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Eventual Regularity of the Two-Dimensional Boussinesq Equations with Supercritical Dissipation

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Abstract This paper studies solutions of the two-dimensional incompressible Boussinesq equations with fractional dissipation. The spatial domain is a periodic box. The Boussinesq equations concerned here govern the coupled evolution of the fluid velocity and the temperature and have applications in fluid mechanics and geophysics. When the dissipation is in the supercritical regime (the sum of the fractional powers of the Laplacians in the velocity and the temperature equations is less than 1), the classical solutions of the Boussinesq equations are not known to be global in time. Leray–Hopf type weak solutions do exist. This paper proves that such weak solutions become eventually regular (smooth after some time $T > 0$) when the fractional Laplacian powers are in a suitable supercritical range. This eventual regularity is established by exploiting the regularity of a combined quantity of the vorticity and the temperature as well as the eventual regularity of a generalized supercritical surface quasi-geostrophic equation.

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1 Introduction

We consider the following two-dimensional (2D) incompressible Boussinesq equations with fractional dissipation in the periodic box $\mathbb{T}^2 \equiv [0, 2\pi]^2$,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^{\alpha} u = -\nabla p + \theta \mathbf{e}_2, \qquad x \in \mathbb{T}^2,\ t > 0, \\ \nabla \cdot u = 0, \qquad x \in \mathbb{T}^2,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^{\beta} \theta = 0, \qquad x \in \mathbb{T}^2,\ t > 0, \\ u(x, 0) = u_0(x),\ \theta(x, 0) = \theta_0(x), \qquad x \in \mathbb{T}^2, \end{cases} \tag{1.1}$$

where $u = u(x, t)$ represents the 2D velocity, $p = p(x, t)$ the pressure, $\theta = \theta(x, t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu > 0$, $\kappa > 0$, $\alpha > 0$ and $\beta > 0$ are real parameters. Here $\Lambda = \sqrt{-\Delta}$ represents the Zygmund operator with Λ^{α} being defined through the Fourier transform, namely

$$\widehat{\Lambda^{\alpha} f}(\xi) = |\xi|^{\alpha}\, \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-ix\cdot\xi}\, f(x)\, \mathrm{d}x.$$

When $\alpha = 2$ and $\beta = 2$, (1.1) becomes the 2D Boussinesq equations with standard dissipation. The standard 2D Boussinesq equations and their fractional Laplacian generalizations have attracted considerable attention recently due to their physical applications and mathematical significance. The Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Raleigh-Bernard convection (see, e.g., Constantin and Doering 1999; Gill 1982; Majda 2003; Pedlosky 1987). Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier–tokes and the Euler equations such as the vortex stretching mechanism. As pointed out in Majda and Bertozzi (2001), the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows.

One main focus of recent research on the 2D Boussinesq equations has been on the global regularity issue when only fractional dissipation is present (see, e.g., Adhikari et al. 2010, 2011; Cao and Wu 2013; Constantin and Vicol 2012; Cui et al. 2012; Danchin and Paicu 2009, 2011; Shu 1994; Hmidi 2011; Hmidi et al. 2010, 2011; Hou and Li 2005; Jiu et al. 0000; Kc et al. 2014, 0000; Lai et al. 2011; Larios et al. 2013; Miao and Xue 2011; Moffatt 2001; Ohkitani 2001; Wu and Wu 0000; Wu 2010; Zhao

2010). The global regularity of solutions to (1.1) relies crucially on the sizes of α and β and it is helpful to divide α and β into three cases:

(i) the subcritical case, $\alpha + \beta > 1$;
(ii) the critical case, $\alpha + \beta = 1$;
(iii) the supercritical case, $\alpha + \beta < 1$.

The smaller the sum $\alpha + \beta$ is, the more difficult the global regularity problem seems to be. The global regularity for several subcritical cases was obtained in Constantin and Vicol (2012), Miao and Xue (2011), and Wu (2010). In Constantin and Vicol (2012), Constantin and Vicol verified the global regularity for the case

$$\nu > 0, \quad \kappa > 0, \quad \alpha \in (0, 2), \quad \beta \in (0, 2), \quad \beta > \frac{2}{2+\alpha}.$$

Miao and Xue in 2011 proved the global existence and uniqueness for (1.1) with $\nu > 0$, $\kappa > 0$ and

$$\alpha \in \left(\frac{6-\sqrt{6}}{4}, 1\right), \quad \beta \in \left(1-\alpha, \min\left\{\frac{7+2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2-2\alpha\right\}\right).$$

Hmidi et al. (2010, 2011) were able to establish the global regularity for two critical cases: (1.1) with $\alpha = 1$ and $\kappa = 0$ and (1.1) with $\nu = 0$ and $\beta = 1$. Jiu et al. (in press) recently examined the general critical case $\alpha + \beta = 1$ and obtained the global existence and uniqueness of classical solutions of (1.1) with $\alpha \geq \alpha_0$, where $\alpha_0 = \frac{23-\sqrt{145}}{12} \approx 0.9132$. The global regularity problem for other critical cases remains open.

Very little is known about the supercritical case $\alpha + \beta < 1$. The global regularity problem appears to be out of reach when α and β are in this regime. This paper shows that Leray-Hopf weak solutions do exist for all time. The major goal of this paper is the eventual regularity of such weak solutions. In fact, we show that weak solutions are actually smooth for $t > T$, where $T > 0$ depends on the initial data and the indices α and β. More precisely, we have the following theorem.

Theorem 1.1 *Consider the initial-value problem* (1.1) *with* $\nu > 0$, $\kappa > 0$, $\alpha > \alpha_0$, $\beta > 0$ *and* $\alpha + \beta < 1$, *where*

$$\alpha_0 = \frac{23-\sqrt{145}}{12} \approx 0.9132. \tag{1.2}$$

Assume that $(u_0, \theta_0) \in H^s(\mathbb{T}^2)$ *with* $s > 2$, *and* u_0 *and* θ_0 *have zero mean. Let* (u, θ) *be a global weak solution of* (1.1). *Then, there exist* $0 < T_1 \leq T_2 < \infty$ *such that* (u, θ) *is actually a classical solution on* $[0, T_1]$ *and on* $[T_2, \infty)$.

This theorem reflects the regularization effect of the dissipation even in the supercritical case. Since the existence of $T_1 > 0$ follows directly from the local well-posedness of classical solutions, the efforts of proving Theorem 1.1 is solely devoted to showing the existence of $T_2 > 0$ such that (u, θ) is actually a classical solution on $[T_2, \infty)$. This is not trivial and the energy method does not provide any global bounds controlling the derivatives of (u, θ). Following Hmidi et al. (2010) and Jiu et al. (in press), we resort to the regularity of a new function

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha} \partial_1,$$

which satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla]\theta + \Lambda^{\beta-\alpha} \partial_1 \theta. \tag{1.3}$$

Here ω denotes the vorticity $\omega = \nabla \times u$ and we have used the standard commutator notation

$$[\mathcal{R}_\alpha, u \cdot \nabla]\theta = \mathcal{R}_\alpha(u \cdot \nabla\theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.$$

