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The resistive magnetohydrodynamic equation near an equilibrium

Ruihong Ji [a,*], Jiahong Wu [b]

[a] Geomathematics Key Laboratory of Sichuan Province, Chengdu University of Technology, Chengdu 610059, PR China
[b] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States

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Abstract

This work intends to understand the stability and large-time behavior of perturbations near a stationary solution of the 2D resistive magnetohydrodynamic (MHD) equation. The stationary solution is taken to be a background magnetic field parallel to the horizontal axis. We obtain three main results. The first result assesses the stability and the precise large-time asymptotic behavior for solutions to the linearzied system satisfied by the perturbation. Due to the lack of viscosity, the standard energy estimates do not work and the proof is achieved by constructing a suitable Lyapunov function. The second result makes use of the special wave structure of the linearization to establish the linear stability and decay rates. The third result obtains the H^1-stability for the full nonlinear system and shows that the L^q-norm ($q \in (2, \infty)$) of the magnetic field perturbation $\widetilde{b}$ and the L^2-norm of the gradient of $\widetilde{b}$ approach zero as $t \to \infty$.

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1. Introduction

The MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They consist of a coupled system of the Navier-Stokes equations

* Corresponding author.
E-mail addresses: jiruihong09@cdut.cn (R. Ji), jiahong.wu@okstate.edu (J. Wu).

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of fluid dynamics and Maxwell's equations of electromagnetism. Since their initial derivation by the Nobel Laureate H. Alfven in 1924, the MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [2,6]).

Besides their wide physical applicability, the MHD equations are also of great interest in mathematics. As a coupled system, the MHD equations contain much richer structures than the Navier-Stokes equations. They are not merely a combination of two parallel Navier-Stokes type equations but an interactive and integrated system. Their distinctive features make analytic studies a great challenge but offer new opportunities.

This paper focuses on the 2D resistive MHD equations with general magnetic diffusion, namely

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t b + u\cdot\nabla b + \eta\,(-\Delta)^\beta b = b\cdot\nabla u, \quad x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \quad x\in\mathbb{R}^2,\ t>0, \end{cases} \tag{1.1}$$

where u denotes the velocity field, p the pressure, b the magnetic field, $\eta > 0$ denotes the magnetic diffusivity (resistivity) and $\beta \geq 0$ is a real parameter. The fractional Laplacian $(-\Delta)^\beta$ is defined in terms of the Fourier transform,

$$\widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta}\widehat{f}(\xi), \qquad \widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx.$$

In addition, for notational convenience, we also write $\Lambda = (-\Delta)^{\frac{1}{2}}$ and use the direction fractional Laplacian operator Λ_1 or Λ_2 defined by

$$\widehat{\Lambda_1^\gamma f}(\xi) = |\xi_1|^\gamma \widehat{f}(\xi), \qquad \widehat{\Lambda_2^\gamma f}(\xi) = |\xi_2|^\gamma \widehat{f}(\xi).$$

(1.1) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence (see, e.g., [17]). This general MHD system in (1.1) includes the standard resistive MHD equation as a special case and has the advantage of allowing simultaneous study of a family of equations.

The stability and the regularity problems are two of the most fundamental problems on the MHD equations. There have been significant developments on these problems, especially on those MHD system with only partial or fractional dissipation. The global regularity problem on (1.1) is not trivial, even in the 2D case. In fact, whether or not smooth solutions of (1.1) with $\beta \leq 1$ can blow up in a finite time remains an outstanding open problem. In particular, the resistive MHD equation with the standard Laplacian dissipation is not yet known to always possess global smooth solutions no matter how smooth the initial data are. Due to extensive efforts in the last few years, this problem is now much better understood (see, e.g., [3,8–12,14, 15,22,24–28]). Especially, we now know that the global regularity problem on (1.1) with $\beta = 1$ in the 2D case is a critical problem and a slight more regularization is sufficient for global regularity.

Our paper will be focusing on the stability and large-time behavior of perturbations near a stationary solution of (1.1). The stability problem on the MHD equations has attracted considerable renewed interests in recent years. The study of the stability problem on the MHD equations has a long history. In his 1942 paper entitled "Existence of electromagnetic-Hydrodynamic waves" [1], Alfvén considered the linear stability of a background magnetic field associated with the MHD equations without resistivity. The study on the nonlinear stability problem is more recent, but significant progress has been made on this issue (see, e.g., [4,13,16,19,21,23,29]). Our focus is on the stability problem when the fluid viscosity can be ignored while the role of resistivity dominates. We also mention that the stability problem on other fluid models such as the Boussinesq equations has also garnered the interests of many researchers (see, e.g., [5,7,18,20]).

We consider the steady solution of (1.1) given by the background magnetic field

$$u^{(0)}(x) = (0,0), \qquad p^{(0)}(x) = 0, \qquad b^{(0)}(x) = (R,0),$$

where R is a real number. The perturbation around this steady solution, namely

$$(\widetilde{u}, \widetilde{p}, \widetilde{b}) = (u, p, b) - (u^{(0)}, p^{(0)}, b^{(0)})$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u} \cdot \nabla \widetilde{u} = -\nabla \widetilde{p} + \widetilde{b} \cdot \nabla \widetilde{b} + R\, \partial_{x_1} \widetilde{b}, \\ \partial_t \widetilde{b} + \widetilde{u} \cdot \nabla \widetilde{b} + \eta\, (-\Delta)^{\beta} \widetilde{b} = \widetilde{b} \cdot \nabla \widetilde{u} + R\, \partial_{x_1} \widetilde{u}, \\ \nabla \cdot \widetilde{u} = 0, \quad \nabla \cdot \widetilde{b} = 0. \end{cases} \tag{1.2}$$

Due to the lack of viscosity and the strong nonlinear coupling, the stability problem on (1.2) is not trivial. Our study appears to be among the very first investigations on the stability of (1.2). We obtain three main results. Due to the significance of understanding the linearization of (1.2), our first two results are on the stability, large-time behavior and the spectra properties of solutions to the linearized system of (1.2). Our first result constructs a suitable Lyapunov function to show the linear stability and obtain explicit decay rates. The second result intends to understand the spectra property of the linearization. We discover that the linearization has a very special structure and can be reduced to a system of decoupled degenerate wave type equations. This allows us to represent the solution to the linearization explicitly in terms of the eigenvalues and the initial data. Explicit decay rates follow as a special consequence of this representation. Our third result is a global H^1-stability result for the full nonlinear system. In particular, we show that the magnetic field perturbation $\widetilde{b}$ of (1.2) obeys the large-time behavior, for any $q \in (2, \infty)$,

$$\|\nabla \widetilde{b}(t)\|_{L^2} \to 0, \quad \|\widetilde{b}(t)\|_{L^q} \to 0 \qquad \text{as } t \to \infty.$$

