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# Viscous approximation and weak solutions of the 3D axisymmetric Euler equations

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**The three-dimensional axisymmetric Euler equations without swirl can be represented by the conservation of $\omega^\theta/r$ along the particle trajectory, where $\omega^\theta$ denotes the swirl component of the vorticity. The two-dimensional Euler equation shares a parallel representation. Delort's work has long settled the global existence of weak solutions corresponding to a vortex sheet data of distinguished sign. In contrast, the parallel global existence problem for the axisymmetric Euler equations remains an outstanding open problem. This paper establishes the global existence of weak solutions to the axisymmetric Euler equations without swirl when the initial vorticity $\omega_0^\theta$ obeys $\omega_0^\theta/r \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for $p \in (1,\infty)$. The approach is the method of viscous approximations. A major step in the proof is to extract a strongly convergent subsequence of solutions to a viscous approximation of the Euler equations. Copyright © 2014 John Wiley & Sons, Ltd.**

**Keywords:** axisymmetric 3D Euler equations ;global existence; viscous approximation; weak solutions

## 1. Introduction

The global (in time) existence of classical solutions to the three-dimensional (3D) incompressible Euler equations

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p, \quad x\in\mathbb{R}^3,\ t>0, \\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^3,\ t>0 \end{cases} \tag{1.1}$$

remains an outstanding open problem (see, e.g., [2, 3]). Even the global weak solution emanating from a $L^2$-initial datum was only recently obtained [4]. Here, $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the velocity of the fluid flow, and the scalar $p = p(x,t)$ denotes the pressure. This paper is concerned with the global existence of weak solutions to the 3D axisymmetric Euler equations with very weak initial data. The velocity $u$ and $p$ representing an axisymmetric flow can be written in the cylindrical coordinate system as

$$u(x,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z, \quad p(x,t) = p(r,z,t),$$

where $e_r$, $e_\theta$ and $e_z$ form an orthogonal basis of the cylindrical coordinates,

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}.$$

The 3D Euler equations in (1.1) modeling axisymmetric flow can be rewritten as

$$\begin{cases} \frac{\widetilde{D}}{Dt}u^r - \frac{(u^\theta)^2}{r} + \partial_r p = 0, \\ \frac{\widetilde{D}}{Dt}\left(ru^\theta\right) = 0, \\ \frac{\widetilde{D}}{Dt}u^z + \partial_z p = 0, \\ \partial_r\left(ru^r\right) + \partial_z\left(ru^z\right) = 0, \end{cases} \tag{1.2}$$

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where $\frac{\widetilde{D}}{Dt} = \partial_t + u^r\,\partial_r + u^z\partial_z$ denotes the material derivative. Writing $\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z$ and noticing that the vorticity components are given by

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_r u^z - \partial_z u^r, \quad \omega^z = \partial_r u^\theta,$$

it is sometimes more efficient to represent the 3D axisymmetric Euler equations in terms of $u^\theta$ and $\omega^\theta$ by the closed system

$$\frac{\widetilde{D}}{Dt}\left(ru^\theta\right) = 0, \qquad \frac{\widetilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = -\frac{1}{r^4}\partial_z\left(ru^\theta\right)^2. \tag{1.3}$$

(1.3) is often compared to the inviscid two-dimensional (2D) Boussinesq equations (see, e.g., [3]).

The issue of global (in time) existence of solutions to (1.3) appears to depend crucially on whether the swirl component $u^\theta$ is zero. If the swirl component $u^\theta$ is not zero, the global existence of classical solutions or weak solutions associated with very weak data remains unsettled, although the local well-posedness and various regularity criteria have been obtained (see, e.g., [5,6]). In the case, when there is no swirl, namely $u^\theta \equiv 0$, (1.3) becomes

$$\frac{\widetilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = 0 \tag{1.4}$$

and $\frac{\omega^\theta}{r}$ is conserved along any particle trajectory. As a consequence, the 3D axisymmetric Euler equations without a swirl always possess a unique global solution if the initial datum is sufficiently smooth ([3,7]).

However, the story appears to be different if the datum is not so smooth. Delort [8] studied the global existence of weak solutions to (1.4) emanating from a vortex sheet data, namely $\frac{\omega_0^\theta}{r} \in M(H)$, where $H \equiv \{(r,z), r > 0, z \in \mathbb{R}\}$ and $M(H)$ denotes the space of finite Radon measures. Although (1.4) resembles the 2D Euler vorticity equation and the global existence of weak solutions corresponding to a vortex sheet data of distinguished sign has long been resolved (see [1, 9, 10]), the global existence of weak solutions to (1.4) remains an outstanding open problem, and Delort's study concludes that any approximation sequence must converge strongly in $L^2$ to the weak solution if the latter exists. This is in contrast to the global existence of vortex sheet with one sign for the 2D Euler equations.

In two papers [11, 12] Jiu and Xin considered a smooth approximation sequence $\omega_0^\epsilon$ of a vortex sheet data $\omega_0$ and examined the limit of the solution $\omega^\epsilon$ to (1.4) and that of the corresponding velocity $u^\epsilon$, as $\epsilon \to 0$. They showed that the strong convergence of a subsequence $u^{\epsilon_j}$ to $u$ on any compact subset away from the symmetry axis implies the strong convergence of $u^{\epsilon_j}$ to $u$ in $L^2_{loc}\left(\mathbb{R}^3\right)$. This result reveals that any potential energy-concentration must contain points outside the symmetry axis.

