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# Vanishing viscosity limits for the degenerate lake equations with Navier boundary conditions

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**Abstract**

This paper concerns the vanishing viscosity limit of the two-dimensional degenerate viscous lake equations when the Navier slip conditions are prescribed on the impermeable boundary of a simply connected bounded regular domain. When the initial vorticity is in the Lebesgue space $L^q$ with $2 < q \leqslant \infty$, we show that the degenerate viscous lake equations possess a unique global solution and the solution converges to a corresponding weak solution of the inviscid lake equations. In a special case when the vorticity is in $L^\infty$, an explicit convergence rate is obtained.

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## 1. Introduction

In [15] Levermore and Sammartino derived a system of shallow water equations that model the large-scale horizontal motion confined to a fixed basin with a slowly varying bottom topograph from three-dimensional incompressible flow with eddy viscosities, which read as

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu b^{-1} \nabla \cdot (2bD(v) - b\nabla \cdot v I) + \nabla h = -\eta v + f, \\ \nabla \cdot (bv) = 0, \end{cases} \tag{1.1}$$

where $x \in \Omega$, $t > 0$, $\Omega \subset \mathbb{R}^2$ is a simply connected bounded domain. The unknown functions are $v(x, t)$, which is the horizontal fluid velocity averaged vertically over $x \in \Omega$ at time $t$, and $h(x, t)$, which means the top surface height. $\mu$ and $\eta$ are a positive eddy viscosity coefficient and a nonnegative turbulent drag coefficient defined over $\Omega$, respectively, and $f(x, t)$ is the wind forcing defined over $\Omega \times [0, \infty)$. $I$ is a $2 \times 2$ identity matrix and $D(v)$ stands for the deformation tensor, defined by

$$D(v) = \frac{\nabla v + (\nabla v)^t}{2}.$$

Physically $b = b(x)$ denotes the depth of the basin and is assumed to be positive in [15], i.e. $b(x) \geqslant b_0 > 0, x \in \bar{\Omega}$ for some positive constant $b_0$, which means that lakes and oceans have vertical lateral boundaries, like swimming pools.

The derived boundary conditions in [15] are

$$v \cdot n = 0, \quad \mu\tau \cdot 2D(v) \cdot n + \alpha v \cdot \tau = 0 \qquad x \in \partial\Omega, \tag{1.2}$$

where $n(x)$ and $\tau(x)$ are the outward unit normal and a unit tangent to $\partial\Omega$, and $\alpha(x)$ is a nonnegative turbulent boundary drag coefficient defined on $\partial\Omega$. (1.2) are usually called the Navier boundary conditions, which were first used by Navier in 1827 and mean that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress. In [15], the authors also studied the well-posedness of the initial–boundary-value problem to the system (1.1)–(1.2) with some given initial data.

At the end of [15], Levermore and Sammartino proposed that '......*it would be natural to investigate the zero viscosity limit of our model equations (i.e., system (1.1)) and prove that the solutions converge to the solution of the model derived in [4]*', which is

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0, \\ \nabla \cdot (b\, u^0) = 0, \end{cases} \tag{1.3}$$

with the corresponding boundary condition

$$bu^0 \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{1.4}$$

and given initial data

$$u^0(x, t)|_{t=0} = u_0, \quad x \in \Omega. \tag{1.5}$$

(1.3) are the known lake equations which have been derived in [4, 5, 8, 13] to model the evolution of the vertically averaged horizontal components of the 3D velocity to the incompressible Euler equations confined to a shallow basin with a varying bottom topography. It is clear that when $b(x)$ has a positive lower bound, (1.4) is equivalent to $u \cdot n = 0, \ x \in \partial\Omega$.

In this paper, we are not assuming that $b$ is nondegenerate, namely that $b$ may be zero on $\partial\Omega$. We neglect the linear term $\eta u$ and the source term $f$ in (1.1) (for simplicity) and consider the following viscous lake equations:

$$\begin{cases} \partial_t u^\mu + u^\mu \cdot \nabla u^\mu - \mu b^{-1} \nabla \cdot (2bD(u^\mu) - b\nabla \cdot u^\mu I) + \nabla p^\mu = 0, \\ \nabla \cdot (bu^\mu) = 0. \end{cases} \tag{1.6}$$

Attention is focused here on the initial- and boundary-value problem (IBVP) for (1.6) with the free boundary condition

$$bu^\mu \cdot n = 0, \quad \nabla \times u^\mu = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.7}$$

and given initial data

$$u^\mu(x, t)\ |_{t=0} = u_0, \qquad x \in \Omega, \tag{1.8}$$

It is remarked that (1.7) is called the free (Navier) boundary condition, which was introduced in [1, 16, 17] and can be regarded as a special case of the general Navier boundary condition

$$bu^\mu \cdot n = 0, \quad 2D(u^\mu)n \cdot \tau + \alpha u^\mu \cdot \tau = 0, \qquad x \in \partial\Omega. \tag{1.9}$$

In detail, (1.9) reduces to (1.7) when $\alpha(x) = 2\kappa(x)$, where $\kappa(x)$ is the curvature of the boundary $\partial\Omega$ (see lemma 2.1 in [7] and corollary 4.3 in [12], see also lemma 2.8 in section 2) The boundary condition $\nabla \times u^\mu = 0$ makes the $L^q(2 < q \leqslant \infty)$-estimates of the vorticity available and it is still unknown so far how to get these estimates using the general Navier boundary (1.9).

In [3] Bresch and Métivier studied the well-posedness of the lake equations (1.3) in the presence of a beach, assuming that $b(x)$ is positive in the interior of the domain and vanishes at the boundary $\partial\Omega$. Motivated by [3], in this paper, we write $\partial\Omega$ as the zero level set of a smooth function. That is,

$$b(x) = \varphi(x)^a, \qquad \Omega = \{\varphi > 0\} \quad \text{and} \quad \partial\Omega = \{\varphi = 0\}, \tag{1.10}$$

where $a > 0$ and $\varphi \in C^2(\overline{\Omega})$. This type of condition (1.10) covers a wide range of functions and is the same as what is required in the degenerate elliptic estimates obtained in [3] (see lemma 2.7 in section 2).

It should be noted that in the derivations of (1.1) in [15] and (1.3) in [4, 5, 8, 13], it is assumed that the amplitude of the surface waves is much smaller than the depth of the lake and therefore it is reasonable to impose that the bottom function $b(x)$ has a positive lower bound. However, it would be interesting to study the case in the presence of a beach, which corresponds to the case that the bottom function $b(x)$ degenerates on the boundary. As remarked by Levermore *et al* in [13], when the lakes and oceans have beaches, the dynamics in the interior of the domain still can be described by the lake equations, but how to model the surface wave near the beach may need further investigation. Although it is not physically relevant in the presence of a beach, similar models are still used by several geophysicists and mathematicians to understand the dynamics of the lakes (see [3]).

