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Optimal Decay Estimates for 2D Boussinesq Equations with Partial Dissipation

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Abstract
Buoyancy-driven fluids such as many atmospheric and oceanic flows and the Rayleigh–Bénard convection are modeled by the Boussinesq systems. By rigorously estimating the large-time behavior of solutions to a special Boussinesq system, this paper reveals a fascinating phenomenon on buoyancy-driven fluids that the temperature can actually stabilize the fluids. The Boussinesq system concerned here governs the motion of perturbations near the hydrostatic equilibrium. When the buoyancy forcing is not present, the velocity of the fluid obeys the 2D Navier–Stokes equation with only vertical dissipation and its Sobolev norm could potentially grow even though its precise large-time behavior remains open. This paper shows that the temperature through the coupling and interaction tames and regularizes the fluids, and causes the velocity (measured in Sobolev norms) to decay in time. Optimal decay rates are obtained.

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1 Introduction

The two-dimensional (2D) incompressible Euler equation given by

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0 \end{cases}$$

is the simplest but one of the most frequently used models for incompressible ideal fluids. Here u denotes the fluid velocity and p the pressure. The precise large-time behavior of its solution has recently attracted considerable interests. One particular issue is whether the vorticity gradient can grow double exponentially in time. Here the vorticity $\omega = \nabla \times u$ is transported by the velocity field,

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad u = \nabla^{\perp} \Delta^{-1} \omega,$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$ and u is recovered from ω from the Biot–Savart law (Majda and Bertozzi 2002). It is not very difficult to show that $\|\nabla \omega(t)\|_{L^q}$ with $1 \leq q \leq \infty$ can grow at most double exponentially, namely

$$\|\nabla \omega(t)\|_{L^q} \leq (\|\nabla \omega_0\|_{L^q})^{e^{C\|\omega_0\|_{L^\infty} t}},$$

where ω_0 is the initial vorticity. A significant problem is whether or not the double exponential growth rate is sharp (Tao, https://terrytao.wordpress.com/2007/03/18/why-global-regularity-for-navier-stokes-is-hard/). Kiselev and Sverak were able to construct an explicit initial vorticity on a unit disk for which the corresponding vorticity gradient indeed grows double exponentially (Kiselev and Sverak 2014). A general bounded domain appears to share this property (Xu 2016). Whether or not such examples can be constructed in $\mathbb{R}^2$ remains an open problem. Other important results on related issues can be found in several references (see, e.g., Chae et al. 2014; Denisov 2015; Zlatoš 2015). As a special consequence of these growth results, perturbations near the trivial solution of the 2D Euler equation are in general not stable in the Sobolev setting.

In contrast, the Sobolev norms of solutions to the 2D incompressible Navier–Stokes equation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0 \end{cases} \tag{1.1}$$

always decay algebraically in time (see, e.g., Schonbek 1985; Schonbek and Wiegner 1996). In particular, perturbations near the trivial solution of (1.1) are always asymptotically stable in the Sobolev space $H^2(\mathbb{R}^2)$. The situation with the partial dissipation is not well understood. The stability and the large-time behavior problem on the solutions to the 2D Navier–Stokes equations with only vertical or horizontal dissipation, say

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \partial_{22} u, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0 \end{cases} \tag{1.2}$$

remains unknown. Here ∂_1 and ∂_2 are abbreviations for ∂_{x_1} and ∂_{x_2}, respectively. The lack of horizontal dissipation in (1.2) makes it impossible to control the vorticity gradient. In fact, if we estimate $\|\nabla\omega(t)\|_{L^2}$ via the standard energy method, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t)\|_{L^2}^2 + 2\nu\|\partial_2\nabla\omega(t)\|_{L^2}^2 = -2\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla u \cdot \nabla\omega \,\mathrm{d}x,$$

the dissipation is no longer sufficient in controlling the nonlinearity on the right-hand side. If we further decompose the nonlinear term into four terms,

$$\begin{aligned} \text{Hard} := & -\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla u \cdot \nabla\omega \,\mathrm{d}x \\ = & -\int_{\mathbb{R}^2} \partial_1 u_1\, (\partial_1\omega)^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} \partial_1 u_2\, \partial_1\omega\, \partial_2\omega \,\mathrm{d}x \\ & -\int_{\mathbb{R}^2} \partial_2 u_1\, \partial_1\omega\, \partial_2\omega \,\mathrm{d}x - \int_{\mathbb{R}^2} \partial_2 u_2\, (\partial_2\omega)^2 \,\mathrm{d}x. \end{aligned} \tag{1.3}$$

The two terms in (1.3), due to the lack of the horizontal dissipation, cannot be bounded suitably.

One goal of this paper is to present an explicit example of partial differential equation (PDE) systems which consists of the 2D Navier–Stokes with only vertical dissipation as a component equation but exhibits stability and large-time decay behavior. This example demonstrates that the coupling and interaction between the component equations in the system can actually stabilize the fluid and drive the fluid velocity to decay in time, even though the stability and large-time behavior of this partially dissipated Navier–Stokes itself remains unknown. To give a more precise account of our study, we shall be more specific. We are concerned with the following 2D Boussinesq system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + \theta e_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla\cdot u = 0, \\ \partial_t \theta + u \cdot \nabla\theta + u_2 + \eta\theta = 0, \end{cases} \tag{1.4}$$

where $u = (u_1, u_2)$ denotes the velocity field, P the pressure, θ the temperature, $e_2 = (0, 1)$, and $\nu > 0$ and $\eta > 0$ are the viscosity and damping coefficients, respectively. (1.4) models the motion of buoyancy-driven fluids. The degenerate Navier–Stokes equation with only vertical dissipation is relevant in certain physical regimes and arises after suitable scaling. The term $\theta\, e_2$ represents the buoyancy forcing generated due to the temperature variation. The third equation in (1.4) governs the evolution of the temperature and involves a thermal damping term instead of the thermal diffusion. The extra term u_2 is generated from the convection when we write the equation of the perturbation near the hydrostatic equilibrium $\theta_{he} = x_2$. In fact, when we treat the temperature as a sum of the equilibrium x_2 and a perturbation θ, the standard convection term becomes $u \cdot \nabla(x_2 + \theta)$, which is $u \cdot \nabla\theta + u_2$. The hydrostatic balance is an important equilibrium state in physics in which the fluid is static and the pressure gradient force is balanced out by the buoyancy.

The Boussinesq equations are important models for geophysical flows as well as for the Rayleigh–Bénard convection (see, e.g., Constantin and Doering 1996; Doering and Gibbon 1995; Majda 2003; Pedlosky 1987; Wen et al. 2012). In addition to their wide applicability in physics and geophysics, the Boussinesq equations are also mathematically significant. The 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations share some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations can be identified with the 3D axisymmetric Euler equations with swirl (Majda and Bertozzi 2002). Furthermore, the Boussinesq systems through coupling and interaction can describe many more phenomena than the hydrodynamic equation alone. As we reveal in this paper, the coupling structure in the Boussinesq system can lead to amazing smoothing and stabilization.

