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Stability and optimal decay for the 3D Navier-Stokes equations with horizontal dissipation

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Abstract

Stability and large-time behavior are essential properties of solutions to many partial differential equations (PDEs) and play crucial roles in many practical applications. When there is full Laplacian, many techniques such as the Fourier splitting method have been created to obtain the large-time decay rates. However, when a PDE is anisotropic and involves only partial dissipation, these methods no longer apply and no effective approach is currently available. This paper aims at the stability and large-time behavior of the 3D anisotropic Navier-Stokes equations. We present a systematic approach to obtain the optimal decay rates of the stable solutions emanating from a small data. We establish that, if the initial velocity is small in the Sobolev space $H^4(\mathbb{R}^3) \cap H_h^{-\sigma}(\mathbb{R}^3)$, then the anisotropic Navier-Stokes equations have a unique global solution, and the solution and its first-order derivatives all decay at the optimal rates. Here $H_h^{-\sigma}$ with $\sigma > 0$ denotes a Sobolev space with negative horizontal index.

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1. Introduction

The goal of this paper is to understand the stability and more importantly the precise large-time behavior of solutions to the 3D Navier-Stokes equations with only horizontal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta_h u, & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the velocity field, $p = p(x,t)$ the pressure and $\nu > 0$ the kinematic viscosity. Here $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$. For notational convenience, we shall write ∂_j for ∂_{x_j} with $j = 1, 2, 3$. In addition, we use $\nabla_h := (\partial_1, \partial_2)$ for the horizontal gradient. (1.1) arises in the modeling of anisotropic geophysical fluids for which the vertical diffusion is much smaller than the horizontal one (see, e.g., [12, Chapter 4])).

Solutions of (1.1) emanating from general large initial data are not known to exist for all time. Whether or not large smooth solutions can blow up in a finite time is an outstanding open problem. However, any sufficiently small initial data with suitable regularity always leads to a unique global solution. Significant progress has been made on the small data global well-posedness and on the scaling invariant regularity criteria for (1.1) (see, e.g., [3–5,7,9–11,20,21]). For the convenience of the readers, we provide a simple proof for the global well-posedness of small data in $H^k(\mathbb{R}^3)$ with any $k \geq 2$ (see Proposition 1.1). The real issue concerned here is the precise large-time behavior. The solutions in some the aforementioned references may grow in time due to the application of Osgood type inequalities.

This paper aims at the exact large-time behavior and optimal decay rates of small global solutions to (1.1). When there is full Laplacian dissipation, Schonbek and her collaborators have developed powerful tools such as the Fourier splitting method to obtain the large-time behavior of solutions to the Navier-Stokes and related equations with full dissipation (see, e.g., [13–16]). However, these tools do not appear to work for the anisotropic Navier-Stokes equations like the one in (1.1). New approaches have to be developed in order to extract the precise large-time behavior for (1.1). This paper offers an efficient but not very sophisticated method for the anisotrophic Navier-Stokes equations. The discoveries of this paper may help understand the large-time behavior of many other anisotropic systems.

Before we describe our main ideas, we explain a basic fact about the decay of the heat equation. For any $v_0 \in L^2$, the solution of the heat equation

$$\partial_t v = \nu \Delta v,$$
$$v(x,0) = v_0(x)$$

is known to decay to zero in the L^2-norm,

$$\|v(t)\|_{L^2} \to 0 \qquad \text{as } t \to \infty.$$

But this decay can be slow and there is no uniform rate [2]. In order to obtain an explicit decay rate, extra assumptions on v_0 must be imposed. Two types of conditions are normally inserted, either $v_0 \in L^q$ with $1 \leq q < 2$ or v_0 in a Sobolev space with negative index, namely $v_0 \in H^{-\sigma}$ with $\sigma > 0$. This explains why we shall choose our initial data to be in the intersection of two Sobolev spaces, one with positive index and one with negative index. Since (1.1) involves only horizontal dissipation, the negative Sobolev setting involves only negative derivatives in the horizontal direction. To be more precise, we define $H_h^{-\sigma}(\mathbb{R}^3)$ with $\sigma > 0$ to be the space of distributions f satisfying

$$\|\Lambda_h^{-\sigma} f\|_{L^2(\mathbb{R}^3)}^2 := \int_{\mathbb{R}^3} |\xi_h|^{-2\sigma} |\widehat{f}(\xi)|^2 \, d\xi < \infty,$$

where $\xi_h = (\xi_1, \xi_2)$ and the fractional Laplacian operator $\Lambda_h^{-\sigma}$ is defined via the Fourier transform

$$\widehat{\Lambda_h^{-\sigma} f}(\xi) = |\xi_h|^{-\sigma} \, \widehat{f}(\xi).$$

The exact functional setting for our initial data u_0 is

$$u_0 \in H^4(\mathbb{R}^3) \cap H_h^{-\sigma}(\mathbb{R}^3), \quad \partial_3 u_0 \in H_h^{-\sigma}(\mathbb{R}^3), \quad \frac{3}{4} \le \sigma < 1.$$

We will explain the range of σ later. Our aim is to achieve the optimal decay rates, namely the rates for the corresponding heat equation,

$$\begin{cases} \partial_t u = \nu \Delta_h u, & x \in \mathbb{R}^3, \, t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

For $u_0 \in H^4(\mathbb{R}^3) \cap H_h^{-\sigma}(\mathbb{R}^3)$, the solution of (1.2) satisfies

$$\|u(t)\|_{H^4} \le \|u_0\|_{H^4}, \quad \|u(t)\|_{H_h^{-\sigma}} \le \|u_0\|_{H_h^{-\sigma}}.$$

Furthermore, using the representation of the solution to (1.2),

$$u(t) = e^{\nu \Delta_h t} \, u_0,$$

we find

$$\|u(t)\|_{L^2} = \|\Lambda_h^{\sigma} \, e^{\nu \Delta_h t} \, \Lambda_h^{-\sigma} u_0\|_{L^2} \le C \, (\nu t)^{-\frac{\sigma}{2}} \, \|\Lambda_h^{-\sigma} u_0\|_{L^2}, \tag{1.3}$$

$$\|\partial_3 u(t)\|_{L^2} \le C \, (\nu t)^{-\frac{\sigma}{2}} \, \|\partial_3 \Lambda_h^{-\sigma} u_0\|_{L^2}, \tag{1.4}$$

$$\|\nabla_h \partial_3 u(t)\|_{L^2} \le C \, (\nu t)^{-\frac{\sigma+1}{2}} \, \|\Lambda_h^{-\sigma} u_0\|_{L^2}. \tag{1.5}$$

We are able to show that the solution of the anisotropic Navier-Stokes equation (1.1) obeys the same decay rates as those for the heat equation (1.2). More precisely, we obtain the following theorem.

