Applied Mathematics Letters 98 (2019) 392–397

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Stability of hydrostatic equilibrium to the 2D Boussinesq systems with partial dissipation

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ARTICLE INFO

Article history:
Received 11 January 2019
Received in revised form 18 June 2019
Accepted 18 June 2019
Available online 21 June 2019

Keywords:
Boussinesq equation
Hydrostatic equilibrium
Partial dissipation
Stability

ABSTRACT

The stability problem on perturbations near the hydrostatic equilibrium of the Boussinesq equations has recently attracted considerable attention and there are substantial developments. This paper establishes the global H^1-stability for a 2D Boussinesq system with partial dissipation. When there is no thermal diffusion, the stability of the temperature gradient remains an open problem. This paper assesses the stability of the temperature gradient when there is a horizontal thermal diffusion.

1. Introduction

This paper concerns itself with the stability problem for the following Boussinesq system with partial dissipation

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 = -\partial_1 P + \nu\,\partial_{22}u_1, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t u_2 + u\cdot\nabla u_2 = -\partial_2 P + \nu\,\partial_{11}u_2 + \Theta, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t \Theta + u\cdot\nabla\Theta = \eta\,\partial_{11}\Theta, \quad x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^2,\ t>0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ denotes the velocity field, P the pressure, Θ the temperature, and $\nu > 0$, $\eta > 0$ are the viscosity and the thermal diffusivity, respectively. Here ∂_1 stands for the partial derivative with respect to x_1 and the notation ∂_2 is similar. It is easy to verify that (1.1) admits the following steady state solution

$$u^{(0)} \equiv (0,0), \quad \Theta^{(0)} = x_2, \quad P^{(0)} = \frac{1}{2}x_2^2. \tag{1.2}$$

This special solution represents the hydrostatic equilibrium. Hydrostatic equilibrium or hydrostatic balance in fluid dynamics refers to the status of a fluid when it is at rest. This occurs when the gravity is balanced

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https://doi.org/10.1016/j.aml.2019.06.019
0893-9659/

out by the pressure-gradient force. Our atmosphere is mostly in the hydrostatic equilibrium. The pressure-gradient force prevents gravity from collapsing Earth's atmosphere into a thin shell, whereas gravity prevents the pressure gradient force from diffusing the atmosphere into space [1,2]. Understanding the stability of perturbations near the hydrostatic balance is important both mathematically and physically.

This paper examines the stability of perturbations near the hydrostatic equilibrium in (1.2). To this end, we consider the perturbation (u, θ) with $\theta = \Theta - x_2$. It is easy to check that, if (u, Θ) satisfies (1.1), then (u, θ) satisfies

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 = -\partial_1 P + \nu\,\partial_{22}u_1, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t u_2 + u\cdot\nabla u_2 = -\partial_2 P + \nu\,\partial_{11}u_2 + \theta, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\,\partial_{11}\theta, \quad x\in\mathbb{R}^2,\ t>0, \\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^2,\ t>0. \end{cases} \tag{1.3}$$

We are able to establish the following global stability result.

Theorem 1.1. *Consider (1.3) with the initial data $(u_0, \theta_0) \in H^1(\mathbb{R}^2)$. Then there exists a constant $\delta > 0$ such that, if $\|(u_0, \theta_0)\|_{H^1} \le \delta$, then (1.3) has a unique global solution (u, θ) in the regularity class*

$$(u,\theta) \in L^\infty(0,\infty; H^1(\mathbb{R}^2)), \quad \partial_2 u_1,\ \partial_1 u_2,\ \partial_1\theta \in L^2(0,\infty; \dot{H}^1(\mathbb{R}^2)). \tag{1.4}$$

Furthermore, (u, θ) satisfies

$$\|(u(t), \theta(t))\|_{H^1} \le C\delta, \tag{1.5}$$

where $C = C(\nu, \eta)$ is a constant.

The Boussinesq equations, especially those with partial dissipation have recently attracted considerable interests. There are substantial developments on two fundamental problems, the global regularity and stability problems (see, e.g., [3–12]). Our motivation of this study is two fold. The first is related to the stability problem on the 2D Navier–Stokes equations with partial dissipation. The 2D Navier–Stokes equations with full dissipation have the stability property that the H^2-norm of any solution does not grow in time. In contrast, the H^2-norm of solutions to the 2D Euler equations may grow double exponentially in time [13]. A natural question is whether solutions to the 2D Navier–Stokes with partial dissipation are stable.