We now explain the difficulties that we would encounter in establishing the aforementioned results. The linearized system of (1.2) is given by

$$\begin{cases} \partial_t \widetilde{u} = R\, \partial_{x_1} \widetilde{b}, \\ \partial_t \widetilde{b} + \eta\, (-\Delta)^{\beta} \widetilde{b} = R\, \partial_{x_1} \widetilde{u}, \\ \nabla \cdot \widetilde{u} = 0, \quad \nabla \cdot \widetilde{b} = 0, \\ \widetilde{u}(x,0) = \widetilde{u}_0(x), \quad \widetilde{b}(x,0) = \widetilde{b}_0(x). \end{cases} \tag{1.3}$$

Even though (1.3) appears to be really simple, but the large-time behavior is not apparent due to the lack of the kinematic viscosity. Powerful tools designed for systems with full dissipation such as Schonbek's Fourier splitting method can no longer be applied here. The L^2 energy equality associated with (1.3) or (1.2) is given by

$$\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t)\|_{L^2}^2 + 2\eta \int_0^t \|\Lambda^\beta \widetilde{b}(\tau)\|_{L^2}^2 \, d\tau = \|\widetilde{u}_0\|_{L^2}^2 + \|\widetilde{b}_0\|_{L^2}^2, \tag{1.4}$$

which does not allow us to extract the desired decay rates for $\|\widetilde{u}(t)\|_{L^2}$ or $\|\widetilde{b}(t)\|_{L^2}$. One approach is to design a suitable Lyapunov function together with an assumption that the initial data is in an appropriate Sobolev space with negative indices. We obtain the following result.

Theorem 1.1. *Consider (1.3) with $\beta \le 1$. Let $\sigma > 0$. Assume that $(\widetilde{u}_0, \widetilde{b}_0)$ satisfies*

$$(\widetilde{u}_0, \widetilde{b}_0) \in L^2(\mathbb{R}^2), \qquad (\Lambda_1^{-\sigma} \widetilde{u}_0, \Lambda_1^{-\sigma} \widetilde{b}_0) \in H^{1+\sigma}(\mathbb{R}^2). \tag{1.5}$$

Then, the corresponding solution $(\widetilde{u}, \widetilde{b})$ of (1.3) satisfies, for a constant C,

$$\|\widetilde{u}(t)\|_{L^2} + \|\widetilde{b}(t))\|_{L^2} + \|\nabla \widetilde{u}(t)\|_{L^2} + \|\nabla \widetilde{b}(t))\|_{L^2} \le C\,(1+t)^{-\frac{\sigma}{2}}. \tag{1.6}$$

Furthermore, if $(\partial_{x_1}^k \widetilde{u}_0, \partial_{x_1}^k \widetilde{b}_0) \in L^2$ for a positive integer k, then

$$\|\partial_{x_1}^k \widetilde{u}(t)\|_{L^2} + \|\partial_{x_1}^k \widetilde{b}(t))\|_{L^2} + \|\partial_{x_1}^k \nabla \widetilde{u}(t)\|_{L^2} + \|\partial_{x_1}^k \nabla \widetilde{b}(t))\|_{L^2} \le C\,(1+t)^{-\frac{\sigma+k}{2}}. \tag{1.7}$$

Theorem 1.1 assesses the linear stability and provides precise decay rates. Due to the lack of the velocity dissipation, the time integral in (1.4) only contains the term involving $\widetilde{b}$ but not $\widetilde{u}$. As a consequence, this energy equality is not useful in seeking the decay rates. The idea of proving Theorem 1.1 is to construct a suitable Lyapunov function so that its time derivative also includes some positive terms involving $\widetilde{u}$. The Lyapunov function $L(t)$ we construct here is given by

$$L(t) = \|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t))\|_{L^2}^2 + \|\nabla \widetilde{u}(t)\|_{L^2}^2 + \|\nabla \widetilde{b}(t))\|_{L^2}^2 + \epsilon \langle \widetilde{j}, \widetilde{u}_2 \rangle,$$

where $\epsilon > 0$ is a small real number, $\widetilde{j} = \nabla \times \widetilde{b}$ denotes the current density, and $\langle \widetilde{j}, \widetilde{u}_2 \rangle$ denotes the inner product,

$$\langle \widetilde{j}, \widetilde{u}_2 \rangle = \int_{\mathbb{R}^2} \widetilde{j}\, \widetilde{u}_2 \, dx$$

For small $\epsilon > 0$, we can show that $L(t)$ remains positive and serves as a suitable Lyapunov function. The term $\epsilon \langle \widetilde{j}, \widetilde{u}_2 \rangle$ generates a positive term involving $\widetilde{u}$ as intended. More details can be found in the proof of Theorem 1.1.

Our second result intends to understand the behavior of the spectra of the linearization and the large-time behavior of $(\widetilde{u}, \widetilde{b})$ of (1.3). The idea here is to diagonalize and decouple the system (1.3). The special structure of (1.3) allows us to avoid the standard eigenvalue and eigen-function approach. Differentiating the equations in (1.3) in time and making suitable substitutions, we find $(\widetilde{u}, \widetilde{b})$ satisfies

$$\begin{cases} \partial_{tt} \widetilde{u} + \eta(-\Delta)^\beta \partial_t \widetilde{u} - R^2\, \partial_{x_1 x_1} \widetilde{u} = 0, \\ \partial_{tt} \widetilde{b} + \eta(-\Delta)^\beta \partial_t \widetilde{b} - R^2\, \partial_{x_1 x_1} \widetilde{b} = 0, \\ \nabla \cdot \widetilde{u} = 0, \quad \nabla \cdot \widetilde{b} = 0, \\ \widetilde{u}(x,0) = \widetilde{u}_0(x), \quad \widetilde{b}(x,0) = \widetilde{b}_0(x). \end{cases} \tag{1.8}$$

This is a decoupled system with $\widetilde{u}$ and $\widetilde{b}$ each satisfying a degenerate and fractionally damped wave equation. The solution $(\widetilde{u}, \widetilde{b})$ can then be explicitly represented in terms of the initial data. More precisely,

$$\widetilde{u} = G_0\left(R\partial_x \widetilde{b}_0 + \frac{1}{2}\eta(-\Delta)^{\beta}\widetilde{u}_0\right) + G_1\widetilde{u}_0, \tag{1.9}$$

$$\widetilde{b} = G_0\left(R\partial_x \widetilde{u}_0 - \frac{1}{2}\eta(-\Delta)^{\beta}\widetilde{b}_0\right) + G_1\widetilde{b}_0, \tag{1.10}$$

where G_0 and G_1 are Fourier multiplier operators

$$\widehat{G}_0(\xi, t) = \frac{e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t}}{\lambda_1(\xi) - \lambda_2(\xi)}, \qquad \widehat{G}_1(\xi, t) = \frac{1}{2}\left(e^{\lambda_1(\xi)t} + e^{\lambda_2(\xi)t}\right) \tag{1.11}$$

with $\lambda_1(k)$ and $\lambda_2(k)$ being the roots of the characteristic equation

$$\lambda^2 + \eta|\xi|^{2\beta}\lambda + R^2\xi_1^2 = 0,$$

or, more precisely,

$$\lambda_1 = -\frac{1}{2}\eta\,|\xi|^{2\beta}\left(1 - \sqrt{1 - \frac{4R^2\,\xi_1^2}{\eta^2|\xi|^{4\beta}}}\right), \tag{1.12}$$

$$\lambda_2 = -\frac{1}{2}\eta\,|\xi|^{2\beta}\left(1 + \sqrt{1 - \frac{4R^2\,\xi_1^2}{\eta^2|\xi|^{4\beta}}}\right). \tag{1.13}$$

These explicit solution representation allows us to prove the following result.