Our goal here is to understand the vanishing viscosity limit of solutions to the IBVP (1.6)–(1.8) when the initial vorticity $\omega_0 = b^{-1}\nabla \times u_0 \in L^q(\Omega)$ for some $q$ satisfying $2 < q \leqslant \infty$, under the assumption of (1.10) in the presence of a beach. To deal with the vanishing viscosity limit problem, we first establish the global existence of solutions to the viscous IBVP (1.6)–(1.8) with $\omega_0 \in L^q(\Omega)$ for $2 < q \leqslant \infty$. For the inviscid IBVP (1.3), (1.4) and (1.5), there is an adequate theory on the existence and uniqueness of weak solutions. For the general case $\omega_0 \in L^q(\Omega)$ with $2 < q \leqslant \infty$, a global weak solution to (1.3)–(1.5) in the distributional sense is obtained in [10, 13] for nondegenerate $b(x)$, namely

$$0 < b_0 \leqslant b(x) \leqslant b_1 \qquad \text{for all } x \in \Omega. \tag{1.11}$$

When $b(x)$ is degenerate, the global weak solution can be obtained by replacing $b(x)$ by $b(x) + \epsilon$ for small $\epsilon > 0$, applying the result for the nondegenerate case in [10] and taking the limit as $\epsilon \to 0$. The weak solutions of (1.3)–(1.5) are in the distribution sense and their uniqueness is unknown if we just have $\omega_0 \in L^q(\Omega)$ with $2 < q < \infty$. If $\omega_0 \in L^\infty(\Omega)$, Bresch and Métivier [3] established the global existence and uniqueness of weak solutions in the class $\omega \in L^\infty(\Omega \times [0, T])$ for any $T > 0$. In this paper, we are able to establish two vanishing viscosity limit results. The first one is the strong convergence (up to a subsequence)

$$u^\mu \to u^0 \qquad \text{in } L^r(0, T; W^{\delta, r'}(\Omega)) \quad \text{as } \mu \to 0,$$

where $u^\mu$ refers to the aforementioned solution of (1.6)–(1.8) and $u^0$ is some weak solution of (1.3)–(1.5) associated with $\omega_0 \in L^q$, and the indices $r$ and $\delta$ will be specified later. When $\omega_0 \in L^\infty$, an explicit rate of convergence can be obtained. More precisely, we have

$$\|\sqrt{b}(u^\mu - u^0)(t)\|_{L^2}^2 \leqslant C\, M^{2(1-\mathrm{e}^{-\tilde{C}t})}\left(\|\sqrt{b}(u^\mu - u^0)(0)\|_{L^2}^2 + \mu t\right)^{\mathrm{e}^{-\tilde{C}t}}.$$

Here $u^0$ is the unique weak solution of (1.3)–(1.5) obtained in [3]. Precise statements of these results will be given in the following section.

To put our results in proper context, we briefly summarize some recent work on the viscous and inviscid lake equations. When $b = 1$, (1.6) and (1.3) become the classical Navier–Stokes and Euler equations, respectively. There is a large amount of literature on the inviscid limit of the Navier–Stokes equations with the Navier boundary conditions (see, e.g., [1, 2, 7, 9, 18, 19]). If $b$ is not a constant but nondegenerate, namely $b$ satisfies (1.11), the global existence and uniqueness of strong solutions to the IBVP (1.6)–(1.8) are obtained in [15] while the global weak solutions to the IBVP (1.3)–(1.5) are studied by Levermore *et al* in [13, 14]. The vanishing viscosity limit of (1.6)–(1.8) in the case when $b$ is nondegenerate was investigated by Jiu and Niu [10], which answered Levermore and Sammartino's question of [15] in the nondegenerate case. They proved that the solution of (1.6)–(1.8) with any initial vorticity in $L^p$ $(1 < p \leqslant \infty)$ converges to a weak solution of (1.3–(1.5). In another recent work [11], Jiu and Niu studied the viscous boundary layer problem for (1.6) with Navier boundary conditions.

We remark that the vanishing viscosity limit problem for the case when $b$ is degenerate is more difficult than the nondegenerate case. In the nondegenerate case, the viscous term $-\int_\Omega u^\mu \cdot \nabla \cdot (2bD(u^\mu) - b\,\mathrm{div}\, u^\mu I)\,\mathrm{d}x$ gives rise to the $H^1$ bound of $u^\mu$ utilizing the Navier boundary conditions and integrating by parts in a straight way (see [10]). In fact the restriction $\kappa \geqslant 0$ in [10] can be removed by adopting the approach by Kelliher [12]. But in the degenerate case, the degeneracy of $b(x)$ will produce an additional term $\int_\Omega (u^\mu \cdot \nabla)u^\mu \cdot \nabla b\,\mathrm{d}x$ during the integration by parts, which cannot be estimated by the term $\int_\Omega |(u^\mu \cdot \nabla)u^\mu \cdot b|\,\mathrm{d}x$. Thus, the $H^1$-estimate of the velocity $u^\mu$ cannot be obtained directly, even under the restriction $\kappa \geqslant 0$. To encounter this difficulty, we first obtain the $L^\infty([0, T]; L^2(\Omega))$ estimate of $\sqrt{b}u^\mu$ in lemma 3.1. The restriction $\kappa \geqslant 0$ is crucial in this case and it is still open now how to remove it. Moreover, a key tool employed here is an elliptic-type estimate for degenerate equations (see [3] and lemma 2.7 below). This estimate allows us to bound the $W^{1,q}$-norm of $u^\mu$ and $u^0$ uniformly with respect to the degenerate $b(x)$. Other techniques involved such as the Yudovich approach will be unfolded in the subsequent sections.

The rest of this paper is divided into three sections. The second section states the main results and provides the tools to be used in the subsequent sections. The third section establishes the existence and uniqueness of solutions to the IBVP (1.6)–(1.8) while the last section presents the inviscid limit results.

## 2. Main results and preparations

This section provides the precise statements of the main results and list some of the tools to be used in the proofs of these theorems.

One of the main theorems asserts the global existence and uniqueness of solutions to the viscous IBVP (1.6)–(1.8). This theorem involves the vorticity formulation. If $u^\mu$ solves the IBVP (1.6)–(1.8), then it can be verified (see [10]) that $\omega^\mu = b^{-1}\nabla \times u^\mu$ solves the following IBVP for the vorticity equation

$$\begin{cases} \partial_t \omega^\mu + u^\mu \cdot \nabla \omega^\mu - \mu \Delta \omega^\mu + 3\mu b^{-1} \nabla b \cdot \nabla \omega^\mu = \mu G(u^\mu, \nabla u^\mu), \\ b\omega^\mu = 0, \qquad x \in \partial\Omega, \\ b\omega^\mu(\cdot, 0) = b\omega_0, \qquad x \in \Omega. \end{cases} \tag{2.1}$$

where $G(u^\mu, \nabla u^\mu)$ involves only the linear terms of the first derivatives of $u^\mu$, and is given by

$$\begin{aligned} G = {} & (b^{-1}\Delta b + |\nabla \ln b|^2)\omega^\mu + b^{-1}\nabla \times ((\nabla u^\mu \cdot) \ln b) \\ & + b^{-1}\nabla \times (\nabla \ln b (u^\mu \cdot \nabla(\ln b))). \end{aligned} \tag{2.2}$$