Due to their physical applications and mathematical importance, the Boussinesq systems have recently attracted a lot of interests. Especially, there have been substantial developments on the Boussinesq systems with only partial or fractional dissipation, or no dissipation at all. Significant progress has been made on several fundamental issues such as the global existence and regularity problem, and the stability problem on perturbations near several physically important steady states and large-time behavior. We briefly describe some of the closely related results. The 2D Boussinesq equations with either Laplacian viscosity or Laplacian thermal diffusion (but not both) are shown to have global classical solutions in Chae (2006) and Hou and Li (2005). Unique weak solution of such equations is obtained in Boardman et al. (2019) and Danchin and Paicu (2009). Danchin and Paicu (2011) and Larios et al. (2013) examined the 2D Boussinesq equations with either horizontal viscosity or horizontal thermal diffusion and established their global regularity. For the 2D Boussinesq equations with both vertical viscosity and vertical thermal diffusion, Cao and Wu (2013) solved the global regularity problem by proving a delicate global bound for the L^r-norm of the vertical component of the velocity, while (Adhikari et al. 2010, 2011), prior to Cao and Wu (2013), established several partial results. Li and Titi (2016) weakened the assumptions of Cao and Wu (2013) on the initial data. Several other partial dissipation cases are investigated by Adhikari et al. (2016). Partially dissipated Boussinesq equations on bounded domains with various boundary conditions are studied in Hu et al. (2015, 2018), Lai et al. (2011) and Zhao (2010). Important progress has also been made on the 2D Boussinesq equations with fractional dissipation. Global existence and regularity theory has been developed for various levels of fractional dissipation including the subcritical case (Miao and Xue 2011; Wu and Xu 2014; Wu et al. 2018; Yang et al. 2014, 2018; Ye and Xu 2016), the critical case (Hmidi et al. 2010, 2011; Jiu et al.

2014; Stefanov and Wu 2019; Wu et al. 2016), and the supercritical case (Chae and Wu 2012; Jiu et al. 2015; Kc et al. 2014; Li et al. 2016). The global regularity problem on the inviscid Boussinesq equations remains a mystery, but a much better understanding has been obtained (Adhikari et al. 2014; Chae et al. 2014; Choi et al. 2015; Constantin et al. 2015; Elgindi and Jeong 2020; Elgindi and Widmayer 2015; Kiselev and Tan 2018; Sarria and Wu 2015). More details on the global regularity problem on the Boussinesq equations can be found in Wu (2016). Rigorous study on the stability problem concerning the Boussinesq equations is more recent. Tao et al. (2020), Doering et al. (2018) established the stability and large-time behavior of perturbations near the hydrostatic equilibrium for the 2D Boussinesq with only velocity dissipation. Castro et al. (2019) obtained the asymptotic stability of the hydrostatic equilibrium for the 2D Boussinesq system with a velocity damping term. More recent work on the hydrostatic equilibrium can be found in Ben Said et al. (2020), Lai et al. (2021), Wan (2019), Wu et al. (2019). There are very significant recent developments on the stability of shear flow to the Boussinesq equations with various partial dissipation (Tao and Wu 2019; Yang and Lin 2018; Zillinger 2020; Deng et al. 2020; Bianchini et al. 2020).

The primary goal of this paper is to obtain optimal decay rates for solutions of the Boussinesq system in (1.4). A previous work (Lai et al. 2021) has established the global existence and stability of (1.4) in the Sobolev space H^2. In addition, the large-time behavior of $\|\nabla u(t)\|_{L^2}$ and $\|\nabla\theta(t)\|_{L^2}$ is also obtained via energy methods in Lai et al. (2021). However, important issues on the large-time behavior such as explicit decay rates for the solution itself and for the high-order derivatives of the solution remain unresolved in Lai et al. (2021). More importantly, a systematic approach on the large-time behavior of perturbations obeying a PDE system with only partial dissipation is still lacking. This paper intends to fill the void by developing an efficient method to extract the optimal large-time decay rates of partially dissipated PDE systems. For the convenience of later references, we first provide an accurate account of the stability result in Lai et al. (2021).

Theorem 1.1 *Assume* $(u_0, \theta_0) \in H^2$ *with* $\nabla\cdot u_0 = 0$. *Then, there exists a constant* $\varepsilon > 0$ *such that, if*

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon,$$

then (1.4) *has a unique global solution* (u, θ) *satisfying, for any* $t > 0$,

$$\|(u, \theta)(t)\|_{H^2}^2 + \int_0^t \left(\|(\partial_2 u, \theta)(\tau)\|_{H^2}^2 + \|(u_2, \partial_1 u_2)(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \le C\,\varepsilon^2. \tag{1.5}$$

Furthermore, we obtain the following decay rate

$$\|\nabla u(t)\|_{L^2} + \|\nabla\theta(t)\|_{L^2} \le C\varepsilon(1+t)^{-\frac{1}{2}}, \tag{1.6}$$

where $C > 0$ *is a positive constant independent of* ε *and* t.

The decay estimate in (1.6) was obtained by careful energy estimates. However, energy estimates are not very efficient in dealing with the decay rates of the solution itself or its high-order derivatives. This paper intends to provide a systematic approach to the large-time behavior problem on PDE systems with only partial dissipation. We obtain the following optimal decay rates by applying this approach to (1.4).

Theorem 1.2 *Consider* (1.4) *with* $\nu > 0$ *and* $\eta > 0$. *Assume the initial data* $(u_0, \theta_0) \in H^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *satisfy* $\nabla \cdot u_0 = 0$. *Then, there exists a sufficiently small constant* $\varepsilon > 0$ *such that, if*

$$\|(u_0, \theta_0)\|_{L^1} + \|(u_0, \theta_0)\|_{H^2} \le \varepsilon, \tag{1.7}$$

then the corresponding solution (u, θ) *of* (1.4) *obtained in Theorem 1.1 obeys*

$$\begin{aligned}
\|(u, \theta)(t)\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{1}{2}}, \quad \|\partial_2 u(t)\|_{L^2} \le C\varepsilon(1+t)^{-1}, \\
\|\partial_1 u(t)\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{3}{4}}, \quad \|\partial_2 \nabla u(t)\|_{L^2} \le C\varepsilon(1+t)^{-\frac{3}{4}}, \\
\|\nabla \theta(t)\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{3}{4}}, \quad \|\partial_{22} u(t)\|_{L^2} \le C\varepsilon(1+t)^{-\frac{5}{4}}.
\end{aligned}$$

The decay rates presented in Theorem 1.2 appear to be sharp for solutions in the regularity class H^2. In addition, these rates also reflect the extra smoothing and damping due to the coupling in the system. The decay rate for (u, θ) in the L^2-norm is the same as that for the 2D heat equation with an initial data in $L^1 \cap L^2$, and is thus optimal. The decay rate for $\partial_2 u$ also coincides with that of the 2D heat equation and is also sharp. Due to the smoothing and stabilizing effect, we are also able to obtain the decay estimate for $\|\partial_1 u(t)\|_{L^2}$ at the rate of $(1+t)^{-\frac{3}{4}}$. This rate is optimal and cannot be improved to $(1+t)^{-1}$ due to the lack of the horizontal dissipation. The decay rate for $\|\nabla\theta(t)\|_{L^2}$ also appears to be sharp. Since our solution is restricted to the regularity class H^2, estimating the decay rates of some of the second-order derivatives is difficult if not impossible when the system is only partially dissipated. Nevertheless, we managed to obtain the decay rates for $\|\partial_2 \nabla u(t)\|_{L^2}$ and for $\|\partial_{22} u(t)\|_{L^2}$.