Theorem 1.1. *Consider (1.1) with* $\nu > 0$. *Let* $\frac{3}{4} \le \sigma < 1$. *Assume*

$$u_0 \in H^4(\mathbb{R}^3), \quad \nabla \cdot u_0 = 0, \quad \Lambda_h^{-\sigma} u_0,\ \Lambda_h^{-\sigma} \partial_3 u_0 \in L^2(\mathbb{R}^3).$$

Then, there is $\varepsilon > 0$ *such that, if*

$$\|u_0\|_{H^4(\mathbb{R}^3)} + \|\Lambda_h^{-\sigma} u_0\|_{L^2(\mathbb{R}^3)} + \|\Lambda_h^{-\sigma} \partial_3 u_0\|_{L^2(\mathbb{R}^3)} \le \varepsilon,$$

then (1.1) has a unique global solution u satisfying

$$\|u(t)\|_{H^4(\mathbb{R}^3)} \le C\varepsilon, \tag{1.6}$$

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2(\mathbb{R}^3)} \le C\,\varepsilon, \tag{1.7}$$

$$\|u(t)\|_{L^2(\mathbb{R}^3)} \le C\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \quad \|\partial_3 u(t)\|_{L^2(\mathbb{R}^3)} \le C\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \tag{1.8}$$

$$\|\nabla_h u(t)\|_{L^2(\mathbb{R}^3)} \le C\,\varepsilon(1+t)^{-\frac{\sigma+1}{2}}. \tag{1.9}$$

The decay rates in (1.8) and (1.9) coincide with those for the corresponding heat equation of (1.1) and are thus optimal.

The decay rates in (1.8) and (1.9) are exact the same as those for the heat equations in (1.3), (1.4) and (1.5), and are thus optimal. In addition, Theorem 1.1 assesses that the solution u remains bounded and small in $H_h^{-\sigma}$, namely (1.7) for all time when $u_0 \in H_h^{-\sigma}$ is small. This especially implies that the anisotropic Navier-Stokes equation concerned here preserves and actually improves the regularity setting of the initial data. This distinguishes Theorem 1.1 from many existing decay results, which provides no information on the boundedness of the solution in the Sobolev space with negative index even though the initial data is required to be in this space. In general it is not trivial to show that the solutions of partially dissipated systems remain in the negative Sobolev setting for all time. Normally the solution regularity of such systems deteriorates as time evolves.

Since the local-in-time well-posedness can be established by standard approaches (see, e.g. [8]), we focus on the global *a priori* bounds on the solution. We adopt the bootstrapping argument (see, e.g., [17, p. 21]). Assuming that $u_0 \in H^4 \cap H_h^{-\sigma}$ satisfies

$$\|u_0\|_{H^4} \le \varepsilon \quad \text{and} \quad \|u_0\|_{H_h^{-\sigma}} \le \varepsilon$$

for sufficiently small $\varepsilon > 0$, the bootstrapping argument starts with the ansatz that, for $t < T$,

$$\|u(t)\|_{H^4} \le C_0\,\varepsilon, \tag{1.10}$$

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2} \le C_0\,\varepsilon, \tag{1.11}$$

$$\|u(t)\|_{L^2}, \|\partial_3 u(t)\|_{L^2} \le C_0\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \tag{1.12}$$

$$\|\partial_1 u(t)\|_{L^2}, \|\partial_2 u(t)\|_{L^2} \le C_0\,\varepsilon(1+t)^{-\frac{\sigma}{2}-\frac{1}{2}} \tag{1.13}$$

for suitably selected $C_0 > 0$. We then show via (1.1), (1.10), (1.11), (1.12) and (1.13) that

$$\|u(t)\|_{H^4} \le \frac{C_0}{2}\,\varepsilon, \tag{1.14}$$

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon, \tag{1.15}$$

$$\|u(t)\|_{L^2}, \|\partial_3 u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \tag{1.16}$$

$$\|\partial_1 u(t)\|_{L^2}, \|\partial_2 u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon(1+t)^{-\frac{\sigma}{2}-\frac{1}{2}}. \tag{1.17}$$

The bootstrapping argument then assesses that $T=\infty$ and (1.14), (1.15), (1.16) and (1.17) hold for all $t<\infty$.

Since the assertion that any small initial data in H^4 yields a unique global small solution $u \in L^\infty(0,\infty; H^4)$ itself represents an important fact, we take out this part and state it as a proposition.

Proposition 1.1. *Consider (1.1) with $\nu>0$. Let $k \ge 2$ be an integer. Assume $u_0 \in H^k(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon>0$ such that, if*

$$\|u_0\|_{H^k(\mathbb{R}^3)} \le \varepsilon,$$

then (1.1) has a unique global solution $u \in L^\infty(0,\infty; H^k(\mathbb{R}^3))$ satisfying

$$\|u(t)\|_{H^k(\mathbb{R}^3)} \le C\,\varepsilon$$

for some constant $C>0$ and for all $t>0$.

As a special consequence of Proposition 1.1, we obtain (1.14). To show (1.15), we perform energy estimates on $\|\Lambda_h^{-\sigma} u(t)\|_{L^2}$. By invoking various anisotropic inequalities, we are able to obtain a suitable upper bound for the nonlinear term

$$\begin{aligned}&\int_{\mathbb{R}^3} \Lambda_h^{-\sigma}(u\cdot\nabla u)\cdot \Lambda_h^{-\sigma} u\,dx \\ &\le C\,\|\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}}\,\|u_3\|_{L^2}^{\sigma-\frac{1}{2}}\,\|\nabla_h u_3\|_{L^2}^{1-\sigma}\,\|\partial_3 u\|_{L^2}\,\|\Lambda_h^{-\sigma} u\|_{L^2}.\end{aligned}$$

As a consequence, the time integral of this bound, together with the ansatz, yields the desired upper bound in (1.15). In order to obtain the decay bounds in (1.16) and (1.17), we make use of the integral representation of (1.1),

$$u(t) = e^{\nu\Delta_h t} u_0 - \int_0^t e^{\nu\Delta_h (t-\tau)}\mathbb{P}(u\cdot\nabla u)(\tau)\,d\tau,$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the Leray projection onto divergence vector fields. This representation helps facilitate the estimates of $\|u(t)\|_{L^2}$, $\|\partial_3 u(t)\|_{L^2}$ and $\|\nabla_h u(t)\|_{L^2}$. The estimates of the nonlinear terms are technical and involve many anisotropic inequalities. We shall leave the technical details to the next section, which provides the proof of Theorem 1.1.

2. Proof of Theorem 1.1

This section is devoted to the proofs of Theorem 1.1 and of Proposition 1.1. We need several tools stated in the following lemmas.

The first lemma provides an upper bound for the L^p-norm of a one-dimensional function, which serves as a basic ingredient for anisotropic upper bounds. A proof can be found in [19].

Lemma 2.1. *Let $2 \le p \le \infty$. Let $s > \frac{1}{2} - \frac{1}{p}$. Then, there exists a constant $C = C(p, s)$ such that, for any 1D functions $f \in H^s(\mathbb{R})$,*

$$\|f\|_{L^p(\mathbb{R})} \le C\, \|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

In particular, if $p = \infty$ and $s = 1$, then any $f = f(x_3) \in H^1(\mathbb{R})$ satisfies

$$\|f\|_{L^\infty(\mathbb{R})} \le C\, \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

The second lemma provides an anisotropic upper bound for the integral of a triple product. It is a very powerful tool in dealing with anisotropic equations. A simple proof of this lemma can be found in [18].