(1.3) can be obtained by taking the difference of the vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega + \nu \Lambda^\alpha \omega = \partial_1 \theta, \\ u = \nabla^\perp \psi, \quad \Delta\psi = \omega \qquad \text{or} \quad u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \tag{1.4}$$

and the resulting equation after applying $\mathcal{R}_\alpha$ to the temperature equation. As shown in Jiu et al. (in press), for $\alpha_0 < \alpha$, G is globally regular in the sense that

$$\widetilde{u} \equiv \nabla^\perp \Delta^{-1} G$$

is actually Lipschitz. As a consequence, the velocity can be decomposed into

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta \equiv \widetilde{u} + v. \tag{1.5}$$

where $v \equiv \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta$ in can be written more explicitly as

$$v = \nabla^\perp \Delta^{-1} \Lambda^{-\alpha}\, \partial_1\, \theta.$$

Therefore, the temperature equation becomes a generalized supercritical surface quasi-geostrophic (SQG) type equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta + \Lambda^\beta \theta = 0, \quad x \in \mathbb{T}^2,\ t > 0, \\ u = \widetilde{u} + v, \quad \Lambda^\alpha v = -\nabla^\perp \Lambda^{-2} \partial_1 \theta, \quad x \in \mathbb{T}^2,\ t > 0, \\ \theta(x, 0) = \theta_0(x), \quad x \in \mathbb{T}^2, \end{cases} \tag{1.6}$$

where $\alpha + \beta < 1$. The SQG equation with critical or supercritical dissipation has been investigated extensively recently (see, e.g., Caffarelli and Vasseur 2010; Chen et al. 2007; Constantin 2006; Constantin et al. 2008, 1994; Constantin and Vicol 2012; Constantin and Wu 2008, 2009; Córdoba and Córdoba 2004; Córdoba and Fefferman

2002; Dabkowski 2011; Dong and Pavlović 2009; Kiselev 2011; Kiselev et al. 2007; Miao et al. 2012; Silvestre 2010). Especially eventual regularity results have been established for the supercritical SQG equation. (1.6) can be treated as a generalized SQG equation with supercritical dissipation. Following the work of Dabkowski (2011), we can show that weak solutions of (1.6) are eventually regular. More precisely, we have the following proposition.

Proposition 1.2 *Consider* (1.6) *with* $\alpha > 0$, $\beta > 0$ *and* $\alpha + \beta < 1$. *Assume that* $\widetilde{u}$ *satisfies*

$$M \equiv \|\nabla \widetilde{u}\|_{L^\infty_{t,loc} L^\infty_x} < \infty. \tag{1.7}$$

Assume $\theta_0 \in H^s(\mathbb{T}^2)$ *with* $s > 2$ *and has zero mean. Let* θ *be a Leray-Hopf weak solution of (1.6). Then there exists* $T > 0$ *such that* $\theta \in L^\infty([T, \infty), C^\sigma(\mathbb{T}^2))$ *for some* $\sigma > 1 - \alpha - \beta$.

Here C^σ denotes the standard Hölder space. It then follows from a regularity result in Constantin and Wu (2008) that $\theta \in C^\infty(\mathbb{T}^2 \times [T, \infty))$. This leads to the regularity of u and ω according to (1.4), and especially Theorem 1.1.

The rest of this paper is divided into three regular sections and one appendix. Section 2 presents the global existence of weak solutions to (1.1). Section 3 proves Theorem 1.1 assuming Proposition 1.2 while Sect. 4 establishes Proposition 1.2. A key component in the proof of Proposition 1.2 is stated as a proposition and proved in Appendix 1.

2 A Global Weak Solution of (1.1)

The statement of Theorem 1.1 involves the global existence of a weak solution of (1.1). This section provides a proof for this fact, which is stated here as a proposition.

Proposition 2.1 *Consider* (1.1) *with* $\nu > 0$, $\kappa > 0$, $\alpha > 0$ *and* $\beta > 0$. *Assume that* $(u_0, \theta_0) \in L^2(\mathbb{T}^2)$ *and* u_0 *and* θ_0 *have zero mean. Then* (1.1) *has a global weak solution* $u \in C_w([0, T]; L^2) \cap L^2([0, T]; H^{\frac{\alpha}{2}})$ *for any* $T > 0$ *and* $\theta \in C_w([0, \infty); L^2) \cap L^2([0, \infty); \dot{H}^{\frac{\beta}{2}})$.

Here C_w denotes the continuity in the weak L^2 sense and $\dot{H}^s$ denotes the homogeneous Sobolev space. The weak solution is defined in the standard sense. We recall it for reader's convenience.

Definition 2.2 Let $T > 0$. A function pair (u, θ) satisfying $u \in C_w([0, T]; L^2) \cap L^2([0, T]; H^{\frac{\alpha}{2}})$ and $\theta \in C_w([0, T]; L^2) \cap L^2([0, T]; \dot{H}^{\frac{\beta}{2}})$ is said to be a weak solution of (1.1) if it obeys the following conditions:

(1) For any vector field $\phi \in C_0^\infty(\mathbb{T}^2 \times [0, T])$ with $\nabla \cdot \phi = 0$, for any scalar function $\varphi \in C_0^\infty(\mathbb{T}^2 \times [0, T])$ and for a.e. $t \in [0, T]$,

$$\begin{aligned}
&\int u(x,t) \cdot \phi(x,t)\, \mathrm{d}x - \int u_0(x) \cdot \phi(x,0)\, \mathrm{d}x - \int_0^t \int u \cdot \partial_\tau \phi\, \mathrm{d}x \mathrm{d}\tau \\
&\quad = -\nu \int_0^t \int u \cdot \Lambda^\alpha \phi\, \mathrm{d}x \mathrm{d}\tau + \int_0^t \int (u \otimes u) : \nabla \phi\, \mathrm{d}x \mathrm{d}\tau \\
&\qquad + \int_0^t \int \phi \cdot (\theta \mathbf{e}_2)\, \mathrm{d}x \mathrm{d}\tau,
\end{aligned}$$

and

$$\begin{aligned}
&\int \theta(x,t)\, \varphi(x,t)\, \mathrm{d}x - \int \theta_0(x)\, \varphi(x,0)\, \mathrm{d}x - \int_0^t \int \theta \cdot \partial_\tau \varphi\, \mathrm{d}x \mathrm{d}\tau \\
&\quad = -\nu \int_0^t \int \theta\, \Lambda^\beta \varphi\, \mathrm{d}x \mathrm{d}\tau - \int_0^t \int \theta u \cdot \nabla \varphi\, \mathrm{d}x \mathrm{d}\tau.
\end{aligned}$$

(2) For any scalar function $\eta \in C_0^\infty(\mathbb{T}^2)$ and for a.e. $t \in [0, T]$,

$$\int u(x,t) \cdot \nabla \eta(x)\, \mathrm{d}x = 0.$$

Proposition 2.1 is proven through the Galerkin approximation. For an integer $N \geq 1$ and for $f \in L^2(\mathbb{T}^2)$, J_N is a Fourier truncation operator defined by

$$J_N f(x) = \sum_{|m| \leq N} \widehat{f}(m)\, e^{im\cdot x},$$

where $\widehat{f}$ denotes the Fourier mode of f,

$$\widehat{f}(m) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-im\cdot x}\, f(x)\, \mathrm{d}x.$$

In addition,

$$L_N^2 \equiv \left\{ f \in L^2(\mathbb{T}^2) : f(x) = \sum_{|m| \leq N} \widehat{f}(m)\, e^{im\cdot x} \right\}.$$

It is clear that $f \in L_N^2$ implies that $f \in H^s(\mathbb{T}^2)$ for any $s > 0$. We also write $\mathbb{P}$ for the Leray projection operator onto divergence-free vector fields, namely

$$\mathbb{P} = I - \nabla(-\Delta)^{-1} \nabla \cdot$$

with I being the identity operator. The following Picard type local existence and extension result will be used in the proof (see, e.g., Majda and Bertozzi 2001, pp. 100–103).