Theorem 1.2. *Consider (1.3) with $\beta \le 1$. Assume that $\widetilde{u}_0$ and $\widetilde{b}_0$ satisfy*

$$(\widetilde{u}_0, \widetilde{b}_0) \in L^2(\mathbb{R}^2), \quad (\Lambda^{-\beta}\widetilde{u}_0, \Lambda^{-\beta}\widetilde{b}_0) \in L^1(\mathbb{R}^2), \quad (\Lambda_1^{-1}\widetilde{u}_0, \Lambda_1^{-1}\widetilde{b}_0) \in H^{\beta}(\mathbb{R}^2).$$

Then the corresponding solution $(\widetilde{u}, \widetilde{b})$ of (1.3) obeys the following decay estimates

$$\|(\widetilde{u}(t), \widetilde{b}(t))\|_{L^2} \le C\,(1+t)^{-1/(2\beta)}\|\Lambda^{-\beta}(\widetilde{u}_0, \widetilde{b}_0)\|_{L^1} + C\,(1+t)^{-1/2}\,\|\Lambda_1^{-1}(\widetilde{u}_0, \widetilde{b}_0)\|_{H^{\beta}},$$

where C is a pure constant.

Our third main result focuses on the full nonlinear system in (1.2) with $\beta = 1$. We establish the global H^1-stability and obtains the large-time behavior of the perturbations.

Theorem 1.3. *Consider (1.2) with $\beta = 1$. Assume $(\widetilde{u}_0, \widetilde{b}_0) \in H^1(\mathbb{R}^2)$. Then (1.2) has a global weak solution $(\widetilde{u}, \widetilde{b}) \in H^1$ with the following properties:*

(1) $(\widetilde{u}, \widetilde{b})$ *is uniformly bounded in* H^1, *namely, for any* $t > 0$,

$$\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t)\|_{L^2}^2 \le \|\widetilde{u}_0\|_{L^2}^2 + \|\widetilde{b}_0\|_{L^2}^2,$$
$$\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{b}(t)\|_{L^2}^2 \le (\|\nabla\widetilde{u}_0\|_{L^2}^2 + \|\nabla\widetilde{b}_0\|_{L^2}^2)\, e^{C\,(\|\widetilde{u}_0\|_{L^2}^2 + \|\widetilde{b}_0\|_{L^2}^2)},$$

which especially implies the global H^1*-stability.*

(2) $\|\widetilde{b}(t)\|_{L^2}^2$ *is a Lipschitz function in* $t \in [0, \infty)$.

(3) *As* $t \to \infty$,

$$\|\nabla\widetilde{b}(t)\|_{L^2}^2 \to 0$$

Especially, for any $2 < q < \infty$, *as* $t \to \infty$,

$$\|\widetilde{b}(t)\|_{L^q} \to 0.$$

The rest of this paper is divided into three sections with each one devoted to the proof of one of the three theorems stated above.

2. Proof of Theorem 1.1

This section proves Theorem 1.1. As aforementioned in the introduction, the idea is to construct a suitable Lyapunov function.

Proof of Theorem 1.1. We set

$$L(t) = \|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t))\|_{L^2}^2 + \|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{b}(t))\|_{L^2}^2 + \epsilon\langle\widetilde{j}, \widetilde{u}_2\rangle$$
$$M(t) = 2\eta\|\Lambda^\beta\widetilde{b}\|_{L^2}^2 + 2\eta\|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}^2 + \epsilon R\|\nabla\widetilde{u}_2\|_{L^2}^2 - \epsilon R\|\nabla\widetilde{b}_2\|_{L^2}^2 + \epsilon\eta\langle\Lambda^{2\beta}\widetilde{j}, \widetilde{u}_2\rangle.$$

By making use of the equations in (1.3), we can check that

$$\frac{d}{dt}L(t) + M(t) = 0. \tag{2.1}$$

In fact, dotting the first two equations of (1.3) with $(\widetilde{u}, \widetilde{b})$ yields

$$\frac{d}{dt}\left(\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t))\|_{L^2}^2\right) + 2\eta\|\Lambda^\beta\widetilde{b}\|_{L^2}^2 = 0, \tag{2.2}$$

where we have used the simple fact that

$$\langle\partial_1\widetilde{b}, \widetilde{u}\rangle + \langle\widetilde{b}, \partial_1\widetilde{u}\rangle = 0.$$

Similarly,

$$\frac{d}{dt}\left(\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{b}(t))\|_{L^2}^2\right) + 2\eta\|\Lambda^{1+\beta}\widetilde{b}(t)\|_{L^2}^2 = 0. \tag{2.3}$$

Furthermore, since $\widetilde{j} = \nabla \times \widetilde{b}$ satisfies

$$\partial_t \widetilde{j} = -\eta(-\Delta)^\beta \widetilde{j} + \partial_1 \widetilde{\omega}$$

with $\widetilde{\omega} = \nabla \times \widetilde{u}$, we find

$$\begin{aligned}
\partial_t \langle \widetilde{j}, \widetilde{u}_2\rangle &= \langle \partial_t \widetilde{j}, \widetilde{u}_2\rangle + \langle \widetilde{j}, \partial_t \widetilde{u}_2\rangle \\
&= \langle -\eta(-\Delta)^\beta \widetilde{j}, \widetilde{u}_2\rangle + \langle \partial_1 \widetilde{\omega}, \widetilde{u}_2\rangle + R\langle \widetilde{j}, \partial_1 \widetilde{b}_2\rangle \\
&= \langle -\eta(-\Delta)^\beta \widetilde{j}, \widetilde{u}_2\rangle - \|\nabla \widetilde{u}_2\|_{L^2}^2 + R\|\nabla \widetilde{b}_2\|_{L^2}^2,
\end{aligned} \tag{2.4}$$

where we have used the simple fact that

$$\langle \partial_1 \widetilde{\omega}, \widetilde{u}_2\rangle = \langle \Delta \widetilde{u}_2, \widetilde{u}_2\rangle = -\|\nabla \widetilde{u}_2\|_{L^2}^2, \quad \langle \widetilde{j}, \partial_1 \widetilde{b}_2\rangle = \|\nabla \widetilde{b}_2\|_{L^2}^2.$$

Combining (2.2), (2.3) and (2.4) leads to (2.1). We remark that, for $\epsilon > 0$ sufficiently small,

$$\begin{aligned}
L(t) &\geq \|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t))\|_{L^2}^2 + \|\nabla \widetilde{u}(t)\|_{L^2}^2 + \|\nabla \widetilde{b}(t))\|_{L^2}^2 - \epsilon \|\widetilde{j}\|_{L^2} \|\widetilde{u}_2\|_{L^2} \\
&\geq \frac{1}{2}\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t))\|_{L^2}^2 + \|\nabla \widetilde{u}(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla \widetilde{b}(t))\|_{L^2}^2.
\end{aligned} \tag{2.5}$$