In several papers, Chae and his collaborators studied the global existence of weak solutions to the 3D axisymmetric Euler without swirl in the Lebesgue space and other functional settings ([6, 13, 14]). In particular, Chae and Kim [14] established that any $\frac{\omega_0}{r} \in L^{\frac{6}{5}}\left(\mathbb{R}^3\right) \cap L^p\left(\mathbb{R}^3\right)$ with $p > 3$ leads to a global weak solution. Chae and Imanuvilov [13] settled the global existence of weak solutions for near-vortex-sheets initial data under the assumption that $\frac{\omega_0}{r}$ belongs to an Orlicz space $L\left(\log L\left(\mathbb{R}^3\right)\right)^\alpha$ with $\alpha > \frac{1}{2}$, and they posed the problem of whether $\frac{\omega_0}{r} \in L^1\left(\mathbb{R}^3\right)$ would guarantee the global existence of a weak solution. This paper partially solves their problem. It should be noted that in [13] and [14], the authors constructed the approximate solutions by regularizing the initial data. In this paper, we employ the approach of viscous approximations to prove the global existence of weak solutions to the axisymmetric Euler equations without swirl when $\omega_0^\theta$ obeys $\omega_0^\theta/r \in L^1\left(\mathbb{R}^3\right) \cap L^p\left(\mathbb{R}^3\right)$ for $p > 1$. Moreover, in order to prove our main result, we prove an inequality involving the radial component of the velocity (Lemma 2.3) for a general initial vorticity. This generalizes a previous corresponding inequality in [11], which was only shown to be true for the viscous approximations under one-sign vorticity.

Our approximate solutions are constructed by solving the following Navier–Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \varepsilon \Delta u, & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, \quad x \in \mathbb{R}^3,\ t > 0 \end{cases} \tag{1.5}$$

To give a precise statement of our result, we first provide the definition of a weak solution.

*Definition 1.1*

Given $T > 0$. A vector field $u = u(x,t) \in L^\infty\left([0,T]; L^2_{loc}\left(\mathbb{R}^3\right)\right)$ is said to be a weak solution of (1.1) with an initial data $u_0 = u_0(x)$ if

(i) For any vector field, $\Phi = \Phi(x,t) \in C_0^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$ with $\nabla \cdot \Phi = 0$,

$$\int_0^\infty \int_{\mathbb{R}^3} (u \cdot \Phi_t + (u \otimes u) : \nabla \Phi) dxdt = \int_{\mathbb{R}^3} u_0(x) \cdot \Phi(x,0) dx,$$

where $u \otimes u$ denotes the standard tensor product and $A : B$ denotes the trace of the matrix product $AB$.

(ii) For any $\varphi(x,t) \in C_0^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$,

$$\int_0^\infty \int_{\mathbb{R}^3} \nabla \varphi \cdot u\, dxdt = 0.$$

In addition, we use $\rho_\epsilon$ to denote the standard mollifier, namely $\rho_\epsilon(x) = \epsilon^{-3}\rho\left(\frac{x}{\epsilon}\right)$ with

$$\rho(x) = \rho(|x|) \in C_0^\infty\left(\mathbb{R}^3\right), \quad \rho \geq 0, \quad \int_{\mathbb{R}^3} \rho(x)\, dx = 1.$$

Our global existence result can be stated as follows.

*Theorem 1.2*
Assume the initial velocity $u_0 \in L^2\left(\mathbb{R}^3\right)$ with the swirl component $u_0^\theta \equiv 0$. Assume the corresponding vorticity $\omega_0$ obeys, for $p > 1$,

$$\frac{\omega_0^\theta}{r} \in L^1\left(\mathbb{R}^3\right) \cap L^p\left(\mathbb{R}^3\right).$$

Set $u_0^\epsilon = \rho_\epsilon * u_0$ and $\omega_0^\epsilon = \nabla \times u_0^\epsilon = \left(\omega_0^\theta\right)^\epsilon e_\theta$. Let $\omega^\epsilon$ be the unique global solution of the Navier–Stokes vorticity equation

$$\begin{cases} \frac{\widetilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = \epsilon\left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\left(\frac{\omega^\theta}{r}\right) + \epsilon\frac{2}{r}\partial_r\left(\frac{\omega^\theta}{r}\right), \quad (r,z) \in [0,\infty)\times\mathbb{R}, \\ \frac{\omega^\theta}{r}\big|_{t=0} = \frac{\left(\omega_0^\theta\right)^\epsilon}{r} \end{cases} \tag{1.6}$$

and denote by $u^\epsilon$ the corresponding velocity determined by the Biot–Savart law, namely $u^\epsilon = \nabla^\perp\Delta^{-1}\omega^\epsilon$. Then, there exists a subsequence $u^{\epsilon_j}$ and a vector field $u \in L^\infty_{loc}\left([0,\infty); L^2_{loc}\left(\mathbb{R}^3\right)\right)$ such that, for any $T > 0$,

$$u^{\epsilon_j} \to u \quad \text{strongly in } L^2\left([0,T]; L^2_{loc}\left(\mathbb{R}^3\right)\right), \quad \text{as } j \to \infty$$

and $u$ is a global weak solution of (1.1) in the sense of Definition 1.1.

A main ingredient in the proof of Theorem 1.2 is the aforementioned convergence result of Jiu and Xin [12]. That is, if a subsequence $u^{\epsilon_j}$ converges strongly in $L^2$ to $u$ on any compact subset away from the symmetry axis, namely

$$u^{\epsilon_j} \to u \quad \text{in } L^2([0,T]; L^2(Q)), \quad \text{as } \epsilon_j \to 0, \tag{1.7}$$

for any $Q \subset\subset \mathbb{R}^3 \setminus \{r = 0\}$, then $u^{\epsilon_j} \to u$ in $L^2\left([0,T]; L^2_{loc}\left(\mathbb{R}^3\right)\right)$. Therefore, it suffices to show the strong convergence in (1.7). For this purpose, we first establish the global uniform bounds for $\frac{\omega^\epsilon}{r}$,

