu\eta^2\|\partial_2 b\|_{\mathring{H}^s}^2\right) \mathrm{d}\tau \\
&\quad \leqslant C(\nu,\eta)\left(\|b_0\|_{\mathring{H}^s}^2 + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s}^2 + \|\partial_2 b_0\|_{\mathring{H}^s}^2\right). \qquad (1.13)
\end{aligned}
$$

*and*

$$
\begin{aligned}
&\|\partial_t u(t)\|_{\mathring{H}^s}^2 + \frac{\eta^2}{4}\|u(t)\|_{\mathring{H}^s}^2 + 2\|\partial_1 u(t)\|_{\mathring{H}^s}^2 + \frac{5}{2}\eta\nu\|\partial_2 u(t)\|_{\mathring{H}^s}^2 \\
&\quad + \int_0^t \left(4\nu\|\partial_2\partial_\tau u\|_{\mathring{H}^s}^2 + 3\eta\|\partial_\tau u\|_{\mathring{H}^s}^2 + \eta\|\partial_1 u\|_{\mathring{H}^s}^2 + \nu\eta^2\|\partial_2 u\|_{\mathring{H}^s}^2\right) \mathrm{d}\tau \\
&\quad \leqslant C(\nu,\eta)\left(\|u_0\|_{\mathring{H}^s}^2 + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s}^2 + \|\partial_2 u_0\|_{\mathring{H}^s}^2 + \|\partial_{22} u_0\|_{\mathring{H}^s}^2\right). \qquad (1.14)
\end{aligned}
$$

*(b)* *$(u,b)$ obeys the following decay properties, as $t \to \infty$,*

$$
\begin{aligned}
&(1+t)\left(\|\partial_t \Phi b(t)\|_{L^2}^2 + \|\Phi b(t)\|_{L^2}^2 + \|\nabla \Phi b(t)\|_{L^2}^2\right) \to 0, \\
&(1+t)\|\nabla \Phi u(t)\|_{L^2}^2 \to 0.
\end{aligned}
$$

*In particular, the following pointwise estimates hold,*

$$
\begin{aligned}
&\|\partial_t \Phi b(t)\|_{L^2} + \|\Phi b(t)\|_{L^2} + \|\nabla \Phi b(t)\|_{L^2} \\
&\quad \leqslant C(\nu,\eta)\left(\|\Phi b_0\|_{L^2} + \|\partial_1 \Phi(u_0,b_0)\|_{L^2} + \|\partial_2 \Phi b_0\|_{L^2}\right)(1+t)^{-\frac{1}{2}}. \qquad (1.15)
\end{aligned}
$$

*and*

$$
\|\nabla \Phi u(t)\|_{L^2} \leqslant C(\nu,\eta)\left(\|\Phi(u_0,b_0)\|_{L^2} + \|\Phi(\nabla u_0, \nabla b_0)\|_{L^2} + \|\partial_{22}\Phi u_0\|_{L^2}\right)(1+t)^{-\frac{1}{2}}. \qquad (1.16)
$$

*When $\Phi = (-\Delta)^{\frac{s}{2}}$, we obtain, as $t \to \infty$,*

$$
\begin{aligned}
&(1+t)\left(\|\partial_t b(t)\|_{\mathring{H}^s}^2 + \|b(t)\|_{\mathring{H}^s}^2 + \|\nabla b(t)\|_{\mathring{H}^s}^2\right) \to 0, \\
&(1+t)\|\nabla u(t)\|_{\mathring{H}^s}^2 \to 0,
\end{aligned}
$$

*which especially imply*

$$\|\partial_t b(t)\|_{\mathring{H}^s} + \|b(t)\|_{\mathring{H}^s} + \|\nabla b(t)\|_{\mathring{H}^s}$$
$$\leqslant C(\nu,\eta)\left(\|b_0\|_{\mathring{H}^s} + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s} + \|\partial_2 b_0\|_{\mathring{H}^s}\right)(1+t)^{-\frac{1}{2}}.$$

*and*

$$\|\nabla u(t)\|_{\mathring{H}^s} \leqslant C(\nu,\eta)\left(\|(u_0,b_0)\|_{\mathring{H}^s} + \|(\nabla u_0,\nabla b_0)\|_{\mathring{H}^s} + \|\partial_{22}u_0\|_{\mathring{H}^s}\right)(1+t)^{-\frac{1}{2}}.$$

We notice from the statement of theorem 1.1 that $u$ and $b$ obey slightly different regularization upper bounds and exhibits slightly different large-time behavior. When $(u_0, b_0)$, $(\nabla u_0, \nabla b_0)$ and $\partial_{22}u_0$ are all in the homogeneous Sobolev space $\mathring{H}^s$ for a real number $s$, then $b$, $\nabla b$ and $\partial_t b$ are all bounded uniformly in $\mathring{H}^s$ and their $\mathring{H}^s$-norms are all square time integrable. The $\mathring{H}^s$-norms of $b$, $\nabla b$ and $\partial_t b$ all decay faster than the rate $(1+t)^{-\frac{1}{2}}$. However, the $\mathring{H}^s$-norm of $u$ itself is not known to be square time integrable and we do not have a decay rate for it. Another remark is that, if the initial data is more regular, we can establish higher time regularity estimates and decay bounds for $\|\partial_t \nabla u(t)\|_{\mathring{H}^s}$.

Next we explore the large-time behavior of the frequency piece of the solution $(u, b)$ to (1.8) that is supported away from the origin. We take advantage of the wave structure in (1.8) to derive energy inequalities that imply an exponential decay rate for the frequency piece away from the origin. These inequalities also allow us to conclude that if the Fourier transform of the initial data $(u_0, b_0)$ is supported away from the origin, then the solution $(u, b)$ decay exponentially in time. To state our result precisely, we define a Fourier cutoff function. Let $a > 0$ be arbitrarily fixed and define

$$\widehat{\phi}(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \geqslant a, \\ 0 & \text{if} \quad |\xi| < a. \end{cases} \tag{1.17}$$

**Theorem 1.2.** *Consider the linearized system in (1.8). Assume that the initial data $(u_0, b_0)$ satisfies*

$$u_0,\ b_0,\ \nabla u_0,\ \nabla b_0,\ \partial_{22}u_0 \in L^2, \qquad \nabla\cdot u_0 = \nabla\cdot b_0 = 0.$$

*Then $(u, b)$ decays exponentially in time in the following sense*

$$\|\partial_t(\phi*u)(t)\|_{L^2} + \|(\phi*u)(t)\|_{H^1} \leqslant C(\nu,\eta)\left(\|((\phi*u_0),(\phi*b_0))\|_{H^1}\right.$$
$$\left. + \|\partial_{22}(\phi*u_0)\|_{L^2}\right)e^{-c_0 t}, \tag{1.18}$$

$$\|\partial_t(\phi*b)(t)\|_{L^2} + \|(\phi*b)(t)\|_{H^1} \leqslant C(\nu,\eta)\|((\phi*u_0),(\phi*b_0))\|_{H^1}e^{-c_0 t}, \tag{1.19}$$

*where $H^1$ denotes the inhomogeneous $H^1$-norm and $c_0 > 0$ is a constant.*

Theorems 1.1 and 1.2 tell us about how much regularity we can extract from the wave structure and how fast the solution decays. To deal with the full nonlinear system in (1.1), we take full advantage of the smoothing and stabilization effect generated by the wave structure to control the nonlinearity. We are able to establish the following nonlinear stability and large-time behavior result.

**Theorem 1.3.** *Let $\eta$ and $\nu > 0$. Consider (1.1) with the initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu,\eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \leqslant \varepsilon,$$

*then* (1.1) *has a unique global classical solution* $(u, b)$ *satisfying, for any* $t > 0$,

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \int_0^t \left(\|\partial_1 u\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|b\|_{H^2}^2\right) \mathrm{d}\tau \leqslant C\varepsilon^2$$

*for some universal constant. In addition, the solution obeys the following large-time decay estimates, for some constant* $C$,

$$\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \leqslant C(1+t)^{-\frac{1}{2}}. \tag{1.20}$$

Theorem 1.3 is a consequence of the smoothing and stabilization effect of the magnetic field. In particular, the time integrability

$$\int_0^\infty \|\partial_1 u(t)\|_{L^2}^2 \,\mathrm{d}t \leqslant C\varepsilon^2$$

is not a consequence of the vertical dissipation in the velocity equation, but an exhibition of the smoothing effect of the magnetic field. We explain why the stability for the 2D Navier–Stokes equation with only vertical dissipation, namely (1.2) remains open and what makes the stability problem for the MHD system solvable. It follows from (1.2) and the corresponding vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega \tag{1.21}$$

that the $H^1$-norm of $u$ is uniformly bounded,

$$\|u(t)\|_{H^1} \leqslant \|u_0\|_{H^1}.$$

The difficulty is how to control the $H^2$-norm of $u$ or $\|\nabla\omega\|_{L^2}$. When we estimate $\|\nabla\omega\|_{L^2}$ via (1.21), the nonlinear term becomes an insurmountable hurdle. In fact, it follows from (1.21) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2 + 2\nu\|\partial_2\nabla\omega\|_{L^2}^2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \,\mathrm{d}x.$$

The right-hand side can be further decomposed into four terms

$$\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \,\mathrm{d}x = \int \partial_1 u_1 (\partial_1\omega)^2 \,\mathrm{d}x + \int \partial_1 u_2 \partial_1\omega \partial_2\omega \,\mathrm{d}x + \int \partial_2 u_1 \partial_1\omega \partial_2\omega \,\mathrm{d}x + \int \partial_2 u_2 (\partial_2\omega)^2 \,\mathrm{d}x. \tag{1.22}$$

Due to the lack of the horizontal dissipation, the first two terms can not be suitably bounded. When we deal with the stability problem on the MHD system (1.1), we need to control exactly the same nonlinearity. It is the coupling and interaction in the MHD system that allows us to have more maneuver. When we estimate the $H^2$-norm via the equations of the vorticity and current density in (1.6), we also encounter the term (1.22). The idea of bounding the first two terms in (1.22) is to replace $\partial_1\omega$ by the equation of $j$,

$$\partial_1\omega = \partial_t j + u \cdot \nabla j + \eta j - b \cdot \nabla\omega - Q.$$

For example, the first term on the right of (1.22) would become

$$\int \partial_1 u_1 (\partial_1\omega)^2 \mathrm{d}x = \int \partial_1 u_1 \partial_1\omega(\partial_t j + u \cdot \nabla j + \eta j - b \cdot \nabla\omega - Q)\mathrm{d}x. \tag{1.23}$$

We further shift the time derivative in the first term in (1.23), namely

$$\int \partial_1 u_1 \partial_1 \omega \partial_t j \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1 u_1 \partial_1 \omega j \,\mathrm{d}x - \int \partial_t \partial_1 u_1 \partial_1 \omega j \,\mathrm{d}x - \int \partial_1 u_1 j \partial_t \partial_1 \omega \,\mathrm{d}x. \quad (1.24)$$

By substituting $\partial_t u_1$ and $\partial_t \omega$ by their corresponding equations in (1.24), we find that the first term in (1.22) is then converted to

$$\begin{aligned}\int \partial_1 u_1 (\partial_1 \omega)^2 \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}&\int \partial_1 u_1 \partial_1 \omega j \,\mathrm{d}x \\ &- \int \partial_1 \omega j \partial_1(-u\cdot\nabla u_1 - \partial_1 P + \nu \partial_{22} u_1 + b\cdot\nabla b_1 + \partial_1 b_1)\mathrm{d}x \\ &- \int \partial_1 u_1 j \partial_1(-u\cdot\nabla\omega + \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j)\mathrm{d}x \\ &+ \int \partial_1 u_1 \partial_1 \omega (u\cdot\nabla j + \eta j - b\cdot\nabla\omega - Q)\mathrm{d}x. \end{aligned} \quad (1.25)$$

Even though the original one term is converted into fourteen terms, but all of the terms can be bounded suitably by applying anisotropic inequalities such as the one stated in the following lemma. The second terms on the right of (1.22) can be treated similarly. Estimating all these terms is a tedious and long process.