For $\beta \leq 1$, we can also show $M(t) > 0$. The key is to bound the last term. It can be bounded by

$$\begin{aligned}
\left|\epsilon\eta\langle \Lambda^{2\beta} \widetilde{j}, \widetilde{u}_2\rangle\right| &\leq \epsilon\eta \|\Lambda^{2\beta-1}\widetilde{j}\|_{L^2} \|\nabla \widetilde{u}_2\|_{L^2} = \epsilon\eta \|\Lambda^{2\beta}\widetilde{b}\|_{L^2} \|\nabla \widetilde{u}_2\|_{L^2} \\
&\leq \epsilon\eta \|\Lambda^\beta \widetilde{b}\|_{L^2}^{1-\beta} \|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}^\beta \|\nabla \widetilde{u}_2\|_{L^2} \\
&\leq \begin{cases} \frac{\epsilon}{2}\|\nabla \widetilde{u}_2\|_{L^2}^2 + C\epsilon\eta \|\Lambda^\beta \widetilde{b}\|_{L^2}^2 + C\epsilon\eta\|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}^2, & \text{if } \beta < 1, \\ \frac{\epsilon}{2}\|\nabla \widetilde{u}_2\|_{L^2}^2 + \frac{1}{2}\epsilon\eta^2 \|\Lambda^2 \widetilde{b}\|_{L^2}^2, & \text{if } \beta = 1. \end{cases}
\end{aligned}$$

In addition,

$$\epsilon \|\nabla b_2\|_{L^2}^2 \leq C\,\epsilon \|\Lambda^\beta \widetilde{b}\|_{L^2}^2 + C\,\epsilon\|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}^2.$$

As a consequence, for $\epsilon > 0$ sufficiently small,

$$M(t) \geq \frac{\eta}{2}\|\Lambda^\beta \widetilde{b}\|_{L^2}^2 + \frac{\eta}{2}\|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla \widetilde{u}_2\|_{L^2}^2 > 0.$$

The energy equality in (2.1) is not sufficient for the large-time behavior. To find the exact decay rates, we further define

$$N(t) = \|\Lambda_1^{-\sigma}\widetilde{u}(t)\|_{L^2}^2 + \|\Lambda_1^{-\sigma}\widetilde{b}(t)\|_{L^2}^2 + \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma}\widetilde{u}(t)\|_{L^2}^2 + \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma}\widetilde{b}(t)\|_{L^2}^2.$$

It is easy to check that

$$\frac{d}{dt}N(t) + 2\eta\|\Lambda^\beta\Lambda_1^{-\sigma}\widetilde{b}(t)\|_{L^2}^2 + 2\eta\|\Lambda^{1+\sigma+\beta}\Lambda_1^{-\sigma}\widetilde{b}(t)\|_{L^2}^2 = 0.$$

Especially, for any $t \geq 0$,

$$N(t) \leq N(0).$$

Next we show that, for a constant $C > 0$,

$$L(t) \leq C\, M(t)^{\frac{\sigma}{1+\sigma}}\, N(t)^{\frac{1}{1+\sigma}}. \tag{2.6}$$

We check that each term of $L(t)$ obeys the bound above. In fact,

$$\|\widetilde{u}\|_{L^2}^2 \leq C\, \|\nabla \widetilde{u}_2\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\, \|\Lambda_1^{-\sigma} \widetilde{u}\|_{L^2}^{\frac{2}{1+\sigma}} \leq C\, M(t)^{\frac{\sigma}{1+\sigma}}\, N(t)^{\frac{1}{1+\sigma}}, \tag{2.7}$$

$$\|\nabla \widetilde{u}\|_{L^2}^2 \leq C\, \|\nabla \widetilde{u}_2\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\, \|\Lambda^{1+\sigma}\, \Lambda_1^{-\sigma} \widetilde{u}\|_{L^2}^{\frac{2}{1+\sigma}} \leq C\, M(t)^{\frac{\sigma}{1+\sigma}}\, N(t)^{\frac{1}{1+\sigma}}. \tag{2.8}$$

(2.7) can be easily verified by resorting to the stream function. Due to $\widetilde{u} = \nabla^\perp \widetilde{\phi}$ and the interpolation

$$|\xi|^2 |\widehat{\widetilde{\phi}}(\xi)|^2 = \left(|\xi|^2 |\xi_1|^2 |\widehat{\widetilde{\phi}}(\xi)|^2\right)^{\frac{\sigma}{1+\sigma}} \left(|\xi|^2 |\xi_1|^{-2\sigma} |\widehat{\widetilde{\phi}}(\xi)|^2\right)^{\frac{1}{1+\sigma}},$$

we find, by Hölder's inequality,

$$\begin{aligned}
\|\widetilde{u}\|_{L^2}^2 = \|\nabla^\perp \widetilde{\phi}\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\xi|^2 |\widehat{\widetilde{\phi}}(\xi)|^2\, d\xi \\
&\leq \left(\int_{\mathbb{R}^2} |\xi|^2 |\xi_1|^2 |\widehat{\widetilde{\phi}}(\xi)|^2\, d\xi\right)^{\frac{\sigma}{1+\sigma}} \left(\int_{\mathbb{R}^2} |\xi|^2 |\xi_1|^{-2\sigma} |\widehat{\widetilde{\phi}}(\xi)|^2\, d\xi\right)^{\frac{1}{1+\sigma}} \\
&= \|\nabla \partial_1 \widetilde{\phi}\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\, \|\Lambda_1^{-\sigma} \nabla^\perp \widetilde{\phi}\|_{L^2}^{\frac{2}{1+\sigma}} = \|\nabla \widetilde{u}_2\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\, \|\Lambda_1^{-\sigma} \widetilde{u}\|_{L^2}^{\frac{2}{1+\sigma}},
\end{aligned}$$

which verifies (2.7). (2.8) can be similarly verified. Similarly,

$$\|\widetilde{b}\|_{L^2}^2 \leq C\, \|\nabla \widetilde{b}_2\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\, \|\Lambda_1^{-\sigma} \widetilde{b}\|_{L^2}^{\frac{2}{1+\sigma}}.$$

Invoking the interpolation inequality

$$\|\nabla \widetilde{b}_2\|_{L^2} \leq C\, \|\Lambda^\beta \widetilde{b}\|_{L^2}^\beta\, \|\Lambda^{1+\beta} \widetilde{b}\|_{L^2}^{1-\beta} \leq C\left(\|\Lambda^\beta \widetilde{b}\|_{L^2} + \|\Lambda^{1+\beta} \widetilde{b}\|_{L^2}\right),$$

we find

$$\|\widetilde{b}\|_{L^2}^2 \leq C\, M(t)^{\frac{\sigma}{1+\sigma}}\, N(t)^{\frac{1}{1+\sigma}}. \tag{2.9}$$