$$\left\|\frac{\left(\omega^\theta(t)\right)^\epsilon}{r}\right\|_{L^1(\mathbb{R}^3)} \le \left\|\frac{\omega_0^\theta}{r}\right\|_{L^1(\mathbb{R}^3)}, \quad \left\|\frac{\left(\omega^\theta(t)\right)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)} \le \left\|\frac{\omega_0^\theta}{r}\right\|_{L^p(\mathbb{R}^3)}, \tag{1.8}$$

which, in particular, implies that $\left(\omega^\theta\right)^\epsilon$ admits a unform bound in $L^1(H_R) \cap L^p(H_R)$, namely

$$\left\|\left(\omega^\theta\right)^\epsilon\right\|_{L^1(H_R)\cap L^p(H_R)} \le C(R),$$

where $C(R)$ is a constant independent of $\epsilon$ and

$$H_R = \left\{(r,z) \in \mathbb{R}^2 : r > \frac{1}{R}, r^2 + z^2 < R^2\right\}. \tag{1.9}$$

The global bounds in (1.8) allow us to show that, for a constant $C(R)$ depending on $R$ only,

$$\left\|\nabla u^\epsilon\right\|_{L^p(H_R)} \le C(R).$$

The detailed proof of this inequality is given in Lemma 2.4 in Section 2. Because of the trivial uniform bound

$$\left\|u^\epsilon\right\|_{L^2(\mathbb{R}^3)} \le \|u_0\|_{L^2(\mathbb{R}^3)}$$

and, for $p \in (1,2)$, we have $\|u^\epsilon\|_{L^p(H_R)} \le C(R)\|u_0\|_{L^p(H_R)}$. Therefore, $u^\epsilon$ is in $W^{1,p}(H_R)$. Let $u$ denote the weak limit of $u^\epsilon$ in $L^2$. By the Rellich–Kondrachov compactness theorem, the embedding $W^{1,p}(H_R) \hookrightarrow L^2(H_R)$ with any $1 < p < 2$ is compact, where $H_R \subset \mathbb{R}^2$ is defined by (1.9). Thus, there exists a subsequence $u^{\epsilon_j}$ (depending on $R$) such that

$$u^{\epsilon_j} \to u \quad \text{in } L^2(H_R).$$

By a diagonal selection process, we can select a subsequence of $u^{\epsilon_j}$ that is independent of $R$ (still denoted by $u^{\epsilon_j}$) such that

$$u^{\epsilon_j} \to u \quad \text{in } L^2(Q)$$

for any $Q \subset\subset \mathbb{R}^3 \setminus \{r = 0\}$. This strong limit together with Aubin–Lions lemma would allow us to conclude that $u$ is the desired weak solution of the Euler equation. We leave more details to the next section, in which we prove Theorem 1.2.

## 2. Proof of Theorem 1.2

The plan for proving Theorem 1.2 is as follows. First, we rigorously state and prove the global *a priori* bound (1.8). Then we state a convergence result parallel to that of Jiu and Xin as aforementioned and prove a lemma that is used in obtaining this convergence result. With these preparation at our disposal, we finally provide the proof of Theorem 1.2.

*Proposition 2.1*
Let $u^\epsilon$ and $\omega^\epsilon$ be defined as in Theorem 1.2. Let $T > 0$ be arbitrarily fixed. Then $u^\epsilon$ and $\omega^\epsilon$ obey the following global bounds:

(i) $u^\epsilon$ is uniformly bounded in $L^\infty\left([0,T];L^2\left(\mathbb{R}^3\right)\right)$;
(ii) $\left(\omega^\theta\right)^\epsilon$ is uniformly bounded in $L^\infty\left([0,T];L^1\left([0,\infty)\times\mathbb{R}, drdz\right)\right)$, namely

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\left(\omega^\theta\right)^\epsilon\right| drdz \le \int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\omega_0^\theta\right| drdz; \tag{2.1}$$

(iii) $\frac{(\omega^\theta)^\epsilon}{r}$ is uniformly bounded in $L^\infty\left([0,\infty);L^1\left(\mathbb{R}^3\right)\cap L^p\left(\mathbb{R}^3\right)\right)$, namely

$$\left\|\frac{\left(\omega^\theta(t)\right)^\epsilon}{r}\right\|_{L^1(\mathbb{R}^3)} \le \left\|\frac{\omega_0^\theta}{r}\right\|_{L^1(\mathbb{R}^3)}, \quad \left\|\frac{\left(\omega^\theta(t)\right)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)} \le \left\|\frac{\omega_0^\theta}{r}\right\|_{L^p(\mathbb{R}^3)}. \tag{2.2}$$

*Proof*
(i) is obvious because $u^\epsilon$ satisfies the Navier–Stokes equations and the basic energy inequality

$$\left\|u^\epsilon(t)\right\|_{L^2(\mathbb{R}^3)} \le \left\|u_0^\epsilon\right\|_{L^2(\mathbb{R}^3)} \le \left\|u_0\right\|_{L^2(\mathbb{R}^3)}$$

holds. We now show (ii). Writing $\sigma = \frac{(\omega^\theta)^\epsilon}{r}$, we have, by (1.6),

$$\frac{\tilde{D}}{Dt}\sigma = \epsilon\left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\sigma + \epsilon\frac{2}{r}\partial_r\sigma, \qquad (r,z)\in\mathbb{R}^+\times\mathbb{R}. \tag{2.3}$$

Let $S_h(\sigma)$ with $h > 0$ be a convex approximation of $S(\sigma) = |\sigma|$, say

$$S_h(\sigma) = \begin{cases} -\sigma, & \sigma \le -h, \\ \frac{\sigma^2}{2h} + \frac{h}{2}, & -h \le \sigma \le h, \\ \sigma, & \sigma \ge h. \end{cases}$$