Many important methods have been obtained to understand the large-time behavior of solutions to fully dissipative PDEs or PDE systems such as Schonbek's Fourier splitting scheme (see, e.g., Schonbek 1985; Schonbek and Wiegner 1996). Unfortunately, these methods do not appear to apply when the PDE system involves only partial dissipation. We develop a general approach to extract the large-time behavior of stable solutions to PDE systems with only partial dissipation. Since the vorticity $\omega = \nabla \times u$ is a scalar and the velocity u can be recovered from ω by the Biot–Savart law $u = \nabla^\perp \Delta^{-1}\omega$, we mainly work with the equivalent system consisting of the vorticity equation and the equation of θ,

$$\begin{cases}
\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22}\omega + \partial_1 \theta, \\
\partial_t \theta + u \cdot \nabla \theta + u_2 + \eta\,\theta = 0, \\
u = \nabla^\perp \Delta^{-1}\omega.
\end{cases} \tag{1.8}$$

The first step of our approach is to exploit the coupling structure in (1.8) to reveal the smoothness and stabilization hidden in the original system. By differentiating the equations in t one more time and making suitable substitutions, we can decouple ω from θ in the linear parts to obtain

$$\begin{cases} \partial_{tt}\omega + (\eta - \nu\partial_{22})\partial_t\omega - (\nu\eta\partial_{22}\omega + \mathcal{R}_1^2\omega) = M_1, \\ \partial_{tt}\theta + (\eta - \nu\partial_{22})\partial_t\theta - (\nu\eta\partial_{22}\theta + \mathcal{R}_1^2\theta) = M_2, \end{cases} \tag{1.9}$$

where M_1 and M_2 are the nonlinear terms

$$\begin{aligned} M_1 &= -(\partial_t + \eta)(u \cdot \nabla\omega) - \partial_1(u \cdot \nabla\theta), \\ M_2 &= (\nu\partial_{22} - \partial_t)(u \cdot \nabla\theta) + (u \cdot \nabla u_2 - \partial_2\Delta^{-1}\nabla \cdot (u \cdot \nabla u)). \end{aligned}$$

Here $\mathcal{R}_1 = \partial_1\Lambda^{-1}$ denotes the standard Riesz transform with $\Lambda = (-\Delta)^{\frac{1}{2}}$, and the fractional Laplace operator $(-\Delta)^\beta$ with $\beta \in \mathbb{R}$ is defined through the Fourier transform,

$$\widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta}\widehat{f}(\xi).$$

In comparison with (1.8), the wave equations in (1.9) reveal more smoothing and regularization due to the coupling and interaction. These regularizing properties will be reflected in the upper bounds on the kernel functions in the integral representation in Sect. 2. The second step is to separate the linear terms from the nonlinear ones in (1.8), solve the linearized system in the Fourier space and represent the system in an integral form via the Duhamel principle. To simplify the calculations, we work with the equations of ω and $\partial_1\theta$ (since the right-hand side of ω contains $\partial_1\theta$),

$$\begin{cases} \partial_t\omega + u \cdot \nabla\omega = \nu\partial_{22}\omega + \partial_1\theta, \\ \partial_t\partial_1\theta + \partial_1(u \cdot \nabla\theta) + \partial_1 u_2 + \eta\partial_1\theta = 0. \end{cases} \tag{1.10}$$

(1.10) can be converted into the integral form as

$$\begin{aligned} \widehat{\omega}(\xi, t) &= \widehat{K_1}(t)\widehat{\omega_0} + \widehat{K_2}(t)\widehat{\partial_1\theta_0} + \int_0^t \big(\widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\big)\,\mathrm{d}\tau, \\ \widehat{\partial_1\theta}(\xi, t) &= \widehat{K_3}(t)\widehat{\omega_0} + \widehat{K_4}(t)\widehat{\partial_1\theta_0} + \int_0^t \big(\widehat{K_3}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_4}(t-\tau)\widehat{N_2}(\tau)\big)\,\mathrm{d}\tau, \end{aligned}$$

where $\widehat{K_1}$, $\widehat{K_2}$, $\widehat{K_3}$ and $\widehat{K_4}$ are kernel functions with their explicit formula given in Sect. 2, and $N_1 = -u \cdot \nabla\omega$ and $N_2 = -\partial_1(u \cdot \nabla\theta)$. These kernel functions are frequency dependent and admit different upper bounds in different subdomains of the frequency space. The third step divides the frequency space into suitable subdomains and establishes sharp upper bounds for each kernel function in these subdomains. The details are provided in Proposition 2.1. With these preparations at our disposal, the fourth step employs a bootstrapping argument to prove the desired upper bounds. An abstract statement on the bootstrapping argument can be found in Tao (2006, p.21).

The argument starts with the ansatz that

$$\begin{aligned}
&\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le C_0\, \varepsilon\, (1+t)^{-\frac{1}{2}},\\
&\|\partial_2 u(t)\|_{L^2} \le C_0\, \varepsilon\, (1+t)^{-1},\\
&\|\partial_1 u(t)\|_{L^2} + \|\partial_2 \nabla u(t)\|_{L^2} \le C_0\, \varepsilon\, (1+t)^{-\frac{3}{4}},
\end{aligned}$$

where C_0 will be specified later in the proof. Making use of the integral representation and the upper bounds in the previous steps, we show that the bounds in the ansatz can actually be reduced by half, namely

$$\begin{aligned}
&\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\, (1+t)^{-\frac{1}{2}},\\
&\|\partial_2 u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\, (1+t)^{-1},\\
&\|\partial_1 u(t)\|_{L^2} + \|\partial_2 \nabla u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\, (1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{1.11}$$

Then, the bootstrapping argument assesses that the upper bounds in (1.11) indeed hold for all time. The verification of (1.11) is a long process involving repeated applications of various anisotropic inequalities and the upper bounds on the kernel functions. The estimates for $\nabla\theta$ and $\partial_{22} u$ are performed independently and take advantage of the rates in (1.11). More details are presented in Sect. 3.

The rest of the paper provides the details outlined above. Section 2 represents (1.4) in the integral form and presents the upper bounds for the kernel functions. Section 3 implements the bootstrapping argument to obtain the desired decay rates and thus finish the proof of Theorem 1.2.

2 Integral Representation

This section achieves two goals. The first is to represent (1.4) in an integral form (in the Fourier space). This is accomplished by separating the linear parts from the nonlinear ones, solving the linearized system in the Fourier space and representing the nonlinear system via Duhamel's principle. The kernel functions are the crucial components in this representation. The second main task is to obtain sharp upper bounds for the kernel functions. Since these kernel functions have a strong dependence on the frequency, we need to divide the frequency space into subdomains in order to obtain the optimal upper bounds. This is crafted in Proposition 2.1.