The second motivation is due to several very recent important results on the stability problem on the Boussinesq equations. Doering, Wu, Zhao and Zheng [6] were able to obtain the stability of the hydrostatic balance of the 2D Boussinesq equations with only dissipation (without thermal diffusion). In addition, [6] showed that all first-order derivatives of the velocity field decay to zero. However, [6] left open the exact decay rates of the velocity and the eventual temperature profile. [10] made further progress by providing precise decay rates of the velocity, and the temperature profile for the solution of the linearized perturbation system. A conditional decay rate for the full nonlinear system is also obtained in [10]. When there is no thermal diffusion, no information on the derivative of the temperature has been obtained and the stability of the temperature gradient remains unknown. Theorem 1.1 is able to assess the stability of the temperature gradient when the 2D Boussinesq system involves the horizontal thermal diffusion. It is also worth mentioning the beautiful work of [4], which studied the stability of the hydrostatic balance for the 2D Boussinesq with a velocity damping term.

We employ the bootstrapping argument to prove the desired H^1-stability. We define the H^1-energy $A(t)$ by

$$A(t) = \|u(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + 2\eta\int_0^t \|\partial_1\theta(\tau)\|_{H^1}^2 d\tau + 2\nu\int_0^t (\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2) d\tau$$

and prove that, for a constant $C > 0$ and any $t \ge 0$,

$$A(t) \le A(0) + C\,A(t)^{\frac{3}{2}}. \tag{1.6}$$

A bootstrapping argument (see, e.g., [14, p. 20]) then yields (1.5). The proof of (1.6) involves the estimates of quite a few triple product terms. The following anisotropic estimate for triple products is extremely useful.

Lemma 1.2. *Assume that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then,*

$$\int\int |fgh|dxdy \leq C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

This lemma can be found in [15]. The rest of this paper is devoted to the proof of Theorem 1.1, which is provided in the next section.

2. Proof of Theorem 1.1

This section proves Theorem 1.1. For the sake of clarity, the proof is divided into two main parts: the H^1-stability and the uniqueness. The local existence can be obtained by a standard approach of Friedrichs' method of cutoff in Fourier space (see, e.g., [16]), we omit the details here.

2.1. The H^1-stability

This subsection proves the stability part of Theorem 1.1. First of all, we have the global L^2 bound

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu\int_0^t (\|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2)d\tau + 2\eta\int_0^t \|\partial_1\theta\|_{L^2}^2\,d\tau \\ &= \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.\end{aligned} \tag{2.1}$$

To bound $\|\nabla u\|_{L^2}$ and $\|\nabla\theta\|_{L^2}$, we resort to the vorticity equation,

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu(\partial_{111}u_2 - \partial_{222}u_1) + \partial_1\theta, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\partial_{11}\theta. \end{cases} \tag{2.2}$$

Taking the gradient of the second equation of (2.2) and dotting by $(\omega, \nabla\theta)$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta\|\partial_1\nabla\theta\|_{L^2}^2 \\ &= \nu\int(\partial_{111}u_2 - \partial_{222}u_1)\omega dx + \int(\partial_1\theta\omega - \nabla u_2\cdot\nabla\theta)dx - \int\nabla\theta\cdot\nabla u\cdot\nabla\theta dx \\ &:= K_1 + K_2 + K_3.\end{aligned} \tag{2.3}$$

Writing $\omega = \partial_1 u_2 - \partial_2 u_1$, we have, by integration by parts,

$$\begin{aligned}K_1 &= \nu\int_{\mathbb{R}^2}(\partial_{111}u_2 - \partial_{222}u_1)(\partial_1 u_2 - \partial_2 u_1)dx \\ &= -\nu\int(\partial_{11}u_2)^2 + (\partial_{22}u_1)^2 dx - \nu\int\partial_{111}u_2\partial_2 u_1 dx - \nu\int\partial_{222}u_1\partial_1 u_2 dx \\ &= -\nu\int(|\partial_2\nabla u_1|^2 + |\partial_1\nabla u_2|^2)dx.\end{aligned}$$

Writing $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$, we have

$$K_2 = \int_{\mathbb{R}^2}(\partial_1\theta\Delta\psi - \nabla\partial_1\psi\cdot\nabla\theta)dx = \int(\nabla\partial_1\psi\cdot\nabla\theta - \nabla\partial_1\psi\cdot\nabla\theta)dx = 0.$$