Lemma 2.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\int_{\mathbb{R}^3} |fgh|\,dx \lesssim \|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^3} |fgh|\,dx \lesssim \|f\|_{L^2}^{\frac{1}{4}} \|\partial_1 f\|_{L^2}^{\frac{1}{4}} \|\partial_2 f\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2}^{\frac{1}{4}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_3 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}.$$

The third lemma states Minkowski's inequality. It is an elementary tool that allows us to estimate the Lebesgue norm with larger index first followed by the Lebesgue norm with a smaller index. The following version is taken from [1, p. 4] and a more general statement can be found in [6, p. 47].

Lemma 2.3. *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces. Let f be a nonnegative measurable function over $X_1 \times X_2$. For all $1 \le p \le q \le \infty$, we have*

$$\Big\| \|f(\cdot, x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \Big\| \|f(x_1, \cdot)\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

In particular, for a nonnegative measurable function f over $\mathbb{R}^m \times \mathbb{R}^n$ and for $1 \le p \le q \le \infty$,

$$\Big\| \|f\|_{L^p(\mathbb{R}^m)} \Big\|_{L^q(\mathbb{R}^n)} \le \Big\| \|f\|_{L^q(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m)}.$$

The next lemma provides an exact $L^p - L^q$ decay estimate for the generalized heat operator associated with a fractional Laplacian.

Lemma 2.4. *Let $\sigma \ge 0$, $\alpha > 0$, $\nu > 0$, $1 \le p \le q \le \infty$. Then*

$$\|\Lambda^{\sigma} e^{-\nu(-\Delta)^{\alpha} t} f\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\frac{\sigma}{2\alpha}-\frac{d}{2\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

We introduce a few notations. We write $\|f\|_{L^p_{x_j}}$ with $j = 1, 2, 3$ for the L^p-norm with respect to x_j on $\mathbb{R}$, and $\|f\|_{L^p_{x_j x_k}}$ with $j, k = 1, 2, 3$ for the L^p-norm with respect to (x_j, x_k) on $\mathbb{R}^2$. We also write $\|f\|_{L^q_h}$ for $\|f\|_{L^q_{x_1 x_2}}$ to shorten the notation. In addition, the anisotropic norm

$$\|f\|_{L^p_h L^q_{x_3}} := \|\|f\|_{L^q_{x_3}}\|_{L^p_h}$$

is also frequently used.

We are ready to prove the proposition and Theorem 1.1.

Proof of Proposition 1.1. First of all, any initial data $u_0 \in H^k$ with $k \ge 2$ leads to a local-in-time solution. This is the consequence of the standard contraction mapping principle and a local *a priori* bound on the norm $\|u\|_{H^k}$. The contraction mapping part can be verified via a standard procedure and can be found in many references such as the book by Majda and Bertozzi [8]. We focus on the global-in-time *a priori* bound for $\|u(t)\|_{H^k}$. Due to the norm equivalence

$$\|f\|_{H^k}^2 \quad \sim \quad \|f\|_{L^2}^2 + \sum_{m=1}^{3} \|\partial_m^k f\|_{L^2}^2 \tag{2.1}$$

and the uniform bound for the L^2-norm of u,

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla_h u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2, \tag{2.2}$$

it suffices to evaluate the last part in (2.1). Applying ∂_m^k to the equation in (1.1) and then taking the inner product with $\partial_m^k u$ yields

$$\frac{d}{dt} \sum_{m=1}^{3} \|\partial_m^k u\|_{L^2}^2 + 2\nu \sum_{m=1}^{3} \|\nabla_h \partial_m^k u\|_{L^2}^2 = -2 \sum_{m=1}^{3} \int \partial_m^k (u \cdot \nabla u) \cdot \partial_m^k u \, dx := I. \tag{2.3}$$

To estimate I, we rewrite it as

$$I = -2 \sum_{m,i,j=1}^{3} \sum_{\alpha=1}^{k} \binom{k}{\alpha} \int \partial_m^{\alpha} u_j \, \partial_j \partial_m^{k-\alpha} u_i \, \partial_m^k u_i \, dx, \tag{2.4}$$

where $\binom{k}{\alpha} = \frac{k!}{\alpha!(k-\alpha)!}$ is the binomial coefficient, and we have used the fact that the term with $\alpha = 0$ vanishes due to $\nabla \cdot u = 0$. We divide the terms in (2.4) into two types, the terms with at least one of m and j being 1 or 2 and the terms with $m = j = 3$. The first type can be handled directly by using the anisotropic inequalities in Lemma 2.2. Without loss of generality, we consider the term with $j = 1$ and $m = 3$. By Lemma 2.2, this term can be bounded by, for any $1 \le \alpha \le k$,

$$
\begin{aligned}
&\left| \int \partial_3^\alpha u_1 \, \partial_3^{k-\alpha} \partial_1 u_i \, \partial_3^k u_i \, dx \right| \\
&\le C \, \|\partial_3^\alpha u_1\|_{L^2}^{\frac12} \, \|\partial_1 \partial_3^\alpha u_1\|_{L^2}^{\frac12} \, \|\partial_3^{k-\alpha} \partial_1 u_i\|_{L^2}^{\frac12} \, \|\partial_3 \partial_3^{k-\alpha} \partial_1 u_i\|_{L^2}^{\frac12} \, \|\partial_3^k u_i\|_{L^2}^{\frac12} \, \|\partial_2 \partial_3^k u_i\|_{L^2}^{\frac12} \\
&\le C \, \|\nabla_h u\|_{H^k}^2 \, \|u\|_{H^k}.
\end{aligned}
$$

For the second type of terms, we have $m = j = 3$ and use the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$. Therefore, these terms can be bounded by

$$
\begin{aligned}
&\left| \int \partial_3^\alpha u_3 \, \partial_3^{k-\alpha} \partial_3 u_i \, \partial_3^k u_i \, dx \right| \\
&= \left| \int \partial_3^{\alpha-1} (\partial_1 u_1 + \partial_2 u_2) \, \partial_3^{k-\alpha} \partial_3 u_i \, \partial_3^k u_i \, dx \right| \\
&\le C \, \|\partial_3^{\alpha-1} \partial_1 u_1\|_{L^2}^{\frac12} \, \|\partial_3^\alpha \partial_1 u_1\|_{L^2}^{\frac12} \, \|\partial_3^{k+1-\alpha} u_i\|_{L^2}^{\frac12} \, \|\partial_1 \partial_3^{k+1-\alpha} u_i\|_{L^2}^{\frac12} \\
&\qquad \times \|\partial_3^k u_i\|_{L^2}^{\frac12} \, \|\partial_2 \partial_3^k u_i\|_{L^2}^{\frac12} \\
&\quad + C \, \|\partial_3^{\alpha-1} \partial_2 u_2\|_{L^2}^{\frac12} \, \|\partial_3^\alpha \partial_2 u_2\|_{L^2}^{\frac12} \, \|\partial_3^{k+1-\alpha} u_i\|_{L^2}^{\frac12} \, \|\partial_1 \partial_3^{k+1-\alpha} u_i\|_{L^2}^{\frac12} \\
&\qquad \times \|\partial_3^k u_i\|_{L^2}^{\frac12} \, \|\partial_2 \partial_3^k u_i\|_{L^2}^{\frac12} \\
&\le C \, \|\nabla_h u\|_{H^k}^2 \, \|u\|_{H^k}.
\end{aligned}
$$