To make the calculations simpler, we work with the system of ω and $\partial_1\theta$ to find their representations. The representations of u and θ then follow as a consequence. Taking the Fourier transform of (1.10) yields

$$\partial_t \begin{pmatrix} \widehat{\omega} \\ \widehat{\partial_1\theta} \end{pmatrix} = \mathbb{A} \begin{pmatrix} \widehat{\omega} \\ \widehat{\partial_1\theta} \end{pmatrix} + \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix},$$

where $\mathbb{A}$ comes from the linear operators, and N_1, N_2 are the nonlinear terms,

$$\mathbb{A} = \begin{pmatrix} -\nu\xi_2^2 & 1 \\ -\frac{\xi_1^2}{|\xi|^2} & -\eta \end{pmatrix}, \quad N_1 = -u \cdot \nabla\omega, \quad N_2 = -\partial_1(u \cdot \nabla\theta).$$

The characteristic polynomial of $\mathbb{A}$ is given by

$$\lambda^2 + \left(\eta + \nu\xi_2^2\right)\lambda + \left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right) = 0 \tag{2.1}$$

and thus, the eigenvalues of $\mathbb{A}$ are

$$\lambda_1 = \frac{-\left(\eta + \nu\xi_2^2\right) - \sqrt{\Gamma}}{2}, \qquad \lambda_2 = \frac{-\left(\eta + \nu\xi_2^2\right) + \sqrt{\Gamma}}{2} \tag{2.2}$$

with

$$\Gamma := \left(\eta + \nu\xi_2^2\right)^2 - 4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right).$$

By computing the corresponding eigenvectors and diagonalizing $\mathbb{A}$, we find

$$\begin{aligned} \widehat{\omega}(\xi, t) &= \widehat{K_1}(t)\widehat{\omega_0} + \widehat{K_2}(t)\widehat{\partial_1\theta_0} \\ &\quad + \int_0^t \left(\widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right) \mathrm{d}\tau, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \widehat{\partial_1\theta}(\xi, t) &= \widehat{K_3}(t)\widehat{\omega_0} + \widehat{K_4}(t)\widehat{\partial_1\theta_0} \\ &\quad + \int_0^t \left(\widehat{K_3}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_4}(t-\tau)\widehat{N_2}(\tau)\right) \mathrm{d}\tau, \end{aligned} \tag{2.4}$$

where the kernel functions $\widehat{K_1}$, $\widehat{K_2}$, $\widehat{K_3}$ and $\widehat{K_4}$ are given by

$$\widehat{K_1} = -\nu\xi_2^2 G_1 + G_3, \quad \widehat{K_2} = G_1, \quad \widehat{K_3} = -\frac{\xi_1^2}{|\xi|^2} G_1, \quad \widehat{K_4} = \nu\xi_2^2 G_1 + G_2, \tag{2.5}$$

with

$$\begin{aligned} G_1 &= \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \; G_2 = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} + \lambda_2 G_1 = e^{\lambda_2 t} + \lambda_1 G_1, \\ G_3 &= \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1. \end{aligned}$$

We now analyze the behavior of $\widehat{K_1}(\xi, t)$, $\widehat{K_2}(\xi, t)$, $\widehat{K_3}(\xi, t)$ and $\widehat{K_4}(\xi, t)$, which clearly rely on the Fourier frequencies ξ. The following proposition provides upper bounds for the kernel functions in different subdomains of the frequency space. The notation Re ρ denotes the real part of a complex number ρ.

Proposition 2.1 *Let S_1 and S_2 be the following subsets of $\mathbb{R}^2$,*

$$S_1 := \left\{ \xi \in \mathbb{R}^2 : \ 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2} \le \frac{1}{4} \quad \text{or} \quad \nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} \ge \frac{3}{16}\left(\eta + \nu\xi_2^2\right)^2 \right\},$$

$$S_2 := \left\{ \xi \in \mathbb{R}^2 : \ 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2} > \frac{1}{4} \quad \text{or} \quad \nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} < \frac{3}{16}\left(\eta + \nu\xi_2^2\right)^2 \right\}.$$

Then, G_1, G_2 and G_3, and $\widehat{K_1}$, $\widehat{K_2}$, $\widehat{K_3}$ and $\widehat{K_4}$ admit the following upper bounds:

(I) There exists some $c = c(\nu, \eta) > 0$ such that, for any $\xi \in S_1$,

$$\begin{aligned}
&\mathrm{Re}\lambda_1 \le -\frac{1}{2}\left(\eta + \nu\xi_2^2\right), \quad \mathrm{Re}\lambda_2 \le -\frac{1}{4}\left(\eta + \nu\xi_2^2\right),\\
&|G_1| \le t e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}, \quad G_2 = e^{\lambda_1 t} + \lambda_2 G_1, \quad G_3 = e^{\lambda_1 t} - \lambda_1 G_1,\\
&|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}|, |\widehat{K_4}| \le C e^{-c\left(1+\xi_2^2\right)t}.
\end{aligned} \tag{2.6}$$

(II) There exists some $c = c(\nu, \eta) > 0$ such that, for any $\xi \in S_2$,

$$\begin{aligned}
&\lambda_1 \le -\frac{3}{4}\left(\eta + \nu\xi_2^2\right), \quad \lambda_2 \le -\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \nu\xi_2^2},\\
&|G_1| \le 2\left(\eta + \nu\xi_2^2\right)^{-1}\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right),\\
&|\widehat{K_1}| \le e^{-c\left(1+\xi_2^2\right)t} + C\left(1 + \xi_2^2\right)^{-1} e^{-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}t},\\
&|\widehat{K_2}|, |\widehat{K_3}| \le C\left(1 + \xi_2^2\right)^{-1}\left(e^{-c(1+\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}t}\right),\\
&|\widehat{K_4}| \le C e^{-c(1+\xi_2^2)t} + C e^{-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}t}.
\end{aligned}$$

We further split S_2 into three subdomains as follows:

$$S_{21} := \left\{ \xi \in S_2, \quad \nu\xi_2^2 \ge \eta \right\},$$

$$S_{22} := \left\{\xi \in S_2, \quad \nu\xi_2^2 < \eta \quad \text{and} \quad |\xi_1| \geq |\xi_2|\right\},$$
$$S_{23} := \left\{\xi \in S_2, \quad \nu\xi_2^2 < \eta \quad \text{and} \quad |\xi_1| < |\xi_2|\right\}.$$

Then, we have the following more concise upper bounds

(a) *For* $\xi \in S_{21}$,

$$\begin{aligned}
&|\widehat{K_1}| \leq C\mathrm{e}^{-c\left(1+\xi_2^2\right)t} + C\left(1+\xi_2^2\right)^{-1}\mathrm{e}^{-ct},\\
&|\widehat{K_2}|, |\widehat{K_3}| \leq C\left(1+\xi_2^2\right)^{-1}\left(\mathrm{e}^{-c(1+\xi_2^2)t} + \mathrm{e}^{-ct}\right),\\
&|\widehat{K_4}| \leq C\mathrm{e}^{-c(1+\xi_2^2)t} + C\mathrm{e}^{-ct}.
\end{aligned} \tag{2.7}$$

(b) *For* $\xi \in S_{22}$,

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}|, |\widehat{K_4}| \leq C\mathrm{e}^{-c\left(1+\xi_2^2\right)t}. \tag{2.8}$$

(c) *For* $\xi \in S_{23}$,

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}|, |\widehat{K_4}| \leq C\left(\mathrm{e}^{-c(1+\xi_2^2)t} + \mathrm{e}^{-c|\xi|^2 t}\right). \tag{2.9}$$

Proof (I) For $\xi \in S_1$, λ_1 and λ_2 given by (2.2) obviously satisfy

$$\mathrm{Re}\lambda_1 \leq -\frac{1}{2}\left(\eta + \nu\xi_2^2\right), \quad \mathrm{Re}\lambda_2 \leq -\frac{1}{4}\left(\eta + \nu\xi_2^2\right).$$

To further the proof, we divide our consideration into two cases:

$$1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2} \geq 0 \quad \text{and} \quad 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2} < 0. \tag{2.10}$$

Both λ_1 and λ_2 are real in the first case and

$$|\lambda_1|, \ |\lambda_2| \leq \frac{1}{2}\left(\eta + \nu\xi_2^2\right)\left(1 + \sqrt{1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2}}\right) \leq \frac{3}{4}\left(\eta + \nu\xi_2^2\right). \tag{2.11}$$