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected and smooth bounded domain with nonnegative curvature $\kappa \geqslant 0$. Consider the IBVP (1.6)–(1.8) with $b = b(x)$ being given by (1.10) for $a \geqslant 2$. Assume $\sqrt{b}u_0 \in L^2(\Omega)$ and $\omega_0 = b^{-1}\nabla \times u_0 \in L^q(\Omega)$ for some $q$ satisfying $2 < q < \infty$. Then (1.6)–(1.8) has a unique solution which satisfies*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega \phi \cdot u^\mu b\,\mathrm{d}x + 2\mu \int_\Omega Du^\mu : D\phi\, b\,\mathrm{d}x - \mu \int_\Omega \operatorname{div} u^\mu \operatorname{div}\phi\, b\,\mathrm{d}x \\ \quad + \displaystyle\int_\Omega u^\mu \cdot \nabla u^\mu \cdot \phi\, b\,\mathrm{d}x + 2\mu \int_{\partial\Omega} \kappa u^\mu \cdot \phi b\,\mathrm{d}S = \int_\Omega \partial_t \phi \cdot u^\mu b\,\mathrm{d}x, \\ bu^\mu \cdot n = 0, \qquad x \in \partial\Omega, \\ u^\mu(x,0) = u_0, \qquad x \in \Omega \end{cases}$$

*for any test function $\phi \in C([0,T); W^{1,\frac{q}{q-1}})$ with $\phi \cdot n = 0$ on $\partial\Omega$.*

*In addition, $\omega^\mu = b^{-1}\nabla \times u^\mu$ is well defined, and satisfies (2.1) in the distribution sense. Furthermore, for any $T > 0$, $b^{\frac{1}{p}}\omega^\mu \in C([0,T]; L^q(\Omega))$ and*

$$\|\sqrt{b}u^\mu\|_{L^\infty(0,T;L^2)} + \|b^{\frac{1}{q}}\omega^\mu\|_{L^\infty(0,T;L^q)} \leqslant C, \tag{2.3}$$

$$\|u^\mu\|_{W^{1,q}} \leqslant C, \tag{2.4}$$

*where $C$ is a constant depending on $a$, $q$, $T$, $\|\varphi\|_{C^2(\overline{\Omega})}$ and the initial norms $\|\sqrt{b}u_0\|_{L^2}$ and $\|\omega_0\|_{L^q}$ only.*

Since $\Omega$ is a bounded domain, $\omega_0 \in L^\infty(\Omega)$ can be treated as a special case of theorem 2.1.

**Corollary 2.2.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected and smooth bounded domain with nonnegative curvature $\kappa \geqslant 0$. Consider the IBVP (1.6)–(1.8) with $b = b(x)$ being given by (1.10) for $a \geqslant 2$. Assume $\sqrt{b}u_0 \in L^2(\Omega)$ and $\omega_0 \in L^\infty(\Omega)$. Then (1.6)–(1.8) has a unique solution $u^\mu$, which obeys (2.3) and (2.4) for any $2 < q < \infty$.*

It is not clear whether the vorticity $\omega^\mu$ is in $L^\infty(\Omega)$. The approach of taking the limit of $\|\omega\|_{L^q}$ as $q \to \infty$ would not work since the bound for $\|\omega\|_{L^q}$ grows with respect to $q$ very quickly (see the bound in lemma 3.2).

Two other main results are the following theorems on inviscid limits. The first one is a strong convergence result without an explicit rate. The second result is that when $\omega_0 \in L^\infty$ an explicit convergence rate can be obtained. Before that, we give a definition of the weak solution of (1.3) in the sense of distributions, which is as follows.

**Definition 2.1.** *For any $T > 0$, we call $(u^0, p^0)$ a weak solution of (1.3)–(1.5) if*

*(i) $\sqrt{b}u^0 \in C([0,T]; L^2(\Omega))$ and for any $\phi \in C^1([0,T] \times \Omega)$ with $\phi(x,T) = 0$,*

$$\int_\Omega \phi u^0 b\,\mathrm{d}x + \int_0^T \int_\Omega u^0 \cdot \nabla u^0 \cdot \phi b\,\mathrm{d}x = \int_\Omega u_0 \phi(0,\cdot) b\,\mathrm{d}x + \int_0^T \int_\Omega \partial_t \phi u^\mu b\,\mathrm{d}x\,\mathrm{d}t;$$

*(ii) the boundary condition (1.4) is satisfied in the weak sense. That is, for any scalar function $\varphi(x) \in C^1(\bar{\Omega}$,*

$$\int_\Omega bu^0 \cdot \nabla\varphi\,\mathrm{d}x = 0.$$

We note that the boundary condition (1.4) can also be explained in the weak trace sense (see lemma 2.1 in [12] for instance). In [3], a definition of the weak solution was given in the vorticity-stream form. It is remarked that these two definitions are equivalent if the weak solution has more regularity, for example, if $\sqrt{b}u^0 \in C([0,T]; L^2(\Omega))$ and $\omega = b^{-1}\nabla \times u \in L^\infty([0,T]; L^2(\Omega))$ (see remark 2.4 in [3]).

In the following theorem $u^0$ denotes a weak solution of the inviscid IBVP (1.3)-(1.5) in the distributional sense. As we explained in the introduction, such weak solutions exist for all time. In particular, for the case when $\omega_0 \in L^\infty(\Omega)$, the existence and uniqueness of weak solutions was obtained by Bresch and Métivier [3], which is as follows.

**Lemma 2.3.** *Consider the inviscid IBVP (1.3)–(1.5) with $b = b(x)$ being given by (1.10) for $a \geqslant 2$. Assume $\sqrt{b}u_0 \in L^2(\Omega)$ and $\omega_0 \in L^\infty(\Omega)$. Then (1.3)–(1.5) has a unique solution $u^0$, which satisfies, for any $2 < p < \infty$ and any $T > 0$,*

$$u^0 \in C([0,T]; W^{1,p}), \qquad \omega^0 \in C([0,T]; L^p) \cap L^\infty([0,T]\times\Omega)$$

*and*

$$\sup_{p\geqslant 3} \frac{1}{p}\left(\int_\Omega |\nabla u^0|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

We now state our first vanishing viscosity limit result.

**Theorem 2.4.** *Suppose that the assumptions of theorem 2.1 hold true. Assume $\sqrt{b}u_0 \in L^2(\Omega)$ and $\omega_0 \in L^q(\Omega)$ for some $2 < q \leqslant \infty$. Let $u^\mu$ be the unique solution established in theorem 2.1. Let $\omega^\mu = b^{-1}\nabla \times u^\mu$. Then, there exist a subsequence of $u^\mu$ (still denoted by itself) and a measurable function $u^0(x,t)$ such that for any $1 < r < \infty$ satisfying $1 < 1/r + 2/q < 3/2$,*

$$u^\mu \longrightarrow u^0 \qquad \text{in } L^r(0,T; W^{\delta,r'}(\Omega)), \tag{2.5}$$

*as $k \longrightarrow \infty$, where $r'$ is the conjugate index of $r$, $1/r + 1/r' = 1$ and $\delta \in (0,1)$ satisfies $1/r' < 1/q - (1-\delta)/2$. Moreover, $u^0$ is a weak solution to (1.3)–(1.5), satisfying, in the case when $2 < q < \infty$*

$$\sqrt{b}u^0 \in L^2(\Omega), \qquad \omega^0 \in L^\infty([0,T], L^q(\Omega))$$

*and, if $q = \infty$, $\omega^0 \in L^\infty([0,T], L^{\tilde q}(\Omega))$ for any $1 \leqslant \tilde q < \infty$.*

When $\omega_0 \in L^\infty(\Omega)$, the weak solution $u^0$ in theorem 2.4 coincides with the unique one presented in lemma 2.3, which is as follows:

**Corollary 2.5.** *Suppose that the assumptions of theorem 2.1 hold true. If $\omega_0 \in L^\infty(\Omega)$, the weak solution $u^0$ in theorem 2.4 coincides with the unique weak solution in lemma 2.3 and the convergence in (2.5) holds true for the whole sequence of $u^\mu$.*

Moreover, when $\omega_0 \in L^\infty(\Omega)$, we obtain an explicit convergence rate.