Thus any term of either type admits the same upper bound. Therefore,

$$
|I| \le C \, \|\nabla_h u\|_{H^k}^2 \, \|u\|_{H^k}.
$$

Inserting this upper bound in (2.3), integrating in time, adding to (2.2) and invoking the equivalence, we find

$$
\|u(t)\|_{H^k}^2 + 2\nu \int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 \le \|u_0\|_{H^k}^2 + C_0 \int_0^t \|\nabla_h u\|_{H^k}^2 \, \|u\|_{H^k} \, d\tau \tag{2.5}
$$

When the initial data u_0 is taken to be sufficiently small, say

$$
\|u_0\|_{H^k} < C_0^{-1} \nu,
$$

then (2.5) implies

$$
\|u(t)\|_{H^k}^2 + \nu \int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 \le \|u_0\|_{H^k}^2,
$$

which yields the desired global uniform bound and the stability for $\|u(t)\|_{H^k}$.

We now briefly explain the uniqueness at the H^2-level, which can be quickly established. Assume that $u^{(1)}$ and $u^{(2)}$ are two solutions of (1.1) in the regularity class, for $T > 0$,

$$u^{(1)},\ u^{(2)} \in L^\infty(0, T; H^2).$$

The difference $\widetilde{u} = u^{(1)} - u^{(2)}$ satisfies

$$\begin{cases} \partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} = -\nabla \widetilde{p} + \nu \Delta_h \widetilde{u}, & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot \widetilde{u} = 0, \\ \widetilde{u}(x, 0) = 0, \end{cases} \tag{2.6}$$

where $\widetilde{p} = p^{(1)} - p^{(2)}$ with $p^{(1)}$ and $p^{(2)}$ being the pressures corresponding to $u^{(1)}$ and $u^{(2)}$, respectively. Taking the inner product of $\widetilde{u}$ with (2.6) yields

$$\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2 + 2\nu\|\nabla_h \widetilde{u}\|_{L^2}^2 = -\int \widetilde{u} \cdot \nabla u^{(2)} \cdot \widetilde{u}\, dx. \tag{2.7}$$

Since the dissipation involves only the horizontal dissipation, we need an anisotropic upper bound for the term on the right. By Hölder's inequality and Lemma 2.1

$$\begin{aligned} -\int \widetilde{u} \cdot \nabla u^{(2)} \cdot \widetilde{u}\, dx &\le \|\widetilde{u}\|_{L_h^4 L_{x_3}^2}^2 \, \|\nabla u^{(2)}\|_{L_h^2 L_{x_3}^\infty} \\ &\le C\, \|\widetilde{u}\|_{L^2} \|\nabla_h \widetilde{u}\|_{L^2}\, \|\nabla u^{(2)}\|_{L^2}^{\frac{1}{2}}\, \|\partial_3 \nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \\ &\le \nu \|\nabla_h \widetilde{u}\|_{L^2}^2 + C(\nu)\, \|u^{(2)}\|_{H^2}^2 \|\widetilde{u}\|_{L^2}^2 \end{aligned}$$

Incorporating this upper bound in (2.7) yields

$$\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2 + \nu\|\nabla_h \widetilde{u}\|_{L^2}^2 \le C(\nu)\, \|u^{(2)}\|_{H^2}^2 \|\widetilde{u}\|_{L^2}^2,$$

which leads to the uniqueness due to $u^{(2)} \in L^\infty(0, T; H^2)$. This completes the proof of Proposition 1.1. □

Proof of Theorem 1.1. The proof focuses on the global bounds in (1.6), (1.7), (1.8) and (1.9). As pointed out in the introduction, the framework of the proof is the bootstrapping argument. Assume that $u_0 \in H^4 \cap H_h^{-\sigma}$ satisfies

$$\|u_0\|_{H^4} + \|u_0\|_{H_h^{-\sigma}} + \|\partial_3 u_0\|_{H_h^{-\sigma}} \le \varepsilon \tag{2.8}$$

for sufficiently small $\varepsilon > 0$. We make the ansatz that, for $t < T$,

$$\|u(t)\|_{H^4} \le C_0\,\varepsilon, \tag{2.9}$$
$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2} \le C_0\,\varepsilon, \tag{2.10}$$
$$\|u(t)\|_{L^2}, \|\partial_3 u(t)\|_{L^2} \le C_0\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \tag{2.11}$$
$$\|\partial_1 u(t)\|_{L^2}, \|\partial_2 u(t)\|_{L^2} \le C_0\,\varepsilon(1+t)^{-\frac{\sigma}{2}-\frac{1}{2}} \tag{2.12}$$

for suitably selected $C_0 > 0$. We then show via (1.1), (2.8), (2.9), (2.10), (2.11) and (2.12) that

$$\|u(t)\|_{H^4} \le \frac{C_0}{2}\,\varepsilon, \tag{2.13}$$
$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon, \tag{2.14}$$
$$\|u(t)\|_{L^2}, \|\partial_3 u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon(1+t)^{-\frac{\sigma}{2}}, \tag{2.15}$$
$$\|\partial_1 u(t)\|_{L^2}, \|\partial_2 u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon(1+t)^{-\frac{\sigma}{2}-\frac{1}{2}}. \tag{2.16}$$

The bootstrapping argument then assesses that $T = \infty$, and (2.13), (2.14), (2.15) and (2.16) hold for all $t < \infty$.