Multiplying (2.3) by $S_h'(\sigma)$ yields

$$\begin{aligned}
&\frac{\partial}{\partial t}S_h(\sigma) + (u^r)^\epsilon\,\partial_r S_h(\sigma) + (u^z)^\epsilon\,\partial_z S_h(\sigma)\\
&= \epsilon\left(\partial_r^2 + \partial_z^2\right)\sigma S_h'(\sigma) + \epsilon\frac{3}{r}\partial_r S_h(\sigma)\\
&= \epsilon\partial_r^2 S_h(\sigma) - \epsilon S_h''(\sigma)\left(\partial_r\sigma\right)^2 + \epsilon\partial_z^2 S_h(\sigma) - \epsilon S_h''(\sigma)\left(\partial_z\sigma\right)^2 + \epsilon\frac{3}{r}\partial_r S_h(\sigma)\\
&\le \epsilon\partial_r^2 S_h(\sigma) + \epsilon\partial_z^2 S_h(\sigma) + \epsilon\frac{3}{r}\partial_r S_h(\sigma).
\end{aligned} \tag{2.4}$$

Multiplying (2.4) by $r$ and integrating with respect to $(r,z)$ over $(0,+\infty)\times(-\infty,+\infty)$, we obtain

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\int_0^{+\infty} S_h(\sigma) r drdz \le -2\epsilon\int_{-\infty}^{+\infty} S_h(\sigma)|_{r=0}\,dz. \tag{2.5}$$

To derive (2.5), we have integrated by parts and used a few simple facts. By integration by parts and the divergence-free condition

$$\partial_r\left(r\left(u^r\right)^\epsilon\right) + \partial_z\left(r\left(u^z\right)^\epsilon\right) = 0, \tag{2.6}$$

we have

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\left[\left(u^r\right)^\epsilon\partial_r S_h(\sigma) + \left(u^z\right)^\epsilon\partial_z S_h(\sigma)\right] r drdz = 0.$$

By integration by parts,

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_z^2 S_h(\sigma) r drdz = 0.$$

Integrating by parts and using the fact that $\left(\omega^\theta\right)^\epsilon|_{r=0} = 0$ and $\frac{(\omega^\theta)^\epsilon}{r} = \partial_r\left(\omega^\theta\right)^\epsilon|_{r=0}$ (see, e.g., [15]), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r^2 S_h(\sigma) r dr dz \\
&= \int_{-\infty}^{+\infty}\left(\partial_r S_h(\sigma) r\right)\Big|_{r=0}^{+\infty} dz - \int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r S_h(\sigma) dr dz \\
&= \int_{-\infty}^{+\infty}\left[S_h'(\sigma)\left(-\frac{\left(\omega^\theta\right)^\epsilon}{r^2} + \frac{\partial_r\left(\omega^\theta\right)^\epsilon}{r}\right) r\right]\Bigg|_{r=0}^{+\infty} dz - \int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r S_h(\sigma) dr dz \\
&= -\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r S_h(\sigma) dr dz.
\end{aligned}$$

Because $S_h(\sigma)|_{r=0} \geq 0$, (2.5) leads to

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\int_0^{+\infty} S_h(\sigma) r dr dz \leq 0.$$

By the definition of $S_h$, this inequality allows us to conclude, after letting $h \to 0$,

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\left(\omega^\theta\right)^\epsilon\right| dr dz \leq \int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\omega_0^\theta\right| dr dz.$$

We now prove (iii). By switching to polar coordinates and by (2.1),

$$\begin{aligned}
\left\|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right\|_{L^1(\mathbb{R}^3)} &= 4\pi^2\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right| r dr dz \\
&= 4\pi^2\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\left(\omega^\theta\right)^\epsilon\right| dr dz \\
&\leq 4\pi^2\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\left(\omega_0^\theta\right)^\epsilon\right| dr dz \\
&\leq \left\|\frac{\omega_0^\theta}{r}\right\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

Multiplying (1.6) by $\frac{(\omega^\theta)^\epsilon}{r}\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} r$ and integrating with respect to $(r, z)$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)_t\frac{\left(\omega^\theta\right)^\epsilon}{r}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz \\
&+ \int_{-\infty}^{+\infty}\int_0^{+\infty}\left[\left(u^r\right)^\epsilon\partial_r\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right) + \left(u^z\right)^\epsilon\partial_z\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\right]\frac{\left(\omega^\theta\right)^\epsilon}{r}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz \\
&= \epsilon\int_{-\infty}^{+\infty}\int_0^{+\infty}\left(\left[\partial_r^2 + \partial_z^2 + \frac{3}{r}\partial_r\right]\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\right)\frac{\left(\omega^\theta\right)^\epsilon}{r}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz.
\end{aligned} \tag{2.7}$$

By integration by parts and the divergence-free condition (2.6),

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\left[\left(u^r\right)^\epsilon\partial_r\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right) + \left(u^z\right)^\epsilon\partial_z\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\right]\frac{\left(\omega^\theta\right)^\epsilon}{r}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz \\
&= \frac{1}{p}\int_{-\infty}^{+\infty}\int_0^{+\infty}\left[r\left(u^r\right)^\epsilon\partial_r\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^p + r\left(u^z\right)^\epsilon\partial_z\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^p\right] dr dz = 0.
\end{aligned} \tag{2.8}$$

By integration by parts,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_z^2\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\frac{\left(\omega^\theta\right)^\epsilon}{r}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz \\
&= -(p-1)\int_{-\infty}^{+\infty}\int_0^{+\infty}\left(\partial_z\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\right)^2\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^{p-2} r dr dz.
\end{aligned} \tag{2.9}$$