**Lemma 1.4.** *Assume that $f$, $g$, $\partial_2 g$, $h$ and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int_{\mathbb{R}^2} |fgh| \,\mathrm{d}x \leqslant C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

This lemma is taken from [6]. It is very useful in dealing with partial differential equations with anisotropic dissipation and allows us to selectively put directional derivatives on the components of a triple product.

To prove the stability part of theorem 1.3, we use the bootstrapping argument (see, e.g., [34, p.21]). It starts with the definition of a suitable energy functional $E(t)$. We set

$$E(t) := \sup_{0\leqslant\tau\leqslant t} \{\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2\} + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2 \,\mathrm{d}\tau + 2\eta\int_0^t \|b\|_{H^2}^2 \,\mathrm{d}\tau.$$

The main efforts are then devoted to proving that for some constants $C$,

$$E(t) \leqslant E(0) + CE^{\frac{3}{2}}(0) + CE^2(t) + CE^{\frac{3}{2}}(t). \quad (1.26)$$

This is a long process including estimating the term (1.22) and making the substitution as in (1.25). The bootstrapping argument applied to (1.26) allows us to conclude that, if $E(0)$ or $\|(u_0, b_0)\|_{H^2}$ is sufficiently small, say

$$E(0) \leqslant \varepsilon^2 \quad \text{or} \quad \|(u_0, b_0)\|_{H^2} \leqslant \varepsilon$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains small for all time $t > 0$ and

$$E(t) \leqslant C\varepsilon^2 \quad (1.27)$$

for some constant $C > 0$.

In order to prove the large-time decay estimates stated in theorem 1.3, we further show that the solution $(u, b)$ obtained above has the following properties,

$$\int_0^\infty \|\partial_1 u(t)\|_{L^2}^2 \, \mathrm{d}t \leqslant C\varepsilon^2. \tag{1.28}$$

and

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \leqslant C\|(\nabla u(s), \nabla b(s))\|_{L^2} \quad \text{for any} \quad 0 \leqslant s \leqslant t \tag{1.29}$$

(1.28) is not a direct consequence of the dissipation in the velocity equation. It is shown by taking into account of the coupling of the system. We replace $\partial_1 u$ by

$$\partial_1 u = \partial_t b + u \cdot \nabla b + \eta b - b \cdot \nabla u$$

in the $L^2$-norm,

$$\int_0^\infty \int \partial_1 u \cdot (\partial_t b + u \cdot \nabla b + \eta b - b \cdot \nabla u) \mathrm{d}x \, \mathrm{d}t.$$

By shifting the time derivative and applying various anisotropic inequalities, we are able to prove (1.28). The generalized monotonicity in (1.29) is established by estimating $\|\omega\|_{L^2}$ and $\|j\|_{L^2}$ via (1.6). Then (1.27) and (1.28) together leads to the time integrability of

$$\int_0^\infty \left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right) \mathrm{d}t \leqslant C\varepsilon^2. \tag{1.30}$$

(1.29) and (1.30) then fulfill the two conditions of lemma 2.1 and the desired decay estimate in (1.20) follows as a consequence.

Finally we remark that there are substantial recent developments on fundamental issues concerning the MHD equations such as the global regularity and stability problems. One recent focus is on the MHD equations with only partial or fractional dissipation. Significant progress has been made (see, e.g., [3–7, 9, 11–14, 17–21, 24–29, 31, 33, 35–49]).

The rest of this paper is divided into two sections. Section 2 presents the proofs of theorems 1.1 and 1.2 while section 3 proves theorem 1.3.

## 2. Proofs of theorems 1.1 and theorem 1.2

This section is devoted to proving theorems 1.1 and 1.2. The proof of theorem 1.1 makes use of the wave structure to construct a suitable Lyapunov functional, which allows us to eliminate some unfavorable terms. The decay estimates are obtained by using a tool lemma stated below and the key components are the verification on the conditions of the lemma. The proof of theorem 1.2 also involves the combination of energy estimates to form a suitable Lyapunov functional. The frequency part of the solution that is supported away from the origin allows the application of Poincare type inequalities.

The following lemma provides a precise decay rate for a nonnegative integrable function, which is also monotonic in a generalized sense.

**Lemma 2.1.** *Let $f = f(t)$ be a nonnegative function satisfying, for two constants $a_0 > 0$ and $a_1 > 0$,*

$$\int_0^\infty f(\tau)\mathrm{d}\tau \leqslant a_0 < \infty \quad \textit{and} \quad f(t) \leqslant a_1 f(s) \quad \textit{for any} \quad 0 \leqslant s < t. \tag{2.1}$$

*Then f(t) decays at a rate faster than* $(1+t)^{-1}$, *or*

$$(1+t)f(t) \to 0 \quad \textit{as} \quad t \to \infty.$$

*In particular, for* $a_2 = \max\{2a_1 f(0), 4a_0 a_1\}$ *and for any* $t > 0$,

$$f(t) \leqslant a_2(1+t)^{-1}.$$

**Proof of Theorem 1.1.** We start with the estimates on the norms of $b$. Let $\Phi$ be the Fourier multiplier defined in (1.10). Applying $\Phi$ to the equation of $b$ in (1.8) and then taking the $L^2$-inner product with $\partial_t \Phi b$, we obtain after integrating by parts and invoking $\nabla \cdot b = 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_t \Phi b\|_{L^2}^2 + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2\right) + \nu\|\partial_2\partial_t \Phi b\|_{L^2}^2 + \eta\|\partial_t \Phi b\|_{L^2}^2 = 0 \tag{2.2}$$

Applying $\Phi$ to the equation of $b$ in (1.8) and then taking the $L^2$-inner product with $\Phi b$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\eta\|\Phi b\|_{L^2}^2 + \nu\|\partial_2 \Phi b\|_{L^2}^2) + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2 + \int \partial_{tt}\Phi b \cdot \Phi b \,\mathrm{d}x = 0.$$

We further rewrite the last term as

$$\begin{aligned}\int \partial_{tt}\Phi b \cdot \Phi b \,\mathrm{d}x &= \int (\partial_t(\partial_t \Phi b \cdot \Phi b) - |\partial_t \Phi b|^2)\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\partial_t \Phi b, \Phi b) - \|\partial_t \Phi b\|_{L^2}^2,\end{aligned}$$

where we have introduced the notation for the $L^2$-inner product,

$$(f, g) = \int_{\mathbb{R}^2} f \cdot g \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta\|\Phi b\|_{L^2}^2 + \nu\|\partial_2 \Phi b\|_{L^2}^2 + 2(\partial_t \Phi b, \Phi b)\right) \\ &\quad + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2 - \|\partial_t \Phi b\|_{L_2}^2 = 0.\end{aligned} \tag{2.3}$$

Let $\lambda > 0$ be a parameter to be determined later. Then, 2.2 $+\lambda$2.3 yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_t \Phi b\|_{L^2}^2 + \|\partial_1 \Phi b\|_{L^2}^2 + (\lambda\nu + \eta\nu)\|\partial_2 \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 + 2\lambda(\partial_t \Phi b, \Phi b)\right) \\ &\quad + \nu\|\partial_2\partial_t \Phi b\|_{L^2}^2 + (\eta - \lambda)\|\partial_t \Phi b\|_{L^2}^2 + \lambda\|\partial_1 \Phi b\|_{L^2}^2 + \lambda\eta\nu\|\partial_2 \Phi b\|_{L^2}^2 = 0.\end{aligned} \tag{2.4}$$

By Hölder's and Young's inequality,

$$\begin{aligned}&\|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 + 2\lambda(\partial_t \Phi b, \Phi b) \\ &\quad \geqslant \|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 - 2\lambda\|\partial_t \Phi b(t)\|_{L^2}\|\Phi b\|_{L^2} \\ &\quad \geqslant \|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 - \left(\frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + 2\lambda^2\|\Phi b\|_{L^2}^2\right) \\ &\quad \geqslant \frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + (\lambda\eta - 2\lambda^2)\|\Phi b\|_{L^2}^2.\end{aligned} \tag{2.5}$$

In particular, for $\lambda = \frac{\eta}{4}$, (2.5) becomes

$$\|\partial_t \Phi b\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b\|_{L^2}^2 + \frac{\eta}{2}(\partial_t \Phi b, \Phi b) \geqslant \frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + \frac{\eta^2}{8}\|\Phi b\|_{L^2}^2. \tag{2.6}$$

Integrating (2.4) in time and invoking (2.6), we find, for any $0 \leqslant s \leqslant t$,

$$\begin{aligned}
&\|\partial_t \Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1 \Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi b(t)\|_{L^2}^2 \\
&\quad + \int_s^t \left(4\nu\|\partial_2\partial_\tau \Phi b\|_{L^2}^2 + 3\eta\|\partial_\tau \Phi b\|_{L^2}^2 + \eta\|\partial_1 \Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2 \Phi b\|_{L^2}^2\right) \mathrm{d}\tau \\
&\leqslant 3\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{3\eta^2}{4}\|\Phi b(s)\|_{L^2}^2 + 2\|\partial_1 \Phi b(s)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi b(s)\|_{L^2}^2,
\end{aligned} \tag{2.7}$$

where we have used the following upper bound to obtain the right-hand side

$$\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(s)\|_{L^2}^2 + \frac{\eta}{2}(\partial_t \Phi b, \Phi b)(s) \leqslant \frac{3}{2}\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{3\eta^2}{8}\|\Phi b(s)\|_{L^2}^2.$$

Since $u$ and $b$ satisfy exactly the same wave equation, the bound above also holds for $u$,

$$\begin{aligned}
&\|\partial_t \Phi u(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi u(t)\|_{L^2}^2 + 2\|\partial_1 \Phi u(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi u(t)\|_{L^2}^2 \\
&\quad + \int_s^t \left(4\nu\|\partial_2\partial_\tau \Phi u\|_{L^2}^2 + 3\eta\|\partial_\tau \Phi u\|_{L^2}^2 + \eta\|\partial_1 \Phi u\|_{L^2}^2 + \nu\eta^2\|\partial_2 \Phi u\|_{L^2}^2\right) \mathrm{d}\tau \\
&\leqslant 3\|(\partial_t \Phi u)(s)\|_{L^2}^2 + \frac{3\eta^2}{4}\|\Phi u(s)\|_{L^2}^2 + 2\|\partial_1 \Phi u(s)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi u(s)\|_{L^2}^2.
\end{aligned} \tag{2.8}$$

Invoking the original linearized system of $(u, b)$, namely (1.9) and letting $t \to 0$, we obtain

$$(\partial_t u)(0) = \nu\partial_{22}u_0 + \partial_1 b_0, \qquad (\partial_t b)(0) = -\eta b_0 + \partial_1 u_0. \tag{2.9}$$

By setting $s = 0$ in (2.7) and (2.8), and using (2.9), we obtain the desired global bound in (1.11) and (1.12). By taking the Fourier multiplier operator $\Phi$ to be the fractional Laplacian operator,

$$\Phi f = (-\Delta)^{\frac{s}{2}} f$$

and identifying the homogeneous $\mathring{H}^s$-norm as the following $L^2$-norm,

$$\|f\|_{\mathring{H}^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2},$$

we can then reduce (1.11)–(1.14), respectively.