By the definition of S_1, there exists a constant C such that, for any $\xi \in S_1$,

$$|\xi_2| \leq C. \tag{2.12}$$

By (2.11) and (2.12),

$$|\lambda_1|, \ |\lambda_2| \leq C.$$

Furthermore, by the mean-value theorem, there is $\zeta \in (\lambda_1, \lambda_2)$ such that

$$G_1 = t e^{\zeta t} \le t e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}.$$

By the definitions of $\widehat{K}_1$ through $\widehat{K}_4$ in (2.5), for $c = \min\{\frac{1}{8}\nu, \frac{1}{8}\eta\} > 0$,

$$\begin{aligned}
|\widehat{K_1}| &= |G_3 - \nu\xi_2^2 G_1| = |e^{\lambda_1 t} - \lambda_1 G_1 - \nu\xi_2^2 G_1| \le C e^{-c\left(1+\xi_2^2\right)t},\\
|\widehat{K_2}| &\le |G_1| \le C e^{-c\left(1+\xi_2^2\right)t}, \quad |\widehat{K_3}| \le |G_1| \le C e^{-c\left(1+\xi_2^2\right)t},\\
|\widehat{K_4}| &= |G_2 + \nu\xi_2^2 G_1| = |e^{\lambda_1 t} + \lambda_2 G_1 + \nu\xi_2^2 G_1| \le C e^{-c\left(1+\xi_2^2\right)t},
\end{aligned}$$

where we have used the simple fact that $x e^{-x} \le C$ for any $x \ge 0$. For the second case in (2.10), λ_1 and λ_2 are a pair of complex conjugates and their norms are

$$|\lambda_1|,\ |\lambda_2| = \sqrt{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}} \le \sqrt{\nu\eta\xi_2^2 + 1}. \tag{2.13}$$

As we explained before, $|\xi_2| \le C$ for any $\xi \in S_1$. Therefore, (2.13) implies

$$|\lambda_1| \le C, \quad |\lambda_2| \le C.$$

In addition, since λ_1 and λ_2 are a pair of complex conjugates,

$$G_1 = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t}\,\frac{2\sin(\frac{1}{2}Qt)}{Q}, \tag{2.14}$$

where

$$Q := (\eta + \nu\xi_2^2)\sqrt{\frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2} - 1}.$$

By the simple fact that $|\sin\rho| \le |\rho|$ for any $\rho \in \mathbb{R}$, (2.14) implies

$$|G_1| \le t\, e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t}.$$

The bounds for $|\widehat{K_1}|$ through $|\widehat{K_4}|$ then follow as before.

(II) For $\xi \in S_2$, λ_1 and λ_2 are real. The bound for λ_1 is obvious. To estimate λ_2, we rewrite it as

$$\lambda_2 = -\frac{1}{2}\left(\eta + \nu\xi_2^2\right)\left(1 - \sqrt{1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2}}\right)$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)+\sqrt{\left(\eta+\nu\xi_2^2\right)^2-4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}}\\
&\le -\frac{\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}.
\end{aligned}
$$

Furthermore,

$$
|\lambda_2| = \frac{1}{2}\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)+\sqrt{\left(\eta+\nu\xi_2^2\right)^2-4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}} \le \frac{4}{3}\frac{\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2} \le C. \tag{2.15}
$$

Due to the lower bound

$$
\lambda_2-\lambda_1 = \left(\eta+\nu\xi_2^2\right)\sqrt{1-\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2}} \ge \frac{1}{2}\left(\eta+\nu\xi_2^2\right),
$$

the upper bound for G_1 then follows:

$$
|G_1| \le 2\left(\eta+\nu\xi_2^2\right)^{-1}\left(e^{\lambda_1 t}+e^{\lambda_2 t}\right).
$$

The upper bounds for $\widehat{K_2}$ and $\widehat{K_3}$ are trivial by the definition in (2.5). Since λ_1 and λ_2 are the roots of (2.1), they satisfy

$$
\lambda_1+\lambda_2 = -(\eta+\nu\xi_2^2).
$$

Invoking the uniform bound for $|\lambda_2|$ in (2.15), we get, for $c=\min\{\frac{3}{4}\nu, \frac{3}{4}\eta\}>0$,

$$
\begin{aligned}
|\widehat{K_1}| &= |G_3-\nu\xi_2^2G_1| = |e^{\lambda_1 t}-\lambda_1 G_1-\nu\xi_2^2 G_1| = |e^{\lambda_1 t}+\lambda_2 G_1+\eta G_1|\\
&\le e^{-c\left(1+\xi_2^2\right)t}+C\left(1+\xi_2^2\right)^{-1}\left(e^{-c(1+\xi_2^2)t}+e^{-\frac{\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}t}\right)\\
&\le e^{-c\left(1+\xi_2^2\right)t}+C\left(1+\xi_2^2\right)^{-1}e^{-\frac{\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2}t}
\end{aligned}
$$

and

$$|\widehat{K_4}| = |G_2 + \nu\xi_2^2 G_1| = |e^{\lambda_1 t} + \lambda_2 G_1 + \nu\xi_2^2 G_1| \le C e^{-c(1+\xi_2^2)t} + C e^{-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \nu\xi_2^2} t}.$$

To make the upper bound for $|\widehat{K_1}|$, $|\widehat{K_2}|$, $|\widehat{K_3}|$ and $|\widehat{K_4}|$ more concise, we further divide S_2 into S_{21}, S_{22} and S_{23}. For $\xi \in S_{21}$,

$$-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \nu\xi_2^2} \le -\frac{\nu\eta\xi_2^2}{\eta + \nu\xi_2^2} \le -\frac{\nu\eta\xi_2^2}{\nu\xi_2^2 + \nu\xi_2^2} = -\frac{\eta}{2}.$$

Applying this upper bound and taking $c = \min\{\frac{3}{4}\nu, \frac{1}{2}\eta\} > 0$ leads to (2.7). For $\xi \in S_{22}$,

$$-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \nu\xi_2^2} \le -\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \eta} = -\frac{\nu}{2}\xi_2^2 - \frac{1}{2\eta}\frac{\xi_1^2}{|\xi|^2} \le -\frac{\nu}{2}\xi_2^2 - \frac{1}{4\eta}.$$

By taking $c = \min\{\frac{1}{2}\nu, \frac{3}{4}\eta, \frac{1}{4\eta}\} > 0$, we obtain (2.8). For $\xi \in S_{23}$,

$$-\frac{\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}}{\eta + \nu\xi_2^2} \le -\frac{\nu\eta\xi_2^2}{\eta + \nu\xi_2^2} \le -\frac{\nu\eta\xi_2^2}{\eta + \eta} = -\frac{\nu}{2}\xi_2^2 \le -\frac{\nu}{4}|\xi|^2,$$

we can take $c = \min\{\frac{1}{4}\nu, \frac{3}{4}\eta\} > 0$ to obtain (2.9). This completes the proof of Proposition 2.1. □

3 The Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. For the sake of clarity, we divide this section into three subsections. The first subsection establishes the desired decay rates for $\|u(t)\|_{L^2}$, $\|\theta(t)\|_{L^2}$, $\|\partial_2 u(t)\|_{L^2}$, $\|\partial_1 u(t)\|_{L^2}$ and $\|\partial_2\nabla u(t)\|_{L^2}$. The framework of the proof is the bootstrapping argument. The integral representation in (2.3) and (2.4) and the upper bounds on the kernel functions obtained in Proposition 2.1 will be used extensively. The second subsection shows that

$$\|\nabla\theta\|_{L^2} \le C\,\varepsilon\,(1+t)^{-\frac{3}{4}}.$$

We make use of the upper bounds in the first subsection. The third subsection proves the upper bound

$$\|\partial_{22} u\|_{L^2} \le C\,\varepsilon\,(1+t)^{-\frac{5}{4}}.$$

Several inequalities will be used repeatedly in the proof. For easy references, we list these inequalities in the following lemmas. The first one is Minkowski's inequality. It is an elementary tool that allows us to estimate the Lebesgue norm with larger index first followed by the Lebesgue norm with a smaller index. The following version is taken from Bahouri et al. (2011, p. 4), and a more general statement can be found in Lieb and Loss (2001, p. 47).