Similarly,

$$\begin{aligned}\|\nabla\widetilde{b}\|_{L^2}^2 &\le C\,\|\nabla\widetilde{b}_2\|_{L^2}^{\frac{2\sigma}{1+\sigma}}\,\|\Lambda^{1+\sigma}\,\Lambda_1^{-\sigma}\widetilde{b}\|_{L^2}^{\frac{2}{1+\sigma}}\\ &\le C\,\left(\|\Lambda^{\beta}\widetilde{b}\|_{L^2}+\|\Lambda^{1+\beta}\widetilde{b}\|_{L^2}\right)^{\frac{2\sigma}{1+\sigma}}\,\|\Lambda^{1+\sigma}\,\Lambda_1^{-\sigma}\widetilde{b}\|_{L^2}^{\frac{2}{1+\sigma}}\\ &\le C\,M(t)^{\frac{\sigma}{1+\sigma}}\,N(t)^{\frac{1}{1+\sigma}}. \end{aligned}\tag{2.10}$$

(2.7), (2.8), (2.9) and (2.10) imply (2.6). It follows from (2.6) that

$$L(t)\le C M(t)^{\frac{\sigma}{1+\sigma}}\,N_0^{\frac{1}{1+\sigma}}\quad\text{or}\quad M(t)\ge C\,N_0^{-\frac{1}{\sigma}}\,L(t)^{1+\frac{1}{\sigma}}.$$

Inserting this inequality in (2.1), we find

$$\frac{d}{dt}L(t)+C\,N_0^{-\frac{1}{\sigma}}\,L(t)^{1+\frac{1}{\sigma}}\le 0,$$

which yields, for a pure constant C,

$$L(t)\le\left(L(0)^{-\frac{1}{\sigma}}+C\,\frac{1}{\sigma N(0)^{1/\sigma}}t\right)^{-\sigma}.$$

This leads to the desired decay rate in (1.6). To prove the decay rate in (1.7), we set

$$U=\partial_{x_1}^k\widetilde{u},\qquad B=\partial_{x_1}^k\widetilde{b}.$$

Clearly, (U,B) satisfies the same equations as (1.3) but with the initial data

$$U(x,0)=U_0(x):=\partial_{x_1}^k\widetilde{u}_0(x),\qquad B(x,0)=B_0(x):=\partial_{x_1}^k\widetilde{b}_0(x).$$

Since (U_0,B_0) satisfies $(U_0,B_0)\in L^2$ and

$$(\Lambda_1^{-(k+\sigma)}U_0,\,\Lambda_1^{-(k+\sigma)}B_0)\in H^{1+\sigma}(\mathbb{R}^2).$$

The argument above can be repeated and the result is (1.7). This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

This section provides the proof of Theorem 1.2. The proof relies crucially on two lemmas. The first lemma presents the solution formula for a fractionally damped degenerate wave type equation while the second lemma provides bounds on the decay rates for Fourier multiplier operators G_0 and G_1 defined in (1.11). The behavior of $\widehat{G}_0$ and $\widehat{G}_1$ depends on the Fourier frequencies ξ.

Lemma 3.1. *Assume that f satisfies the degenerate wave type equation*

$$\begin{cases}\partial_{tt}f+\eta(-\Delta)^{\beta}\partial_t f-R^2\,\partial_{x_1x_1}f=0, & x\in\mathbb{R}^2,\ t>0,\\ f(x,0)=f_0(x), \qquad (\partial_t f)(x,0)=f_1(x).\end{cases}\tag{3.1}$$

Then f can be explicitly represented as

$$f = G_0\left(f_1 + \frac{1}{2}\eta(-\Delta)^\beta f_0\right) + G_1 f_0$$

where G_0 and G_1 are as in (1.11).

Proof. Taking the Fourier transform of the equation of f, we have

$$\partial_{tt}\widehat{f} + \eta|\xi|^{2\beta}\partial_t\widehat{f} + R^2\xi_1^2\widehat{f} = 0.$$

Using the method of operator splitting, we find

$$\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} + \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right)\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} - \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right)\widehat{f} = 0.$$

This equation can be written into two different systems of equations,

$$\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} + \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right) g = 0, \tag{3.2}$$

$$\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} - \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right)\widehat{f} = g \tag{3.3}$$

and

$$\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} - \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right) h = 0, \tag{3.4}$$

$$\left(\partial_t + \frac{1}{2}\eta|\xi|^{2\beta} + \frac{1}{2}\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}\right)\widehat{f} = h \tag{3.5}$$

Taking the difference of (3.5) and (3.3), we find

$$\widehat{f}(\xi,t) = \frac{h(\xi,t) - g(\xi,t)}{\sqrt{\eta^2|\xi|^{4\beta} - 4R^2\xi_1^2}}$$

By (3.2) and (3.4), and also (1.12) and (1.13),

$$g(\xi,t) = g(\xi,0)\, e^{\lambda_2(\xi)t}, \qquad h(\xi,t) = h(\xi,0)\, e^{\lambda_1(\xi)t}$$

The initial data can be obtained via (3.3) and (3.5),

$$g(\xi,0) = \widehat{f_1} - \lambda_1\widehat{f_0}, \qquad h(\xi,0) = \widehat{f_1} - \lambda_2\widehat{f_0}.$$

Therefore,

$$\widehat{f}(\xi,t) = \frac{(\widehat{f_1} - \lambda_2 \widehat{f_0})\, e^{\lambda_1(\xi)t} - (\widehat{f_1} - \lambda_1 \widehat{f_0}) e^{\lambda_2(\xi)t}}{\lambda_1 - \lambda_2}$$
$$= \widehat{G}_0 \left(\widehat{f_1} + \frac{1}{2}\eta|\xi|^{2\beta} \widehat{f_0} \right) + \widehat{G}_1 \widehat{f_0}.$$

This completes the proof of Lemma 3.1. □

Clearly the behavior of $\widehat{G}_0(\xi,t)$ and $\widehat{G}_1(\xi,t)$ depends on the Fourier frequencies ξ. The second lemma provides upper bounds for $\widehat{G}_0$ and $\widehat{G}_1$ in different frequency domains. We will use ReA to denote the real part of a complex number A.

Lemma 3.2. *Let S_1 and S_2 be the following subsets of $\mathbb{R}^2$,*

$$S_1 := \left\{ k \in \mathbb{R}^2 : \xi_1^2 \geq \frac{\eta^2}{4R^2}|\xi|^{4\beta} \quad or \quad \sqrt{1 - \frac{4R^2\xi_1^2}{\eta^2|\xi|^{4\beta}}} \leq \frac{1}{2} \right\}$$
$$S_2 := \left\{ k \in \mathbb{R}^2 : \xi_1^2 < \frac{\eta^2}{4R^2}|\xi|^{4\beta} \quad or \quad \sqrt{1 - \frac{4R^2\xi_1^2}{\eta^2|\xi|^{4\beta}}} > \frac{1}{2} \right\}.$$

Then $\widehat{G}_0$ and $\widehat{G}_1$ obey the following bounds:

(1) *For any $\xi \in S_1$,*

$$Re\,\lambda_2 \leq -\frac{1}{2}\eta|\xi|^{2\beta}, \qquad Re\,\lambda_1 \leq -\frac{1}{4}\eta|\xi|^{2\beta},$$
$$|\widehat{G}_0| \leq t\, e^{-\frac{1}{4}\eta|\xi|^{2\beta}t}, \qquad |\widehat{G}_0| \leq |\xi|^{-2\beta} e^{-C\eta|\xi|^{2\beta}t},$$
$$|\widehat{G}_1| \leq C\, e^{-\frac{1}{4}\eta|\xi|^{2\beta}t},$$

where $C > 0$ is a constant.