Next we show the decay rates in (1.15) and (1.16). The idea is to apply lemma 2.1. We set

$$F(t) := \|\partial_t \Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1 \Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\|\partial_2 \Phi b(t)\|_{L^2}^2$$

and verify that $F(t)$ obeys the conditions in (2.1). It is clear from (2.7) that, for any $0 \leqslant s \leqslant t < \infty$, there is a constant $C$ independent of $s$ and $t$ satisfying

$$F(t) \leqslant CF(s). \tag{2.10}$$

In addition, by taking $s = 0$ in (2.7) and invoking (2.9), we have

$$\int_0^\infty \left(3\eta\|\partial_t\Phi b\|_{L^2}^2 + \eta\|\partial_1\Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2\Phi b\|_{L^2}^2\right) \mathrm{d}t \\ \leqslant \frac{15\eta^2}{4}\|\Phi b_0\|_{L^2}^2 + 3\|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \frac{5}{2}\nu\eta\|\partial_2\Phi b_0\|_{L^2}^2. \tag{2.11}$$

In addition, a simple $L^2$-energy estimate on (1.9) leads to

$$\|\Phi(u(t), b(t))\|_{L^2}^2 + 2\int_0^t (\nu\|\Phi\partial_2 u\|_{L^2}^2 + \eta\|\Phi b\|_{L^2}^2)\mathrm{d}\tau = \|\Phi(u_0, b_0)\|_{L^2}^2.$$

In particular,

$$\eta\int_0^\infty \|\Phi b\|_{L^2}^2 \ \mathrm{d}t \leqslant \|\Phi(u_0, b_0)\|_{L^2}^2. \tag{2.12}$$

Adding (2.11) and (2.12) yields

$$\int_0^\infty F(t)\mathrm{d}t \leqslant C(\nu, \eta)\left(\|\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right). \tag{2.13}$$

(2.10) and (2.13) then verify (2.1). Lemma 2.1 then implies

$$(1 + t)F(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.14}$$

As a special consequence,

$$F(t) \leqslant C(\nu, \eta)\left(\|\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right)(1 + t)^{-1},$$

which is (1.15). The process of showing the decay rate for $b$ does not work for $u$. The reason is that we do not have the corresponding time integrability bound (2.12) for $u$. We do not know if $\|\Phi u(t)\|_{L^2}$ decays or not. What we can obtain is an explicit decay rate for $\|\nabla\Phi u(t)\|_{L^2}$. According to (1.9) and (2.14), we have

$$(1 + t)\|\partial_1\Phi u(t)\|_{L^2}^2 \leqslant C(1 + t)\left(\|\partial_t\Phi b(t)\|_{L^2}^2 + \eta\|\Phi b(t)\|_{L^2}^2\right) \to 0 \quad \text{as} \quad t \to \infty$$

and

$$\|\partial_1\Phi u(t)\|_{L^2} \leqslant \|\partial_t\Phi b(t)\|_{L^2} + \eta\|\Phi b(t)\|_{L^2} \\ \leqslant C(\nu, \eta)\left(\|\Phi(u_0, b_0)\|_{L^2} + \|\partial_1\Phi(u_0, b_0)\|_{L^2} + \|\partial_2\Phi b_0\|_{L^2}\right)(1 + t)^{-\frac{1}{2}}. \tag{2.15}$$

To obtain the decay rate for $\partial_2\Phi u$, we apply $\partial_2\Phi$ to (1.9) and then dot with $(\partial_2\Phi u, \partial_2\Phi b)$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_2\Phi u\|_{L^2}^2 + \|\partial_2\Phi b\|_{L^2}^2) + \nu\|\partial_{22}\Phi u\|_{L^2}^2 + \eta\|\partial_2\Phi b\|_{L^2}^2 = 0.$$

Therefore, for $0 \leqslant s \leqslant t$,

$$\|\partial_2\Phi u(t)\|_{L^2}^2 + \|\partial_2\Phi b(t)\|_{L^2}^2 \leqslant \|\partial_2\Phi u(s)\|_{L^2}^2 + \|\partial_2\Phi b(s)\|_{L^2}^2.$$

Furthermore, (2.8) with $s = 0$, together with (2.9), gives

$$\int_0^\infty \|\partial_2 \Phi u(t)\|_{L^2}^2 \, \mathrm{d}t$$
$$\leqslant C(\nu,\eta)\left(\|\Phi u_0\|_{L^2}^2 + \|\nabla\Phi u_0\|_{L^2}^2 + \|\partial_1\Phi b_0\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right). \quad (2.16)$$

Combining (2.11) and (2.16) leads to

$$\int_0^\infty \left(\|\partial_2 \Phi u(t)\|_{L^2}^2 + \|\partial_2\Phi b(t)\|_{L^2}^2\right) \mathrm{d}t$$
$$\leqslant C(\nu,\eta)\left(\|\Phi(u_0,b_0)\|_{L^2}^2 + \|\Phi(\nabla u_0,\nabla b_0)\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right).$$

It then follows from lemma 2.1 that

$$(1+t)\left(\|\partial_2\Phi u(t)\|_{L^2}^2 + \|\partial_2\Phi b(t)\|_{L^2}^2\right) \to 0 \quad \text{as} \quad t\to\infty$$

and

$$\|\partial_2\Phi u(t)\|_{L^2}^2 + \|\partial_2\Phi b(t)\|_{L^2}^2$$
$$\leqslant C(\nu,\eta)\left(\|\Phi(u_0,b_0)\|_{L^2}^2 + \|\Phi(\nabla u_0,\nabla b_0)\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right)(1+t)^{-1}. \quad (2.17)$$

(2.15) and (2.17) yield (1.16). This completes the proof of theorem 1.1. □

We now turn to the proof of theorem 1.2.

**Proof of Theorem 1.2.** We make use of some of the estimates from the proof of theorem 1.1. Recall the definition of $\phi$ in (1.17). By taking $\Phi$ to be the convolution operator $\phi$, namely

$$\Phi f = \phi * f \quad \text{or} \quad \widehat{\Phi f}(\xi) = \widehat{\phi}(\xi)\widehat{f}(\xi),$$

we obtain from (2.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + 2\nu\|\partial_2\partial_t(\phi*b)\|_{L^2}^2 + 2(\eta-\lambda)\|\partial_t(\phi*b)\|_{L^2}^2$$
$$+ 2\lambda\|\partial_1(\phi*b)\|_{L^2}^2 + 2\lambda\eta\nu\|\partial_2(\phi*b)\|_{L^2}^2 = 0,$$

where

$$G(t) = \|\partial_t(\phi*b)\|_{L^2}^2 + \|\partial_1(\phi*b)\|_{L^2}^2 + (\lambda\nu+\eta\nu)\|\partial_2(\phi*b)\|_{L^2}^2$$
$$+ \lambda\eta\|(\phi*b)\|_{L^2}^2 + 2\lambda(\partial_t(\phi*b),(\phi*b)).$$

By setting $\lambda = \frac{\eta}{4}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + \frac{3\eta}{2}\|\partial_t(\phi*b)\|_{L^2}^2 + \frac{\eta}{2}\|\partial_1(\phi*b)\|_{L^2}^2 + \frac{\nu\eta^2}{2}\|\partial_2(\phi*b)\|_{L^2}^2 \leqslant 0. \quad (2.18)$$

In particular, if we set

$$C_1 = \min\left\{\frac{3\eta}{2}, \frac{\eta}{2}, \frac{\nu\eta^2}{2}\right\},$$

then (2.18) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + C_1\left(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2\right) \leqslant 0.$$

By Plancherel's theorem and the definition of $\widehat{\phi}$,

$$\begin{aligned}\|\phi * b\|_{L^2}^2 = \|\widehat{\phi}\cdot\widehat{b}\|_{L^2}^2 &= \int_{|\xi|\geqslant a}|\widehat{\phi}|^2|\widehat{b}|^2\,\mathrm{d}x\\ &\leqslant \int_{|\xi|\geqslant a}\frac{|\xi|^2}{a^2}|\widehat{\phi}|^2|\widehat{b}|^2\,\mathrm{d}x \leqslant \frac{1}{a^2}\|\nabla(\phi * b)\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + C_1\left(\|\partial_t(\phi * b)\|_{L^2}^2 + \frac{1}{2}\|\nabla(\phi * b)\|_{L^2}^2 + \frac{a^2}{2}\|(\phi * b)\|_{L^2}^2\right) \leqslant 0.$$

If we write

$$C_2 = \min\left\{\frac{C_1}{2}, \frac{C_1}{2}a^2\right\},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + C_2(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{L^2}^2) \leqslant 0. \tag{2.19}$$

Clearly, for $\lambda = \frac{\eta}{4}$,

$$G(t) \leqslant \frac{3}{2}\|\partial_t(\phi * b)\|_{L^2}^2 + \frac{3\eta^2}{8}\|(\phi * b)\|_{L^2}^2 + \|\partial_1(\phi * b)\|_{L^2}^2 + \frac{5\eta\nu}{4}\|\partial_2(\phi * b)\|_{L^2}^2. \tag{2.20}$$

For any constant $C_0$ satisfying

$$0 < C_0 \leqslant \min\left\{\frac{2C_2}{3}, \frac{8C_2}{3\eta^2}, \frac{4C_2}{5\nu\eta}\right\},$$

(2.20) implies

$$C_2(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{L^2}^2) \geqslant C_0 G(t).$$

(2.19) then implies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + C_0 G \leqslant 0 \quad \text{or} \quad G(t) \leqslant G(0)\mathrm{e}^{-C_0 t}. \tag{2.21}$$

By the definition of $G$,

$$\begin{aligned}G(0) = \|(\partial_t(\phi * b))(0)\|_{L^2}^2 &+ \|\partial_1(\phi * b_0)\|_{L^2}^2 + \frac{5\eta\nu}{4}\|\partial_2(\phi * b_0)\|_{L^2}^2\\ &+ \frac{\eta^2}{4}\|(\phi * b_0)\|_{L^2}^2) + \frac{\eta}{2}((\partial_t(\phi * b))(0), (\phi * b_0)).\end{aligned}$$

By (1.9),

$$(\partial_t(\phi * b))(0) = -\eta(\phi * b_0) + \partial_1(\phi * u_0).$$

Setting $\lambda = \frac{\eta}{4}$ and applying Hölder's and Young's inequalities, we have

$$
\begin{aligned}
G(0) \leqslant & \frac{3}{2}\|(\partial_t(\phi * b))(0)\|_{L^2}^2 + \frac{3\eta^2}{8}\|(\phi * b_0)\|_{L^2}^2 + \|\partial_1(\phi * b_0)\|_{L^2}^2 \\
& + \frac{5\eta\nu}{4}\|\partial_2(\phi * b_0)\|_{L^2}^2 \leqslant C_3(\nu, \eta)\|(\phi * u_0, \phi * b_0)\|_{H^1}^2. \qquad (2.22)
\end{aligned}
$$

Clearly, for $\lambda = \frac{\eta}{4}$, $G(t)$ admits the lower bound

$$
G(t) \geqslant C_4(\|\partial_t(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{H^1}^2), \qquad (2.23)
$$

where $C_4 = C_4(\nu, \eta)$ is a constant. Hence, (2.21)–(2.23) lead to

$$
\|\partial_t(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{H^1}^2 \leqslant C_5(\|(\phi * u_0)\|_{H^1}^2 + \|(\phi * b_0)\|_{H^1}^2)\mathrm{e}^{-C_0 t},
$$

which is (1.19). The proof for (1.18) is very similar and we omit the details. This completes the proof of theorem 1.2. □

## 3. Proof of theorem 1.3

This section proves theorem 1.3. This theorem consists of two main parts, the stability and the large-time behavior estimate. Naturally our proof is divided into two main parts with the first devoted to the stability and the second to the proof of (1.20). Due to the lack of the horizontal dissipation in the velocity equation, the main difficulty in the proof of the stability is how to bound the velocity nonlinear term, namely (1.22). This is the reason that the 2D Navier–Stokes with degenerate dissipation is not known to be stable. We fully exploit the smoothing and stabilization effect of the magnetic field to overcome this difficulty.

The proof of the decay estimate (1.20) focuses on the time integrability

$$
\int_0^\infty \|\partial_1 u\|_{L^2}^2 \,\mathrm{d}t \leqslant C\varepsilon^2,
$$

which is not a consequence of the vertical dissipation in the velocity equation. It is established by making use of the regularization effect of the magnetic field through the coupling and interaction.