Lemma 2.3 *Let Y be a Banach space and let $O \subset Y$ be an open subset. Let $F : O \to Y$ be a mapping satisfying*

(1) *F maps O to Y;*
(2) *F is locally Lipschitz, namely, for any $f \in O$, there exists $L = L(f)$ and a neighborhood $U(f)$ such that*

$$\|F(f) - F(g)\|_Y \le L(f)\|f - g\|_Y \quad \textit{for any } g \in U(f).$$

Then, for any $f_0 \in O$, the ODE

$$\frac{\mathrm{d}f}{\mathrm{d}t} = F(f), \quad f(0) = f_0$$

has a unique local solution $f \in C^1([0, T]; O)$ for some $T > 0$. Furthermore, either $f = f(t)$ exists for all time or $T < \infty$ and $f(t)$ leaves O as $t \to T$.

We will now prove Proposition 2.1.

Proof The proof is divided into three main steps. The first step establishes the global existence of an approximate solution. Fix an integer $N \ge 1$ and consider $(u^N, \theta^N) \in (L^2_N \times L^2_N) \times L^2_N$ (abbreviated as L^2_N later) solving the following system of equations

$$\begin{cases} \partial_t u^N + \mathbb{P}J_N(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N) + \nu\, \Lambda^\alpha \mathbb{P}J_N u^N = \mathbb{P}J_N(\theta^N \mathbf{e}_2), \\ \partial_t \theta^N + J_N(\mathbb{P}J_N u^N \cdot \nabla J_N \theta^N) + \kappa\, \Lambda^\beta J_N \theta^N = 0, \\ u^N(x, 0) = u_0^N(x) \equiv J_N u_0(x), \quad \theta^N(x, 0) = \theta_0^N(x) = J_N \theta_0(x). \end{cases} \tag{2.1}$$

We apply Lemma 2.3 with $Y = O = L^2_N$, $f = (u^N, \theta^N)$ and

$$F(f) = \begin{bmatrix} -\mathbb{P}J_N(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N) - \nu\, \Lambda^\alpha \mathbb{P}J_N u^N + \mathbb{P}J_N(\theta^N \mathbf{e}_2) \\ -J_N(\mathbb{P}J_N u^N \cdot \nabla J_N \theta^N) - \kappa\, \Lambda^\beta J_N \theta^N \end{bmatrix}.$$

We check that $F(f)$ defined above indeed verifies the conditions in Lemma 2.3. For any $f = (u^N, \theta^N) \in L^2_N$, we have

$$\begin{aligned} \|F(f)\|_{L^2} &= \|\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N\|_{L^2} + \nu \|\Lambda^\alpha \mathbb{P}J_N u^N\|_{L^2} + \|\mathbb{P}J_N(\theta^N \mathbf{e}_2)\|_{L^2} \\ &\quad + \|J_N(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N \theta^N)\|_{L^2} + \kappa\, \|\Lambda^\beta J_N \theta^N\|_{L^2} \\ &\le \left(1 + \nu N^\alpha + \kappa N^\beta\right)\|f\|_{L^2} + C\, N^2 \|f\|_{L^2}^2, \end{aligned}$$

where we have used the simple estimates that, for a constant independent of N,

$$\begin{aligned}\|\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N\|_{L^2} &\le \|\mathbb{P}J_N u^N\|_{L^\infty}\, \|\nabla \mathbb{P}J_N u^N\|_{L^2}\\ &\le \|\widehat{\mathbb{P}J_N u^N}\|_{L^1}\, N\, \|\mathbb{P}J_N u^N\|_{L^2}\\ &\le C\,N^2\, \|\widehat{\mathbb{P}J_N u^N}\|_{L^2}\, \|\mathbb{P}J_N u^N\|_{L^2}\\ &\le C\,N^2\, \|f\|_{L^2}^2\end{aligned}$$

together with $\|\widehat{\mathbb{P}J_N u^N}\|_{L^1} \le CN\, \|\widehat{\mathbb{P}J_N u^N}\|_{L^2}$, which follows from

$$\|\widehat{\mathbb{P}J_N u^N}\|_{L^1} = \sum_{|m|\le N} |\widehat{\mathbb{P}u^N}(m)| \le CN \left[\sum_{|m|\le N} |\widehat{\mathbb{P}u^N}(m)|^2\right]^{1/2} = CN\, \|\widehat{\mathbb{P}J_N u^N}\|_{L^2}.$$

Therefore, F maps Y to Y. In addition, for $g = (U^N, \Theta^N)$, we can check that

$$\begin{aligned}\|F(f) - F(g)\|_Y &\le \nu N^\alpha \|u^N - U^N\|_{L^2} + (1+\kappa N^\beta)\|\theta^N - \Theta^N\|_{L^2}\\ &\quad + C\,N^2 \left(\|u^N\|_{L^2} + \|\theta^N\|_{L^2} + \|U^N\|_{L^2} + \|\Theta^N\|_{L^2}\right)\\ &\quad \times \left(\|u^N - U^N\|_{L^2} + \|\theta^N - \Theta^N\|_{L^2}\right)\\ &\le \left(1 + \nu N^\alpha + \kappa N^\beta + C\,N^2(\|f\|_{L^2} + \|g\|_{L^2})\right) \|f-g\|_{L^2},\end{aligned}$$

which verifies the local Lipschitz continuity. Thus we obtain a unique local solution $(u^N, \theta^N) \in L^2_N$ of (2.1) on a time interval $[0, T_0]$ for some $T_0 > 0$. In addition, noticing that $\mathbb{P}^2 = \mathbb{P}$ and $J_N^2 = J_N$, we easily see that $(\mathbb{P}u^N, \theta^N)$ and $(J_N u^N, J_N \theta^N)$ are also solutions. The uniqueness then implies that

$$\mathbb{P}u^N = u^N, \quad u^N = J_N u^N, \quad \theta^N = J_N \theta^N.$$

Consequently, u^N is divergence-free, $\nabla \cdot u^N = 0$, and (2.1) is reduced to

$$\partial_t u^N + \mathbb{P}J_N(u^N \cdot \nabla u^N) + \nu\, \Lambda^\alpha u^N = \mathbb{P}J_N(\theta^N \mathbf{e}_2), \tag{2.2}$$

$$\partial_t \theta^N + J_N(u^N \cdot \nabla \theta^N) + \kappa\, \Lambda^\beta \theta^N = 0, \tag{2.3}$$

$$u^N(x,0) = u_0^N(x) \equiv J_N u_0(x), \quad \theta^N(x,0) = \theta_0^N(x) = J_N \theta_0(x).$$

Therefore, we obtain after taking the inner product of (2.2) with u^N and (2.3) with θ^N and integrating by parts

$$\|\theta^N(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\Lambda^{\frac{\beta}{2}} \theta^N(\tau)\|_{L^2}^2\, d\tau = \|J_N \theta_0\|_{L^2}^2 \le \|\theta_0\|_{L^2}^2, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^N(t)\|_{L^2}^2 + 2\nu \|\Lambda^{\frac{\alpha}{2}} u^N(\tau)\|_{L^2}^2 \le 2\|u^N(t)\|_{L^2}\, \|\theta^N(t)\|_{L^2}$$

or

$$\|u^N(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^{\frac{\alpha}{2}} u^N(\tau)\|_{L^2}^2 \, d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2. \tag{2.5}$$

Therefore $(u^N(t), \theta^N(t)) \in L_N^2$ at any time $t > 0$. The extension part of Lemma 2.3 then implies that $(u^N(t), \theta^N(t)) \in L_N^2$ is a global solution for all time.