To bound K_3, we write out the four terms in K_3 explicitly,

$$\begin{aligned} K_3 &= -\int_{\mathbb{R}^2} (\partial_1 u_1 (\partial_1\theta)^2 + \partial_1 u_2 \partial_1\theta\partial_2\theta + \partial_2 u_1 \partial_1\theta\partial_2\theta + \partial_2 u_2 (\partial_2\theta)^2) dx \\ &= K_{31} + K_{32} + K_{33} + K_{34}. \end{aligned}$$

By Lemma 1.2,

$$|K_{31}| \le C\|\partial_1 u_1\|_{L^2} \|\partial_1\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \|\partial_1\theta\|_{L^2}^{\frac12} \|\partial_1\partial_1\theta\|_{L^2}^{\frac12} \le C\,\|u\|_{H^1}\,\|\partial_1\theta\|_{H^1}^2,$$

$$|K_{32}| \le C\|\partial_1 u_2\|_{L^2} \|\partial_1\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \|\partial_2\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \le C\,\|\theta\|_{H^1} \|\partial_1 u_2\|_{H^1} \|\partial_1\theta\|_{H^1},$$

$$|K_{33}| \le C\|\partial_2 u_1\|_{L^2} \|\partial_1\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \|\partial_2\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \le C\,\|\theta\|_{H^1} \|\partial_2 u_1\|_{H^1} \|\partial_1\theta\|_{H^1}.$$

To bound K_{34}, we integrate by parts and apply Lemma 1.2,

$$\begin{aligned} K_{34} &= \int \partial_1 u_1 (\partial_2\theta)^2 dx = -2\int u_1 \partial_2\theta\partial_1\partial_2\theta dx \\ &\le C\,\|\partial_1\partial_2\theta\|_{L^2} \|\partial_2\theta\|_{L^2}^{\frac12} \|\partial_1\partial_2\theta\|_{L^2}^{\frac12} \|u_1\|_{L^2}^{\frac12} \|\partial_2 u_1\|_{L^2}^{\frac12} \\ &\le C\,\|u\|_{H^1}^{\frac12}\,\|\theta\|_{H^1}^{\frac12}\,\|\partial_2 u_1\|_{H^1}^{\frac12}\,\|\partial_1\theta\|_{H^1}^{\frac32}. \end{aligned}$$

We now set

$$A(t) = \|u(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + 2\eta\int_0^t \|\partial_1\theta(\tau)\|_{H^1}^2 d\tau + 2\nu\int_0^t (\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2) d\tau.$$

The time integral of the bounds for K_{31}, K_{32}, K_{33} and K_{34} satisfies

$$\int_0^t |K_{31}|\,d\tau \le C \sup_{\tau\in[0,t]} \|u(t)\|_{H^1} \int_0^t \|\partial_1\theta\|_{H^1}^2\,d\tau \le C\,A(t)^{\frac32}$$

and

$$\int_0^t |K_{32}|\,d\tau,\ \int_0^t |K_{33}|\,d\tau,\ \int_0^t |K_{34}|\,d\tau \le C\,A(t)^{\frac32}.$$

Using the fact that $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$, integrating (2.3) in time and combining with (2.1), we find

$$A(t) \le A(0) + CA(t)^{\frac32}.$$

A bootstrapping argument implies that, there is $\delta > 0$, such that, if $A(0) < \delta$, then

$$A(t) \le C\delta$$

for a pure constant C and for all $t > 0$. This implies H^1-stability.

2.2. Uniqueness

This subsection proves the uniqueness part of Theorem 1.1. We show that two solutions $(u^{(1)}, P^{(1)}, \theta^{(1)})$ and $(u^{(2)}, P^{(2)}, \theta^{(2)})$ of (1.3) in the regularity class (1.4) must coincide. Their difference $(\widetilde{u}, \widetilde{P}, \widetilde{\theta})$ with

$$\widetilde{u} = u^{(1)} - u^{(2)}, \quad \widetilde{P} = P^{(1)} - P^{(2)}, \quad \widetilde{\theta} = \theta^{(1)} - \theta^{(2)}$$