By integration by parts,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r^2\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\frac{(\omega^\theta)^\epsilon}{r}\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdz\\
&\quad=\frac{1}{p}\int_{-\infty}^{+\infty}\left[r\,\partial_r\left(\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^p\right)\right]\Big|_{r=0}^{\infty}dz-\frac{1}{p}\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r\left(\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^p\right)drdz\\
&\qquad-(p-1)\int_{-\infty}^{+\infty}\int_0^{+\infty}\left[\partial_r\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\right]^2\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdz.
\end{aligned}$$

Using the facts that $(\omega^\theta)^\epsilon|_{r=0}=0$ and $\frac{(\omega^\theta)^\epsilon}{r}=\partial_r(\omega^\theta)^\epsilon|_{r=0}$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r^2\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\frac{(\omega^\theta)^\epsilon}{r}\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdz\\
&\quad=-\frac{1}{p}\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r\left(\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^p\right)drdz\\
&\qquad-(p-1)\int_{-\infty}^{+\infty}\int_0^{+\infty}\left[\partial_r\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\right]^2\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdz.
\end{aligned}\tag{2.10}$$

Inserting (2.8), (2.9), and (2.10) in (2.7), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\left\|\frac{(\omega^\theta)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)}^p+\epsilon(p-1)\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\nabla\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\right|^2\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdz\\
&\quad+\frac{2\epsilon}{p}\int_{-\infty}^{+\infty}\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^p\Bigg|_{r=0}dz=0.
\end{aligned}$$

Integrating with respect to $t$ leads to

$$\begin{aligned}
&\frac{1}{p}\left\|\frac{(\omega^\theta)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)}^p+\epsilon(p-1)\int_0^t\int_{-\infty}^{+\infty}\int_0^{+\infty}\left|\nabla\left(\frac{(\omega^\theta)^\epsilon}{r}\right)\right|^2\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^{p-2} rdrdzdt\\
&\quad+\frac{2\epsilon}{p}\int_0^t\int_{-\infty}^{+\infty}\left|\frac{(\omega^\theta)^\epsilon}{r}\right|^p\Bigg|_{r=0}dzdt=\frac{1}{p}\left\|\frac{(\omega_0^\theta)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)}^p.
\end{aligned}$$

Especially,

$$\left\|\frac{(\omega^\theta)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)}\le\left\|\frac{\omega_0^\theta}{r}\right\|_{L^p(\mathbb{R}^3)}.$$

This completes the proof of Proposition 2.1. □

We now state a convergence result, a key ingredient in the proof of Theorem 1.2. This result is parallel to a theorem of Jiu and Xin [12].

*Proposition 2.2*
Let $u^\epsilon$ and $\omega^\epsilon$ be defined as in Theorem 1.2. Let $T>0$ be arbitrarily fixed. If there exists a subsequence $\{u^{\epsilon_j}\}\subset\{u^\epsilon\}$ such that, for any $Q\subset\subset\{x\in\mathbb{R}^3|r>0\}$,

$$u^{\epsilon_j}\longrightarrow u\quad strongly\ \ in\ \ L^2\left([0,T];L^2(Q)\right)$$

as $j\to\infty$, then there exists a further subsequence of $\{u^{\epsilon_j}\}$, still denoted by itself, such that, as $j\to\infty$,

$$u^{\epsilon_j}\longrightarrow u\quad strongly\ \ in\ \ L^2\left([0,T];L^2_{loc}\left(\mathbb{R}^3\right)\right).$$

The proof of Proposition 2.2 is similar to that in [12], and we shall not provide the full details. Instead, we point out a lemma used in the proof of Proposition 2.2. This lemma is similar to a result due to Chae and Imanuvilov [13].

*Lemma 2.3*
Let $u^\epsilon$ and $\omega^\epsilon$ be defined as in Theorem 1.2. Then, for any $T>0$,

$$\int_0^T\int_{\mathbb{R}^3}\frac{1}{1+z^2}\left(\frac{(u^r)^\epsilon}{r}\right)^2dxdt\le C\left(\|u_0\|_{L^2(\mathbb{R}^3)}^2+\left\|\frac{\omega_0^\epsilon}{r}\right\|_{L^1\cap L^p(\mathbb{R}^3)}\right),\tag{2.11}$$

where $z$ denotes the third component of $x$ and $C=C(T)$ is a constant independent of $\epsilon$.

*Proof*
The proof is parallel to that of Chae and Imanuvilov [13]. It is provided here for the sake of completeness. Set $\rho(z) = \int_{-\infty}^{z} \frac{1}{1+\tau^2} d\tau$. Multiplying (1.6) by $2\pi r\rho(z)$ and integrating with respect to $(r,z,t)$ over $\mathbb{R}^2_+ \times [0,T]$, we have

$$\begin{aligned} 2\pi \int_0^T \int_{\mathbb{R}^2_+} \partial_t \left( \frac{(\omega^\theta)^\epsilon}{r} \right) r\rho(z) drdzdt \\ + 2\pi \int_0^T \int_{\mathbb{R}^2_+} \left[ (u^r)^\epsilon \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) + (u^z)^\epsilon \partial_z \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \right] \rho(z) rdrdzdt \\ = 2\pi\epsilon \int_0^T \int_{\mathbb{R}^2_+} \left[ \partial_r^2 + \partial_z^2 + \frac{3}{r}\partial_r \right] \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \rho(z) rdrdzdt, \end{aligned} \tag{2.12}$$