The second step is to show that, for a subsequence of (u^N, θ^N) (still denoted by (u^N, θ^N)),

$$u^N - u \to 0 \quad \text{in } L^2([0,T]; L^2), \quad \theta^N - \theta \to 0 \quad \text{in } L^2([0,T]; L^2).$$

First, the global bounds in (2.4) and (2.5) allow us to extract a subsequence of (u^N, θ^N), still denoted by (u^N, θ^N), which converges weakly to $(u, \theta) \in L^\infty([0,T]; L^2)$ for $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^{\frac{\alpha}{2}})$ and $\theta \in L^\infty([0,\infty); L^2) \cap L^2([0,\infty); \dot{H}^{\frac{\beta}{2}})$. Now we show that, for any $s > 2$,

$$\partial_t u^N, \ \partial_t \theta^N \in L^\infty([0,T]; H^{-s}). \tag{2.6}$$

To verify (2.6), we test $\partial_t u^N$ against $\phi \in H^s$ with $s > 2$,

$$\begin{aligned} \int \partial_t u^N \cdot \phi \mathrm{d}x &= -\int u^N \cdot \nabla u^N \cdot \mathbb{P} J_N \phi \mathrm{d}x - \nu \int \Lambda^\alpha u^N \cdot \phi \mathrm{d}x + \int \theta^N \mathbf{e}_2 \cdot \mathbb{P} J_N \phi \mathrm{d}x \\ &= \int u^N \cdot \nabla \mathbb{P} J_N \phi \cdot u^N \mathrm{d}x - \nu \int u^N \cdot \Lambda^\alpha \phi \mathrm{d}x + \int \theta^N \mathbf{e}_2 \cdot \mathbb{P} J_N \phi \mathrm{d}x. \end{aligned}$$

By Hölder's inequality and the Sobolev embedding,

$$\begin{aligned} \left| \int \partial_t u^N \cdot \phi \mathrm{d}x \right| &\le \nu \|u^N\|_{L^2} \|\Lambda^\alpha \phi\|_{L^2} + \|u^N\|_{L^2}^2 \|\mathbb{P} J_N \nabla \phi\|_{L^\infty} + \|\theta^N\|_{L^2} \|\phi\|_{L^2} \\ &\le \nu \|u^N\|_{L^2} \|\phi\|_{H^1} + C \|u^N\|_{L^2}^2 \|\phi\|_{H^s} + \|\theta^N\|_{L^2} \|\phi\|_{L^2} < \infty. \end{aligned}$$

That is, for $s > 2$,

$$\begin{aligned} \|\partial_t u^N\|_{H^{-s}} &\le \nu \|u^N\|_{L^2} + C \|u^N\|_{L^2}^2 + \|\theta^N\|_{L^2} \\ &\le C (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2 + \nu(\|u_0\|_{L^2} + t\|\theta_0\|_{L^2}) + \|\theta_0\|_{L^2}. \end{aligned}$$

Similarly, we can prove that $\partial_t \theta^N \in H^{-s}$. This verifies (2.6). Combining this with the global bounds in (2.4) and (2.5), especially $u^N \in L^2([0,T]; \dot{H}^{\frac{\alpha}{2}})$ and $\theta \in L^2([0,\infty); \dot{H}^{\frac{\beta}{2}})$, we obtain, by applying the Aubin-Lions Lemma, the strong convergence

$$u^N - u \to 0 \quad \text{in } L^2([0,T]; L^2), \quad \theta^N - \theta \to 0 \quad \text{in } L^2([0,T]; L^2).$$

This strong convergence allows us to pass the limit in the weak formulation of (2.1) to obtain the global weak solution defined in Definition 2.2. This completes the proof of Proposition 2.1. □

3 Proof of Theorem 1.1

This section proves Theorem 1.1. We recall two results to be used in the proof. The first result states that $\widetilde{u}$ (defined in the introduction) is actually Lipschitz at any time. This fact is established in the work of Jiu et al. (in press).

Proposition 3.1 *Consider* (1.1) *with* $\nu > 0$, $\kappa > 0$, $\alpha > \alpha_0$, $\beta > 0$ *and* $\alpha + \beta < 1$, *where* α_0 *is given by* (1.2). *Assume that* $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$. *Let* (u, θ) *be a global weak solution of* (1.1). *Then,* $\widetilde{u}$ *defined by the Biot–Savart law*

$$\widetilde{u} = \nabla^\perp \Delta^{-1} G \quad \textit{with} \quad G = \omega - \Lambda^{-\alpha} \partial_1 \theta$$

is actually Lipschitz at any time, namely

$$\|\nabla \widetilde{u}(\cdot, t)\|_{L^\infty(\mathbb{T}^2)} < \infty$$

for any $t \in (0, \infty)$.

We also need a regularization result stating that, if a weak solution of (1.6) is Hölder continuous with an index bigger than $1 - \alpha - \beta$, then it is actually a smooth solution. This result can be proven by following the lines in the paper by Constantin and Wu (2008).

Proposition 3.2 *Consider* (1.6) *with* $\alpha > 0$, $\beta > 0$ *and* $\alpha + \beta < 1$. *Assume that* $\widetilde{u}$ *satisfies*

$$\|\nabla \widetilde{u}(\cdot, t)\|_{L^\infty(\mathbb{T}^2)} < \infty$$

for any $t \in (0, \infty)$. *Let* $\theta_0 \in H^s(\mathbb{T}^2)$ *with* $s > 2$ *and let* θ *be a corresponding weak solution of* (1.6). *If, for* $\sigma > 1 - \alpha - \beta$ *and* $0 < t_1 < t_2 \leq \infty$,

$$\theta \in L^\infty([t_1, t_2); C^\sigma(\mathbb{T}^2)),$$

then

$$\theta \in C^\infty(\mathbb{T}^2 \times (t_1, t_2)).$$

We now turn to the proof of our main theorem.

Proof of Theorem 1.1 The existence of $T_1 > 0$ such that

$$(u, \theta) \in C([0, T_1]; H^s(\mathbb{T}^2))$$

is a consequence of the local well-posedness of (1.1) in H^s with $s > 2$. By Propositions 3.1 and 1.2, there exists $T_2 > 0$ such that

$$\theta \in C^\sigma(\mathbb{T}^2 \times [T_2, \infty)).$$

By Proposition 3.2,

$$\theta \in C^\infty(\mathbb{T}^2 \times [T_2, \infty)).$$

Then it follows from the vorticity equation that

$$\omega \in L^\infty(\mathbb{T}^2 \times [T_2, \infty)),$$

which further implies

$$u \in C([T_2, \infty); H^s).$$

Then a result similar to Proposition 3.2 on the 2D Navier–Stokes equations with a smooth forcing term implies

$$u \in C^\infty(\mathbb{T}^2 \times [T_2, \infty)).$$

This completes the proof of Theorem 1.1. □

4 Proof of Proposition 1.2

This section presents the proof of Proposition 1.2. We draw ideas from Dabkowski (2011). For the convenience of the reader and for future references on the generalized SQG type equations, we provide a complete proof.

The proof of Proposition 1.2 relies on an equivalent definition of the standard Hölder space, as the dual of a special class of functions. We first define the dual class and then state the equivalence result. This equivalence holds in $\mathbb{T}^d$ for general dimension d.

Definition 4.1 Let $r \in (0, 1]$ and $p \geq 2$. The class, denoted by $\mathcal{U}(r)$, consists of functions ψ satisfying the following two conditions:

(1) There exist $A > 0$ such that

$$\|\psi\|_{L^p(\mathbb{T}^d)} \leq A^{\frac{1}{p}} r^{-\frac{d}{q}}$$

where q is the conjugate index of p, namely, $\frac{1}{p} + \frac{1}{q} = 1$.

(2) For any function $f \in C^\infty \cap \mathrm{Lip}(1)$,

$$\left| \int_{\mathbb{T}^d} f(x)\, \psi(x)\, dx \right| \leq r, \tag{4.1}$$

where $\mathrm{Lip}(B)$ denotes the set of Lipschitz functions with the Lipschitz coefficient equal to B, namely, $|f(x) - f(y)| \le B\,|x - y|$ for any $x, y \in \mathbb{T}^2$.