**Theorem 2.6.** *Suppose that the assumptions of theorem 2.1 hold true. Assume $\sqrt{b}u_0 \in L^2(\Omega)$ and $\omega_0 \in L^\infty(\Omega)$. Let $u^\mu$ be the unique solution established in theorem 2.1 and let $u^0$ be the unique weak solution of the IBVP (1.3)–(1.5). Then, for any $T > 0$ and $t \leqslant T$,*

$$\|\sqrt{b}(u^\mu - u^0)(t)\|_{L^2}^2 \leqslant C\, M^{2(1-\mathrm{e}^{-\tilde C t})}(\|\sqrt{b}(u^\mu - u^0)(0)\|_{L^2}^2 + \mu t)^{\mathrm{e}^{-\tilde C t}},$$

*where $C$, $\tilde C$ and $M$ are constants depending on $a$, $T$, $\|\varphi\|_{C^2(\overline{\Omega})}$ and the norms $\|\sqrt{b}u_0\|_{L^2}$ and $\|\omega_0\|_{L^\infty}$ only. In particular, if $\|\sqrt{b}(u^\mu - u^0)(0)\|_{L^2} \to 0$, then $\|\sqrt{b}(u^\mu - u^0)(t)\|_{L^2} \to 0$ with an explicit rate, as $\mu \to 0$.*

We now list some of the tools to be used in the proofs of the theorems stated above. The first one is an estimate for solutions of degenerate elliptic equations. This estimate was obtained in [3, theorem 2.3].

**Lemma 2.7.** *Let $\Omega \subset \mathbb{R}^d$ be a simply connected bounded domain with a smooth boundary and let $b = b(x)$ be given by (1.10). Consider*

$$\nabla \cdot (bv) = 0, \quad \nabla \times v = f \quad \text{in } \Omega \quad \text{and} \quad (bv) \cdot n = 0 \quad \text{on } \partial\Omega.$$

*If, for $2 < p < \infty$,*

$$bv \in L^2(\Omega) \qquad \text{and} \qquad f \in L^p(\Omega),$$

*then*

$$v \in C^{1-\frac{d}{p}}(\overline{\Omega}), \quad \nabla v \in L^p(\Omega), \quad v \cdot n|_{\partial\Omega} = 0$$

*and, for a constant $C_p$ depending on $p$ only,*

$$\|v\|_{C^{1-\frac{d}{p}}} \leqslant C_p\,(\|f\|_{L^p} + \|bv\|_{L^2}).$$

*In particular,*

$$\|v\|_{L^p} \leqslant C\|v\|_{L^\infty} \leqslant C_p\,(\|f\|_{L^p} + \|bv\|_{L^2}). \tag{2.6}$$

*In addition, for any $p_0 > 2$ and $p_0 < p < \infty$, there is a constant $C$ depending on $p_0$ only such that*

$$\|\nabla v\|_{L^p} \leqslant Cp\,(\|f\|_{L^p} + \|bv\|_{L^2}). \tag{2.7}$$

**Remark 2.1.** The estimates in lemma 2.7 bound the $W^{1,p}$-norm of $v$ uniformly with respect to $b$. The estimates in (2.6) and (2.7) actually hold for $p = 2$, namely the $H^1$-norm of $v$ is bounded by $C(\|f\|_{L^2} + \|bv\|_{L^2})$.

The following lemma reformulates the Navier friction condition in terms of vorticity (see, e.g., [18]).

**Lemma 2.8.** *Suppose $v \in H^2(\Omega)$ with $v \cdot n = 0$ on $\partial\Omega$. Then,*

$$D(v)n \cdot \tau = -\kappa(v \cdot \tau) + \tfrac{1}{2}\nabla \times v \qquad \text{on } \partial\Omega,$$

*where $\tau$ denotes the unit tangent vector and $\kappa$ the curvature of $\partial\Omega$. In particular, if $\nabla \times v = 0$ on $\partial\Omega$, then*

$$D(v)n \cdot \tau = -\kappa(v \cdot \tau) \qquad \text{on } \partial\Omega.$$

We will also need the following Osgood-type inequality(see, e.g., [6]).

**Lemma 2.9.** *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geqslant 0$ satisfies*

$$\int_0^\infty \frac{1}{\omega(r)}\,\mathrm{d}r = \infty.$$

*Suppose that $\rho(t) > 0$ satisfies*

$$\rho(t) \leqslant a + \int_{t_0}^t \alpha(s)\omega(\rho(s))\,\mathrm{d}s$$

*for some constant $a \geqslant 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then*

$$-\Omega(\rho(t)) + \Omega(a) \leqslant \int_{t_0}^t \alpha(\tau)\,\mathrm{d}\tau,$$

*where*

$$\Omega(x) = \int_x^1 \frac{\mathrm{d}r}{\omega(r)}.$$

## 3. Global solutions of the viscous equations

This section is devoted to the proof of theorem 2.1. For this purpose, we first establish several *a priori* estimates including a global $L^2$-bound for the velocity, a global $L^q$-bound for the vorticity and a global $L^2_t H^1_x$ bound for the velocity.

We start with the $L^2$-bound for the velocity.

**Lemma 3.1** (**$L^2$-estimate**). *Suppose that the assumptions of theorem 2.1 hold and let $u^\mu$ be a smooth solution of (1.6). Then, for any $T > 0$,*

$$\|\sqrt{b}u^\mu\|^2_{L^\infty((0,T);L^2(\Omega))} + \int_0^T \int_{\partial\Omega} \kappa |u^\mu \cdot \tau|^2 b \,\mathrm{d}S \leqslant \|\sqrt{b}u_0\|^2_{L^2(\Omega)}, \tag{3.1}$$

*where $\kappa \geqslant 0$ is the curvature of $\partial\Omega$.*

**Proof.** We take the inner product of the first equation of (1.6) with $bu^\mu$ and integrate by parts. Due to the divergence-free condition $\nabla \cdot (bu^\mu) = 0$, the contribution from the nonlinear term and the pressure term is zero. The inner product with the dissipative term is

$$\begin{aligned}
\mu \int_\Omega u^\mu \cdot \nabla \cdot (2bDu^\mu - b\nabla \cdot u^\mu I)\,\mathrm{d}x
&= -\mu \int_{\partial\Omega} (2u^\mu \cdot Du^\mu n - (u^\mu \cdot n)\nabla \cdot u^\mu) b \,\mathrm{d}S \\
&\quad + 2\mu \int_\Omega \nabla u^\mu : Du^\mu b \,\mathrm{d}x - \mu \int_\Omega (\nabla \cdot u^\mu)^2 b \,\mathrm{d}x.
\end{aligned}$$