where we have written $\mathbb{R}^2_+ = \{(r,z) \in \mathbb{R}^2 | r \geq 0, z \in \mathbb{R}\}$. By integration by parts,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2_+} \left[ (u^r)^\epsilon \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) + (u^z)^\epsilon \partial_z \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \right] \rho(z) rdrdzdt \\ = - \int_0^T \int_{\mathbb{R}^2_+} \rho'(z) (u^z)^\epsilon \left( \omega^\theta \right)^\epsilon drdzdt \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2_+} \left[ \partial_r^2 + \partial_z^2 + \frac{3}{r}\partial_r \right] \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \rho(z) rdrdzdt \\ &= \int_0^T \int_{-\infty}^{+\infty} \left[ \rho(z) r\, \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \right] \Big|_{r=0}^{+\infty} dzdt - \int_0^T \int_{\mathbb{R}^2_+} \rho(z)\, \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) drdzdt \\ &\quad + \int_0^T \int_0^{+\infty} \left[ \rho(z) r \partial_z \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \right] \Big|_{z=-\infty}^{+\infty} drdt - \int_0^T \int_{\mathbb{R}^2_+} \rho'(z) r\, \partial_z \left( \frac{(\omega^\theta)^\epsilon}{r} \right) drdzdt \\ &\quad + 3 \int_0^T \int_{\mathbb{R}^2_+} \rho(z) \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) drdzdt \\ &= - \int_0^T \int_{\mathbb{R}^2_+} \partial_z \left( \omega^\theta \right)^\epsilon \rho'(z) drdzdt + 2 \int_0^T \int_{\mathbb{R}^2_+} \partial_r \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \rho(z) drdzdt \\ &= \int_0^T \int_{\mathbb{R}^2_+} \left( \omega^\theta \right)^\epsilon \rho''(z) drdzdt - 2 \int_0^T \int_{-\infty}^{+\infty} \left[ \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \rho(z) \right] \Big|_{r=0} dzdt. \end{aligned}$$

Therefore, by (2.12),

$$\begin{aligned} &2\pi\epsilon \int_0^T \int_{\mathbb{R}^2_+} \left( \omega^\theta \right)^\epsilon \rho''(z) drdzdt - 4\pi\epsilon \int_0^T \int_{-\infty}^{+\infty} \left[ \left( \frac{(\omega^\theta)^\epsilon}{r} \right) \rho(z) \right] \Big|_{r=0} dzdt. \\ &= \int_{\mathbb{R}^3} \frac{(\omega^\theta)^\epsilon}{r} \rho(z) dx \Big|_0^T - 2\pi \int_0^T \int_{\mathbb{R}^2_+} \rho'(z) (u^z)^\epsilon \left( \omega^\theta \right)^\epsilon drdzdt \\ &= \int_{\mathbb{R}^3} \frac{(\omega^\theta)^\epsilon}{r} \rho(z) dx \Big|_0^T - 2\pi \int_0^T \int_{\mathbb{R}^2_+} \rho'(z) (u^z)^\epsilon \left( \partial_r (u^z)^\epsilon - \partial_z (u^r)^\epsilon \right) drdzdt \\ &= \int_{\mathbb{R}^3} \frac{(\omega^\theta)^\epsilon}{r} \rho(z) dx \Big|_0^T + \pi \int_0^T \int_{-\infty}^{+\infty} \left[ \rho'(z) \left( (u^z)^\epsilon \right)^2 \right] |_{r=0} dzdt \\ &\quad - 2\pi \int_0^T \int_{\mathbb{R}^2_+} \left( \rho''(z) (u^z)^\epsilon (u^r)^\epsilon + \rho'(z) (u^r)^\epsilon \partial_z (u^z)^\epsilon \right) drdzdt. \end{aligned}$$

Using the divergence-free condition, namely (2.6), we obtain

$$
\begin{aligned}
&2\pi\epsilon\int_0^T\int_{\mathbb{R}^2_+}\left(\omega^\theta\right)^\epsilon\rho''(z)drdzdt-4\pi\epsilon\int_0^T\int_{-\infty}^{+\infty}\left[\left(\frac{\left(\omega^\theta\right)^\epsilon}{r}\right)\rho(z)\right]\Bigg|_{r=0}dzdt.\\
&=\int_{\mathbb{R}^3}\frac{\left(\omega^\theta\right)^\epsilon}{r}\rho(z)dx\Bigg|_0^T-2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho''(z)\left(u^z\right)^\epsilon\left(u^r\right)^\epsilon drdzdt\\
&\quad-2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\left(u^r\right)^\epsilon\left(-\frac{\left(u^r\right)^\epsilon}{r}-\partial_r\left(u^r\right)^\epsilon\right)drdzdt\\
&=\int_{\mathbb{R}^3}\frac{\left(\omega^\theta\right)^\epsilon}{r}\rho(z)dx\Bigg|_0^T-2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho''(z)\left(u^z\right)^\epsilon\left(u^r\right)^\epsilon drdzdt\\
&+2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt.
\end{aligned}
$$

Because $\rho'(z)=\frac{1}{1+z^2}>0$ and $|\rho(z)|\le C$ for all $z\in\mathbb{R}$, we have