Lemma 4.2 *Let $\sigma \in (0, 1]$. Then, a function $g \in C^\sigma(\mathbb{T}^d)$ if and only if*

$$\left|\int_{\mathbb{T}^d} g(x)\,\psi(x)\,\mathrm{d}x\right| \le C r^\sigma$$

for some constant $C > 0$, any $\psi \in \mathcal{U}(r)$ and any $r \in (0, 1)$.

Proposition 1.2 is proven through an inductive process and one key step is the following proposition that assesses a property on the evolution of the class functions.

Proposition 4.3 *Let $0 < \alpha, \beta, \sigma < 1$, $\alpha + \beta < 1$ and $\alpha + \beta + \sigma - \frac{2}{q} > 1$. Assume that $\theta_0 \in H^s$ with $s > 2$ and let θ be a weak solution of (1.6). Then, there exist two small parameters $\delta > 0$ and $r_0 > 0$ such that the following conclusion holds: Fix any $t > 0$, $0 < r \le r_0$ and $0 < s < r^\beta$. Assume that*

(1) *θ satisfies*

$$\left|\int_{\mathbb{T}^2} \theta(x,\tau)\,\phi(x,\tau)\,\mathrm{d}x\right| \le R^\sigma \tag{4.2}$$

for any $\tau \in [t - s, t]$, $R \ge r\,e^\delta$ and $\phi \in \mathcal{U}(R)$; and

(2) *$\psi(x, t) \in \mathcal{U}(r)$ and $\psi(x, \tau)$ with $\tau \in [t - s, t]$ solves the following equation*

$$\begin{cases} \partial_\tau \psi + u \cdot \nabla \psi - \Lambda^\beta \psi = 0, & x \in \mathbb{T}^2, \\ u = \widetilde{u} + v, \quad \Lambda^\alpha v = -\nabla^\perp \Lambda^{-2} \partial_1 \theta, & x \in \mathbb{T}^2, \\ \psi(x,\tau)|_{\tau = t} = \psi(x, t), & x \in \mathbb{T}^2, \end{cases} \tag{4.3}$$

that is, $\psi(x, \tau)$ evolves backward in time starting from $\psi(x, t)$.

Then

$$\psi(x, t - s)/(e^{-\delta s r^{-\beta}}) \in \mathcal{U}\big(r e^{\delta\,\sigma^{-1} s r^{-\beta}}\big)$$

or

$$\psi(x, t - s) \in e^{-\delta s r^{-\beta}}\,\mathcal{U}\big(r e^{\delta\,\sigma^{-1} s r^{-\beta}}\big). \tag{4.4}$$

We also need a decay property on the solution of (1.6) in Lebesgue spaces.

Lemma 4.4 *Consider the generalized supercritical SQG equation (1.6). Assume that $\theta_0 \in H^s$ with $s > 2$ and let θ be a weak solution of (1.6). If θ_0 has zero mean, then $\|\theta(\cdot, t)\|_{L^q}$ with any $q \in (1, \infty)$ decays exponentially in time,*

$$\|\theta(\cdot, t)\|_{L^q} \le \|\theta_0\|_{L^q}\, e^{-t}.$$

If θ_0 does not have zero mean, then $\|\theta(\cdot, t)\|_{L^q}$ decays at least algebraically,

$$\|\theta(\cdot, t)\|_{L^q} \le \frac{\|\theta_0\|_{L^q}}{\big(1 + C\,a\,t\|\theta_0\|_{L^q}^{a\,q}\big)^{\frac{1}{a\,q}}}, \qquad a = \frac{\beta}{2(q-1)}.$$

The exponential decay rate easily follows from (1.6) while the algebraical decay can be found in Córdoba and Córdoba (2004).

With these preparations at our disposal, we now prove Proposition 1.2.

Proof of Proposition 1.2 The proof is achieved through an inductive process. The goal here is to show that there exist small parameters $r_0 > 0$ and $\delta > 0$, and a sequence of times T_n such that for $t \geq T_n$,

$$\left| \int_{\mathbb{T}^2} \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq r^\sigma \qquad \text{for any } r \geq r_0 e^{-n\delta} \text{ and } \psi \in \mathcal{U}(r) \tag{4.5}$$

and

$$T \equiv \sum_{n=0}^{\infty} T_n < \infty. \tag{4.6}$$

It then follows that, for $t \geq T$,

$$\left| \int_{\mathbb{T}^2} \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq r^\sigma \qquad \text{for any } r \in (0,1]. \tag{4.7}$$

According to Lemma 4.2, (4.7) implies that $\theta(\cdot, t) \in C^\sigma(\mathbb{T}^2)$ for any $t \geq T$.

We start with the case $n = 0$. The existence of $T_0 > 0$ relies on the large-time decay result stated in Lemma 4.4. By Hölder's inequality and Lemma 4.4,

$$\left| \int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq \|\theta\|_{L^q}\, \|\psi\|_{L^p} \leq \widetilde{C}\, t^{-\frac{2}{p\beta}} r^{-\frac{2}{q}},$$

where $\widetilde{C}$ is a constant. Therefore, for any fixed $r_0 > 0$, there is $T_0 > 0$ such that

$$\left| \int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq r^\sigma \qquad \text{for any } r \geq r_0 \text{ and } t \geq T_0.$$

In particular, we choose r_0 and $\delta > 0$ as in Proposition 4.3. We make the inductive assumption that there exists $T_n > 0$ such that, for $t \geq T_0 + T_1 + \cdots + T_n$,

$$\left| \int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq r^\sigma \qquad \text{for any } r \geq r_0\, e^{-n\delta} \text{ and } \psi \in \mathcal{U}(r) \tag{4.8}$$

and show that, there exists $T_{n+1} > 0$ such that, for $t \geq T_0 + T_1 + \cdots + T_n + T_{n+1}$,

$$\left| \int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \leq r^\sigma \qquad \text{for any } r \geq r_0\, e^{-(n+1)\delta} \text{ and } \psi \in \mathcal{U}(r). \tag{4.9}$$

The proof of (4.9) is slightly involved. Assume $\psi \in \mathcal{U}(r)$ with $r \geq r_0\, e^{-(n+1)\delta}$. Consider the backward in time evolution starting from $\psi(x,t)$, namely $\psi(x,\tau)$ with $\tau \in [t-s, t]$ satisfying (4.3). Here $0 < s \leq r^\beta$, as in Proposition 4.3. Noticing the negative sign in the front of $\Lambda^\beta \psi$ and applying the divergence-free condition, $\nabla \cdot u = 0$, we can check that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int \theta(x,\tau)\, \psi(x,\tau)\, \mathrm{d}x = 0.$$

Therefore, for $\tau \in [t-s, t]$,

$$\int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x = \int \theta(x,\tau)\, \psi(x,\tau)\, \mathrm{d}x. \tag{4.10}$$

Since the inductive assumption (4.8) fulfills the conditions of Proposition 4.3, we have, by Proposition 4.3 with $r \in [r_0\, e^{-(n+1)\delta}, r_0\, e^{-n\delta})$ and $T_{n+1} = s = \sigma r^\beta$,

$$\psi(\cdot, t - T_{n+1}) \in e^{-\delta\sigma}\, \mathcal{U}(re^\delta). \tag{4.11}$$

Then, if

$$t \geq T_0 + T_1 + \cdots + T_n + T_{n+1} \quad \text{and} \quad r \geq r_0\, e^{-(n+1)\delta},$$

we have

$$t - T_{n+1} \geq T_0 + T_1 + \cdots + T_n \quad \text{and} \quad re^\delta \geq r_0\, e^{-n\delta}.$$

Therefore, by (4.10) and (4.11),

$$\begin{aligned} \left| \int \theta(x,t)\, \psi(x,t)\, \mathrm{d}x \right| &= \left| \int \theta(x, t - T_{n+1})\, \psi(x, t - T_{n+1})\, \mathrm{d}x \right| \\ &\leq e^{-\delta\sigma}\, (re^\delta)^\sigma = r^\sigma. \end{aligned}$$

That is, (4.9) is true. Finally we verify that (4.6) is true. According to our choice of T_n above,

$$T = T_0 + \sum_{n=1}^{\infty} \sigma\, (r_0 e^{-n\delta})^\beta = T_0 + \sigma\, r_0^\beta e^{-\delta\beta} \frac{1}{1 - e^{-\delta\beta}} < \infty.$$

We have thus completed the proof of Proposition 1.2. □

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Appendix 1: Proof of Proposition 4.3

This appendix provides a detailed proof of Proposition 4.3. The proof draws ideas from Dabkowski (2011), but the framework is generalized to deal with the general form of the SQG type equation and may be useful for future work. In addition, some technical details are simplified here, for example, the proof of (5.7).