Writing $u^\mu = (u^\mu \cdot n)n + (u^\mu \cdot \tau)\tau$, applying the boundary condition in (1.7) and the basic identity $\nabla u^\mu : Du^\mu = Du^\mu : Du^\mu$, and invoking lemma 2.8, namely $D(u^\mu)n \cdot \tau = -\kappa(u^\mu \cdot \tau)$ on $\partial\Omega$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u^\mu|^2 b \,\mathrm{d}x + 2\mu \int_\Omega Du^\mu : Du^\mu \, b \,\mathrm{d}x - \mu \int_\Omega (\nabla \cdot u^\mu)^2 \, b \,\mathrm{d}x \\
&\quad + 2\mu \int_{\partial\Omega} \kappa |u^\mu \cdot \tau|^2 b \,\mathrm{d}S = 0,
\end{aligned} \tag{3.2}$$

where $\kappa$ is the curvature of $\partial\Omega$ which is nonnegative by assumption. Since

$$2Du^\mu : Du^\mu - (\nabla \cdot u^\mu)^2 = (\partial_1 u_2^\mu + \partial_2 u_1^\mu)^2 + (\partial_1 u_1^\mu - \partial_2 u_2^\mu)^2 \geqslant 0,$$

(3.1) then follows from (3.2). The proof of the lemma is then finished.

For the vorticity $\omega^\mu = b^{-1}\nabla \times u^\mu$, we have the following estimate.

**Lemma 3.2 (Estimate of vorticity).** *Suppose that the assumptions of theorem 2.1 hold and let $u^\mu$ be a smooth solution of (1.6). Let $\omega^\mu = b^{-1}\nabla \times u^\mu$. Then, for any $T > 0$,*

$$\|(b)^{\frac{1}{q}} \omega^\mu\|^q_{L^\infty(0,T;L^q)} \leqslant (\|\sqrt{b}u_0\|^q_{L^2} + \|\omega_0\|^q_{L^q}) \mathrm{e}^{\mu(Cq)^{q+1}T}, \tag{3.3}$$

*where $C$ is a constant depending on $a$, $q$, $T$ and $\|\varphi\|_{C^2(\overline{\Omega})}$.*

**Proof.** As stated in section 2, $\omega^\mu$ satisfies (2.1). Taking the inner product of $|\omega^\mu|^{q-2}\omega^\mu b$ with the first equation of (2.1), integrating by parts and using the zero boundary condition for $b\omega^\mu$, we have

$$\begin{aligned}
&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|b^{\frac{1}{q}}\omega^\mu\|^q_{L^q} + \frac{4(q-1)}{q^2}\mu \int_\Omega |\nabla(\omega^\mu)^{\frac{q}{2}}|^2 b \,\mathrm{d}x \\
&\quad \leqslant \mu \left| \int_\Omega |G(u^\mu, \nabla u^\mu)| |\omega^\mu|^{q-2}\omega^\mu b \,\mathrm{d}x \right| + 4\mu \left| \int_\Omega \nabla b \cdot \nabla \omega^\mu |\omega^\mu|^{q-2}\omega^\mu \,\mathrm{d}x \right|.
\end{aligned}$$

To bound the first term, we first notice from (2.2) that

$$\|bG(u^\mu,\nabla u^\mu)\|_{L^q}\leqslant \|u^\mu\|_{W^{1,q}}.$$

It then follows from Hölder's inequality that

$$\mu\left|\int_\Omega |G(u^\mu,\nabla u^\mu)||\omega^\mu|^{q-2}\omega^\mu b\,\mathrm{d}x\right|\leqslant C\mu\|u^\mu\|_{W^{1,q}}\|\omega^\mu\|_{L^q}^{q-1}.$$

To bound the last term, we recall that $b=\varphi^a$ with $\varphi\in C^2(\overline{\Omega})$ and $\varphi\geqslant 0$. Therefore, for $a\geqslant 2$,

$$|\nabla b|^2=|a\varphi^{a-1}\nabla\varphi|^2\leqslant C\varphi^{2a-2}\leqslant C\varphi^a=Cb. \tag{3.4}$$

Thus, by Hölder's and Young's inequalities,

$$\int_\Omega |\nabla b\cdot\nabla\omega^\mu||\omega^\mu|^{q-2}\omega^\mu\,\mathrm{d}x\leqslant\frac{\mu}{q}\int|\nabla(\omega^\mu)^{\frac{q}{2}}|^2 b\,\mathrm{d}x+\frac{C\mu}{q}\|\omega^\mu\|_{L^q}^q,$$

where $C$ is independent of $q$. Therefore, we obtain

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|b^{\frac{1}{q}}\omega^\mu\|_{L^q}^q&+\frac{3q-4}{q^2}\mu\int_\Omega|\nabla(\omega^\mu)^{\frac{q}{2}}|^2b\,\mathrm{d}x\\&\leqslant\frac{C\mu}{q}\|\omega^\mu\|_{L^q}^q+\mu\|u^\mu\|_{W^{1,q}}\|\omega^\mu\|_{L^q}^{q-1}.\end{aligned}$$

By the estimates in lemma 2.7,

$$\|\omega^\mu\|_{L^q}\leqslant\|\nabla u^\mu\|_{L^q}\leqslant Cq\,(\|b\,\omega^\mu\|_{L^q}+\|bu^\mu\|_{L^2}).$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|b^{\frac{1}{q}}\omega^\mu\|_{L^q}^q&+\frac{3q-4}{q}\mu\int_\Omega|\nabla(\omega^\mu)^{\frac{q}{2}}|^2b\,\mathrm{d}x\\&\leqslant\mu(Cq)^q(\|b\,\omega^\mu\|_{L^q}+\|bu^\mu\|_{L^2})^q\\&\leqslant\mu(Cq)^{q+1}(\|b\omega^\mu\|_{L^q}^q+\|bu^\mu\|_{L^2}^q).\end{aligned}$$

Noting that $\|b\omega^\mu\|_{L^q}\leqslant\|b^{1/q}\omega^\mu\|_{L^q}$ and applying lemma 3.1, we have

$$\|b^{\frac{1}{q}}\omega^\mu\|_{L^\infty(0,T;L^q)}^q\leqslant(\|\sqrt{b}u_0\|_{L^2}^q+\|\omega_0\|_{L^q}^q)\mathrm{e}^{\mu(Cq)^{q+1}T},$$

which is (3.3). The proof of the lemma is complete.

The following lemma provides a bound for $\|\sqrt{b}\nabla u\|_{L^2(\Omega\times[0,T])}$. In addition, its proof is also useful in proving theorem 2.6.