$$
\begin{aligned}
&2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt\\
&\le 2\pi\int_0^T\int_{\mathbb{R}^2_+}\left|\rho''(z)\left(u^z\right)^\epsilon\left(u^r\right)^\epsilon\right|drdzdt+\int_{\mathbb{R}^3}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|dx\,\Big|_0^T\\
&\quad+4\pi\epsilon\int_0^T\int_{-\infty}^{+\infty}\left[\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|\rho(z)\right]\Bigg|_{r=0}dzdt+4\pi\epsilon\int_0^T\int_{\mathbb{R}^2_+}\left|\left(\omega^\theta\right)^\epsilon\right|drdzdt.
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2_+}\left|\rho''(z)\left(u^z\right)^\epsilon\left(u^r\right)^\epsilon\right|drdzdt\\
&\le\left[\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt\right]^{\frac12}\left[\int_0^T\int_{\mathbb{R}^2_+}\left(\left(u^z\right)^\epsilon\right)^2\frac{|\rho''(z)|^2}{\rho'(z)}rdrdzdt\right]^{\frac12}\\
&\le\frac12\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt+C\int_0^T\int_{\mathbb{R}^3}\left|u^\epsilon\right|^2dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T\int_{-\infty}^{+\infty}\left[\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|\rho(z)\right]\Bigg|_{r=0}dzdt\\
&\le\left[\int_0^T\int_{-\infty}^{+\infty}\left|\frac{\left(\omega^\theta\right)^\epsilon}{r}\right|^p\Big|_{r=0}dzdt\right]^{\frac1p}\left[\int_0^T\int_{-\infty}^{+\infty}|\rho(z)|^{\frac{p}{p-1}}dzdt\right]^{\frac{p-1}{p}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2\pi\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt&\le\pi\int_0^T\int_{\mathbb{R}^2_+}\rho'(z)\frac{\left(\left(u^r\right)^\epsilon\right)^2}{r}drdzdt\\
&+C\|u_0\|^2_{L^2(\mathbb{R}^3)}+C\left\|\frac{\left(\omega_0^\theta\right)^\epsilon}{r}\right\|_{L^p(\mathbb{R}^3)}+C\left\|\frac{\left(\omega_0^\theta\right)^\epsilon}{r}\right\|_{L^1(\mathbb{R}^3)},
\end{aligned}
$$

which immediately implies (2.11). This completes the proof of Lemma 2.3. □

We now prove Theorem 1.2.

*Proof*

The proof follows the lines presented in the introduction. Let $H_R\subset\mathbb{R}^2$ be defined as in (1.9). Clearly, $H_1\subset H_2\subset\cdots$ and, as $R\to\infty$,

$$H_R\to\cup_{R=1}^\infty H_R=\{(r,z):r\in(0,\infty),z\in(-\infty,+\infty)\}.$$

Because $\|u^\epsilon(t)\|_{L^2(\mathbb{R}^3)}\le\|u_0\|_{L^2(\mathbb{R}^3)}$, it follows by Hölder's inequality that, for $p\in(1,2)$,

$$\int_{H_R}\left|u^\epsilon(r,z,t)\right|^p drdz\le C(R)\left\|u^\epsilon(t)\right\|_{L^2(\mathbb{R}^3)}\le C(R)\|u_0\|_{L^2(\mathbb{R}^3)},$$

where $C(R)$ is a constant depending on $R$ only. By Proposition 2.1, we have

$$\|\omega^\epsilon\|_{L^p(H_R)} = \left\|\frac{\omega^\epsilon}{r} r\right\|_{L^p(H_R)} \le R\left\|\frac{\omega^\epsilon}{r}\right\|_{L^p(H_R)} \le C(R).$$

By Lemma 2.4,

$$\|\nabla u^\epsilon\|_{L^p(H_R)} \le C_1 \|\omega^\epsilon\|_{L^p(H_R)} + C_2(R)\left\|\frac{\omega_0^\epsilon}{r}\right\|_{L^1\cap L^p(\mathbb{R}^3)} \le C(R).$$

Therefore, $\{u^\epsilon\}$ is uniformly bounded in $L^\infty\left([0,T]; W^{1,p}(H_R, drdz)\right)$. By the Rellich–Kondrachov compactness theorem, the embedding $W^{1,p}(H_R, drdz) \hookrightarrow L^2(H_R, drdz)$ with any $1 < p < 2$ is compact. By the definition of $H_R$,

$$\frac{1}{R}\int_{H_R} |u^\epsilon(r,z,t)|^2 drdz \le \int_{H_R} |u^\epsilon(r,z,t)|^2 rdrdz \le \|u^\epsilon(t)\|^2_{L^2(\mathbb{R}^3)} \le \|u_0\|^2_{L^2(\mathbb{R}^3)},$$

that is,

$$\|u^\epsilon(t)\|_{L^2(H_R)} \le C(R)\|u_0\|_{L^2(\mathbb{R}^3)},$$

Moreover, for any $\varphi \in H^3_{0,\sigma}(H_R)$, which is the closure of $\left\{f \in C_0^\infty(H_R) |\ \mathrm{div}\, f = 0\right\}$ in $H^3(H_R)$, we have

$$\begin{aligned} -\int_0^T \int_{H_R} u_j^\epsilon \partial_j u_i^\epsilon \varphi_i drdzdt = & \int_0^T \int_{H_R} u_j^\epsilon u_i^\epsilon \partial_j \varphi_i drdzdt \\ \le & \ C\|u^\epsilon\|_{L^2([0,T];L^2(H_R))} \|\nabla\varphi\|_{L^2([0,T],L^\infty(H_R))} \\ \le & \ C(T)\|u_0\|_{L^2(H_R)}\|\varphi\|_{L^2\left([0,T],H^3_{0,\sigma}(H_R)\right)}. \end{aligned}$$

Using the Navier–Stokes equations (1.5), we obtain

$$\|\partial_t u^\epsilon\|_{L^2\left([0,T],\left(H^3_{0,\sigma}(H_R)\right)^*\right)} \le C(R),$$

where $\left(H^3_{0,\sigma}(H_R)\right)^*$ is the dual space of $H^3_{0,\sigma}(H_R)$. By the Aubin–Lions lemma, there exists a subsequence $u^{\epsilon_j}$ (depending on $R$) such that

$$u^{\epsilon_j} \to u \quad \text{in } L^2\left([0,T]; L^2(H_R)\right).$$

By a diagonal selection process, we can select a subsequence of $u^{\epsilon_j}$ that is independent of $R$ (still denoted by $u^{\epsilon_j}$) such that

$$u^{\epsilon_j} \to u \quad \text{in } L^2\left([0,T]; L^2(Q)\right)$$

for any $Q \subset\subset \mathbb{R}^3 \setminus \{r = 0\}$. By Proposition 2.2,

$$u^{\epsilon_j} \to u \quad \text{in } L^2\left([0,T]; L^2_{loc}(\mathbb{R}^3)\right). \tag{2.13}$$

Equation (2.13) would allow us to conclude that $u$ is a weak solution of (1.1). In fact, for any $\Phi(x,t) \in C_0^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$ and $\nabla\cdot\Phi = 0$, we have, as $j \to \infty$,

$$\int_0^T \int_{\mathbb{R}^3} u^{\epsilon_j} \otimes u^{\epsilon_j} : \nabla\Phi\, dxdt \longrightarrow \int_0^\infty \int_{R^3} u \otimes u : \nabla\Phi\, dxdt$$

by the Dominated Convergence Theorem. The convergence of the linear terms can be easily shown. This completes the proof of Theorem 1.2. □

In the proof of Theorem 1.2, we have used an inequality in the following lemma.