Proof of Proposition 4.3 According to Definition 4.1, it suffices to verify that ψ possesses the following properties:

(1) For any $f_0 \in Lip(1)$, namely f_0 Lipschitz with Lipschitz constant 1,

$$\left| \int_{\mathbb{T}^2} f_0(x)\, \psi(x, t-s)\, \mathrm{d}x \right| \le r\, e^{(\sigma^{-1}-1)\delta\, s\, r^{-\beta}}; \tag{5.1}$$

(2) For A and p as defined in Definition 4.1, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\psi(x, t-s)\|_{L^p} \le A^{\frac{1}{p}} r^{-\frac{2}{q}}\, e^{-(1+\frac{2}{q\sigma})\delta\, s\, r^{-\beta}}. \tag{5.2}$$

The proof of (5.1) is involved. The idea is to consider the evolution of f_0 according to an equation close to (1.6) so that we can use the condition $\psi(x, t) \in \mathcal{U}(r)$. Let u_r to be the mollification of u (u is defined in (1.6)), namely

$$u_r = \rho_r * u,$$

where ρ_r represents the standard mollifier, namely

$$\rho \ge 0,\ \ \rho \in C_0^\infty,\ \ \mathrm{supp}\rho \subset B(0,1),\ \ \int_{\mathbb{T}^2} \rho(x)\, dx = 1,\ \ \rho_r(x) = r^{-2}\rho(r^{-1}x). \tag{5.3}$$

Here r is assumed to be small. Assume $f = f(x, \tau)$ with $\tau \in [t-s, t]$ solves the linear equation

$$\partial_\tau f + u_r \cdot \nabla f + \Lambda^\beta f = 0, \qquad f(x, t-s) = f_0(x). \tag{5.4}$$

It is then easily checked using (4.3) and (5.4) that

$$\begin{aligned} \int f_0(x)\, \psi(x, t-s)\, \mathrm{d}x &= \int f(x, t)\, \psi(x, t)\, \mathrm{d}x \\ &\quad + \int_{t-s}^{t} \int (u(x,\tau) - u_r(x,\tau)) \cdot \nabla f(x,\tau)\, \psi(x,\tau)\, \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{5.5}$$

To bound the first term on the right, we show that

$$f(\cdot, t) \in Lip(C_1\, e^{C\, s\, r^{\alpha+\sigma-1}}). \tag{5.6}$$

This can be achieved by simple energy estimates. In fact, ∇f satisfies

$$\partial_\tau(\nabla f) + u_r \cdot \nabla(\nabla f) + \Lambda^\beta(\nabla f) = -(\nabla u_r)(\nabla f).$$

To bound $\|\nabla f\|_{L^\infty}$, we first bound $\|\nabla f\|_{L^\gamma}$ for large γ and then let $\gamma \to \infty$. Multiplying each side by $\nabla f|\nabla f|^{\gamma-2}$, integrating in space and applying $\nabla \cdot u_r = 0$, we find

$$\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|_{L^\gamma}^\gamma \le \|\nabla u_r\|_{L^\infty}\,\|\nabla f\|_{L^\gamma}^\gamma,$$

where we have applied the following inequality involving fractional Laplacian operator (see Córdoba and Córdoba 2004),

$$\int |\nabla f|^{\gamma-2}\nabla f \cdot \Lambda^\beta(\nabla f)\,\mathrm{d}x \ge 0.$$

Therefore,

$$\|\nabla f(\cdot,\tau)\|_{L^\gamma(\mathbb{T}^2)} \le \|\nabla f_0\|_{L^\gamma(\mathbb{T}^2)}\, e^{\int_{t-s}^{\tau}\|\nabla u_r(\cdot,\zeta)\|_{L^\infty(\mathbb{T}^2)}\mathrm{d}\zeta}.$$

Due to $\|\nabla f_0\|_{L^\gamma(\mathbb{T}^2)} \le (4\pi^2)^{\frac{1}{\gamma}}\,\|\nabla f_0\|_{L^\infty}$ and $f_0 \in Lip(1)$, we obtain by letting $\gamma \to \infty$,

$$\|\nabla f(\cdot,t)\|_{L^\infty} \le e^{\int_{t-s}^{t}\|\nabla u_r(\cdot,\zeta)\|_{L^\infty(\mathbb{T}^2)}\mathrm{d}\zeta}. \tag{5.7}$$

Recall that

$$\nabla u_r = \nabla(\rho_r * (\widetilde{u} + v)) = \rho_r * \nabla\widetilde{u} + \nabla\rho_r * v.$$

Due to (1.7), namely $\widetilde{u} \in Lip(M)$,

$$\|\rho_r * \nabla\widetilde{u}\|_{L^\infty} \le \|\rho_r\|_{L^1}\|\nabla\widetilde{u}\|_{L^\infty} \equiv M < \infty. \tag{5.8}$$

It is not difficult to verify that $\nabla\rho_r(x) = r^{-3}(\nabla\rho)(r^{-1}x) \in r^{-1}\mathcal{U}(2r)$ using the fact that $\|\nabla\rho_r\|_{L^1} \le r^{-1}$, $\|\nabla\rho_r\|_{L^\infty} \le C\,r^{-3}$ and $\nabla\rho_r$ has mean zero. Thus, by (4.2),

$$|\nabla\rho_r * \theta| = \left|\int \theta(y)\,\nabla\rho_r(x-y)\,\mathrm{d}y\right| \le Cr^{-1}\,(2r)^\sigma = C\,r^{\sigma-1}, \tag{5.9}$$

where we have used the fact that $\nabla\rho_r(x-y) = \nabla\rho_r(y-x) \in r^{-1}\mathcal{U}(2r)$ (the translation of a test function is still a test function). Since

$$v = -\nabla^\perp\Lambda^{-2}\Lambda^{-\alpha}\partial_1\theta,$$

we have $\|v\|_{C^\alpha} \le \|\theta\|_{L^\infty}$ by using the fact that Riesz transforms $\nabla^\perp\Lambda^{-2}\partial_1$ are bounded in Hölder space. By 5.9 and Lemma 4.2 (relating a Hölder function to the power of r when acting on a test function),

$$|\nabla \rho_r * v| \le C\, r^{\alpha+\sigma-1}. \tag{5.10}$$

Inserting (5.8) and (5.10) in (5.7) leads to (5.6). By (5.6) and the fact that $\psi(x,t) \in \mathcal{U}(r)$,

$$\left| \int f(x,t)\, \psi(x,t)\, \mathrm{d}x \right| \le C_1\, r\, e^{C\, s\, r^{\alpha+\sigma-1}}. \tag{5.11}$$