As in the statement and proof of Proposition 1.1, if

$$\|u_0\|_{H^4} \le C_1\,\nu$$

for some pure constant $C_1 > 0$, then, for all time $t > 0$,

$$\|u(t)\|_{H^4} \le \|u_0\|_{H^4}.$$

As a special consequence, if

$$\varepsilon \le C_1 \nu \quad \text{and} \quad \|u_0\|_{H^4} \le \varepsilon,$$

then, for $C_0 \ge 2$,

$$\|u(t)\|_{H^4} \le \|u_0\|_{H^4} \le \varepsilon \le \frac{C_0}{2}\varepsilon,$$

which is (2.13). We now show (2.14). Applying $\Lambda_h^{-\sigma}$ to (1.1) and dotting with $\Lambda_h^{-\sigma} u$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\Lambda_h^{-\sigma} u\|_{L^2}^2 + 2\nu\|\Lambda_h^{1-\sigma} u\|_{L^2}^2 &= -2\int \Lambda_h^{-\sigma}(u\cdot\nabla u)\cdot \Lambda_h^{-\sigma} u\,dx \\ &= M. \end{aligned}\tag{2.17}$$

M can be written as

$$M = -2\int \Lambda_h^{-\sigma}(u_1\partial_1 u + u_2\partial_2 u)\cdot \Lambda_h^{-\sigma} u\,dx - 2\int \Lambda_h^{-\sigma}(u_3\partial_3 u)\cdot \Lambda_h^{-\sigma} u\,dx. \tag{2.18}$$

Clearly the first two terms in (2.18) are better than the last term in (2.18) in the sense that they contain the favorable horizontal derivatives. Therefore, the worst term is

$$M_3 := -\int \Lambda_h^{-\sigma}(u_3\partial_3 u_1)\cdot \Lambda_h^{-\sigma} u_1\, dx.$$

We set

$$\frac{1}{2}+\frac{\sigma}{2}=\frac{1}{q} \quad \text{or} \quad q=\frac{2}{1+\sigma}.$$

Clearly, for $\frac{3}{4}\le \sigma < 1$, we have

$$1<q<2.$$

By Hölder's inequality, the Hardy-Littlewood-Sobolev inequality and Lemma 2.3,

$$\begin{aligned}
|M_3| &\le \|\Lambda_h^{-\sigma}(u_3\partial_3 u_1)\|_{L^2}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}\\
&= \Big\|\,\|\Lambda_h^{-\sigma}(u_3\partial_3 u_1)\|_{L^2_h}\Big\|_{L^2_{x_3}}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}\\
&= \Big\|\,\|u_3\partial_3 u_1\|_{L^q_h}\Big\|_{L^2_{x_3}}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}\\
&\le \Big\|\,\|u_3\partial_3 u_1\|_{L^2_{x_3}}\Big\|_{L^q_h}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}\\
&\le \Big\|\,\|u_3\|_{L^\infty_{x_3}}\|\partial_3 u_1\|_{L^2_{x_3}}\Big\|_{L^q_h}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}\\
&\le \|u_3\|_{L_h^{\frac{2}{\sigma}}L^\infty_{x_3}}\,\|\partial_3 u_1\|_{L^2_hL^2_{x_3}}\,\|\Lambda_h^{-\sigma}u_1\|_{L^2}.
\end{aligned} \tag{2.19}$$

The first part on the right-hand side can be further bounded as follows. By Hölder's inequality with $\frac{\sigma}{2}=\frac{1}{4}+\frac{2\sigma-1}{4}$,

$$\begin{aligned}
\|u_3\|_{L_h^{\frac{2}{\sigma}}L^\infty_{x_3}} &\le C\,\Big\|\,\|u_3\|^{\frac12}_{L^2_{x_3}}\,\|\partial_3 u_3\|^{\frac12}_{L^2_{x_3}}\Big\|_{L_h^{\frac{2}{\sigma}}}\\
&\le C\,\Big\|\,\|u_3\|^{\frac12}_{L^2_{x_3}}\Big\|_{L_h^{\frac{4}{2\sigma-1}}}\,\Big\|\,\|\partial_3 u_3\|^{\frac12}_{L^2_{x_3}}\Big\|_{L^4_h}\\
&\le C\,\|\partial_3 u_3\|^{\frac12}_{L^2}\,\|u_3\|^{\frac12}_{L_h^{\frac{2}{2\sigma-1}}L^2_{x_3}}\\
&\le C\,\|\nabla_h\cdot u_h\|^{\frac12}_{L^2}\,\|u_3\|^{\frac12}_{L^2_{x_3}L_h^{\frac{2}{2\sigma-1}}}\\
&\le C\,\|\nabla_h\cdot u_h\|^{\frac12}_{L^2}\,\|u_3\|^{\sigma-\frac12}_{L^2}\,\|\nabla_h u_3\|^{1-\sigma}_{L^2}.
\end{aligned}$$

Thus we have obtained the following bound

$$|M_3| \le C\,\|\nabla_h\cdot u_h\|_{L^2}^{\frac12}\,\|u_3\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_3\|_{L^2}^{1-\sigma}\,\|\partial_3 u_1\|_{L^2}\,\|\Lambda_h^{-\sigma} u_1\|_{L^2}.$$

Similarly, the first two terms in (2.18) can be bounded by

$$\begin{aligned}
&-2\int \Lambda_h^{-\sigma}(u_1\partial_1 u + u_2\partial_2 u)\cdot \Lambda_h^{-\sigma} u\,dx\\
&\le C\,\|\partial_3 u_1\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_1\|_{L^2}^{1-\sigma}\,\|\partial_1 u\|_{L^2}\,\|\Lambda_h^{-\sigma} u\|_{L^2}\\
&\quad + C\,\|\partial_3 u_2\|_{L^2}^{\frac12}\,\|u_2\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_2\|_{L^2}^{1-\sigma}\,\|\partial_2 u\|_{L^2}\,\|\Lambda_h^{-\sigma} u\|_{L^2}.
\end{aligned}$$

Integrating in time in (2.17) yields

$$\|\Lambda_h^{-\sigma} u\|_{L^2}^2 + 2\nu\int_0^t \|\Lambda_h^{1-\sigma} u\|_{L^2}^2 d\tau \le N, \tag{2.20}$$

where

$$\begin{aligned}
N := C&\int_0^t \|\partial_3 u_1\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_1\|_{L^2}^{1-\sigma}\,\|\partial_1 u\|_{L^2}\,\|\Lambda_h^{-\sigma} u\|_{L^2}\,d\tau\\
&+ C\int_0^t \|\partial_3 u_2\|_{L^2}^{\frac12}\,\|u_2\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_2\|_{L^2}^{1-\sigma}\,\|\partial_2 u\|_{L^2}\,\|\Lambda_h^{-\sigma} u\|_{L^2}\,d\tau\\
&+ C\int_0^t \|\nabla_h\cdot u_h\|_{L^2}^{\frac12}\,\|u_3\|_{L^2}^{\sigma-\frac12}\,\|\nabla_h u_3\|_{L^2}^{1-\sigma}\,\|\partial_3 u_1\|_{L^2}\,\|\Lambda_h^{-\sigma} u_1\|_{L^2}\,d\tau.
\end{aligned}$$

We then invoke the ansatz in (2.9) through (2.12) to bound N.