**Lemma 3.3.** *Suppose that the assumptions of theorem 2.1 hold and let $u^\mu$ be a smooth solution of (1.6). Then, for any $T>0$,*

$$\begin{aligned}\|\sqrt{b}u^\mu\|_{L^\infty((0,T);L^2(\Omega))}^2&+\mu\int_0^T\|\sqrt{b}\nabla u^\mu(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t+\mu\int_0^T\int_{\partial\Omega}\kappa|u^\mu\cdot\tau|^2b\,\mathrm{d}S\,\mathrm{d}t\\&\leqslant C(\|\sqrt{b}u_0\|_{L^2(\Omega)}^2+\|\omega_0\|_{L^2(\Omega)}^2).\end{aligned}\tag{3.5}$$

**Proof.** Substituting the identity

$$2Du^\mu:Du^\mu-(\nabla\cdot u^\mu)^2=|\nabla u^\mu|^2+2(\partial_1u_2^\mu\partial_2u_1^\mu-\partial_1u_1^\mu\partial_2u_2^\mu)$$

into (3.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^\mu|^2 b\,\mathrm{d}x + \mu\int_\Omega |\nabla u^\mu|^2 b\,\mathrm{d}x - 2\mu\int_\Omega (\partial_1 u_1^\mu \partial_2 u_2^\mu - \partial_1 u_2^\mu \partial_2 u_1^\mu) b\,\mathrm{d}x$$
$$+2\mu\int_{\partial\Omega} \kappa |u^\mu\cdot\tau|^2 b\,\mathrm{d}S = 0. \tag{3.6}$$

It is easy to check that

$$\begin{aligned}
J &\equiv 2\int_\Omega (\partial_1 u_1^\mu \partial_2 u_2^\mu - \partial_1 u_2^\mu \partial_2 u_1^\mu) b\,\mathrm{d}x \\
&= \int_\Omega \nabla\cdot(u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu) b\,\mathrm{d}x \\
&= \int_\Omega \nabla\cdot[(u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu) b]\,\mathrm{d}x \\
&\quad - \int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x.
\end{aligned}$$

Writing

$$\begin{aligned}
&(u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu) \\
&\quad = u_1^\mu(\partial_2 u_2^\mu, -\partial_1 u_2^\mu) - u_2^\mu(\partial_2 u_1^\mu, -\partial_1 u_1^\mu)
\end{aligned}$$

and applying the divergence theorem, we have

$$\begin{aligned}
J &= \int_{\partial\Omega} (u_1^\mu \tau\cdot\nabla u_2^\mu - u_2^\mu \tau\cdot\nabla u_1^\mu) b\,\mathrm{d}S \\
&\quad - \int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x.
\end{aligned}$$

Since $bu^\mu\cdot n = 0$ on $\partial\Omega$, we have $bu^\mu = (bu^\mu\cdot\tau)\tau$ on $\partial\Omega$. Writing $u_1^\mu \tau\cdot\nabla u_2^\mu - u_2^\mu \tau\cdot\nabla u_1^\mu = -\tau\cdot\nabla u^\mu\cdot(u_2^\mu, -u_1^\mu)$, we find

$$\begin{aligned}
J &= -\int_{\partial\Omega} (\tau\cdot\nabla u^\mu\cdot n)(u^\mu\cdot\tau) b\,\mathrm{d}S \\
&\quad - \int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x.
\end{aligned}$$

By lemma 2.8,

$$\begin{aligned}
J &= \int_{\partial\Omega} \kappa |u^\mu\cdot\tau|^2 b\,\mathrm{d}S \\
&\quad - \int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x.
\end{aligned}$$

Then (3.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^\mu|^2 b\,\mathrm{d}x + 2\mu\int_\Omega |\nabla u^\mu|^2 b\,\mathrm{d}x + \mu\int_{\partial\Omega} \kappa |u^\mu\cdot\tau|^2 b\,\mathrm{d}S$$
$$= -\mu\int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x. \tag{3.7}$$

Applying Hölder's inequality and using (3.4), we have

$$
\begin{aligned}
\mu & \left| \int_\Omega (u_1^\mu \partial_2 u_2^\mu - u_2^\mu \partial_2 u_1^\mu, u_2^\mu \partial_1 u_1^\mu - u_1^\mu \partial_1 u_2^\mu) \cdot \nabla b \,\mathrm{d}x \right| \\
& \leqslant \frac{1}{2}\mu \int_\Omega |\nabla u^\mu|^2 b \,\mathrm{d}x + C\,\mu \|u^\mu\|_{L^2}^2 \\
& \leqslant \frac{1}{2}\mu \int_\Omega |\nabla u^\mu|^2 b \,\mathrm{d}x + C\mu(\|b^{1/2}\omega^\mu\|_{L^2}^2 + \|b^{1/2}u^\mu\|_{L^2}^2).
\end{aligned}
\tag{3.8}
$$

Combining (3.7) with (3.8), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^\mu|^2 b \,\mathrm{d}x + \mu \int_\Omega |\nabla u^\mu|^2 b \,\mathrm{d}x \leqslant C\mu(\|b^{1/2}\omega^\mu\|_{L^2}^2 + \|b^{1/2}u^\mu\|_{L^2}^2).
$$

Applying (3.3) and the Gronwall inequality, we obtain (3.5) and thus finish the proof of this lemma.

We are now ready to prove theorem 2.1.

**Proof of theorem 2.1.** Let $\epsilon > 0$ be a small parameter. We construct the approximate solutions $(u^{\epsilon,\mu}, \omega^{\epsilon,\mu})$ to the nondegenerate viscous lake equations with $b^\epsilon = b + \epsilon$, namely

$$
\begin{cases}
\partial_t u^{\epsilon,\mu} + u^{\epsilon,\mu} \cdot \nabla u^{\epsilon,\mu} \\
\qquad -\mu (b^\epsilon)^{-1} \nabla \cdot (2b^\epsilon D(u^{\epsilon,\mu}) - b^\epsilon \nabla \cdot u^{\epsilon,\mu} I) + \nabla p^{\epsilon,\mu} = 0, \\
\nabla \cdot (b^\epsilon u^\mu) = 0, \\
b^\epsilon u^{\epsilon,\mu} \cdot n = 0,\ b^\epsilon \omega^\mu = 0 \qquad \text{on } \partial\Omega, \\
u^{\epsilon,\mu}(x,t)\,|_{t=0} = u_0.
\end{cases}
\tag{3.9}
$$

Since $b^\epsilon$ is nondegenerate, the global existence and uniqueness of such solutions can be obtained by a similar approach as in [10]. Moreover, $u^{\epsilon,\mu}$ satisfies (3.9) in the sense of distribution

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi \cdot u^{\epsilon,\mu} b^\epsilon \,\mathrm{d}x + 2\mu \int_\Omega D u^{\epsilon,\mu} : D\phi \; b^\epsilon \,\mathrm{d}x \\
& \qquad -\mu \int_\Omega \operatorname{div} u^{\epsilon,\mu} \operatorname{div}\phi \; b^\epsilon \,\mathrm{d}x + \int_\Omega u^{\epsilon,\mu} \cdot \nabla u^{\epsilon,\mu} \cdot \phi \; b^\epsilon \,\mathrm{d}x \\
& \qquad + \mu \int_{\partial\Omega} \kappa (u^{\epsilon,\mu} \cdot \phi) b^\epsilon \,\mathrm{d}S = \int_\Omega \partial_t \phi \cdot u^\mu b^\epsilon \,\mathrm{d}x
\end{aligned}
\tag{3.10}
$$

for any test function $\phi \in C([0,T]; W^{1,\frac{p}{p-1}}(\Omega))$ with $\phi \cdot n = 0$ on $\partial\Omega$. Thanks to lemmas 3.1 and 3.2, we deduce the uniform estimates, for any $T > 0$,

$$
\|\sqrt{b^\epsilon} u^{\epsilon,\mu}\|_{L^\infty(0,T;L^2)} + \|(b^\epsilon)^{\frac{1}{q}} \omega^{\epsilon,\mu}\|_{L^\infty(0,T;L^q)} \leqslant C.
\tag{3.11}
$$