Next we bound the second term on the right of (5.5). Applying Hölder's inequality in space and integrating in time, we obtain

$$\begin{aligned} &\left| \int_{t-s}^{t} \int (u - u_r) \cdot \nabla f\, \psi(x,\tau)\, \mathrm{d}x \mathrm{d}\tau \right| \\ &\quad \le s \sup_{\tau \in [t-s,t]} \|(u-u_r)(\cdot,\tau)\|_{L^q} \|\nabla f\|_{L^\infty} \|\psi(\cdot,\tau)\|_{L^p}. \end{aligned} \tag{5.12}$$

Recall that $\|\nabla f\|_{L^\infty}$ is bounded according to (5.6) and also $\frac{1}{p} + \frac{1}{q} = 1$. Since $\psi(\cdot,t) \in \mathcal{U}(r)$ and $\|\psi(\cdot,\tau)\|_{L^p}$ increases in time (due to the evolution equation (4.3)), we have

$$\|\psi(\cdot,\tau)\|_{L^p} \le \|\psi(\cdot,t)\|_{L^p} \le A^{\frac{1}{p}} r^{-\frac{2}{q}}. \tag{5.13}$$

It then suffices to bound $\|(u-u_r)(\cdot,\tau)\|_{L^q}$. We claim that

$$\|(u-u_r)(\cdot,\tau)\|_{L^q} \le C\, r^{\sigma+\alpha}. \tag{5.14}$$

Since $\widetilde{u} \in Lip(M)$,

$$\begin{aligned} \|(u-u_r)(\cdot,\tau)\|_{L^q} &\le \|(\widetilde{u}-\widetilde{u}_r)(\cdot,\tau)\|_{L^q} + \|(v-v_r)(\cdot,\tau)\|_{L^q} \\ &\le C\, r\, \|\nabla \widetilde{u}\|_{L^\infty} + \|(v-v_r)(\cdot,\tau)\|_{L^q} \\ &\le C\, r\, M + \|(v-v_r)(\cdot,\tau)\|_{L^q}. \end{aligned}$$

Recall $v = -\nabla^\perp \Lambda^{-2} \Lambda^{-\alpha} \partial_1 \theta$. By the boundedness of Riesz transforms on L^q ($1 < q < \infty$) and a simple analysis of Fourier series (Stein 1970),

$$\|(v-v_r)(\cdot,\tau)\|_{L^q(\mathbb{T}^2)} \le C\, \|\Lambda^{-\alpha}(\theta-\theta_r)\|_{L^q(\mathbb{T}^2)} \le C r^\alpha\, \|\theta-\theta_r\|_{L^q(\mathbb{T}^2)}. \tag{5.15}$$

To bound $\|\theta-\theta_r\|_{L^q(\mathbb{T}^2)}$, we cover $\mathbb{T}^2$ by B_r, disks of radius r and the number of such disks is of order r^{-2}. First we bound $\|\theta-\theta_r\|_{L^q(B_r)}$. For any constant c,

$$\|\theta-\theta_r\|_{L^q(B_r)} \le \|\theta-c\|_{L^q(B_r)} + \|c-\theta_r\|_{L^q(B_r)} \le 2\|\theta-c\|_{L^q(B_{3r})}. \tag{5.16}$$

To further the estimate, we choose c such that $\mathrm{sign}(\theta - c)\,|\theta - c|^{q-1}$ has mean zero on B_{3r}, namely

$$\int_{B_{3r}} \mathrm{sign}(\theta - c)\,|\theta - c|^{q-1}\,\mathrm{d}x = 0.$$

Then,

$$\begin{aligned}
\|\theta - c\|^q_{L^q(B_{3r})} &= \int_{B_{3r}} (\theta - c)\,\mathrm{sign}(\theta - c)\,|\theta - c|^{q-1}\,\mathrm{d}x \\
&= \int_{B_{3r}} \theta\,\mathrm{sign}(\theta - c)\,|\theta - c|^{q-1}\,\mathrm{d}x \\
&= r^{\frac{2}{q}}\,\|\theta - c\|^{q-1}_{L^{p(q-1)}(B_{3r})} \int_{\mathbb{T}^2} \theta(x,\tau)\,\phi(x,\tau)\,\mathrm{d}x, \qquad (5.17)
\end{aligned}$$

where we have set

$$\phi(x,\tau) = r^{-\frac{2}{q}}\,\|\theta - c\|^{-(q-1)}_{L^{p(q-1)}(B_{3r})}\,\chi_{B_{3r}}\,\mathrm{sign}(\theta - c)\,|\theta - c|^{q-1}.$$

Since ϕ has mean zero, it is easily checked that ϕ satisfies, for any $f \in Lip(1)$,

$$\|\phi\|_{L^p(\mathbb{T}^2)} \le A^{\frac{1}{p}}\,(3r)^{-\frac{2}{q}}, \qquad \left|\int_{\mathbb{T}^2} f(x)\,\phi(x,\tau)\,\mathrm{d}x\right| \le C\,(3r)$$

for some constants A and C. That is, $\phi \in \mathcal{U}(3r)$. Therefore, by (4.2) with $e^{\delta} \le 3$,

$$\left|\int_{\mathbb{T}^2} \theta(x,\tau)\,\phi(x,\tau)\,\mathrm{d}x\right| \le (3r)^{\sigma}. \qquad (5.18)$$

Inserting (5.18) in (5.17) and realizing that $p(q-1) = q$, we have

$$\|\theta - c\|^q_{L^q(B_{3r})} \le C\,r^{\frac{2}{q}+\sigma}\,\|\theta - c\|^{q-1}_{L^q(B_{3r})}$$

or

$$\|\theta - c\|_{L^q(B_{3r})} \le C\,r^{\frac{2}{q}+\sigma}. \qquad (5.19)$$

Since the number of disks needed to cover $\mathbb{T}^2$ is of the order r^{-2}, we combine (5.15), (5.16) and (5.19) to obtain

$$\|v - v_r\|_{L^q(\mathbb{T}^2)} \le C\,r^{-\frac{2}{q}}\,r^{\frac{2}{q}+\sigma+\alpha} = C\,r^{\alpha+\sigma}. \qquad (5.20)$$

Therefore, by inserting (5.6), (5.13) and (5.20) in (5.12), we have

$$\left|\int_{t-s}^{t}\int (u(x,\tau)-u_r(x,\tau))\cdot\nabla f(x,\tau)\,\psi(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\right| \le C\,A^{\frac{1}{p}}\,r^{\alpha+\sigma-\frac{2}{q}}\,s\,e^{C s\, r^{\alpha+\sigma-1}}. \tag{5.21}$$

Combining the bounds in (5.11) and (5.21) and recalling that

$$s \le r^{\beta}, \qquad \alpha+\beta+\sigma-\frac{2}{q} > 1,$$

we have, by taking $C\,r_0^{\alpha+\beta+\sigma-1} \le C\,r_0^{\frac{2}{q}} \le (\sigma^{-1}-1)\delta$,

$$\left|\int_{\mathbb{T}^2} f_0(x)\,\psi(x,t-s)\,\mathrm{d}x\right| \le C\,r\,e^{C s\, r^{\alpha+\sigma-1}} \le r\,e^{(\sigma^{-1}-1)\delta\, s\, r^{-\beta}}.$$

We have thus completed the proof of (5.1).