$$\begin{aligned}
N \le C&\int_0^t \Big(C_0\varepsilon(1+\tau)^{-\frac{\sigma}{2}}\Big)^{\sigma}\,\Big(C_0\varepsilon(1+\tau)^{-\frac{\sigma+1}{2}}\Big)^{2-\sigma}\,C_0\varepsilon\,d\tau\\
&+ C\int_0^t \Big(C_0\varepsilon(1+\tau)^{-\frac{\sigma+1}{2}}\Big)^{\frac12}\,\Big(C_0\varepsilon(1+\tau)^{-\frac{\sigma}{2}}\Big)^{\sigma-\frac12}\\
&\qquad\cdot\Big(C_0\varepsilon(1+\tau)^{-\frac{\sigma+1}{2}}\Big)^{1-\sigma}\,C_0\varepsilon(1+\tau)^{-\frac{\sigma}{2}}\,C_0\varepsilon\,d\tau
\end{aligned}$$

$$= C\,C_0^3\varepsilon^3\int_0^t (1+\tau)^{-\frac{\sigma}{2}-1}\,d\tau \,+ C\,C_0^3\varepsilon^3\int_0^t (1+\tau)^{-\frac{\sigma}{2}-\frac34}\,d\tau \tag{2.21}$$

$$\le C\,C_0^3\varepsilon^3, \tag{2.22}$$

where we have used the fact that the time integrals in (2.21) are bounded for any $\frac{3}{4} \leq \sigma < 1$. If we choose $\varepsilon > 0$ to be sufficiently small such that

$$CC_0\varepsilon \leq \frac{1}{4}, \tag{2.23}$$

then (2.22) and (2.23) implies

$$N \leq \frac{1}{4}C_0^2\varepsilon^2.$$

Inserting this bound in (2.20) yields

$$\|\Lambda_h^{-\sigma}u\|_{L^2}^2 \leq \frac{1}{4}C_0^2\varepsilon^2.$$

This completes (2.14).

To prove (2.15), we represent (1.1) in the integral form

$$u(t) = e^{\nu\Delta_h t}u_0 - \int_0^t e^{\nu\Delta_h(t-\tau)}\mathbb{P}(u\cdot\nabla u)(\tau)\,d\tau, \tag{2.24}$$

where $\mathbb{P}$ denotes the Leray projection. Taking the L^2-norm of (2.24) and noticing the boundedness of $\mathbb{P}$ on L^2 functions, we have

$$\|u(t)\|_{L^2} \leq \|e^{\nu\Delta_h t}u_0\|_{L^2} + \int_0^t \|e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}\,d\tau. \tag{2.25}$$

The linear part in (2.25) can be easily bounded. In fact, by Lemma 2.4,

$$\begin{aligned}\|e^{\nu\Delta_h t}u_0\|_{L^2} &\leq C\,(1+t)^{-\frac{\sigma}{2}}\left(\|u_0\|_{H_h^{-\sigma}} + \|u_0\|_{L^2}\right)\\ &\leq \frac{C_0}{4}\varepsilon(1+t)^{-\frac{\sigma}{2}},\end{aligned} \tag{2.26}$$

where C_0 is selected to satisfy $C_0 \geq 4C$. We bound the nonlinear part. Writing $u\cdot\nabla u = u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, we realize that the worst terms are $u_3\partial_3 u_1$ and $u_3\partial_3 u_2$, which would yield the worst decay rate. We should estimate them first. By Lemma 2.1 and Lemma 2.4,

$$\begin{aligned}&\int_0^t \|e^{\nu\Delta_h(t-\tau)}u_3\partial_3 u_1(\tau)\|_{L^2}\,d\tau\\ &\leq \int_0^t \|\|e^{\nu\Delta_h(t-\tau)}u_3\partial_3 u_1(\tau)\|_{L_h^2}\|_{L_{x_3}^2}\,d\tau\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \|(t-\tau)^{-\frac{1}{2}}\|u_3\partial_3 u_1(\tau)\|_{L_h^1}\|_{L_{x_3}^2}\,d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{1}{2}}\Big\|\|u_3(\tau)\|_{L_h^2}\|\partial_3 u_1(\tau)\|_{L_h^2}\Big\|_{L_{x_3}^2}\,d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{1}{2}}\|u_3(\tau)\|_{L_{x_3}^\infty L_h^2}\|\partial_3 u_1(\tau)\|_{L_{x_3}^2 L_h^2}\,d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{1}{2}}\|u_3(\tau)\|_{L_h^2 L_{x_3}^\infty}\|\partial_3 u_1(\tau)\|_{L^2}\,d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{1}{2}}\|u_3(\tau)\|^{\frac{1}{2}}_{L_h^2 L_{x_3}^2}\|\partial_3 u_3(\tau)\|^{\frac{1}{2}}_{L_h^2 L_{x_3}^2}\|\partial_3 u_1(\tau)\|_{L^2}\,d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{1}{2}}\|u_3(\tau)\|^{\frac{1}{2}}_{L_h^2 L_{x_3}^2}\|(\partial_1 u_1+\partial_2 u_2)(\tau)\|^{\frac{1}{2}}_{L_h^2 L_{x_3}^2}\|\partial_3 u_1(\tau)\|_{L^2}\,d\tau.
\end{aligned}
$$

Invoking the ansatz in (2.9) through (2.12) yields

$$
\begin{aligned}
&\int_0^t \|e^{\nu\Delta_h(t-\tau)}u_3\partial_3 u_1(\tau)\|_{L^2}\,d\tau \\
&\le C_0^2\,\varepsilon^2\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{\sigma}{4}}\,(1+\tau)^{-\frac{\sigma}{4}-\frac{1}{4}}(1+\tau)^{-\frac{\sigma}{2}}\,d\tau \\
&= C_0^2\,\varepsilon^2\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\sigma-\frac{1}{4}}\,d\tau \\
&\le \begin{cases} C_0^2\,\varepsilon^2\,(1+t)^{\frac{1}{4}-\sigma} & \text{if } \sigma<\frac{3}{4} \\ C_0^2\,\varepsilon^2\,(1+t)^{-\frac{1}{2}} & \text{if } \sigma>\frac{3}{4} \\ C_0^2\,\varepsilon^2\,(1+t)^{-\frac{1}{2}}\ln(1+t) & \text{if } \sigma=\frac{3}{4} \end{cases} \\
&\le C_0^2\,\varepsilon^2\,(1+t)^{-\frac{\sigma}{2}}
\end{aligned}
$$

for any $\frac{1}{2}\le\sigma<1$. If ε is taken to be small such that

$$
C_0\,\varepsilon \le \frac{1}{128}, \tag{2.27}
$$

then

$$\int_0^t \|e^{\nu\Delta_h(t-\tau)} u_3\partial_3 u_1(\tau)\|_{L^2}\, d\tau \le \frac{C_0}{128}\,\varepsilon\,(1+t)^{-\frac{\sigma}{2}}. \tag{2.28}$$

Similarly,

$$\int_0^t \|e^{\nu\Delta_h(t-\tau)} u_3\partial_3 u_2(\tau)\|_{L^2}\, d\tau \le \frac{C_0}{128}\,\varepsilon\,(1+t)^{-\frac{\sigma}{2}}. \tag{2.29}$$