In order to make efficient use of the anisotropic dissipation, we employ several anisotropic tools to control the nonlinear terms. One of them is lemma 1.4 stated in the introduction. Another anisotropic inequality we also use extensively is given in the following lemma. A proof is also presented for the convenience of readers.

**Lemma 3.1.** *The following estimates hold when the right-hand sides are all bounded.*

$$
\|f\|_{L^\infty(\mathbb{R}^2)} \leqslant C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.
$$

*Consequently,*

$$
\|f\|_{L^\infty} \leqslant C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}},
$$

$$
\|f\|_{L^\infty} \leqslant C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}.
$$

**Proof.** We recall the following inequality, for a one-dimensional function $g \in H^1(\mathbb{R})$,

$$\|g\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{3.1}$$

By (3.1) and Minkowski's inequality,

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^2)} &= \|\|f\|_{L^\infty_{x_2}(\mathbb{R})}\|_{L^\infty_{x_1}(\mathbb{R})} \\
&\leqslant \sqrt{2}\|\|f\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}}\|\partial_2 f\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}}\|_{L^\infty_{x_1}(\mathbb{R})} \\
&\leqslant \sqrt{2}\|\|f\|_{L^2_{x_2}(\mathbb{R})}\|_{L^\infty_{x_1}(\mathbb{R})}^{\frac{1}{2}}\|\|\partial_2 f\|_{L^2_{x_2}(\mathbb{R})}\|_{L^\infty_{x_1}(\mathbb{R})}^{\frac{1}{2}} \\
&\leqslant \sqrt{2}\|\|f\|_{L^\infty_{x_1}(\mathbb{R})}\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}}\|\|\partial_2 f\|_{L^\infty_{x_1}(\mathbb{R})}\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \\
&\leqslant 2\|\|f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\|\partial_1 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}}\|\|\partial_2 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\|\partial_1\partial_2 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \\
&\leqslant 2\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.
\end{aligned}$$

Here we have written $\|f\|_{L^\infty_{x_j}(\mathbb{R})}$ with $j = 1, 2$ to denote the $L^\infty$-norm of $f$ in terms of $x_j$ on $\mathbb{R}$, and, similarly, $\|f\|_{L^2_{x_j}(\mathbb{R})}$ denotes the $L^2$-norm. □

We are ready to prove theorem 1.3.

**Proof of Theorem 1.3.** The framework of the proof is the bootstrapping argument. We define the energy functional to be

$$E(t) = \sup_{0\leqslant\tau\leqslant t}\{\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2\} + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2\,\mathrm{d}\tau + 2\eta\int_0^t \|b\|_{H^2}^2\,\mathrm{d}\tau \tag{3.2}$$

an show that

$$E(t) \leqslant E(0) + C_1 E^{\frac{3}{2}}(0) + C_2 E^2(t) + C_3 E^{\frac{3}{2}}(t). \tag{3.3}$$

(3.3) is established by estimating the $H^2$-norm of $(u, b)$. As aforementioned in the introduction, it is extremely difficult to obtain suitable upper bounds for some of the terms such as the nonlinear term in the momentum equation. We can only control them through the coupling with the equation of the magnetic field. Equivalently we exploit the regularization and damping effects of the wave structure derived from the coupling and interaction of the velocity and the magnetic fields. The estimates of $\|(u, b)\|_{H^2}$ will involves various operations such as repeated substitutions to take the full advantage of the wave structure.

Due to the equivalence of the inhomogeneous norm $\|(u, b)\|_{H^2}$ with the sum of the $L^2$-norm and the homogeneous norm $\|(u, b)\|_{\mathring{H}^2}$, it suffices to bound the homogeneous norm $\|(u, b)\|_{\mathring{H}^2}$. The uniform $L^2$-bound is an easy consequence of the system in (1.1) itself. Taking the inner product of (1.1) with $(u, b)$, we obtain, after integrating by parts and using $\nabla \cdot u = \nabla \cdot b = 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_2 u\|_{L^2}^2\,\mathrm{d}\tau + 2\eta\int_0^t \|b\|_{L^2}^2\,\mathrm{d}\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{3.4}$$

To estimate the homogeneous norm $\|(u,b)\|_{\mathring{H}^2}$, we make use of the equations of the vorticity $\omega = \nabla\times u$ and the current density $j = \nabla\times b$, namely (1.6),

$$\begin{cases}\partial_t\omega + u\cdot\nabla\omega = \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j,\\ \partial_t j + u\cdot\nabla j + \eta j = b\cdot\nabla\omega + Q + \partial_1\omega,\end{cases}\tag{3.5}$$

where $Q\colon = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$. Due to

$$\|(u,b)\|_{\mathring{H}^2} = \|(\nabla\omega,\nabla j)\|_{L^2},$$

we focus on $\|(\nabla\omega,\nabla j)\|_{L^2}$. Applying the gradient $\nabla$ to (3.5) and taking the inner product of the resultant with $(\nabla\omega,\nabla j)$, we find, after integration by parts and the divergence-free conditions,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2\\ &\quad= -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x - \int\nabla j\cdot\nabla u\cdot\nabla j\,\mathrm{d}x + \int\nabla\omega\cdot\nabla b\cdot\nabla j\,\mathrm{d}x\\ &\qquad+\int\nabla j\cdot\nabla b\cdot\nabla\omega\,\mathrm{d}x + \int\nabla Q\cdot\nabla j\,\mathrm{d}x\\ &\quad:= J + K + L + M + N.\end{aligned}\tag{3.6}$$

$J$ is the most difficult term and its estimate is long and tedious. We start with the easy terms. Even though $L$ and $M$ are not exactly the same, they obviously admit the same upper bound. To bound $L$, we further decompose it into four terms in order to make use of the anisotropic dissipation,

$$\begin{aligned}L &= \int\nabla\omega\cdot\nabla b\cdot\nabla j\,\mathrm{d}x\\ &= \int\partial_1\omega\partial_1 b_1\partial_1 j\,\mathrm{d}x + \int\partial_1\omega\partial_1 b_2\partial_2 j\,\mathrm{d}x + \int\partial_2\omega\partial_2 b_1\partial_1 j\,\mathrm{d}x + \int\partial_2\omega\partial_2 b_2\partial_2 j\,\mathrm{d}x\\ &:= L_1 + L_2 + L_3 + L_4.\end{aligned}$$

By lemma 1.4,

$$\begin{aligned}L_1 &= \int\partial_1\omega\partial_1 b_1\partial_1 j\,\mathrm{d}x \leqslant C\|\partial_1 j\|_{L^2}\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_1^2 b_1\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\\ &\leqslant C\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32},\\ L_2 &= \int\partial_1\omega\partial_1 b_2\partial_2 j\,\mathrm{d}x \leqslant C\|\partial_2 j\|_{L^2}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_1^2 b_2\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\\ &\leqslant C\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32},\end{aligned}$$

where we have used the basic facts,

$$\begin{aligned}&\|\partial_1 j\|_{L^2} = \|\partial_1\nabla b\|_{L^2} \leqslant \|b\|_{H^2},\quad \|\partial_1\omega\|_{L^2} = \|\partial_1\nabla u\|_{L^2} \leqslant \|u\|_{H^2},\\ &\|\partial_2\partial_1\omega\|_{L^2} = \|\partial_2\partial_1\nabla u\|_{L^2} \leqslant \|\partial_2 u\|_{H^2}.\end{aligned}$$

$L_3$ and $L_4$ can be bounded similarly. Therefore,

$$L \leqslant C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}. \tag{3.7}$$

Similarly,

$$M \leqslant C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}. \tag{3.8}$$

We now turn to $K$. Again, in order to make efficient use of the anisotropic dissipation, we decompose $K$ into four terms,

$$\begin{aligned} K &= -\int \nabla j \cdot \nabla u \cdot \nabla j \, \mathrm{d}x \\ &= -\int \partial_1 j \partial_1 u_1 \partial_1 j \, \mathrm{d}x - \int \partial_1 j \partial_1 u_2 \partial_2 j \, \mathrm{d}x - \int \partial_2 j \partial_2 u_1 \partial_1 j \, \mathrm{d}x - \int \partial_2 j \partial_2 u_2 \partial_2 j \, \mathrm{d}x \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned}$$

By Hölder's inequality and lemma 3.1,

$$\begin{aligned} K_1 &= -\int \partial_1 j \partial_1 u_1 \partial_1 j \, \mathrm{d}x \leqslant \|\partial_1 u_1\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_1 j\|_{L^2} \\ &\leqslant \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|b\|_{H^2}^2 \leqslant \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}. \end{aligned}$$

The other three terms $K_2$, $K_3$ and $K_4$ all admit the same upper bound. Therefore,

$$K \leqslant C\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}. \tag{3.9}$$

We now bound $N$. We write out all the component terms in $Q$ explicitly,

$$\begin{aligned} N &= \int \nabla Q \cdot \nabla j \, \mathrm{d}x \\ &= 2\int \big(\partial_1^2 b_1 \partial_2 u_1 \partial_1^2 b_2 + \partial_1^2 b_1 \partial_1 u_2 \partial_1^2 b_1 - \partial_1^2 b_1 \partial_2 u_1 \partial_1 \partial_2 b_1 - \partial_2^2 b_1 \partial_1 u_2 \partial_1 \partial_2 b_1 \\ &\quad + \partial_1 b_1 \partial_1 \partial_2 u_1 \partial_1^2 b_2 + \partial_1 b_1 \partial_1^2 u_1 \partial_1^2 b_2 - \partial_1 b_1 \partial_1 \partial_2 u_1 \partial_1 \partial_2 b_1 - \partial_1 b_1 \partial_1^2 u_2 \partial_1 \partial_2 b_1 \\ &\quad - \partial_1^2 u_1 \partial_2 b_1 \partial_1^2 b_2 - \partial_1^2 u_2 \partial_1 b_2 \partial_1^2 b_2 + \partial_1^2 u_1 \partial_2 b_1 \partial_1 \partial_2 b_1 + \partial_1^2 u_1 \partial_1 b_2 \partial_1 \partial_2 b_1 \\ &\quad - \partial_1 u_1 \partial_1 \partial_2 b_1 \partial_1^2 b_2 - \partial_1 u_1 \partial_1^2 b_2 \partial_1^2 b_2 + \partial_1 u_1 \partial_1 \partial_2 b_1 \partial_1 \partial_2 b_1 + \partial_1 u_1 \partial_1^2 b_2 \partial_1 \partial_2 b_1\big) \, \mathrm{d}x. \end{aligned}$$

Even though $N$ contains sixteen terms, but all of them can be bounded suitably using Hölder's inequality, lemmas 1.4 and 3.1. Since the details are quite similar to those in the estimates of $K$, we omit them for conciseness. The upper bound is

$$N \leqslant C\|b\|_{H^2}^2\|u\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}. \tag{3.10}$$

We now turn to the most difficult term $J$. Again, we split $J$ into four terms,

$$
\begin{aligned}
J &= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\ \mathrm{d}x\\
&= -\int \partial_1\omega\partial_1 u_1\partial_1\omega\ \mathrm{d}x - \int \partial_1\omega\partial_1 u_2\partial_2\omega\ \mathrm{d}x - \int \partial_2\omega\partial_2 u_1\partial_1\omega\ \mathrm{d}x - \int \partial_2\omega\partial_2 u_2\partial_2\omega\ \mathrm{d}x\\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.11}
$$