Lemma 3.1 *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces. Let f be a nonnegative measurable function over $X_1 \times X_2$. For all $1 \le p \le q \le \infty$, we have*

$$\Big\| \|f(\cdot, x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \Big\| \|f(x_1, \cdot)\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

In particular, for a nonnegative measurable function f over $\mathbb{R}^m \times \mathbb{R}^n$ and for $1 \le p \le q \le \infty$,

$$\Big\| \|f\|_{L^p(\mathbb{R}^m)} \Big\|_{L^q(\mathbb{R}^n)} \le \Big\| \|f\|_{L^q(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m)}.$$

By combining the basic one-dimensional inequality $\|g\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}$ with Hölder's inequality, we have the elementary inequalities in the following two lemmas.

Lemma 3.2 *Assume f, $\partial_1 f$, g and $\partial_2 g$ are all in $L^2(\mathbb{R}^2)$. Then, for a pure constant C,*

$$\|fg\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.$$

Lemma 3.3 *The following estimate holds when the right-hand sides are all bounded,*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

The next lemma provides the exact decay rate for the solution operator associated with a fractional Laplacian when acting upon Lebesgue spaces. Its proof can be found in many references (see, e.g., Schonbek and Schonbek 2005; Wu 2001).

Lemma 3.4 *Let $\alpha \ge 0$ and $\beta > 0$ are real numbers, $1 \le q \le p \le \infty$. Then, for any $t > 0$,*

$$\|(-\Delta)^\alpha e^{-(-\Delta)^\beta t} f\|_{L^p(\mathbb{R}^d)} \le C\, t^{-\frac{\alpha}{\beta} - \frac{d}{2\beta}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^q(\mathbb{R}^d)}.$$

The two lemmas below offer upper bounds with optimal decay rates for two special integrals (see, e.g., Lai et al. 2019; Zhang and Zhao 2010).

Lemma 3.5 *If $0 < s_1 \le s_2$, then*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\, d\tau \le \begin{cases} C(1+t)^{-s_1}, & \text{if } \ s_2 > 1; \\ C(1+t)^{-s_1}\ln(1+t), & \text{if } \ s_2 = 1; \\ C(1+t)^{1-s_1-s_2}, & \text{if } \ s_2 < 1. \end{cases}$$

Lemma 3.6 *For any $c > 0$ and $s > 0$,*

$$\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-s}\,\mathrm{d}\tau \le C(1+t)^{-s}.$$

We are now ready to prove Theorem 1.2.

3.1 Decay Estimates for $\|(u, \theta)\|_{L^2}$, $\|\partial_2 u\|_{L^2}$ and $\|(\partial_1 u, \partial_2 \nabla u)\|_{L^2}$

This subsection establishes the decay rates for $\|(u, \theta)\|_{L^2}$, $\|\partial_2 u\|_{L^2}$ and $\|(\partial_1 u, \partial_2 \nabla u)\|_{L^2}$. The tool is the bootstrapping argument, which starts with the ansatz that

$$\begin{aligned}
&\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le C_0 \varepsilon\ (1+t)^{-\frac{1}{2}},\\
&\|\partial_2 u(t)\|_{L^2} \le C_0 \varepsilon\ (1+t)^{-1},\\
&\|\partial_1 u(t)\|_{L^2} + \|\partial_2 \nabla u(t)\|_{L^2} \le C_0 \varepsilon\ (1+t)^{-\frac{3}{4}},
\end{aligned} \tag{3.1}$$

where C_0 is a constant to be specified later in the following proof. We show by using the ansatz and the integral representation in (2.3) and (2.4) that

$$\begin{aligned}
&\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\ (1+t)^{-\frac{1}{2}},\\
&\|\partial_2 u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\ (1+t)^{-1},\\
&\|\partial_1 u(t)\|_{L^2} + \|\partial_2 \nabla u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon\ (1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{3.2}$$

The bootstrapping argument then implies that (3.2) indeed holds for all time $t > 0$.

The implementation relies crucially on the upper bounds in Proposition 2.1. As a special consequence of Proposition 2.1, we have

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}|, |\widehat{K_4}| \le C\mathrm{e}^{-ct}, \quad \text{if} \quad \xi \in S_1 \cup S_{21} \cup S_{22}, \tag{3.3}$$

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}|, |\widehat{K_4}| \le C\left(\mathrm{e}^{-ct} + \mathrm{e}^{-c|\xi|^2 t}\right), \quad \text{if} \quad \xi \in S_{23}. \tag{3.4}$$

By the Biot–Savart law

$$u = \nabla^\perp \Delta^{-1}\omega = (-\partial_2 \Delta^{-1}\omega, \partial_1 \Delta^{-1}\omega) \tag{3.5}$$

and the integral representation of $\widehat{\omega}$ in (2.3), we obtain

$$\|\widehat{u}\|_{L^2} = \left\| |\xi|^{-1}\widehat{\omega} \right\|_{L^2} \le I_1 + I_2 + I_3 + I_4, \tag{3.6}$$

where

$$I_1 = \left\| |\xi|^{-1}\widehat{K_1}(t)\widehat{\omega_0} \right\|_{L^2},$$

$$I_2 = \left\| |\xi|^{-1} \widehat{K_2}(t) \widehat{\partial_1 \theta_0} \right\|_{L^2},$$
$$I_3 = \int_0^t \left\| |\xi|^{-1} \widehat{K_1}(t-\tau) \widehat{N_1}(\tau) \right\|_{L^2} \mathrm{d}\tau,$$
$$I_4 = \int_0^t \left\| |\xi|^{-1} \widehat{K_2}(t-\tau) \widehat{N_2}(\tau) \right\|_{L^2} \mathrm{d}\tau.$$

By (3.3) and (3.4), Lemma 3.4 and the simple fact that $(1+x)^{\frac{1}{2}} \mathrm{e}^{-ax} \le C(a)$ for any constant $a > 0$,

$$\begin{aligned} I_1 &= \left\| |\xi|^{-1} \widehat{K_1}(t) \widehat{\omega_0} \right\|_{L^2(S_1 \cup S_{21} \cup S_{22})} + \left\| |\xi|^{-1} \widehat{K_1}(t) \widehat{\omega_0} \right\|_{L^2(S_{23})} \\ &\le C \mathrm{e}^{-ct} \|\widehat{u_0}\|_{L^2} + C \left\| \mathrm{e}^{-c|\xi|^2 t} |\widehat{u_0}| \right\|_{L^2} \\ &\le C(1+t)^{-\frac{1}{2}} \|u_0\|_{L^2 \cap L^1}. \end{aligned}$$