*Lemma 2.4*
Let $H_R$ be defined as in (1.9). Let $u^\epsilon$ and $\omega^\epsilon$ be as in the proof of Theorem 1.2. Then

$$\|\nabla u^\epsilon\|_{L^p(H_R)} \le C_1 \|\omega^\epsilon\|_{L^p(H_R)} + C_2(R)\left\|\frac{\omega_0^\epsilon}{r}\right\|_{L^1\cap L^p(\mathbb{R}^3)}, \tag{2.14}$$

where $C_1$ is a constant independent of $R$ and $\epsilon$ and $C_2$ is a constant depending on $R$ but independent of $\epsilon$.

*Proof*
According to Proposition 2.20 in [3, p.76], we can write

$$\nabla u^\epsilon(x,t) = C\,\omega^\epsilon(x,t) + P_3\omega^\epsilon(x,t), \tag{2.15}$$

where $C$ is a pure constant (independent of $\epsilon$) and $P_3$ is a singular integral operator defined by the Cauchy principal-value integral

$$P_3\omega^\epsilon(x,t) = P.V.\int_{\mathbb{R}^3} \nabla K_3(x-y)\,\omega^\epsilon(y,t)\,dy. \tag{2.16}$$

In addition, $|\nabla K_3(x-y)| \leq C\,|x-y|^{-3}$ for any $x, y \in \mathbb{R}^3$ with $x \neq y$. Let $x = (x_1, x_2, z) \in \mathbb{R}^3$ with $(r, z) \in H_R$, where $r = \sqrt{x_1^2 + x_2^2}$. To split the integral in (2.16) into two parts, we write

$$\Omega_1 = \left\{ y = (y_1, y_2, \widetilde{z}) \in \mathbb{R}^3 : \widetilde{r} = \sqrt{y_1^2 + y_2^2} \leq 2R,\ |\widetilde{z}| \leq 2R \right\}, \quad \Omega_2 = \mathbb{R}^3 \setminus \Omega_1.$$

Clearly, $|x - y| \geq R$ for any $y \in \Omega_2$. Thus, for $x = (x_1, x_2, z) \in \mathbb{R}^3$ with $(r, z) \in H_R$,

$$\begin{aligned} \left| P_3 \omega^\epsilon(x,t) \right| \leq & \left| P.V. \int_{\Omega_1} \nabla K_3(x-y)\, \omega^\epsilon(y,t)\, dy \right| \\ & + C \int_{\Omega_2} \frac{1}{|x-y|^3} \left| \omega^\epsilon(y) \right| dy \\ \leq & \left| P.V. \int_{\mathbb{R}^3} \nabla K_3(x-y)\, \chi_{\Omega_1} \omega^\epsilon(y,t)\, dy \right| \\ & + \frac{1}{R^2} \int_{-\infty}^{+\infty} \int_0^\infty \left| \omega^\epsilon(r,z) \right| drdz, \end{aligned} \tag{2.17}$$

where $\chi_{\Omega_1}$ denotes the characteristic function on $\Omega_1$. By Proposition 2.1,

$$\int_{-\infty}^{+\infty} \int_0^\infty \left| \omega^\epsilon(r,z) \right| drdz \leq \int_{-\infty}^{+\infty} \int_0^\infty \left| \omega_0^\epsilon(r,z) \right| drdz. \tag{2.18}$$

Combining (2.15), (2.17), and (2.18) and using the boundedness of the singular integral operators on $L^p\left(\mathbb{R}^3\right)$ with $p \in (1, \infty)$, we obtain

$$\left\| \nabla u^\epsilon \right\|_{L^p(H_R)} \leq C \left\| \omega^\epsilon \right\|_{L^p(H_R)} + C \left\| \chi_{\Omega_1} \omega^\epsilon \right\|_{L^p(\mathbb{R}^3)} + C(R) \left\| \frac{\omega_0^\epsilon}{r} \right\|_{L^1(\mathbb{R}^3)}. \tag{2.19}$$

Furthermore, recalling the definition of $\Omega_1$ and by Proposition 2.1,

$$\begin{aligned} \left\| \chi_{\Omega_1} \omega^\epsilon \right\|_{L^p(\mathbb{R}^3)}^p = & \int_{\mathbb{R}^3} \left| \chi_{\Omega_1} \omega^\epsilon(x,t) \right|^p dx = 2\pi \int_{-2R}^{2R} \int_0^{2R} \left| \omega^\epsilon(r,z,t) \right|^p rdrdz \\ \leq & \ 2\pi (2R)^p \int_{-2R}^{2R} \int_0^{2R} \left| \frac{\omega^\epsilon(r,z,t)}{r} \right|^p rdrdz \\ \leq & \ (2R)^p \left\| \frac{\omega_0^\epsilon}{r} \right\|_{L^p(\mathbb{R}^3)}^p. \end{aligned} \tag{2.20}$$

Inserting (2.20) in (2.19) yields the desired bound in (2.14). This completes the proof of Lemma 2.4. □

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