The terms with $u_1\partial_1 u$ or $u_2\partial_2 u$ actually produce better decay rates. In fact,

$$\begin{aligned}
&\int_0^t \|e^{\nu\Delta_h(t-\tau)} u_1\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le \int_0^t \|\|e^{\Delta_h(t-\tau)} u_1\partial_1 u(\tau)\|_{L^2_h}\|_{L^2_{x_3}}\, d\tau \\
&\le \int_0^t \|(t-\tau)^{-\frac12}\|u_1\partial_1 u(\tau)\|_{L^1_h}\|_{L^2_{x_3}}\, d\tau \\
&\le \int_0^t (t-\tau)^{-\frac12}\|\|u_1(\tau)\|_{L^2_h}\|\partial_1 u(\tau)\|_{L^2_h}\|_{L^2_{x_3}}\, d\tau \\
&\le \int_0^t (t-\tau)^{-\frac12}\|u_1(\tau)\|_{L^\infty_{x_3}L^2_h}\|\partial_1 u(\tau)\|_{L^2_{x_3}L^2_h}\, d\tau \\
&\le \int_0^t (t-\tau)^{-\frac12}\|u_1(\tau)\|_{L^2_h L^\infty_{x_3}}\|\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le \int_0^t (t-\tau)^{-\frac12}\|u_1(\tau)\|^{\frac12}_{L^2_h L^2_{x_3}}\|\partial_3 u_1(\tau)\|^{\frac12}_{L^2_h L^2_{x_3}}\|\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le \int_0^t (t-\tau)^{-\frac12}\|u_1\|^{\frac12}_{L^2}\|\partial_3 u_1\|^{\frac12}_{L^2}\|\partial_1 u(\tau)\|_{L^2}\, d\tau.
\end{aligned}$$

Invoking the ansatz in (2.9) through (2.12) yields, for any $\sigma > \frac12$,

$$\int_0^t \|e^{\nu\Delta_h(t-\tau)} u_1\partial_1 u(\tau)\|_{L^2}\, d\tau$$

$$
\begin{aligned}
&\le C_0^2\,\varepsilon^2 \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{\sigma}{4}}\,(1+\tau)^{-\frac{\sigma}{4}}(1+\tau)^{-\frac{\sigma}{2}-\frac{1}{2}}\,d\tau \\
&= C_0^2\,\varepsilon^2 \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\sigma-\frac{1}{2}}\,d\tau \\
&\le C_0^2\,\varepsilon^2 \left((1+t)^{-\frac{1}{2}}+(1+t)^{-\sigma}\right) \\
&\le C_0^2\,\varepsilon^2\,(1+t)^{-\frac{\sigma}{2}} \\
&\le \frac{C_0}{128}\,\varepsilon\,(1+t)^{-\frac{\sigma}{2}},
\end{aligned}
\tag{2.30}
$$

where we have invoked (2.27). Inserting the upper bounds in (2.26), (2.28), (2.29) and (2.30) in (2.25), we find

$$
\|u(t)\|_{L^2} \le \frac{C_0}{2}\,\varepsilon\,(1+t)^{-\frac{\sigma}{2}}. \tag{2.31}
$$

We now estimate $\|\partial_3 u(t)\|_{L^2}$. Applying ∂_3 to (2.24) and then taking the L^2-norm, we have

$$
\|\partial_3 u(t)\|_{L^2} \le \|e^{\nu\Delta_h t}\partial_3 u_0\|_{L^2} + \int_0^t \|e^{\nu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}\,d\tau. \tag{2.32}
$$

As in (2.26), we have

$$
\begin{aligned}
\|e^{\nu\Delta_h t}\partial_3 u_0\|_{L^2} &\le C\,(1+t)^{-\frac{\sigma}{2}}\left(\|\partial_3 u_0\|_{H_h^{-\sigma}} + \|u_0\|_{L^2}\right) \\
&\le \frac{C_0}{4}\varepsilon\,(1+t)^{-\frac{\sigma}{2}}.
\end{aligned}
\tag{2.33}
$$

To estimate the second part in (2.32), we write

$$
\begin{aligned}
\partial_3(u\cdot\nabla u) &= \partial_3 u_1\,\partial_1 u + u_1\partial_3\partial_1 u + \partial_3 u_2\partial_2 u + u_2\partial_3\partial_2 u \\
&\quad + \partial_3 u_3\partial_3 u + u_3\partial_{33} u
\end{aligned}
\tag{2.34}
$$

and realize that $u_3\partial_{33}u_1$ and $u_3\partial_{33}u_2$ are the terms with the worst possible decay rates. We deal with them first.

$$
\begin{aligned}
&\int_0^t \|e^{\nu\Delta_h(t-\tau)}u_3\,\partial_{33}u(\tau)\|_{L^2}\,d\tau \\
&\le \int_0^t \left\| \|e^{\nu\Delta_h(t-\tau)}u_3\,\partial_{33}u(\tau)\|_{L_h^2} \right\|_{L_{x_3}^2}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \left\| \|u_3\, \partial_{33} u(\tau)\|_{L^1_h} \right\|_{L^2_{x_3}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \left\| \|u_3\|_{L^2_h} \, \|\partial_{33} u(\tau)\|_{L^2_h} \right\|_{L^2_{x_3}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_3\|_{L^2_h L^\infty_{x_3}} \|\partial_{33} u\|_{L^2} \, d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\partial_{33} u\|_{L^2} \, d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_1 + \partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{2}{3}} \|\partial_3^4 u\|_{L^2}^{\frac{1}{3}} \, d\tau.
\end{aligned}
$$

We now invoke the ansatz in (2.9) through (2.12) to obtain, for $\frac{3}{4} \le \sigma < 1$,

$$
\begin{aligned}
&\int_0^t \|e^{\nu \Delta_h (t-\tau)} u_3\, \partial_{33} u(\tau)\|_{L^2} \, d\tau \\
&\le C\, C_0^2\, \varepsilon^2 \int_0^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{\sigma}{4}} \, (1+\tau)^{-\frac{1}{4}-\frac{\sigma}{4}} \, (1+\tau)^{-\frac{\sigma}{3}} \, d\tau \\
&= C\, C_0^2\, \varepsilon^2 \int_0^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{5}{6}\sigma - \frac{1}{4}} \, d\tau \\
&\le C\, C_0^2\, \varepsilon^2 \, (1+t)^{-\frac{5}{6}\sigma + \frac{1}{4}} \\
&\le C\, C_0^2\, \varepsilon^2 \, (1+t)^{-\frac{\sigma}{2}}.
\end{aligned}
$$