Similarly,

$$I_2 \le C(1+t)^{-\frac{1}{2}} \|\theta_0\|_{L^2 \cap L^1}.$$

Using (3.3) and (3.4), and writing

$$\widehat{N_1} = \widehat{u \cdot \nabla \omega} = \widehat{\nabla \cdot (u\omega)} = i\xi \cdot \widehat{u\omega} \tag{3.7}$$

and

$$\begin{aligned} |\widehat{N_1}| &= \left| \widehat{u \cdot \nabla \omega} \right| = \left| \nabla \times \widehat{(u \cdot \nabla u)} \right| = \left| \nabla \times (\widehat{\nabla \cdot (u \otimes u)}) \right| \\ &\le |\xi_1 \xi| \, |\widehat{u u_2}| + |\xi_2 \xi| \, |\widehat{u u_1}| \le |\xi|^2 \left(|\widehat{u u_1}| + |\widehat{u u_2}| \right), \end{aligned} \tag{3.8}$$

we obtain

$$\begin{aligned} I_3 &= \int_0^t \left\| |\xi|^{-1} \widehat{K_1}(t-\tau) \widehat{u \cdot \nabla \omega}(\tau) \right\|_{L^2} \mathrm{d}\tau \\ &\le C \int_0^t \mathrm{e}^{-c(t-\tau)} \|\widehat{u\omega}(\tau)\|_{L^2} \mathrm{d}\tau \\ &\quad + C \int_0^t \left\| |\xi|^{-1} \mathrm{e}^{-c|\xi|^2(t-\tau)} \nabla \times (\widehat{\nabla \cdot (u} \otimes u)) \right\|_{L^2(S_{23})} \mathrm{d}\tau \\ &:= I_{31} + I_{32}. \end{aligned}$$

By (1.5) and (1.6), and Lemma 3.6,

$$I_{31} \le C \int_0^t \mathrm{e}^{-c(t-\tau)} \|u\omega\|_{L^2} \mathrm{d}\tau$$

$$
\begin{aligned}
&\leq C \int_0^t \mathrm{e}^{-c(t-\tau)} \|u\|_{L^\infty} \|\omega\|_{L^2} \mathrm{d}\tau \\
&\leq C\varepsilon^2 \int_0^t \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-\frac{1}{2}} \mathrm{d}\tau \\
&\leq C\varepsilon^2 (1+t)^{-\frac{1}{2}}.
\end{aligned}
$$

By the definition of S_{23}, for $\xi \in S_{23}$,

$$
|\xi| \leq C. \tag{3.9}
$$

By (1.6), (3.1), (3.8), (3.9), Lemma 3.5 and the simple fact that $(1+t)^{-a}\ln(1+t) \leq C(a)$ for any $a > 0$, we have

$$
\begin{aligned}
I_{32} &= C \int_0^t \left\| |\xi|^{-1} \mathrm{e}^{-c|\xi|^2(t-\tau)} \nabla \times (\widehat{\nabla \cdot (u \otimes u)}) \right\|_{L^2(S_{23})} \mathrm{d}\tau \\
&\leq C \int_0^t \left\| |\xi| \mathrm{e}^{-c|\xi|^2(t-\tau)} (|\widehat{uu_2}| + |\widehat{uu_1}|) \right\|_{L^2(S_{23})} \mathrm{d}\tau \\
&\leq C \int_0^t (1+t-\tau)^{-1} (\|u \otimes u\|_{L^2} + \|u \otimes u\|_{L^1}) \mathrm{d}\tau \\
&\leq C \int_0^t (1+t-\tau)^{-1} (\|u\|_{L^2} \|\nabla u\|_{L^2} + \|u\|_{L^2}^2) \mathrm{d}\tau \\
&\leq C \int_0^t (1+t-\tau)^{-1} (C_0\varepsilon^2(1+\tau)^{-1} + C_0^2\varepsilon^2(1+\tau)^{-1}) \mathrm{d}\tau \\
&\leq C(C_0 + C_0^2)\varepsilon^2 (1+t)^{-1} \ln(1+t) \\
&\leq C(C_0 + C_0^2)\varepsilon^2 (1+t)^{-\frac{1}{2}}.
\end{aligned}
$$

As in the estimate of I_3, we can bound I_4 by

$$
\begin{aligned}
I_4 &= \int_0^t \left\| |\xi|^{-1} \widehat{K_2}(t-\tau) \partial_1 \widehat{(u \cdot \nabla\theta)}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\leq C \int_0^t \mathrm{e}^{-c(t-\tau)} \left\| \widehat{u \cdot \nabla\theta}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\quad + C \int_0^t \left\| |\xi|^{-1} \mathrm{e}^{-c|\xi|^2(t-\tau)} \partial_1 \widehat{\nabla \cdot (u\theta)}(\tau) \right\|_{L^2(S_{23})} \mathrm{d}\tau \\
&\leq C \int_0^t \mathrm{e}^{-c(t-\tau)} \|u\|_{L^\infty} \|\nabla\theta\|_{L^2} \mathrm{d}\tau + C \int_0^t \left\| |\xi| \mathrm{e}^{-c|\xi|^2(t-\tau)} \widehat{u\theta}(\tau) \right\|_{L^2(S_{23})} \mathrm{d}\tau \\
&\leq C\varepsilon^2 \int_0^t \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-\frac{1}{2}} \mathrm{d}\tau + C \int_0^t (1+t-\tau)^{-1} (\|u\theta\|_{L^2} + \|u\theta\|_{L^1}) \mathrm{d}\tau \\
&\leq C\varepsilon^2 (1+t)^{-\frac{1}{2}} + C \int_0^t (1+t-\tau)^{-1} (C_0\varepsilon^2(1+\tau)^{-1} + C_0^2\varepsilon^2(1+\tau)^{-1}) \mathrm{d}\tau \\
&\leq C(C_0^2 + C_0 + 1)\varepsilon^2 (1+t)^{-\frac{1}{2}}.
\end{aligned}
$$

Collecting the bounds from I_1 to I_4 and inserting them in (3.6), we obtain, after applying Plancherel's theorem and (1.7),

$$\|u\|_{L^2} = \|\widehat{u}\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}} + C(C_0^2 + C_0 + 1)\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

The estimate for $\|\theta\|_{L^2}$ using (2.4) is very similar and we omit the details. Therefore,

$$\|(u,\theta)(t)\|_{L^2} = \|(\widehat{u},\widehat{\theta})\|_{L^2} \leq C_1\varepsilon(1+t)^{-\frac{1}{2}} + C_2(C_0^2 + C_0 + 1)\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

If we choose C_0 and ε satisfying

$$C_1 \leq \frac{C_0}{4}, \quad \varepsilon \leq \frac{C_0}{4C_2(C_0^2 + C_0 + 1)}, \tag{3.10}$$

then

$$\|(u,\theta)(t)\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}. \tag{3.11}$$

We now turn to $\|\partial_2 u\|_{L^2}$ and verify the upper bound for $\|\partial_2 u\|_{L^2}$ in (3.2). Applying ∂_2 to (3.5), we obtain

$$\partial_2 u = (-\partial_2^2\Delta^{-1}\omega, \partial_2\partial_1\Delta^{-1}\omega).$$

By the integral representation of $\widehat{\omega}$ in (2.3),

$$\begin{aligned}
\|\partial_2 u\|_{L^2} &= \|\widehat{\partial_2 u}\|_{L^2} = \left\| |\xi_2||\xi|^{-1}\widehat{\omega} \right\|_{L^2} \\
&\leq \left\| |\xi_2||\xi|^{-1}\widehat{K_1}(t)\widehat{\omega_0} \right\|_{L^2} + \left\| |\xi_2||\xi|^{-1}\widehat{K_2}(t)\widehat{\partial_1\theta_0} \right\|_{L^2} \\
&\quad + \int_0^t \left\| |\xi_2||\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{N_1}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\quad + \int_0^t \left\| |\xi_2||\xi|^{-1}\widehat{K_2}(t-\tau)\widehat{N_2}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.12}$$