As we have explained in the introduction, due to the lack of the horizontal dissipation, $J_1$ and $J_2$ can not be bounded suitably. It is the smoothing and stabilization effect of the magnetic field that makes it possible to control these two terms. To incorporate this effect, we make use of the equation for the magnetic field. By replacing $\partial_1\omega$ by the second equation in (3.5), namely

$$
\partial_1\omega = \partial_t j + u\cdot\nabla j + \eta j - b\cdot\nabla\omega - Q,
$$

then $J_1$ is converted into five terms,

$$
\begin{aligned}
J_1 &= -\int \partial_1 u_1\,(\partial_t j + u\cdot\nabla j + \eta j - b\cdot\nabla\omega - Q)\,\partial_1\omega\ \mathrm{d}x\\
&:= J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}.
\end{aligned}
$$

We shift the time derivative in $J_{1,1}$, namely

$$
\begin{aligned}
J_{1,1} &= -\int \partial_1 u_1\partial_t j\partial_1\omega\ \mathrm{d}x\\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1 u_1 j\partial_1\omega\ \mathrm{d}x + \int \partial_1(\partial_t u_1) j\partial_1\omega\ \mathrm{d}x + \int \partial_1 u_1 j\partial_1(\partial_t\omega)\ \mathrm{d}x\\
&:= J_{1,1,1} + J_{1,1,2} + J_{1,1,3}.
\end{aligned}
$$

Replacing $\partial_t u_1$ by the first equation of (1.1), we have

$$
\begin{aligned}
J_{1,1,2} &= \int j\partial_1\omega\,(-\partial_1(u\cdot\nabla u_1) - \partial_{11}P + \nu\partial_{221}u_1 + \partial_1(b\cdot\nabla b_1) + \partial_{11}b_1)\,\mathrm{d}x\\
&:= J_{1,1,2,1} + J_{1,1,2,2} + J_{1,1,2,3} + J_{1,1,2,4} + J_{1,1,2,5}.
\end{aligned}
$$

Similarly, we replace $\partial_t\omega$ by the first equation in (3.5),

$$
\begin{aligned}
J_{1,1,3} &= \int \partial_1 u_1 j\,(-\partial_1(u\cdot\nabla\omega) + \nu\partial_{221}\omega + \partial_1(b\cdot\nabla j) + \partial_{11}j)\,\mathrm{d}x\\
&= J_{1,1,3,1} + J_{1,1,3,2} + J_{1,1,3,3} + J_{1,1,3,4}
\end{aligned}
$$

We thus have rewritten $J_1$ as

$$
\begin{aligned}
J_1 &= J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}\\
&= J_{1,1,1} + J_{1,1,2} + J_{1,1,3} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}\\
&= J_{1,1,2,1} + J_{1,1,2,2} + J_{1,1,2,3} + J_{1,1,2,4} + J_{1,1,2,5} + J_{1,1,3,1} + J_{1,1,3,2} + J_{1,1,3,3} + J_{1,1,3,4}\\
&\quad + J_{1,1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}.
\end{aligned} \tag{3.12}
$$

We estimate the fourteen terms on the right of (3.12). By Hölder's inequality, lemmas 1.4 and 3.1,

$$
\begin{aligned}
J_{1,1,2,1} &= -\int j\partial_1\omega\partial_1 u\cdot\nabla u_1\ \mathrm{d}x - \int j\partial_1\omega u\cdot\nabla\partial_1 u_1\ \mathrm{d}x\\
&\lesssim \|\partial_1 u\|_{L^\infty}\|\partial_1\omega\|_{L^2}\|\nabla u_1\|_{L^2}^{\frac12}\|\partial_2\nabla u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12}\\
&\quad + \|u\|_{L^\infty}\|\partial_1\omega\|_{L^2}\|\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1 u\|_{H^1}^{\frac12}\|\partial_2\partial_1 u\|_{H^1}^{\frac12}\|\partial_1\omega\|_{L^2}\|\nabla u_1\|_{L^2}^{\frac12}\|\partial_2\nabla u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12}\\
&\quad + \|u\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^1}^{\frac12}\|\partial_1\omega\|_{L^2}\|\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12}\\
&\lesssim \|u\|_{H^2}^2\|\partial_2 u\|_{H^2}\|b\|_{H^2}.
\end{aligned}
$$

Applying the divergence operator $\nabla\cdot$ to the first equation of 1.1 and invoking $\nabla\cdot u = 0$, we have

$$
\partial_{11}P = \partial_{11}(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b) \tag{3.13}
$$

By substituting (3.13) into $J_{1,1,2,2}$,

$$
\begin{aligned}
J_{1,1,2,2} &= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b)\mathrm{d}x\\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_1(u\cdot\nabla u_1)\mathrm{d}x - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_2(u\cdot\nabla u_2)\mathrm{d}x\\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_1(b\cdot\nabla b_1)\mathrm{d}x + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_2(b\cdot\nabla b_2)\mathrm{d}x\\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u\cdot\nabla u_1)\mathrm{d}x - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(u\cdot\nabla\partial_1 u_1)\mathrm{d}x\\
&\quad - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 u\cdot\nabla u_2)\mathrm{d}x - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(u\cdot\nabla\partial_2 u_2)\mathrm{d}x\\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b\cdot\nabla b_1)\mathrm{d}x + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(b\cdot\nabla\partial_1 b_1)\mathrm{d}x\\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 b\cdot\nabla b_2)\mathrm{d}x + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(b\cdot\nabla\partial_2 b_2)\mathrm{d}x\\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u\cdot\nabla u_1)\mathrm{d}x - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 u\cdot\nabla u_2)\mathrm{d}x\\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b\cdot\nabla b_1)\mathrm{d}x + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 b\cdot\nabla b_2)\mathrm{d}x\\
&= -2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u_1\partial_1 u_1)\mathrm{d}x - 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u_2\partial_2 u_1)\mathrm{d}x\\
&\quad + 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b_1\partial_1 b_1)\mathrm{d}x + 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b_2\partial_2 b_1)\mathrm{d}x.
\end{aligned}
$$

The four terms on the right can be estimated as follows. We use Hölder's inequality, lemmas 1.4 and 3.1, and the fact that the double Riesz transform $\partial_{11}\Delta^{-1}$ is bounded on $L^p$ for any $1 < p < \infty$.

$$\begin{aligned}
J_{1,1,2,2} &\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{11}(-\Delta)^{-1}(\partial_1 u_1\partial_1 u_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{11}(-\Delta)^{-1}(\partial_1 u_2\partial_2 u_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{11}(-\Delta)^{-1}(\partial_1 b_1\partial_1 b_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{11}(-\Delta)^{-1}(\partial_1 b_2\partial_2 b_1)\|_{L^2} \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \left(\|\partial_1 u_1\partial_1 u_1\|_{L^2} + \|\partial_1 u_2\partial_2 u_1\|_{L^2} + \|\partial_1 b_1\partial_1 b_1\|_{L^2} + \|\partial_1 b_2\partial_2 b_1\|_{L^2}\right) \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{L^\infty}\|\partial_2 u\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^4}^2 \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}.
\end{aligned}$$

By lemma 1.4,

$$\begin{aligned}
J_{1,1,2,3} &= \nu\int j\partial_1\omega\partial_{221}u_1\ \mathrm{d}x \lesssim \|\partial_{221}u_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}
\end{aligned}$$

By lemma 1.4 and Sobolev's inequality,

$$\begin{aligned}
J_{1,1,2,4} &= \int j\partial_1\omega\partial_1(b\cdot\nabla b_1)\mathrm{d}x = \int j\partial_1\omega\partial_1 b\cdot\nabla b_1\ \mathrm{d}x + \int j\partial_1\omega b\cdot\partial_1\nabla b_1\ \mathrm{d}x \\
&\lesssim \|\partial_1 b\cdot\nabla b_1\|_{L^2}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|b\cdot\partial_1\nabla b_1\|_{L^2}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim (\|\nabla b\|_{L^4}^2 + \|b\|_{L^\infty}\|\partial_1\nabla b_1\|_{L^2})\|b\|_{H^2}\|u\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^2}^3\|u\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}.
\end{aligned}$$

The last term $J_{1,1,2,5}$ can also be bounded via lemma 1.4,

$$\begin{aligned}
J_{1,1,2,5} &= \int j\partial_1\omega\partial_{11}b_1\ \mathrm{d}x \lesssim \|\partial_{11}b_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}$$

We rewrite $J_{1,1,3,1}$ as

$$\begin{aligned}J_{1,1,3,1} &= -\int \partial_1 u_1 j \partial_1 u \cdot \nabla\omega \ \mathrm{d}x - \int \partial_1 u_1 j u \cdot \partial_1 \nabla\omega \ \mathrm{d}x \\ &= J_{1,1,3,1,1} + J_{1,1,3,1,2}.\end{aligned}$$

By Hölder's inequality, lemmas 1.4 and 3.1,

$$\begin{aligned}J_{1,1,3,1,1} &= \int \partial_1 u_1 j \partial_1 u \nabla\omega \ \mathrm{d}x \\ &\lesssim \|\partial_1 u\|_{L^\infty} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.\end{aligned}$$

By integration by parts, lemmas 1.4 and 3.1,

$$\begin{aligned}J_{1,1,3,1,2} &= -\int \partial_1 u_1 j u \cdot \partial_1 \nabla\omega \ \mathrm{d}x \\ &= \int \partial_{11} u_1 j u \cdot \nabla\omega \ \mathrm{d}x + \int \partial_1 u_1 \partial_1 j u \cdot \nabla\omega \ \mathrm{d}x + \int \partial_1 u_1 j \partial_1 u \cdot \nabla\omega \ \mathrm{d}x \\ &\lesssim \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla\omega\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_1 j\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla\omega\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}J_{1,1,3,2} &= \nu \int \partial_1 u_1 j \partial_{221}\omega \ \mathrm{d}x = -\nu \int \partial_{11} u_1 j \partial_{22}\omega \ \mathrm{d}x - \nu \int \partial_1 u_1 \partial_1 j \partial_{22}\omega \ \mathrm{d}x \\ &\lesssim \|\partial_{22}\omega\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_1 j\|_{L^2} \|\partial_{22}\omega\|_{L^2} \\ &\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}.\end{aligned}$$

By integration by parts, lemmas 1.4 and 3.1,

$$\begin{aligned}J_{1,1,3,3} &= \int \partial_1 u_1 j \partial_1 (b \cdot \nabla j) \mathrm{d}x \\ &= -\int \partial_{11} u_1 j b \cdot \nabla j \ \mathrm{d}x - \int \partial_1 u_1 \partial_1 j b \cdot \nabla j \ \mathrm{d}x \\ &\lesssim \|b\|_{L^\infty} \|\nabla j\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} + \|\partial_1 u_1\|_{L^\infty} \|b\|_{L^\infty} \|\partial_1 j\|_{L^2} \|\nabla j\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|b\|_{H^1}^{\frac{1}{2}}\|\partial_1 b\|_{H^1}^{\frac{1}{2}}\|\nabla j\|_{L^2}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_{11}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}u_1\|_{L^2}^{\frac{1}{2}}\\
&\quad + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|b\|_{H^1}^{\frac{1}{2}}\|\partial_2 b\|_{H^1}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}\|\nabla j\|_{L^2}\\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}.
\end{aligned}
$$

By integration by parts and lemma 1.4,

$$
\begin{aligned}
J_{1,1,3,4} &= \int \partial_1 u_1 j\partial_{11} j \ \mathrm{d}x = -\int \partial_{11}u_1 j\partial_1 j \ \mathrm{d}x - \int \partial_1 u_1 \partial_1 j \partial_1 j \ \mathrm{d}x\\
&\lesssim \|\partial_1 j\|_{L^2}\|\partial_{11}u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_{11}u_1\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} + \|\partial_1 u_1\|_{L^\infty}\|\partial_1 j\|_{L^2}^2\\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}} + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|b\|_{H^2}^2\\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

The next term in (3.12) is $J_{1,1,1}$, which involves the time derivative. Its handling is easy and it will be bounded after we take the time integral. We turn to the next term in (3.12), namely $J_{1,2}$. By lemma 1.4 and then lemma 3.1,

$$
\begin{aligned}
J_{1,2} &= \int \partial_1 u_1 u\cdot\nabla j\partial_1\omega \ \mathrm{d}x\\
&\lesssim \|u\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|u\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^1}^{\frac{1}{2}}\|\nabla j\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2.
\end{aligned}
$$