By the estimates in lemma 2.7,

$$
\begin{aligned}
\|u^{\epsilon,\mu}\|_{W^{1,q}} & \leqslant C(\|\sqrt{b^\epsilon} u^{\epsilon,\mu}\|_{L^\infty(0,T;L^2)} \\
& \qquad + \|(b^\epsilon)^{\frac{1}{q}} \omega^{\epsilon,\mu}\|_{L^\infty(0,T;L^q)}) \leqslant C.
\end{aligned}
\tag{3.12}
$$

In these inequalities $C$ are constants depending on $T$ and $q$ but not on $\epsilon$ or $\mu$. Furthermore, using (3.10), we can prove that $\partial_t u^{\epsilon,\mu}$ is uniformly bounded in $L^\infty((0,T); H^{-s}_{\mathrm{loc}}(\Omega))$ for some $s > 2$. Thus (3.11) and (3.12) yield the compactness of $\sqrt{b^\epsilon} u^{\epsilon,\mu}$ in $L^2(0,T; L^2_{\mathrm{loc}}(\Omega))$ by the Aubin–Lions lemma. This allows one to pass to the limit $\epsilon \to 0$ in (3.10) (up to a subsequence) to get the existence of weak solutions of (1.6)–(1.8). Moreover, the solution $u^\mu, \omega^\mu$ satisfy the estimates of (3.11) and (3.12). Using similar estimates of (3.9) and (3), we can prove uniqueness of the weak solutions and we omit further details. The proof of the theorem is now finished.

## 4. Vanishing viscosity limits

This section proves theorem 2.4, and theorem 2.6, the vanishing viscosity limit results. In addition, a proof of corollary 2.5 is also provided at the end of this section.

**Proof of theorem 2.4.** According to theorem 2.1 and its proof given in the previous section, the unique solution $u^\mu$ of the IBVP (1.6)–(1.8) satisfies

$$\sqrt{b}u^\mu \in C(0,T;L^2)\cap L^2(0,T;H^1(\Omega)),$$
$$u^\mu \in L^\infty(0,T;W^{1,q}(\Omega)), \qquad b^{\frac{1}{q}}\omega^\mu \in L^\infty(0,T;L^q(\Omega))$$

and, for any test function $\phi \in C([0,T);W^{1,\frac{q}{q-1}})$ with $\phi\cdot n = 0$ on $\partial\Omega$,

$$\begin{aligned}\int_\Omega \phi u^\mu b\,\mathrm{d}x &+ 2\mu\int_0^T\int_\Omega Du^\mu : D\phi b\,\mathrm{d}x + \mu\int_0^T\int_\Omega \nabla\cdot u^\mu \mathrm{div}\phi b\,\mathrm{d}x\\ &+\int_0^T\int_\Omega u^\mu\cdot\nabla u^\mu\cdot\phi b\,\mathrm{d}x + \mu\int_0^T\int_{\partial\Omega}\kappa(u^\mu\cdot\phi)b\,\mathrm{d}S\\ &=\int_\Omega u_0\phi(0,\cdot)b\,\mathrm{d}x + \int_0^T\int_\Omega \partial_t\phi u^\mu b\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then there exist a measurable function $u^0(x,t)$ and a subsequence, denoted by $u^{\mu_k}$, such that

$$\begin{aligned}&u^{\mu_k} \rightharpoonup u^0 \qquad \text{in } w*-L^\infty(0,T;W^{1,q}(\Omega))\cap L^\infty(0,T;L^2(\Omega)),\\ &\omega^{\mu_k} \rightharpoonup \omega^0 \qquad \text{in } w*-L^\infty(0,T;L^q(\Omega)),\end{aligned}$$

as $k \longrightarrow \infty$. Therefore, for any $1<r<\infty$ satisfying $1<1/r+2/q<3/2$ and $\alpha\in(0,1)$ satisfying $1/r' < 1/q-(1-\alpha)/2$,

$$u^\mu \longrightarrow u^0 \quad \text{in } L^r(0,T;W^{\alpha,r'}(\Omega)),$$

where $r'$ is the conjugate index of $r$, $1/r+1/r'=1$.

In addition, the limiting function $u^0$ satisfies the weak form of the inviscid lake equations, that is,

$$\int_\Omega \phi u^0 b\,\mathrm{d}x + \int_0^T\int_\Omega u^0\cdot\nabla u^0\cdot\phi b\,\mathrm{d}x = \int_\Omega u_0\phi(0,\cdot)b\,\mathrm{d}x + \int_0^T\int_\Omega \partial_t\phi u^\mu b\,\mathrm{d}x\,\mathrm{d}t.$$

This completes the proof.

We now turn to the proof of theorem 2.6.

**Proof of theorem 2.6.** The differences $v = u^\mu - u^0$ and $p = p^\mu - p^0$ formally satisfy

$$\begin{cases}\partial_t v + v\cdot\nabla u^0 + u^\mu\cdot\nabla v\\ \quad -\mu b^{-1}\nabla\cdot(2bDu^\mu - b\nabla\cdot u^\mu) + \nabla p = 0,\\ \nabla\cdot(bv) = 0,\end{cases} \tag{4.1}$$

with the boundary condition $bv \cdot n = 0$. Taking the inner product of (4.1) with $bv$, integrating by parts and applying the boundary conditions, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{b}v\|^2_{L^2(\Omega)} + \int_\Omega v \cdot \nabla u^0 \cdot vb\,\mathrm{d}x + 2\mu\int_{\partial\Omega}\kappa v \cdot vb\,\mathrm{d}S + 2\mu\int_\Omega D(v) : D(v)b\,\mathrm{d}x$$
$$-\mu\int_\Omega(\nabla\cdot v)^2 b\,\mathrm{d}x = -2\mu\int_{\partial\Omega}\kappa u^0 \cdot vb\,\mathrm{d}S - 2\mu\int_\Omega D(u^0) : D(v)b\,\mathrm{d}x$$
$$+\mu\int_\Omega(\nabla\cdot u^0)(\nabla\cdot v)b\,\mathrm{d}x. \tag{4.2}$$

We remark that (4.2) can be obtained rigorously by using the weak form of the equations. We then combine the terms

$$2\mu\int_\Omega D(v) : D(v)b\,\mathrm{d}x - \mu\int_\Omega(\nabla\cdot v)^2 b\,\mathrm{d}x$$

and bound them as in the proof of lemma 3.3. More explicitly, as calculations in lemma 3.5, we write

$$\begin{aligned}2\mu&\int_\Omega D(v) : D(v)b\,\mathrm{d}x - \mu\int_\Omega(\nabla\cdot v)^2 b\,\mathrm{d}x\\
&= \mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x - 2\mu\int_\Omega(\partial_1 v_1\partial_2 v_2 - \partial_1 v_2\partial_2 v)b\,\mathrm{d}x\\
&= \mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x - \mu\int_{\partial\Omega}\kappa|u^\mu\cdot\tau|^2 b\mathrm{d}S\\
&\quad+\mu\int_\Omega(u_1^\mu\partial_2 u_2^\mu - u_2^\mu\partial_2 u_1^\mu, u_2^\mu\partial_1 u_1^\mu - u_1^\mu\partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x.\end{aligned}$$