(2) *For any $\xi \in S_2$,*

$$Re\,\lambda_2 \leq -\frac{3}{4}\eta|\xi|^{2\beta}, \qquad Re\,\lambda_1 \leq -\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}},$$
$$|\widehat{G}_0| \leq \frac{1}{\eta|\xi|^{2\beta}}\, e^{-\frac{1}{2}\eta|\xi|^{2\beta}t} + \frac{1}{\eta|\xi|^{2\beta}}\, e^{-\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}}t},$$
$$|\widehat{G}_1| \leq C\, e^{-\frac{1}{2}\eta|\xi|^{2\beta}t} + C\, e^{-\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}}t}.$$

Proof. The proof is not difficult. For $k \in S_1$, λ_1 and λ_2 clearly satisfy the bounds specified above. The bound for $\widehat{G}_0$ follows from the mean-value theorem, which implies that there is $\rho \in (Re\lambda_2, Re\lambda_1)$ such that

$$|\widehat{G}_0| = te^{\rho t} \leq t\, e^{-\frac{1}{4}\eta|\xi|^{2\beta}t}.$$

When $\xi \in S_2$, the bound for λ_2 is obvious. To obtain the bound for λ_1, we write λ_1 as

$$\lambda_1 = -\frac{1}{2}\eta\,|\xi|^{2\beta}\left(1-\sqrt{1-\frac{4R^2\,\xi_1^2}{\eta^2|\xi|^{4\beta}}}\right) = -\frac{\frac{2R^2\xi_1^2}{\eta|\xi|^{2\beta}}}{1+\sqrt{1-\frac{4R^2\xi_1^2}{\eta^2|\xi|^{4\beta}}}} \le -\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}}.$$

The bounds for $|\widehat{G}_0|$ and $|\widehat{G}_1|$ follow directly from the bounds for λ_1 and λ_2. This completes the proof of Lemma 3.2. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. We compute the L^2-norm of $\widetilde{u}$ via (1.9),

$$\|\widetilde{u}(t)\|_{L^2}^2 \le \int |\widehat{G}_0(\xi,t)|^2\,\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi + C\int |\widehat{G}_0(\xi,t)|^2\,|\xi|^{4\beta}\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi$$
$$+\int |\widehat{G}_1(\xi,t)|^2\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi.$$

Since $\widehat{G}_0(\xi,t)$ and $\widehat{G}_1(\xi,t)$ behave differently for different ξ, we split each of the summations above into two parts and apply Lemma 3.2, which implies

$$\begin{aligned}
&\int |\widehat{G}_0(\xi,t)|^2\,\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi\\
&= \int_{\xi\in S_1} |\widehat{G}_0(\xi,t)|^2\,\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi + \int_{\xi\in S_2} |\widehat{G}_0(\xi,t)|^2\,\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi\\
&\le C\int |\xi|^{-4\beta}\,e^{-C_0|\xi|^{2\beta}t}\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi + C\int |\xi|^{-4\beta}e^{-\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}}t}\,\xi_1^2|\widehat{\widetilde{b}}_0(\xi)|^2\,d\xi\\
&\le C\,(1+t)^{-\frac{2}{\beta}+1}\,\||\xi|^{-\beta}\widehat{\widetilde{b}}_0(\xi)\|_{L^\infty}^2 + C\,(1+t)^{-1}\int |\xi|^{-2\beta}|\widehat{\widetilde{b}}_0(k)|^2\,d\xi\\
&\le C\,(1+t)^{-\frac{2}{\beta}+1}\,\|\Lambda^{-\beta}\widetilde{b}_0\|_{L^1}^2 + C\,(1+t)^{-1}\|\Lambda^{-\beta}\widetilde{b}_0\|_{L^2}^2\\
&\le C\,(1+t)^{-\frac{1}{\beta}}\,\|\Lambda^{-\beta}\widetilde{b}_0\|_{L^1}^2 + C\,(1+t)^{-1}\|\Lambda^{-\beta}\widetilde{b}_0\|_{L^2}^2
\end{aligned}$$

where we have used $\beta \le 1$ and the simple fact that $\||\xi|^{-\beta}\widehat{\widetilde{b}}_0(\xi)\|_{L^\infty} \le \|\Lambda^{-\beta}\widetilde{b}_0\|_{L^1}$. Similarly,

$$\begin{aligned}
&\int |\widehat{G}_0(\xi,t)|^2\,|\xi|^{4\beta}\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi\\
&\le C\int e^{-C_0|\xi|^{2\beta}t}|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi + C\int e^{-\frac{4R^2\xi_1^2}{3\eta|\xi|^{2\beta}}t}\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi\\
&\le C\,(1+t)^{-\frac{1}{\beta}}\,\|\Lambda^{-\beta}\widetilde{u}_0\|_{L^1}^2 + C\,(1+t)^{-1}\|\Lambda_1^{-1}\widetilde{u}_0\|_{L^2}^2.
\end{aligned}$$

Applying the bounds on $\widehat{G}_1$ in Lemma 3.2, we have

$$\begin{aligned}
&\int |\widehat{G}_1(\xi,t)|^2\,|\widehat{\widetilde{u}}_0|^2\,d\xi \\
&= \int_{\xi\in S_1} |\widehat{G}_1(\xi,t)|^2\,|\widehat{\widetilde{u}}_0|^2\,d\xi + \int_{\xi\in S_2} |\widehat{G}_1(\xi,t)|^2\,|\widehat{\widetilde{u}}_0|^2\,d\xi \\
&\le C\int e^{-C|\xi|^{2\beta}t}\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi + C\int e^{-C\frac{\xi_1^2}{|\xi|^{2\beta}}t}\,|\widehat{\widetilde{u}}_0(\xi)|^2\,d\xi \\
&\le C\,(1+t)^{-1}\,\|\Lambda^{-\beta}\widetilde{u}_0\|_{L^2}^2 + C\,(1+t)^{-1}\,\|\Lambda^{\beta}\Lambda_1^{-1}\widetilde{u}_0\|_{L^2}^2.
\end{aligned}$$

Combining the estimates above, we find

$$\begin{aligned}
\|\widetilde{u}(t)\|_{L^2} &\le C\,(1+t)^{-1/(2\beta)}\,\|\Lambda^{-\beta}(\widetilde{u}_0,\widetilde{b}_0)\|_{L^1} \\
&\quad + C\,(1+t)^{-1/2}\left(\|\Lambda_1^{-1}(\widetilde{u}_0,\widetilde{b}_0)\|_{L^2} + \|\Lambda^{-\beta}(\widetilde{u}_0,\widetilde{b}_0)\|_{L^2} + \|\Lambda^{\beta}\Lambda_1^{-1}(\widetilde{u}_0,\widetilde{b}_0)\|_{L^2}\right).
\end{aligned}$$

The bound for $\|\widetilde{b}(t)\|_{L^2}$ is very similar. This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

This section proves Theorem 1.3. We need the following lemma, whose proof can be found in [7].