By integration by parts, $\nabla\cdot u = 0$, and lemma 1.4,

$$
\begin{aligned}
J_{1,3} &= -\int \eta\partial_1 u_1 j\partial_1\omega \ \mathrm{d}x\\
&= \int \eta\partial_1^2 u_1 j\omega \ \mathrm{d}x + \int \eta\partial_2 u_2\partial_1 j\omega \ \mathrm{d}x\\
&\lesssim \|j\|_{L^2}\|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}} + \|\partial_1 j\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}.
\end{aligned}
$$

By lemmas 1.4 and 3.1,

$$
\begin{aligned}
J_{1,4} &= \int \partial_1 u_1 b\cdot\nabla\omega\partial_1\omega \ \mathrm{d}x\\
&\lesssim \|\partial_1 u_1\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2.
\end{aligned}
$$

$J_{1,5}$ is written more explicitly into four pieces by the definition of $Q$,

$$\begin{aligned}
J_{1,5} &= \int \partial_1 u_1 Q \partial_1 \omega \ \mathrm{d}x \\
&= \int \partial_1 u_1 \partial_1 \omega \left(2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)\right) \mathrm{d}x \\
&:= J_{1,5,1} + J_{1,5,2} + J_{1,5,3} + J_{1,5,4}.
\end{aligned}$$

By lemmas 1.4 and 3.1,

$$\begin{aligned}
J_{1,5,1} &= 2\int \partial_1 u_1 \partial_1 b_1 \partial_1 u_2 \partial_1 \omega \ \mathrm{d}x \\
&\lesssim \|\partial_1 u_2\|_{L^\infty} \|\partial_1 \omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 b_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u_2\|_{H^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_2\|_{H^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 b_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}$$

It is easy to check that $J_{1,5,2}$, $J_{1,5,3}$ and $J_{1,5,4}$ all obey the same bound. Therefore,

$$|J_{1,5}| \lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.$$

We have completed the estimates of the terms of $J_1$ in (3.12). Collecting the upper bounds leads to

$$\begin{aligned}
J_1 \leqslant -\frac{\mathrm{d}}{\mathrm{d}t} \int \partial_1 u_1 j \partial_1 \omega \ \mathrm{d}x + C\|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2 + C\|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \\
+ C\|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2} + C\|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}. \qquad (3.14)
\end{aligned}$$

We now turn to the second term $J_2$ in (3.11). As we have explained before, we need to invoke the smoothing and regularization effect of the magnetic field in order to bound this term suitably. By replacing $\partial_1 u_2$ by (1.1), namely

$$\partial_1 u_2 = \partial_t b_2 + u \cdot \nabla b_2 + \eta b_2 - b \cdot \nabla u_2,$$

we can write

$$\begin{aligned}
J_2 &= -\int \partial_1 \omega \left(\partial_t b_2 + u \cdot \nabla b_2 + \eta b_2 - b \cdot \nabla u_2\right) \partial_2 \omega \ \mathrm{d}x \\
&:= J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}.
\end{aligned}$$

We bound $J_{2,2}, J_{2,3}$ and $J_{2,4}$ first. By lemmas 1.4 and 3.1,

$$\begin{aligned}
J_{2,2} &= -\int \partial_1 \omega u \cdot \nabla b_2 \partial_2 \omega \ \mathrm{d}x \\
&\lesssim \|u\|_{L^\infty} \|\partial_2 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^2 \|b\|_{H^2} \|u\|_{H^2}.
\end{aligned}$$

By lemma 1.4,

$$
\begin{aligned}
J_{2,3} &= -\eta \int \partial_1 \omega b_2 \partial_2 \omega \ \mathrm{d}x \\
&\lesssim \|\partial_2\omega\|_{L^2} \|\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

Again, by lemmas 1.4 and 3.1,

$$
\begin{aligned}
J_{2,4} &= \int \partial_1 \omega b \cdot \nabla u_2 \partial_2 \omega \ \mathrm{d}x \\
&\lesssim \|b\|_{L^\infty} \|\partial_2\omega\|_{L^2} \|\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\nabla u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^1}^{\frac{1}{2}} \|\partial_2 b\|_{H^1}^{\frac{1}{2}} \|\partial_2\omega\|_{L^2} \|\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\nabla u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

To deal with $J_{2,1}$, we shift the time derivative,

$$
\begin{aligned}
J_{2,1} &= -\int \partial_1 \omega \partial_t b_2 \partial_2 \omega \ \mathrm{d}x \\
&= -\frac{d}{dt} \int \partial_1 \omega b_2 \partial_2 \omega \ \mathrm{d}x + \int \partial_1(\partial_t \omega) b_2 \partial_2 \omega \ \mathrm{d}x + \int \partial_1 \omega b_2 \partial_2(\partial_t \omega) \mathrm{d}x \\
&= J_{2,1,1} + J_{2,1,2} + J_{2,1,3}.
\end{aligned}
$$

By invoking the vorticity equation in (3.5), we can write

$$
\begin{aligned}
J_{2,1,2} &= \int \partial_1(-u \cdot \nabla\omega + \nu \partial_{22}\omega + b \cdot \nabla j + \partial_1 j) b_2 \partial_2 \omega \ \mathrm{d}x \\
&:= J_{2,1,2,1} + J_{2,1,2,2} + J_{2,1,2,3} + J_{2,1,2,4}.
\end{aligned}
$$

By integration by parts, and lemmas 1.4 and 3.1,

$$
\begin{aligned}
J_{2,1,2,1} &= \int u \cdot \nabla\omega \partial_1 b_2 \partial_2 \omega \ \mathrm{d}x + u \cdot \nabla \omega b_2 \partial_1 \partial_2 \omega \ \mathrm{d}x \\
&\lesssim \|u\|_{L^\infty} \|\nabla\omega\|_{L^2} \|\partial_2\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|u\|_{L^\infty} \|\partial_1\partial_2\omega\|_{L^2} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\nabla\omega\|_{L^2}^{\frac{1}{2}} \|b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 \|b\|_{H^2} \|u\|_{H^2},
\end{aligned}
$$

and

$$
\begin{aligned}
J_{2,1,2,2} &= -\nu \int \partial_{22}\omega \partial_1 b_2 \partial_2 \omega \ \mathrm{d}x - \nu \int \partial_{22}\omega b_2 \partial_1 \partial_2 \omega \ \mathrm{d}x \\
&\lesssim \|\partial_{22}\omega\|_{L^2} \|\partial_2\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} + \|b_2\|_{L^\infty} \|\partial_{22}\omega\|_{L^2} \|\partial_1\partial_2\omega\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^2 \|b\|_{H^2} + \|b_2\|_{H^1}^{\frac{1}{2}} \|\partial_1 b_2\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^2}^2 \\
&\lesssim \|\partial_2 u\|_{H^2}^2 \|b\|_{H^2}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}J_{2,1,2,3} &= -\int b\cdot\nabla j\partial_1 b_2\partial_2\omega\ \mathrm{d}x - \int b\cdot\nabla j b_2\partial_1\partial_2\omega\ \mathrm{d}x\\&\lesssim \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}}\\&\quad+\|b\|_{L^\infty}\|b_2\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_1\partial_2\omega\|_{L^2}\\&\lesssim \|b\|_{H^1}^{\frac{1}{2}}\|\partial_1 b\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}\|b\|_{H^2}^2+\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|b\|_{H^2}\|\partial_2 u\|_{H^2}\\&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^3\end{aligned}$$

and

$$\begin{aligned}J_{2,1,2,4} &= \int \partial_{11} j b_2\partial_2\omega\ \mathrm{d}x\\&= -\int \partial_1 j\partial_1 b_2\partial_2\omega\ \mathrm{d}x - \int \partial_1 j b_2\partial_1\partial_2\omega\ \mathrm{d}x\\&\lesssim \|\partial_1 j\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}}+\|b_2\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_1\partial_2\omega\|_{L^2}\\&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2+\|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_2\|_{H^1}^{\frac{1}{2}}\|b\|_{H^2}\|\partial_2 u\|_{H^2}\\&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2.\end{aligned}$$

To bound $J_{2,1,3}$, we invoke the vorticity equation in (3.5) again,

$$\begin{aligned}J_{2,1,3} &= \int \partial_2(-u\cdot\nabla\omega+\nu\partial_{22}\omega+b\cdot\nabla j+\partial_1 j)b_2\partial_1\omega\ \mathrm{d}x\\&:= J_{2,1,3,1}+J_{2,1,3,2}+J_{2,1,3,3}+J_{2,1,3,4}.\end{aligned}$$

By integration by parts, and lemmas 1.4 and 3.1,

$$\begin{aligned}J_{2,1,3,1} &= \int u\cdot\nabla\omega\partial_2 b_2\partial_1\omega+u\cdot\nabla\omega b_2\partial_2\partial_1\omega\ \mathrm{d}x\\&\lesssim \|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\\&\quad+\|u\|_{L^\infty}\|\partial_2\partial_1\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\omega\|_{L^2}^{\frac{1}{2}}\|b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\\&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\|b\|_{H^2}\|u\|_{H^2},\end{aligned}$$

$$\begin{aligned}J_{2,1,3,2} &= -\nu\int \partial_{22}\omega\partial_2 b_2\partial_1\omega\ \mathrm{d}x-\nu\int \partial_{22}\omega b_2\partial_2\partial_1\omega\ \mathrm{d}x\\&\lesssim \|\partial_{22}\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\\&\quad+\|b_2\|_{L^\infty}\|\partial_{22}\omega\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2}\\&\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}+\|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_2\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^2,\end{aligned}$$

$$
\begin{aligned}
J_{2,1,3,3} &= -\int b\cdot\nabla j\partial_2 b_2\partial_1\omega\ \mathrm{d}x - \int b\cdot\nabla j b_2\partial_2\partial_1\omega\ \mathrm{d}x \\
&\lesssim \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac12} \\
&\quad + \|b\|_{L^\infty}\|b_2\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2} \\
&\lesssim \|b\|_{H^1}^{\frac12}\|\partial_1 b\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^2 + \|b\|_{H^1}\|\partial_1 b\|_{H^1}\|b\|_{H^2}\|\partial_2 u\|_{H^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32} + \|\partial_2 u\|_{H^2}\|b\|_{H^2}^3,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{2,1,3,4} &= \int \partial_{12} j b_2\partial_1\omega\ \mathrm{d}x \\
&= -\int \partial_1 j\partial_2 b_2\partial_1\omega\ \mathrm{d}x - \int \partial_1 j b_2\partial_2\partial_1\omega\ \mathrm{d}x \\
&\lesssim \|\partial_1 j\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac12} + \|b_2\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^2 + \|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12}\|b\|_{H^2}\|\partial_2 u\|_{H^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12} + \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2.
\end{aligned}
$$

Collecting the estimates for $J_2$, we obtain

$$
\begin{aligned}
J_2 &\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1\omega b_2\partial_2\omega\ dx + C\|\partial_2 u\|_{H^2}^2\|b\|_{H^2}\|u\|_{H^2} + C\|\partial_2 u\|_{H^2}^2\|b\|_{H^2} \\
&\quad + C\|\partial_2 u\|_{H^2}^2\|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12} + C\|\partial_2 u\|_{H^2}^{\frac32}\|b\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12} \\
&\quad + C\|\partial_2 u\|_{H^2}^{\frac32}\|b\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac32}\|b\|_{H^2}^{\frac12} + C\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 \\
&\quad + C\|\partial_2 u\|_{H^2}\|b\|_{H^2}^3 + C\|\partial_2 u\|_{H^2}\|b\|_{H^2}^2 + C\|\partial_2 u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac32}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12}.
\end{aligned} \tag{3.15}
$$