The last inequality is exactly where we need $\sigma \ge \frac{3}{4}$. That is, $\sigma \ge \frac{3}{4}$ is imposed to ensure that

$$
-\frac{5}{6}\sigma + \frac{1}{4} \le -\frac{\sigma}{2}.
$$

The other terms in (2.34) can be dealt with similarly. For example, the first term can be bounded by

$$
\int_0^t \|e^{\nu \Delta_h (t-\tau)} \partial_3 u_1 \, \partial_1 u(\tau)\|_{L^2} \, d\tau
$$

$$
\begin{aligned}
&\le \int_0^t \Big\| \|e^{\nu\Delta_h(t-\tau)}\partial_3 u_1\,\partial_1 u(\tau)\|_{L^2_h}\Big\|_{L^2_{x_3}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac12}\Big\| \|\partial_3 u_1\,\partial_1 u(\tau)\|_{L^1_h}\Big\|_{L^2_{x_3}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac12}\Big\| \|\partial_3 u_1\|_{L^2_h}\,\|\partial_1 u(\tau)\|_{L^2_h}\Big\|_{L^2_{x_3}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac12}\|\partial_3 u_1\|_{L^2_h L^\infty_{x_3}}\|\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac12}\|\partial_3 u_1\|_{L^2}^{\frac12}\,\|\partial_{33} u_1\|_{L^2}^{\frac12}\,\|\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac12}\|\partial_3 u_1\|_{L^2}^{\frac56}\,\|\partial_3^4 u_1\|_{L^2}^{\frac16}\,\|\partial_1 u(\tau)\|_{L^2}\, d\tau \\
&\le C\, C_0^2\,\varepsilon^2\int_0^t (t-\tau)^{-\frac12}(1+\tau)^{-\frac{5}{12}\sigma}(1+\tau)^{-\frac12-\frac{\sigma}{2}}\, d\tau \\
&\le C\, C_0^2\,\varepsilon^2\int_0^t (t-\tau)^{-\frac12}(1+\tau)^{-\frac{11}{12}\sigma-\frac12}\, d\tau \\
&\le C\, C_0^2\,\varepsilon^2\,(1+t)^{-\frac{11}{12}\sigma}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_0^t \|e^{\nu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}\, d\tau &\le C\, C_0^2\,\varepsilon^2\,(1+t)^{-\frac{\sigma}{2}} \\
&\le \frac{C_0}{4}\varepsilon\,(1+t)^{-\frac{\sigma}{2}}, \qquad (2.35)
\end{aligned}
$$

when ε is take to be sufficiently small. Combining (2.33) and (2.35) yields

$$
\|\partial_3 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{\sigma}{2}}. \qquad (2.36)
$$

(2.31) and (2.36) together verify (2.15).

It remains to prove (2.16). Applying ∇_h to (2.24) and then taking the L^2-norm, we have

$$\|\nabla_h u(t)\|_{L^2} \le \|\nabla_h e^{\nu\Delta_h t} u_0\|_{L^2} + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}\, d\tau. \tag{2.37}$$

As in (2.26), we have, for ε sufficiently small,

$$\begin{aligned}\|\nabla_h e^{\nu\Delta_h t} u_0\|_{L^2} &\le C\,(1+t)^{-\frac{\sigma+1}{2}} \left(\|u_0\|_{H_h^{-\sigma}} + \|u_0\|_{L^2}\right)\\ &\le \frac{C_0}{4}\varepsilon\,(1+t)^{-\frac{\sigma+1}{2}}.\end{aligned}$$

To estimate the second part in (2.37), we first apply Lemma 2.4 to obtain

$$\begin{aligned}&\int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}\, d\tau\\ &\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\, \|e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}\, d\tau.\end{aligned} \tag{2.38}$$

To distinguish between the horizontal and the vertical derivatives, we write $u\cdot\nabla u = u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$. We remark that we cannot directly invoke the same estimates as those in (2.25). For example, if we use the bound

$$\|e^{\nu\Delta_h(t-\tau)} u_3\partial_3 u\|_{L^2} \le C\,(t-\tau)^{-\frac{1}{2}} \left\| \|u_3\partial_3 u\|_{L_h^1} \right\|_{L_{x_3}^2}$$

as before, the integrand in (2.38) would involve $(t-\tau)^{-1}$, which is not integrable! To avoid this, we perform different estimates. Let q satisfy

$$\frac{1}{q} = \frac{1}{2} + \frac{\sigma}{2} \quad \text{or} \quad q = \frac{2}{1+\sigma}.$$

For $\frac{3}{4} \le \sigma < 1$, we have $1 < q < 2$. We bound the worst term $u_3\partial_3 u$ in (2.38). Applying Lemma 2.4 yields,

$$\begin{aligned}&\int_0^t (t-\tau)^{-\frac{1}{2}}\, \|e^{\nu\Delta_h(t-\tau)} u_3\partial_3 u(\tau)\|_{L^2}\, d\tau\\ &= \int_0^t (t-\tau)^{-\frac{1}{2}} \left\| \|e^{\nu\Delta_h(t-\tau)} u_3\partial_3 u(\tau)\|_{L_h^2} \right\|_{L_{x_3}^2} d\tau\\ &\le \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}} \left\| \|u_3\partial_3 u(\tau)\|_{L_h^q} \right\|_{L_{x_3}^2} d\tau.\end{aligned}$$

We then bound $\left\| \|u_3 \partial_3 u(\tau)\|_{L^q_h} \right\|_{L^2_{x_3}}$ as in (2.19) to obtain

$$\left\| \|u_3 \partial_3 u(\tau)\|_{L^q_h} \right\|_{L^2_{x_3}} \le C \, \|\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \, \|u_3\|_{L^2}^{\sigma-\frac{1}{2}} \, \|\nabla_h u_3\|_{L^2}^{1-\sigma} \, \|\partial_3 u\|_{L^2}.$$

The term with $u_3 \partial_3 u$ in (2.38) is thus bounded by, for any $\frac{3}{4} \le \sigma < 1$,

$$\begin{aligned}
&\int_0^t \|\nabla_h e^{\nu \Delta_h (t-\tau)} u_3 \partial_3 u(\tau)\|_{L^2} \, d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}} \, \|\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \, \|u_3\|_{L^2}^{\sigma-\frac{1}{2}} \, \|\nabla_h u_3\|_{L^2}^{1-\sigma} \, \|\partial_3 u\|_{L^2} \, d\tau \\
&\le C \, C_0^2 \, \varepsilon^2 \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}} \, (1+\tau)^{-(\frac{3}{2}-\sigma)(\frac{1}{2}+\frac{\sigma}{2})} \, (1+\tau)^{-\frac{\sigma}{2}(\sigma+\frac{1}{2})} \, d\tau \\
&= C \, C_0^2 \, \varepsilon^2 \int_0^t (t-\tau)^{-\frac{1+\sigma}{2}} \, (1+\tau)^{-\frac{3}{4}-\frac{\sigma}{2}} \, d\tau \\
&\le C \, C_0^2 \, \varepsilon^2 \, (1+t)^{-\frac{1+\sigma}{2}} \\
&\le \frac{1}{128} C_0 \, \varepsilon \, (1+t)^{-\frac{1+\sigma}{2}}.
\end{aligned}$$

The terms with $u_1 \partial_1 u$ and $u_2 \partial_2 u$ in (2.38) can be bounded similarly and they admit the same upper bound. Therefore, we have verified that

$$\|\nabla_h u(t)\|_{L^2} \le \frac{C_0}{2} \, \varepsilon \, (1+t)^{-\frac{1+\sigma}{2}}.$$

This completes the proof of Theorem 1.1. □

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