Lemma 4.1. *Assume $f \in L^1(0,\infty)$ is a nonnegative and uniformly continuous function. Then,*

$$f(t) \to 0 \quad \text{as } t \to \infty.$$

Especially, if $f \in L^1(0,\infty)$ is nonnegative and satisfies, for a constant C and any $0 \le s < t < \infty$,

$$|f(t) - f(s)| \le C\,|t-s|,$$

then $f(t) \to 0$ as $t \to \infty$.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Dotting (1.2) with $(\widetilde{u},\widetilde{b})$ and integrating in time yield

$$\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{b}(t)\|_{L^2}^2 + 2\eta\int_0^t \|\nabla\widetilde{b}(\tau)\|_{L^2}^2\,d\tau = \|\widetilde{u}_0\|_{L^2}^2 + \|\widetilde{b}_0\|_{L^2}^2, \tag{4.1}$$

where we have used the following identities, due to $\nabla\cdot\widetilde{u} = 0$ and $\nabla\cdot\widetilde{b} = 0$,

$$\int \widetilde{u}\cdot\nabla\widetilde{u}\cdot\widetilde{u}\,dx=0,\qquad \int \widetilde{u}\cdot\nabla\widetilde{b}\cdot\widetilde{b}\,dx=0,$$

$$\int \widetilde{b}\cdot\nabla\widetilde{b}\cdot\widetilde{u}\,dx+\int \widetilde{b}\cdot\nabla\widetilde{u}\cdot\widetilde{b}\,dx=0,\quad \int \partial_{x_1}\widetilde{b}\cdot\widetilde{u}\,dx+\int \partial_{x_1}\widetilde{u}\cdot\widetilde{b}\,dx=0.$$

To estimate the H^1-norm, we recall that $\widetilde{\omega}=\nabla\times\widetilde{u}$ and $\widetilde{j}=\nabla\times\widetilde{b}$ satisfy

$$\begin{cases}\partial_t\widetilde{\omega}+\widetilde{u}\cdot\nabla\widetilde{\omega}=\widetilde{b}\cdot\nabla\widetilde{j}+\partial_{x_1}\widetilde{j},\\ \partial_t\widetilde{j}+\widetilde{u}\cdot\nabla\widetilde{j}=\eta\Delta\widetilde{j}+\widetilde{b}\cdot\nabla\widetilde{\omega}+Q(\nabla\widetilde{u},\nabla\widetilde{b})+R\partial_{x_1}\widetilde{\omega},\end{cases}\tag{4.2}$$

where

$$Q(\nabla\widetilde{u},\nabla\widetilde{b})=2\partial_1\widetilde{b}_1(\partial_2\widetilde{u}_1+\partial_1\widetilde{u}_2)-2\partial_1\widetilde{u}_1(\partial_2\widetilde{b}_1+\partial_1\widetilde{b}_2).$$

A simple energy estimate yields

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{\omega}\|_{L^2}^2+\|\widetilde{j}\|_{L^2}^2)+\eta\|\nabla\widetilde{j}\|_{L^2}^2=\int Q\,\widetilde{j}\,dx$$

The term on the right can be bounded as follows,

$$\int Q\,\widetilde{j}\,dx\le\|\nabla\widetilde{u}\|_{L^2}\,\|\nabla\widetilde{b}\|_{L^4}\,\|\widetilde{j}\|_{L^4}\le C\,\|\widetilde{\omega}\|_{L^2}\,\|\widetilde{j}\|_{L^2}\|\nabla\widetilde{j}\|_{L^2}$$

$$\le\frac{\eta}{2}\|\nabla\widetilde{j}\|_{L^2}^2+C\,\|\widetilde{j}\|_{L^2}^2\|\widetilde{\omega}\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}(\|\widetilde{\omega}\|_{L^2}^2+\|\widetilde{j}\|_{L^2}^2)+\eta\|\nabla\widetilde{j}\|_{L^2}^2\le C\,\|\widetilde{j}\|_{L^2}^2\|\widetilde{\omega}\|_{L^2}^2$$

or

$$\|\widetilde{\omega}(t)\|_{L^2}^2+\|\widetilde{j}(t)\|_{L^2}^2\le(\|\widetilde{\omega}_0\|_{L^2}^2+\|\widetilde{j}_0\|_{L^2}^2)\,e^{C\int_0^t\|\widetilde{j}\|_{L^2}^2\,d\tau}$$

Noticing (4.1), we have

$$\|\widetilde{\omega}(t)\|_{L^2}^2+\|\widetilde{j}(t)\|_{L^2}^2\le(\|\widetilde{\omega}_0\|_{L^2}^2+\|\widetilde{j}_0\|_{L^2}^2)\,e^{C\,(\|\widetilde{u}_0\|_{L^2}^2+\|\widetilde{b}_0\|_{L^2}^2)}.$$

Since $\|\widetilde{\omega}\|_{L^2}=\|\nabla\widetilde{u}\|_{L^2}$ and $\|\widetilde{j}\|_{L^2}=\|\nabla\widetilde{b}\|_{L^2}$, we have completed the first part of Theorem 1.3. Now we show the time Lipschitz property of $\|\widetilde{b}(t)\|_{L^2}$ and $\|\nabla\widetilde{b}(t)\|_{L^2}$. For any $0\le t_1\le t_2$, we take the dot product of $\widetilde{b}$ with the equation of $\widetilde{b}$ and integrate in $t\in[t_1,t_2]$ to obtain

$$\|\widetilde{b}(t_2)\|_{L^2}^2 - \|\widetilde{b}(t_1)\|_{L^2}^2 = -2\eta \int_{t_1}^{t_2} \|\nabla \widetilde{b}(\tau)\|_{L^2}^2 \, d\tau + \int_{t_1}^{t_2} \int \widetilde{b} \cdot \nabla \widetilde{u} \cdot \widetilde{b} \, dx + \int_{t_1}^{t_2} \int R \partial_{x_1} \widetilde{u} \cdot \widetilde{b} \, dx.$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
&\int_{t_1}^{t_2} \int \widetilde{b} \cdot \nabla \widetilde{u} \cdot \widetilde{b} \, dx \, d\tau \le \int_{t_1}^{t_2} \|\widetilde{b}\|_{L^4}^2 \|\nabla \widetilde{u}\|_{L^2} \, d\tau \\
&\le \int_{t_1}^{t_2} \|\widetilde{b}\|_{L^2} \, \|\nabla \widetilde{b}\|_{L^2} \|\widetilde{\omega}\|_{L^2} \, d\tau \\
&\le \eta \int_{t_1}^{t_2} \|\nabla \widetilde{b}\|_{L^2}^2 \, d\tau + C\, |t_2 - t_1| \sup_{t \in [t_1, t_2]} \|\widetilde{\omega}(t)\|_{L^2}^2 \, \|\widetilde{b}(t)\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| \|\widetilde{b}(t_2)\|_{L^2}^2 - \|\widetilde{b}(t_1)\|_{L^2}^2 \right| \\
&\le \eta |t_2 - t_1| \sup_{t \in [t_1, t_2]} \|\widetilde{j}(t)\|_{L^2}^2 + C\, |t_2 - t_1| \sup_{t \in [t_1, t_2]} \|\widetilde{\omega}(t)\|_{L^2}^2 \|\widetilde{b}(t)\|_{L^2}^2 \\
&\quad + C\, |t_2 - t_1| \sup_{t \in [t_1, t_2]} \|\widetilde{\omega}(t)\|_{L^2} \, \|\widetilde{b}(t)\|_{L^2}.
\end{aligned}$$