satisfies, according to (1.3),

$$\begin{cases} \partial_t\widetilde{u}_1 + \widetilde{u}\cdot\nabla u_1^{(1)} + u^{(2)}\cdot\nabla\widetilde{u}_1 = -\partial_1\widetilde{P} + \nu\,\partial_{22}\widetilde{u}_1, \\ \partial_t\widetilde{u}_2 + \widetilde{u}\cdot\nabla u_2^{(1)} + u^{(2)}\cdot\nabla\widetilde{u}_2 = -\partial_2\widetilde{P} + \nu\,\partial_{11}\widetilde{u}_2 + \widetilde{\theta}, \\ \partial_t\widetilde{\theta} + \widetilde{u}\cdot\nabla\widetilde{\theta}^{(1)} + \widetilde{u}^{(2)}\cdot\nabla\widetilde{\theta} + \widetilde{u}_2 = \eta\,\partial_{11}\widetilde{\theta}, \\ \nabla\cdot\widetilde{u} = 0. \end{cases}$$

Simple energy estimates show that

$$\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{\theta}(t)\|_{L^2}^2\right)+\nu(\|\partial_2\widetilde{u}_1\|_{L^2}^2+\|\partial_1\widetilde{u}_2\|_{L^2}^2)+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2$$
$$=-\int_{\mathbb{R}^2}\widetilde{u}\cdot\nabla u^{(1)}\cdot\widetilde{u}dx-\int_{\mathbb{R}^2}\widetilde{u}\cdot\nabla\theta^{(1)}\widetilde{\theta}dx:=I_1+I_2.$$

The two terms on the right-hand side can be bounded as follows.

$$\begin{aligned}|I_1|&\le\|\nabla u^{(1)}\|_{L^2}\|\widetilde{u}\|_{L^4}^2\le C\|\nabla u^{(1)}\|_{L^2}\|\widetilde{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2}\\&\le\frac{\nu}{4}(\|\partial_2\widetilde{u}_1\|_{L^2}^2+\|\partial_1\widetilde{u}_2\|_{L^2}^2)+C\|\nabla u^{(1)}\|_{L^2}^2\|\widetilde{u}\|_{L^2}^2.\end{aligned}$$

By Lemma 1.2,

$$\begin{aligned}I_2\;&=\;-\int_{\mathbb{R}^2}\widetilde{u}_1\partial_1\theta^{(1)}\widetilde{\theta}dx-\int_{\mathbb{R}^2}\widetilde{u}_2\partial_2\theta^{(1)}\widetilde{\theta}dx\\&\le\;C\|\widetilde{u}_1\|_{L^2}\|\partial_1\theta^{(1)}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\theta^{(1)}\|_{L^2}^{\frac{1}{2}}\|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\\&\quad+C\|\partial_2\theta^{(1)}\|_{L^2}\|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\|\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\\&\le\;\frac{\eta}{2}\|\partial_1\widetilde{\theta}\|_{L^2}^2+\frac{\nu}{4}(\|\partial_1\widetilde{u}_2\|_{L^2}^2+\|\partial_2\widetilde{u}_1\|_{L^2}^2)\\&\quad+C(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)(\|\partial_1\theta^{(1)}\|_{L^2}^{\frac{2}{3}}\|\partial_1\nabla\theta^{(1)}\|_{L^2}^{\frac{2}{3}}+\|\partial_2\theta^{(1)}\|_{L^2}^2).\end{aligned}$$

Combining the estimates above yields, for $Y(t)=\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{\theta}(t)\|_{L^2}^2$,

$$\frac{d}{dt}Y(t)+\nu(\|\partial_2\widetilde{u}_1\|_{L^2}^2+\|\partial_1\widetilde{u}_2\|_{L^2}^2)+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2\le a(t)Y(t),\tag{2.4}$$

where

$$a(t)=C\|\nabla u^{(1)}\|_{L^2}^2+C\|\partial_1\theta^{(1)}\|_{L^2}^{\frac{2}{3}}\|\partial_1\nabla\theta^{(1)}\|_{L^2}^{\frac{2}{3}}+C\|\partial_2\theta^{(1)}\|_{L^2}^2.$$

Since $(u^{(1)},\theta^{(1)})$ is in the regularity class (1.4), we have, for any $T>0$,

$$\int_0^T a(t)dt\le C(T)<+\infty.$$

Gronwall's inequality applied to (2.4) implies that, for any $T>0$,

$$Y(t)\equiv 0\quad for\quad t\in[0,T].$$

The uniqueness result follows.

Acknowledgments

Ji is partially supported by the National Natural Science Foundation of China NSFC (Grant No. 11571243). Wu is partially supported by National Science Foundation, United States grant DMS 1624146 and the AT&T Foundation at Oklahoma State University.

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