The last two terms in (3.11) are $J_3$ and $J_4$. We now evaluate them. By lemma 1.4,

$$
\begin{aligned}
J_3 &= -\int \partial_2\omega\partial_2 u_1\partial_1\omega\ \mathrm{d}x \\
&\lesssim \|\partial_2\omega\|_{L^2}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_1\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12} \\
&\lesssim \|\partial_2 u\|_{H^2}^2\|u\|_{H^2}
\end{aligned} \tag{3.16}
$$

and

$$
\begin{aligned}
J_4 &= -\int \partial_2\omega\partial_2 u_2\partial_2\omega\ \mathrm{d}x \\
&\lesssim \|\partial_2\omega\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_2\partial_2\omega\|_{L^2}^{\frac12} \\
&\lesssim \|\partial_2 u\|_{H^2}^2\|u\|_{H^2}.
\end{aligned} \tag{3.17}
$$

Adding (3.4) and (3.6), integrating in time, and recalling the definition of $E$ in (3.2), we have

$$E(t) \leqslant E(0) + \int_0^t (J+K+L+M+N)\mathrm{d}\tau.$$

Collecting the upper bounds in (3.7)–(3.10), (3.14)–(3.17), we find

$$E(t) \leqslant E(0) - \int \partial_1 u_1 j \partial_1 \omega \ \mathrm{d}x + \int \partial_1 u_{01} j_0 \partial_1 \omega_0 \ \mathrm{d}x \tag{3.18}$$

$$- \int \partial_1 \omega b_2 \partial_2 \omega \ \mathrm{d}x + \int \partial_1 \omega_0 b_{02} \partial_2 \omega_0 \ \mathrm{d}x \tag{3.19}$$

$$+ CE^2(t) + CE^{\frac{3}{2}}(t). \tag{3.20}$$

The last two terms on (3.18) come from the time integral of the first term in (3.14), and the two terms on (3.19) are from the time integral of the first term in (3.15). The two terms on (3.20) are obtained by integrating the aforementioned upper bounds and applying Hölder's inequality. For example, when we integrate one of the upper bounds in (3.14), say $C\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2$,

$$\begin{aligned}\int_0^t C\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 \mathrm{d}\tau &\leqslant C \sup_{0\leqslant\tau\leqslant t} \|u(\tau)\|_{H^2}^2 \int_0^t \|\partial_2 u\|_{H^2}\|b\|_{H^2}\mathrm{d}\tau \\ &\leqslant CE(t)\left(\int_0^t \|\partial_2 u\|_{H^2}^2 \ \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t \|b\|_{H^2}^2 \ \mathrm{d}\tau\right)^{\frac{1}{2}}, \\ &\leqslant CE^2(t).\end{aligned}$$

The four terms on (3.18) and (3.19) can be further bounded as follows. By Hölder's inequality and lemma 1.4,

$$\begin{aligned}&- \int \partial_1 u_1 j \partial_1 \omega \ \mathrm{d}x + \int \partial_1 u_{01} j_0 \partial_1 \omega_0 \ \mathrm{d}x \\ &\quad\leqslant C\|\partial_1\omega\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\qquad + C\|\partial_1\omega_0\|_{L^2}\|\partial_1 u_{01}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_{01}\|_{L^2}^{\frac{1}{2}}\|j_0\|_{L^2}^{\frac{1}{2}}\|\partial_1 j_0\|_{L^2}^{\frac{1}{2}} \\ &\quad\leqslant CE^{\frac{3}{2}}(t) + CE^{\frac{3}{2}}(0).\end{aligned}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}&- \int \partial_1 \omega b_2 \partial_2 \omega \ \mathrm{d}x + \int \partial_1 \omega_0 b_{02} \partial_2 \omega_0 \mathrm{d}x \\ &\quad\leqslant C\|b_2\|_{L^\infty}\|\partial_1\omega\|_{L^2}\|\partial_2\omega\|_{L^2} + C\|b_{02}\|_{L^\infty}\|\partial_1\omega_0\|_{L^2}\|\partial_2\omega_0\|_{L^2} \\ &\quad\leqslant C\|b\|_{H^2}\|\partial_1\omega\|_{L^2}\|\partial_2\omega\|_{L^2} + C\|b_0\|_{H^2}\|\partial_1\omega_0\|_{L^2}\|\partial_2\omega_0\|_{L^2} \\ &\quad\leqslant CE^{\frac{3}{2}}(t) + CE^{\frac{3}{2}}(0).\end{aligned}$$

We have finally obtained (3.3), namely

$$E(t) \leqslant E(0) + C_1 E^{\frac{3}{2}}(0) + C_2 E^2(t) + C_3 E^{\frac{3}{2}}(t). \tag{3.21}$$

A bootstrapping argument applied to (3.21) would lead to the desired stability. We show, by the bootstrapping argument, that if the initial data is sufficiently small, say

$$\|(u_0, b_0)\|_{H^2} \leqslant \varepsilon,$$

with $\varepsilon$ satisfying

$$4\varepsilon^2 + 4C_1\varepsilon^3 \leqslant \delta_0 := \min\left\{\frac{1}{4C_2}, \frac{1}{(4C_3)^2}\right\},$$

then, for any $0 \leqslant t \leqslant \infty$,

$$\|(u(t), b(t))\|_{H^2}^2 \leqslant E(t) \leqslant \delta_0.$$

In fact, if we make the ansatz that, for $0 \leqslant t \leqslant T$,

$$E(t) \leqslant \delta_0,$$

then (3.21) implies

$$\begin{aligned} E(t) &\leqslant \varepsilon^2 + C_1\varepsilon^3 + C_2E(t)E(t) + C_3E^{\frac{1}{2}}(t)E(t) \\ &\leqslant \varepsilon^2 + C_1\varepsilon^3 + C_2\frac{1}{4C_2}E(t) + C_3\frac{1}{4C_3}E(t) \end{aligned}$$

or

$$\frac{1}{2}E(t) \leqslant \varepsilon^2 + C_1\varepsilon^3 \quad \text{or} \quad E(t) \leqslant 2\varepsilon^2 + 2C_1\varepsilon^3 = \frac{1}{2}\delta_0.$$

The bootstrapping argument then implies that $T = \infty$ and $E(t) \leqslant \delta_0$. This completes the proof for the stability part of theorem 1.3.

Next we prove the large-time behavior estimates stated in theorem 1.3, namely (1.20). We make use of lemma 2.1. The main efforts are devoted to verifying that

$$f(t) := \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2$$

satisfies (2.1), namely

$$\int_0^\infty f(t)\mathrm{d}t \leqslant C\varepsilon^2 < \infty \tag{3.22}$$

and, for any $0 \leqslant s < t$,

$$f(t) \leqslant Cf(s). \tag{3.23}$$

The proof of (3.23) is relatively easy while the proof of (3.22) is more complex. Since

$$\|\nabla u(t)\|_{L^2} = \|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b(t)\|_{L^2} = \|j\|_{L^2},$$

we resort to the equations of $\omega$ and $j$ in (3.5). By taking the inner product of (3.5) with $(\omega, j)$, we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \nu\|\partial_2\omega\|_{L^2}^2 + \eta\|j\|_{L^2}^2 = \int Qj\,\mathrm{d}x \\ &\quad = 2\int \partial_1b_1\partial_2u_1j\,\mathrm{d}x + 2\int \partial_1b_1\partial_1u_2j\,\mathrm{d}x - 2\int \partial_1u_1\partial_2b_1j\,\mathrm{d}x - 2\int \partial_1u_1\partial_1b_2j\,\mathrm{d}x. \end{aligned} \tag{3.24}$$

The four terms on the right-hand side can be estimated very similarly. We bound the second term as an example. By lemma 1.4,

$$\begin{aligned}2\int \partial_1 b_1 \partial_1 u_2 j\,\mathrm{d}x &\leqslant C\|\partial_1 b_1\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\\ &\leqslant \frac{\nu}{16}\|\partial_2\partial_1 u_2\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}}\\ &\leqslant \frac{\nu}{16}\|\partial_2\omega\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}}.\end{aligned}$$

The other three terms obey the same bound. Invoking these bounds in (3.24) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \frac{3\nu}{2}\|\partial_2\omega\|_{L^2}^2 + 2\left(\eta - C\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}}\right)\|j\|_{L^2}^2 \leqslant 0. \tag{3.25}$$

According to the first part of our proof, if the initial data $(u_0, b_0)$ satisfies

$$\|(u_0, b_0)\|_{H^2} \leqslant \varepsilon$$

for sufficiently small $\varepsilon > 0$, the solution $(u, b)$ remains small,

$$\|(u(t), b(t))\|_{H^2} \leqslant C\varepsilon.$$

When $\varepsilon > 0$ is taken to be small enough such that

$$\eta - C\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}} \geqslant \eta - C\varepsilon^{\frac{4}{3}} \geqslant 0,$$

then (3.25) implies, for any $0 \leqslant s < t$,

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \leqslant \|\omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2 \quad \text{or} \quad f(t) \leqslant f(s).$$

We now prove (3.22). We have shown in the previous part that

$$\int_0^\infty \|\partial_2 u(t)\|_{H^2}^2\,\mathrm{d}t \leqslant C\varepsilon^2, \qquad \int_0^\infty \|b(t)\|_{H^2}^2\,\mathrm{d}t \leqslant C\varepsilon^2. \tag{3.26}$$

To prove (3.22), it remains to prove

$$\int_0^\infty \|\partial_1 u\|_{L^2}^2\,\mathrm{d}t \leqslant C\varepsilon^2. \tag{3.27}$$

The proof for this upper bound is not trivial. We need to take advantage of the regularization of the magnetic field. We replace one of $\partial_1 u$ in (3.27) by the equation of the magnetic field

$$\partial_1 u = \partial_t b + u\cdot\nabla b + \eta b - b\cdot\nabla u$$

and obtain

$$\begin{aligned}\|\partial_1 u\|_{L^2}^2 &= \int \partial_1 u\cdot\partial_1 u\,\mathrm{d}x = \int \partial_1 u\cdot(\partial_t b + u\cdot\nabla b + \eta b - b\cdot\nabla u)\mathrm{d}x\\ &:= N_1 + N_2 + N_3 + N_4.\end{aligned} \tag{3.28}$$

We first estimate $N_2$, $N_3$ and $N_4$ and then come back to $N_1$. We write out all component terms explicitly,

$$\begin{aligned} N_2 &= \int \partial_1 u \cdot (u \cdot \nabla b) \mathrm{d}x = \int (\partial_1 u_1 (u \cdot \nabla) b_1 + \partial_1 u_2 u \cdot \nabla b_2) \, \mathrm{d}x \\ &= \int ((-\partial_2 u_2)(u \cdot \nabla) b_1 + \partial_1 u_2 (u_1 \partial_1 b_2 + u_2 \partial_2 b_2)) \, \mathrm{d}x \\ &= N_{2,1} + N_{2,2} + N_{2,3}. \end{aligned}$$

By lemma 1.4,

$$\begin{aligned} N_{2,1} &= \int (-\partial_2 u_2)(u \cdot \nabla) b_1 \ \mathrm{d}x \\ &\leqslant C \|\partial_2 u_2\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\nabla b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_1\|_{L^2}^{\frac{1}{2}} \leqslant C \|u\|_{H^2} \|\partial_2 u\|_{H^2} \|b\|_{H^2}. \end{aligned}$$

By integration by parts and lemma 1.4,

$$\begin{aligned} N_{2,2} &= -\int (u_2 \partial_1 u_1 \partial_1 b_2 \ \mathrm{d}x + u_2 u_1 \partial_{11} b_2) \mathrm{d}x \\ &\leqslant C \|u_2\|_{L^2} \|\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_{11} b_2\|_{L^2} \|\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|u_2\|_{L^2}^{\frac{1}{2}} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \\ &\leqslant C \|u\|_{H^2} \|\partial_2 u\|_{H^2} \|b\|_{H^2}. \end{aligned}$$