Together with the uniform-in-time global bounds for $\|\widetilde{b}(t)\|_{L^2}$, $\|\widetilde{j}(t)\|_{L^2}$ and $\|\widetilde{\omega}(t)\|_{L^2}$, the inequality above implies the Lipschitz property of $\|\widetilde{b}(t)\|_{L^2}^2$. We now show the uniform continuity of $\|\widetilde{j}(t)\|_{L^2}^2$. It follows from the equation of $\widetilde{j}$ that

$$\frac{d}{dt} \|\widetilde{j}(t)\|_{L^2}^2 + 2\eta \|\nabla \widetilde{j}(t)\|_{L^2}^2 = \int \widetilde{b} \cdot \nabla \widetilde{\omega} \widetilde{j} \, dx + \int Q \, \widetilde{j} \, dx + \int R \partial_{x_1} \widetilde{\omega} \, \widetilde{j} \, dx$$

For any $0 \le t_1 \le t_2$, we integrate the equation above in $t \in [t_1, t_2]$ to obtain

$$\|\widetilde{j}(t_2)\|_{L^2}^2 - \|\widetilde{j}(t_1)\|_{L^2}^2 = -2\eta \int_{t_1}^{t_2} \|\nabla \widetilde{j}(t)\|_{L^2}^2 \, dt + K_1 + K_2 + K_3, \tag{4.3}$$

where

$$K_1 = \int_{t_1}^{t_2} \int \widetilde{b} \cdot \nabla \widetilde{\omega} \widetilde{j} \, dx, \quad K_2 = \int_{t_1}^{t_2} \int Q \, \widetilde{j} \, dx, \quad K_3 = \int_{t_1}^{t_2} \int R \partial_{x_1} \widetilde{\omega} \, \widetilde{j} \, dx.$$

K_2 can be bounded as before,

$$|K_2| \le C \int_{t_1}^{t_2} \|\widetilde{j}\|_{L^4}^2 \, \|\widetilde{\omega}\|_{L^2} \, dt \le \frac{\eta}{2} \int_{t_1}^{t_2} \|\nabla \widetilde{j}\|_{L^2}^2 \, dt + C \int_{t_1}^{t_2} \|\widetilde{j}\|_{L^2}^2 \, \|\omega\|_{L^2}^2 \, dt.$$

Integrating by parts and applying Sobolev's inequality lead to

$$|K_1| \le \int_{t_1}^{t_2} \|\widetilde{b}\|_{L^\infty} \, \|\nabla \widetilde{j}\|_{L^2} \, \|\widetilde{\omega}\|_{L^2} \, dt.$$

By Sobolev's inequality, for $0 < a < \frac{1}{2}$,

$$\|\widetilde{b}\|_{L^\infty} \le C \|\widetilde{b}\|_{L^2}^a \, \|\nabla \widetilde{b}\|_{L^2}^{1-2a} \, \|\nabla \widetilde{j}\|_{L^2}^a.$$

Therefore,

$$\begin{aligned} |K_1| &\le C \int_{t_1}^{t_2} \|\widetilde{b}\|_{L^2}^a \, \|\nabla \widetilde{b}\|_{L^2}^{1-2a} \, \|\nabla \widetilde{j}\|_{L^2}^{1+a} \, \|\widetilde{\omega}\|_{L^2} \, dt \\ &\le \frac{\eta}{2} \int_{t_1}^{t_2} \|\nabla \widetilde{j}\|_{L^2}^2 \, dt + C \int_{t_1}^{t_2} \|\widetilde{\omega}\|_{L^2}^{\frac{2}{1-a}} \, \|\widetilde{b}\|_{L^2}^{\frac{2a}{1-a}} \, \|\nabla \widetilde{b}\|_{L^2}^{2-\frac{2a}{1-a}} \, dt. \end{aligned}$$

By integrating by parts, K_3 can be bounded by

$$|K_3| \le R \int_{t_1}^{t_2} \|\widetilde{\omega}\|_{L^2} \|\nabla \widetilde{j}\|_{L^2} \, dt \le \frac{\eta}{2} \int_{t_1}^{t_2} \|\nabla \widetilde{j}\|_{L^2}^2 \, dt + C \int_{t_1}^{t_2} \|\widetilde{\omega}\|_{L^2}^2 \, dt.$$

Inserting the estimates for K_1, K_2 and K_3 in (4.3) yields

$$\begin{aligned} \|\widetilde{j}(t_2)\|_{L^2}^2 - \|\widetilde{j}(t_1)\|_{L^2}^2 \le & -\frac{\eta}{2} \int_{t_1}^{t_2} \|\nabla \widetilde{j}\|_{L^2}^2 \, dt \\ & + C|t_2 - t_1| \sup_{t \in [t_1, t_2]} (1 + \|\widetilde{j}(t)\|_{L^2}^2) \|\widetilde{\omega}(t)\|_{L^2}^2 \\ & + C\,|t_2 - t_1| \sup_{t \in [t_1, t_2]} \|\widetilde{\omega}(t)\|_{L^2}^{\frac{2}{1-a}} \, \|\widetilde{b}(t)\|_{L^2}^{\frac{2a}{1-a}} \, \|\nabla \widetilde{b}(t)\|_{L^2}^{2-\frac{2a}{1-a}}. \end{aligned}$$

Combining this inequality with the integrability

$$\int_0^\infty \|\widetilde{j}(t)\|_{L^2}^2 dt \le \|(\widetilde{u}_0, \widetilde{b}_0)\|_{L^2}^2$$

and Lemma 4.1, we can show that, as $t \to \infty$,

$$\|\widetilde{j}(t)\|_{L^2} \to 0$$

By Sobolev's embedding and the simple fact $\|\widetilde{j}(t)\|_{L^2} = \|\nabla\widetilde{b}(t)\|_{L^2}$,

$$\|\widetilde{b}(t)\|_{L^q} \leq C\|\widetilde{b}(t)\|_{L^2}^{\frac{2}{q}} \|\nabla\widetilde{b}(t)\|_{L^2}^{1-\frac{2}{q}} \to 0.$$

This completes the proof of Theorem 1.3. □

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