Next we prove (5.2). To bound $\|\psi(\cdot,\tau)\|_{L^p}$ for $\tau\in[t-s,t]$, we multiply (4.3) by $\Psi\equiv\psi\,|\psi|^{p-2}$, integrate in space and apply $\nabla\cdot u=0$ to obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(\cdot,\tau)\|_{L^p}^p = \int_{\mathbb{T}^2}\Psi\Lambda^{\beta}\psi\,\mathrm{d}x. \tag{5.22}$$

Applying the integral representation for $\Lambda^{\beta}\psi$ (see Córdoba and Córdoba 2004), we have

$$\begin{aligned}\int\Psi\Lambda^{\beta}\psi\,\mathrm{d}x &= C_\beta\int_{\mathbb{T}^2}\Psi(x,\tau)\sum_{n\in\mathbb{Z}^2} p.v.\int_{\mathbb{T}^2}\frac{\psi(x,\tau)-\psi(y,\tau)}{|x-y-n|^{2+\beta}}\,\mathrm{d}y\,\mathrm{d}x\\ &= C_\beta\sum_{n\in\mathbb{Z}^2} p.v.\int_{\mathbb{T}^2\times\mathbb{T}^2}\frac{\Psi(x,\tau)(\psi(x,\tau)-\psi(y,\tau))}{|x-y-n|^{2+\beta}}\,\mathrm{d}y\mathrm{d}x\\ &= \frac{C_\beta}{2}\sum_{n\in\mathbb{Z}^2} p.v.\int_{\mathbb{T}^2\times\mathbb{T}^2}\frac{(\Psi(x,\tau)-\Psi(y,\tau))(\psi(x,\tau)-\psi(y,\tau))}{|x-y-n|^{2+\beta}}\,\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where $p.v.$ denotes principal value. Since the integrand on the right-hand side is non-negative, we have by keeping only the term with $n=(0,0)$,

$$\begin{aligned}\int\Psi\Lambda^{\beta}\psi\,\mathrm{d}x &\ge \frac{C_\beta}{2}p.v.\int_{\mathbb{T}^2\times\mathbb{T}^2}\frac{(\Psi(x,\tau)-\Psi(y,\tau))(\psi(x,\tau)-\psi(y,\tau))}{|x-y|^{2+\beta}}\,\mathrm{d}x\mathrm{d}y\\ &\ge \frac{C_\beta}{2}\int_{|x-y|\le r}\frac{(\Psi(x,\tau)-\Psi(y,\tau))(\psi(x,\tau)-\psi(y,\tau))}{|x-y|^{2+\beta}}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For $|x-y|\le r$,

$$|x-y|^{-2-\beta} \ge r^{-\beta}r^{-2} \ge r^{-\beta}\,\rho_r(x-y),$$

where ρ_r is the standard mollifier defined in (5.3). Since $\Psi\psi = |\psi|^p$,

$$\begin{aligned}
&\int \Psi \Lambda^\beta \psi \, \mathrm{d}x \\
&\quad \geq \frac{C_\beta}{2} \int_{\mathbb{T}^2\times\mathbb{T}^2} (\Psi(x,\tau)-\Psi(y,\tau))(\psi(x,\tau)-\psi(y,\tau))\, r^{-\beta} \rho_r(x-y)\, \mathrm{d}x\mathrm{d}y \\
&\quad = C_\beta \int_{\mathbb{T}^2\times\mathbb{T}^2} |\psi(x)|^p \, r^{-\beta} \rho_r(x-y) \mathrm{d}x\mathrm{d}y \\
&\qquad - C_\beta \int_{\mathbb{T}^2\times\mathbb{T}^2} \Psi(y,\tau)\, \psi(x,\tau)\, r^{-\beta} \rho_r(x-y)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{5.23}$$

Using the fact that $\|\rho_r\|_{L^1} = 1$, we have

$$\begin{aligned}
&\int_{\mathbb{T}^2\times\mathbb{T}^2} |\psi(x,\tau)|^p \, r^{-\beta} \rho_r(x-y)\mathrm{d}x\mathrm{d}y \\
&\quad = r^{-\beta} \int_{\mathbb{T}^2} |\psi(x,\tau)|^p \int_{\mathbb{T}^2} \rho_r(x-y)\mathrm{d}y\mathrm{d}x \\
&\quad = r^{-\beta} \|\psi(\tau)\|_{L^p}^p.
\end{aligned} \tag{5.24}$$

Furthermore,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^2\times\mathbb{T}^2} \Psi(y,\tau)\, \psi(x,\tau)\, r^{-\beta} \rho_r(x-y)\mathrm{d}x\mathrm{d}y \right| \\
&\quad = r^{-\beta} \left| \int_{\mathbb{T}^2} \psi(x,\tau) \int_{\mathbb{T}^2} \rho_r(x-y)\Psi(y,\tau)\mathrm{d}y\, \mathrm{d}x \right| \\
&\quad = r^{-\beta} \left| \int_{\mathbb{T}^2} \psi(x,\tau)\, (\rho_r * \Psi)(x)\, \mathrm{d}x \right|.
\end{aligned}$$

This term is bounded by invoking (5.1). By taking suitable δ and σ such that, for $s \leq r^\beta$,

$$e^{\delta((\sigma^{-1}-1)sr^{-\beta}} \leq 2.$$

If we set $F = \rho_r * \Psi$ and $f = F/\|\nabla F\|_{L^\infty}$, then $f \in Lip(1)$ and we obtain by applying (5.1)

$$\begin{aligned}
\left| \int_{\mathbb{T}^2\times\mathbb{T}^2} \Psi(y,\tau)\, \psi(x,\tau)\, r^{-\beta} \rho_r(x-y) dxdy \right| &\leq r^{-\beta}\, 2r\, \|\nabla F\|_{L^\infty} \\
&\leq 2r^{1-\beta} \|\nabla \rho_r * \Psi\|_{L^\infty} \\
&\leq 2r^{1-\beta} \|\nabla \rho_r\|_{L^p}\, \|\Psi\|_{L^q} \\
&\leq C\, r^{1-\beta}\, r^{-1-\frac{2}{q}} \|\psi\|_{L^p}^{p-1}.
\end{aligned}$$

If $\|\psi\|_{L^p} \le \frac{1}{2}A^{\frac{1}{p}}r^{-\frac{2}{q}}$, then (5.2) is already proven. If $\|\psi\|_{L^p} \ge \frac{1}{2}A^{\frac{1}{p}}r^{-\frac{2}{q}}$, then

$$\left|\int_{\mathbb{T}^2\times\mathbb{T}^2} \Psi(y,\tau)\,\psi(x,\tau)\,r^{-\beta}\rho_r(x-y)dxdy\right| \le C\,A^{-\frac{1}{p}}\,r^{-\beta}\|\psi\|_{L^p}^p. \quad (5.25)$$

Inserting (5.24) and (5.25) in (5.23), we obtain

$$\int \Psi(x,\tau)\Lambda^\beta\psi(x,\tau)\,dx \ge (1-C\,A^{-\frac{1}{p}})\,r^{-\beta}\|\psi\|_{L^p}^p.$$

It then follows from integrating (5.22) that

$$\|\psi(\cdot,t-s)\|_{L^p} \le e^{-s(1-C\,A^{-\frac{1}{p}})\,r^{-\beta}}\,\|\psi(\cdot,t)\|_{L^p}.$$

Since $\psi(x,t)\in\mathcal{U}(r)$, in particular, $\|\psi(\cdot,t)\|_{L^p} \le A^{\frac{1}{p}}r^{-\frac{2}{q}}$, we have

$$\|\psi(\cdot,t-s)\|_{L^p} \le A^{\frac{1}{p}}r^{-\frac{2}{q}}e^{-(1-C\,A^{-\frac{1}{p}})\,s\,r^{-\beta}} \le A^{\frac{1}{p}}r^{-\frac{2}{q}}\,e^{-(1+\frac{2}{q\sigma})\delta s r^{-\beta}}$$

when $(1+\frac{2}{q\sigma})\delta \le (1-C\,A^{-\frac{1}{p}})$. This proves (5.2). We thus have completed the proof of Proposition 4.3. □

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