Again, by integration by parts and lemma 1.4,

$$\begin{aligned} N_{2,3} &= \int \partial_1 u_2 u_2 \partial_2 b_2 \ \mathrm{d}x = \int \partial_1 (\frac{u_2^2}{2}) \partial_2 b_2 \ \mathrm{d}x \\ &= \int \partial_2 (\frac{u_2^2}{2}) \partial_1 b_2 \ \mathrm{d}x = \int u_2 \partial_2 u_2 \partial_1 b_2 \ \mathrm{d}x \\ &\leqslant C \|u\|_{H^2} \|\partial_2 u\|_{H^2} \|b\|_{H^2}. \end{aligned}$$

Clearly,

$$N_3 = \eta \int \partial_1 u \cdot b \ \mathrm{d}x \leqslant C \|\partial_1 u\|_{L^2} \|b\|_{L^2} \leqslant \frac{1}{4} \|\partial_1 u\|_{L^2}^2 + C \|b\|_{L^2}^2.$$

To bound $N_4$, we again write out the component terms explicitly,

$$\begin{aligned} N_4 &= -\int \partial_1 u \cdot (b \cdot \nabla u) \mathrm{d}x = -\int \partial_1 u_1 (b \cdot \nabla) u_1 \ \mathrm{d}x - \int \partial_1 u_2 (b \cdot \nabla) u_2 \ \mathrm{d}x \\ &= \int \partial_2 u_2 (b \cdot \nabla) u_1 \ \mathrm{d}x - \int \partial_1 u_2 b_1 \partial_1 u_2 \ \mathrm{d}x - \int \partial_1 u_2 b_2 \partial_2 u_2 \ \mathrm{d}x \\ &= N_{4,1} + N_{4,2} + N_{4,3}. \end{aligned}$$

By lemma 1.4,

$$\begin{aligned}
N_{4,1}+N_{4,3}&=\int \partial_2 u_2(b\cdot\nabla)u_1\ \mathrm{d}x-\int \partial_1 u_2 b_2\partial_2 u_2\ \mathrm{d}x\\
&\leqslant C\|\partial_2 u_2\|_{L^2}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u_1\|_{L^2}^{\frac{1}{2}}\\
&\quad+C\|\partial_2 u_2\|_{L^2}\|b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\\
&\leqslant C\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|b\|_{H^2}.
\end{aligned}$$

By Sobolev's inequality,

$$\begin{aligned}
N_{4,2}&=-\int(\partial_1 u_2)^2 b_1\ \mathrm{d}x\leqslant C\|b\|_{L^2}\|\partial_1 u_2\|_{L^4}^2\\
&\leqslant C\|b\|_{L^2}\|\partial_1 u_2\|_{L^2}\|\nabla\partial_1 u_2\|_{L^2}\leqslant\frac{1}{4}\|\partial_1 u_2\|_{L^2}^2+C\|b\|_{H^2}^2\|u\|_{H^2}^2.
\end{aligned}$$

We now return to estimate $N_1$. Shifting the time integral and invoking (1.1), we obtain

$$\begin{aligned}
N_1&=\int\partial_1 u\cdot\partial_t b\ \mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int\partial_1 u\cdot b\ \mathrm{d}x+\int\partial_1(u\cdot\nabla u+\nabla P-\nu\partial_{22}u-b\cdot\nabla b-\partial_1 b)\cdot b\ \mathrm{d}x\\
&:=N_{1,1}+N_{1,2}+N_{1,3}+N_{1,4}+N_{1,5}+N_{1,6}.
\end{aligned}$$

$N_{1,1}$ is the time derivative term and we bound it later after we integrate it in time. To estimate $N_{1,2}$, we rewrite it into sums of component terms to reveal the terms with favorable partial derivative such as $\partial_2 u$,

$$\begin{aligned}
N_{1,2}&=\int\partial_1(u\cdot\nabla u)\cdot b\ \mathrm{d}x=\int(\partial_1 u\cdot\nabla u)\cdot b\ \mathrm{d}x+\int(u\cdot\nabla\partial_1 u)\cdot b\ \mathrm{d}x\\
&=\int\partial_1 u_1\partial_1 u\cdot b\ \mathrm{d}x+\int\partial_1 u_2\partial_2 u\cdot b\ \mathrm{d}x+\int u_1\partial_{11}u_1 b_1\ \mathrm{d}x\\
&\quad+\int u_1\partial_{11}u_2 b_2\ \mathrm{d}x+\int u_2\partial_2\partial_1 u\cdot b\ \mathrm{d}x\\
&=\int(-\partial_2 u_2)\partial_1 u\cdot b\ \mathrm{d}x+\int\partial_1 u_2\partial_2 u\cdot b\ \mathrm{d}x+\int u_1(-\partial_{21}u_2)b_1\ \mathrm{d}x\\
&\quad+\int(\partial_1(u_1\partial_1 u_2)b_2-\partial_1 u_1\partial_1 u_2 b_2)\ \mathrm{d}x+\int u_2\partial_2\partial_1 u\cdot b\ \mathrm{d}x\\
&=\int(-\partial_2 u_2)\partial_1 u\cdot b\ \mathrm{d}x+\int\partial_1 u_2\partial_2 u\cdot b\ \mathrm{d}x+\int u_1(-\partial_{21}u_2)b_1\ \mathrm{d}x\\
&\quad-\int(u_1\partial_1 u_2)\partial_1 b_2\ \mathrm{d}x+\int\partial_2 u_2\partial_1 u_2 b_2\ \mathrm{d}x+\int u_2\partial_2\partial_1 u\cdot b\ \mathrm{d}x.
\end{aligned}$$

By Sobolev's inequality and lemma 1.4,

$$\begin{aligned}
N_{1,2}&\leqslant\|\partial_2 u_2\|_{L^2}\|\partial_1 u\|_{L^4}\|b\|_{L^4}+\|\partial_1 u_2\|_{L^4}\|\partial_2 u\|_{L^2}\|b\|_{L^4}\\
&\quad+\|u_1\|_{L^4}\|\partial_{21}u_2\|_{L^2}\|b_1\|_{L^4}+\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

$$\times \|\partial_1 b_2\|_{L^2} + \|\partial_2 u_2\|_{L^2}\|\partial_1 u_2\|_{L^4}\|b_2\|_{L^4} + \|u_2\|_{L^4}\|\partial_2\partial_1 u\|_{L^2}\|b\|_{L^4}$$
$$\leqslant C\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}.$$

$N_{13}$ contains the pressure term $P$. By (3.13),

$$\begin{aligned}
N_{1,3} &= \int \partial_1 \nabla P \cdot b \, \mathrm{d}x = -\int \nabla P \cdot \partial_1 b \, \mathrm{d}x \\
&= -\int \nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b)\cdot\partial_1 b \, \mathrm{d}x \\
&= -\int \nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u)\cdot\partial_1 b \, \mathrm{d}x + \int \nabla(-\Delta)^{-1}\nabla\cdot(b\cdot\nabla b)\cdot\partial_1 b \, \mathrm{d}x \\
&= -\int \nabla(-\Delta)^{-1}(\partial_1\partial_1(u_1^2) + 2\partial_1\partial_2(u_1u_2) + \partial_2\partial_2(u_2^2))\cdot\partial_1 b \, \mathrm{d}x \\
&\quad + \int \nabla(-\Delta)^{-1}\nabla\cdot(b\cdot\nabla b)\cdot\partial_1 b \, \mathrm{d}x.
\end{aligned}$$

By Hölder's inequality and using the fact that the singular integral operators are bounded on $L^p$ for $1 < p < \infty$, namely

$$\|\nabla(-\Delta)^{-1}\partial_1 f\|_{L^p} \leqslant C\|f\|_{L^p}, \qquad \|\nabla(-\Delta)^{-1}\partial_2 f\|_{L^p} \leqslant C\|f\|_{L^p},$$

we have

$$\begin{aligned}
N_{1,3} &\leqslant C(\|\partial_1(u_1^2)\|_{L^2} + \|\partial_2(u_1u_2)\|_{L^2} + \|\partial_2(u_2^2)\|_{L^2})\|\partial_1 b\|_{L^2} + C\|b\cdot\nabla b\|_{L^2}\|\partial_1 b\|_{L^2} \\
&\leqslant C(\|u_1\|_{L^\infty}\|\partial_1 u_1\|_{L^2} + \|u\|_{L^\infty}\|\partial_2 u\|_{L^2})\|\partial_1 b\|_{L^2} + C\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|\partial_1 b\|_{L^2} \\
&\leqslant C\|\partial_2 u_2\|_{H^2}\|u\|_{H^2}\|b\|_{H^2} + C\|b\|_{H^2}^3.
\end{aligned}$$

We now estimate the rest of the terms. By integration by parts,

$$\begin{aligned}
N_{1,4} &= -\nu\int \partial_1\partial_{22} u\cdot b \, \mathrm{d}x = \nu\int \partial_{22} u\cdot\partial_1 b \, \mathrm{d}x \\
&\leqslant C\|\partial_2 u\|_{H^1}\|b\|_{H^2}, \\
N_{1,5} &= -\int \partial_1(b\cdot\nabla b)\cdot b \, \mathrm{d}x = -\int (\partial_1 b\cdot\nabla b)\cdot b \, \mathrm{d}x - (b\cdot\partial_1\nabla b)\cdot b \, \mathrm{d}x \\
&\leqslant C\|b\|_{H^2}^3, \\
N_{1,6} &= -\int \partial_{11} b\cdot b \, \mathrm{d}x \leqslant C\|b\|_{H^2}^2.
\end{aligned}$$

Collecting the upper bounds for $N_1$ through $N_4$ and inserting them in (3.28), we find

$$\begin{aligned}
\|\partial_1 u(t)\|_{L^2}^2 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1 u\cdot b \, \mathrm{d}x + C\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2} + C\|b\|_{H^2}^3 \\
+ C\|\partial_2 u\|_{H^1}\|b\|_{H^2} + C\|b\|_{H^2}^2 + \frac{1}{2}\|\partial_1 u\|_{L^2}^2 + C\|b\|_{H^2}^2\|u\|_{H^2}^2.
\end{aligned}$$

Combining some of the terms and integrating in time, we obtain, for any $T > 0$,

$$\begin{aligned}\int_0^T \|\partial_1 u(t)\|_{L^2}^2\,\mathrm{d}t &\leqslant 2\int(\partial_1 u\cdot b)(x,T)\mathrm{d}x - 2\int \partial_1 u_0\cdot b_0\,\mathrm{d}x\\ &\quad + C\int_0^T \big(\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2} + \|b\|_{H^2}^3 + \|\partial_2 u\|_{H^1}\|b\|_{H^2}\\ &\quad + \|b\|_{H^2}^2 + \|b\|_{H^2}^2\|u\|_{H^2}^2\big)\,\mathrm{d}t\\ &\leqslant 2\|\partial_1 u(T)\|_{L^2}\|b(T)\|_{L^2} + 2\|\partial_1 u_0\|_{L^2}\|b_0\|_{L^2}\\ &\quad + C\sup_{0\leqslant t\leqslant T}\|(u,b)(t)\|_{H^2}\int_0^T(\|\partial_2 u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2)\mathrm{d}t\\ &\quad + C(1+\sup_{0\leqslant t\leqslant T}\|u(t)\|_{H^2}^2)\int_0^T(\|\partial_2 u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2)\mathrm{d}t\\ &\leqslant C(\varepsilon^2+\varepsilon^3+\varepsilon^4). \end{aligned} \tag{3.29}$$

Since the upper bound in (3.29) is uniform in time, we have thus verified (3.27), which, together with (3.26), confirms (3.22). This completes the proof of theorem 1.3. □

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## ORCID iDs

Wen Feng https://orcid.org/0000-0002-9904-9183
Jiahong Wu https://orcid.org/0000-0001-9496-9709

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