and then bound the last term above as in (3.8), namely

$$\begin{aligned}\mu&\left|\int_\Omega(u_1^\mu\partial_2 u_2^\mu - u_2^\mu\partial_2 u_1^\mu, u_2^\mu\partial_1 u_1^\mu - u_1^\mu\partial_1 u_2^\mu)\cdot\nabla b\,\mathrm{d}x\right|\\
&\leqslant \frac{1}{2}\mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x + C\mu(\|b^{1/2}u^\mu\|^2_{L^2} + \|b^{1/2}\omega^\mu\|^2_{L^2})\\
&\leqslant \frac{1}{2}\mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x + C\mu(\|b^{1/2}u^0\|^2_{L^2} + \|b^{1/2}u^\mu\|^2_{L^2})\\
&\quad+C\mu(\|b^{1/2}\omega^0\|^2_{L^2} + \|b^{1/2}\omega^\mu\|^2_{L^2}) \leqslant \frac{1}{2}\mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x + C\mu,\end{aligned}$$

where $C$ depend on the initial norms $\|b^{\frac{1}{2}}u_0\|_{L^2}$ and $\|\omega_0\|_{L^\infty}$ only. Since the boundary of $\Omega$ is smooth, there is a constant $\kappa_0 \geqslant 0$ such that $\kappa(x) \leqslant \kappa_0$ for any $x \in \partial\Omega$. Applying Hölder's inequality and lemma 2.7, we have, for any $T > 0$ and $t \leqslant T$,

$$\begin{aligned}\frac{1}{2}&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{b}v\|^2_{L^2(\Omega)} + \frac{1}{2}\mu\int_\Omega|\nabla v|^2 b\,\mathrm{d}x + \mu\int_{\partial\Omega}\kappa\, v\cdot vb\,\mathrm{d}S\\
&\leqslant C\mu + \left|\int_\Omega v\cdot\nabla u^0\cdot vb\,\mathrm{d}x\right| + 2\mu(\kappa_0)^{\frac{1}{2}}\|b^{1/2}u^0\|_{L^2(\partial\Omega)}\left(\int_{\partial\Omega}\kappa|v|^2 b\,\mathrm{d}S\right)^{1/2}\\
&\quad+2\mu\|\nabla u^0\|_{L^2(\Omega)}\left(\int_\Omega|D(v)|^2 b\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad+\mu\|\nabla\cdot u^0\|_{L^2(\Omega)}\left(\int_\Omega(\nabla\cdot v)^2 b\,\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned} \tag{4.3}$$

Applying the bounds $\|b^{1/2}u^0\|_{L^2} \leqslant C$ for $C$ independent of $\mu$ and by lemma 2.7,

$$\begin{aligned}\|\nabla u^0\|_{L^2(\Omega)} &\leqslant C\,\|\nabla u^0\|_{L^3(\Omega)}\\ &\leqslant C(\|b\omega^0\|_{L^3(\Omega)} + \|bu^0\|_{L^2(\Omega)})\\ &\leqslant C,\end{aligned}$$

where $C$ depend on the initial norms $\|b^{\frac{1}{2}}u_0\|_{L^2}$ and $\|\omega_0\|_{L^\infty}$ only, we have from (4.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{b}v\|^2_{L^2(\Omega)} + \frac{\mu}{2}\int_\Omega |\nabla v|^2 b\,\mathrm{d}x \leqslant 2\left|\int_\Omega v\cdot\nabla u^0\cdot vb\,\mathrm{d}x\right| + C\mu, \tag{4.4}$$

where $C$ is independent of $\mu$. Since $\nabla u^0$ is not known to be bounded in $L^\infty$, we follow the Yudovich approach to deal with the nonlinear term (see, e.g., [3, 20]). For this purpose, we set

$$L := \sup_{p\geqslant 3}\frac{1}{p}\left(\int_\Omega |\nabla u^0|^p\,\mathrm{d}x\right)^{\frac{1}{p}},$$

$$M := \|u^0\|_{L^\infty} + \|u^\mu\|_{L^\infty}.$$

By lemma 2.3, $L < \infty$ and by lemma 2.7, $M < \infty$. Now, for $\delta > 0$, let

$$\Gamma_{\mu,\delta}(t) = \|\sqrt{b}v\|^2_{L^2(\Omega)} + \delta.$$

Applying Hölder's inequality to the nonlinear term in (4.4), we have, for any $p \geqslant 3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mu,\delta}(t) \leqslant pLM^{\frac{2}{p}}\Gamma_{\mu,\delta}(t)^{1-\frac{1}{p}} + C\mu. \tag{4.5}$$

Optimizing the bound on the right of (4.5) with respect to $p \geqslant 3$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mu,\delta}(t) \leqslant Ce(\ln M^2 - \ln\Gamma_{\mu,\delta}(t))\Gamma_{\mu,\delta}(t) + C\mu.$$

Integrating in time leads to

$$\Gamma_{\mu,\delta}(t) \leqslant \Gamma_{\mu,\delta}(0) + C\mu t + Ce\int_0^t \rho(\Gamma_{\mu,\delta}(\tau))\,\mathrm{d}\tau,$$

where $\rho(x) = x(\ln M^2 - \ln x)$. Let

$$\begin{aligned}\Omega(x) = \int_x^1 \frac{\mathrm{d}y}{\rho(y)} &= \int_x^1 \frac{\mathrm{d}y}{y(\ln M^2 - \ln y)}\\ &= \ln(\ln M^2 - \ln x) - \ln\ln M^2.\end{aligned}$$

Applying lemma 2.9, we get

$$-\Omega(\Gamma_{\mu,\delta}(t)) + \Omega(\Gamma_{\mu,\delta}(0) + C\mu t) \leqslant \tilde{C}t,$$

where $C$ and $\tilde{C}$ are constants independent of $\mu$. Therefore,

$$-\ln(\ln M^2 - \ln\Gamma_{\mu,\delta}(t)) + \ln(\ln M^2 - \ln(\Gamma_{\mu,\delta}(0) + C\mu t)) \leqslant \tilde{C}t.$$

That is,

$$\Gamma_{\mu,\delta}(t) \leqslant M^{2(1-\mathrm{e}^{-\tilde{C}t})}(\Gamma_{\mu,\delta}(0) + \mu t)^{\mathrm{e}^{-\tilde{C}t}}.$$

Letting $\delta \to 0$, we obtain

$$\|\sqrt{b}(u^\mu - u^0)(t)\|^2_{L^2} \leqslant C\,M^{2(1-\mathrm{e}^{-\tilde{C}t})}\left(\|\sqrt{b}(u^\mu - u^0)(0)\|^2_{L^2} + \mu t\right)^{\mathrm{e}^{-\tilde{C}t}}.$$

This completes the proof of theorem 2.6.

We finally prove corollary 2.5.

**Proof of corollary 2.5.** Let $\omega_0 \in L^\infty(\Omega)$ and let $u_1^0$ and $u_2^0$ be weak solutions given by lemma 2.3 and theorem 2.4, respectively. Then, the difference

$$\bar{u}^0 = u_1^0 - u_2^0$$

satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\bar{u}^0|^2 b\,\mathrm{d}x \leqslant 2\int_0^T\int_\Omega |\bar{u}|^2|\nabla u_1^0| b\,\mathrm{d}x.$$

A Yudovich-type argument as in the previous proof would lead to $\bar{u}^0 = 0$, or $u_1^0 = u_2^0$. We have thus completed the proof.

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