By the upper bounds for $\widehat{K_1}$ in (2.6), (2.7), (2.8) and (2.9) and Lemma 3.4,

$$\begin{aligned}
J_1 &= \left\| |\xi_2||\xi|^{-1}\widehat{K_1}(t)\widehat{\omega_0} \right\|_{L^2(S_1\cup S_{21}\cup S_{22})} + \left\| |\xi_2||\xi|^{-1}\widehat{K_1}(t)\widehat{\omega_0} \right\|_{L^2(S_{23})} \\
&\leq C\mathrm{e}^{-ct}\|\omega_0\|_{L^2} + C\left\| |\xi|\mathrm{e}^{-c|\xi|^2 t}|\widehat{u_0}| \right\|_{L^2} \leq C(1+t)^{-1}\|u_0\|_{L^1\cap H^1}.
\end{aligned} \tag{3.13}$$

Similarly,

$$J_2 \leq C(1+t)^{-1}\|\theta_0\|_{L^1\cap H^1}. \tag{3.14}$$

By (2.6), (2.7), (2.8), (2.9), (3.7) and (3.8),

$$
\begin{aligned}
J_3 &= \int_0^t \left\| |\xi_2||\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&\le C\int_0^t \left\| |\xi_2| \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)}\widehat{u\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau + C\int_0^t \left\| \frac{|\xi_2|}{|1+\xi_2^2|}\mathrm{e}^{-c(t-\tau)}\widehat{u\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&\quad + C\int_0^t \left\| \mathrm{e}^{-c|\xi|^2(t-\tau)}\nabla\times(\widehat{\nabla\cdot(u\otimes u)})\right\|_{L^2(S_{23})} \mathrm{d}\tau \\
&:= J_{31}+J_{32}+J_{33}.
\end{aligned}
$$

We divide J_{31} into two parts:

$$
J_{31} := J_{311} + J_{312},
$$

where

$$
\begin{aligned}
J_{311} &= C\int_0^{\frac{t}{2}} \left\| |\xi_2| \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)}\widehat{u\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau, \\
J_{312} &= C\int_{\frac{t}{2}}^{t} \left\| |\xi_2| \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)}\widehat{u\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau.
\end{aligned}
$$

By (1.5) and the simple fact that $(1+x)\mathrm{e}^{-ax}\le C(a)$ for any constant $a>0$,

$$
\begin{aligned}
J_{311} &\le C\int_0^{\frac{t}{2}} \mathrm{e}^{-c(t-\tau)}\, \|\partial_2(u\omega)(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\le C\int_0^{\frac{t}{2}} \mathrm{e}^{-c(t-\tau)}\left(\|u\|_{L^\infty}\|\partial_2\omega\|_{L^2}+\|\partial_2 u\|_{L^4}\|\omega\|_{L^4}\right)\mathrm{d}\tau \\
&\le C\frac{t}{2}\mathrm{e}^{-\frac{t}{2}c}\varepsilon^2 \le C\varepsilon^2(1+t)^{-1}.
\end{aligned}
\tag{3.15}
$$

By Lemma 3.3, (1.6) and (3.1),

$$
\|u\|_{L^\infty} \le C\,\|u\|_{L^2}^{\frac14}\,\|\partial_1 u\|_{L^2}^{\frac14}\,\|\partial_2 u\|_{L^2}^{\frac14}\,\|\partial_1\partial_2 u\|_{L^2}^{\frac14} \le CC_0^{\frac14}\varepsilon(1+t)^{-\frac12}. \tag{3.16}
$$

By (1.6), (3.16) and

$$
\int_{\frac{t}{2}}^{t} \mathrm{e}^{-c(t-\tau)}(t-\tau)^{-\frac12}d\tau = \int_0^{\frac{t}{2}} \mathrm{e}^{-cs}s^{-\frac12}\mathrm{d}s \le C,
$$

we derive

$$
J_{312} \le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac12}\mathrm{e}^{-c(t-\tau)}\,\|u\omega\|_{L^2}\,\mathrm{d}\tau
$$

$$\leq CC_0^{\frac{1}{4}}\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-1} d\tau$$
$$\leq CC_0^{\frac{1}{4}}\varepsilon^2 (1+\frac{t}{2})^{-1} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} \mathrm{d}\tau \leq CC_0^{\frac{1}{4}}\varepsilon^2 (1+t)^{-1},$$

which, together with (3.15), yields

$$J_{31} \leq C\varepsilon^2(1+t)^{-1} + CC_0^{\frac{1}{4}}\varepsilon^2(1+t)^{-1}. \tag{3.17}$$

Next we proceed to estimate J_{32}. By Lemma 3.6, (1.6) and (3.16),

$$J_{32} = C\int_0^t \left\| \frac{|\xi_2|}{|1+\xi_2^2|} \mathrm{e}^{-c(t-\tau)} \widehat{u\omega}(\tau) \right\|_{L^2} \mathrm{d}\tau \leq C\int_0^t \mathrm{e}^{-c(t-\tau)} \|u\omega\|_{L^2} \mathrm{d}\tau$$
$$\leq CC_0^{\frac{1}{4}}\varepsilon^2 \int_0^t \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-1} \mathrm{d}\tau \leq CC_0^{\frac{1}{4}}\varepsilon^2(1+t)^{-1}. \tag{3.18}$$

By (3.9), the ansatz in (3.1), and Lemma 3.5,

$$J_{33} = C\int_0^t \left\| \mathrm{e}^{-c|\xi|^2(t-\tau)} \nabla\times(\widehat{\nabla\cdot(u\otimes u)}) \right\|_{L^2(S_{23})} \mathrm{d}\tau$$
$$\leq C\int_0^t \left\| |\xi|^2 \mathrm{e}^{-c|\xi|^2(t-\tau)} (|\widehat{uu_2}| + |\widehat{uu_1}|) \right\|_{L^2(S_{23})} \mathrm{d}\tau$$
$$\leq C\int_0^t (1+t-\tau)^{-\frac{3}{2}} (\|u\otimes u\|_{L^1} + \|u\otimes u\|_{L^2}) \mathrm{d}\tau$$
$$\leq C\int_0^t (1+t-\tau)^{-\frac{3}{2}} (\|u\|_{L^2}^2 + \|u\|_{L^2}\|\nabla u\|_{L^2}) \mathrm{d}\tau$$
$$\leq C\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-1} \mathrm{d}\tau \leq C\varepsilon^2(1+t)^{-1}. \tag{3.19}$$

Collecting (3.17), (3.18) and (3.19) leads to

$$J_3 \leq C\varepsilon^2(1+t)^{-1} + CC_0^{\frac{1}{4}}\varepsilon^2(1+t)^{-1}. \tag{3.20}$$

By the definition of S_1, S_{22} and S_{23}, there exists a constant C such that, for any $\xi \in S_1 \cup S_{22} \cup S_{23}$,

$$|\xi_2| \leq C. \tag{3.21}$$

By (1.6), (3.9), (3.16), (3.21) and the ansatz in (3.1),

$$J_4 = \int_0^t \left\| |\xi_2||\xi|^{-1} \widehat{K_2}(t-\tau) \partial_1 \widehat{(u\cdot\nabla\theta)}(\tau) \right\|_{L^2} \mathrm{d}\tau$$
$$\leq C\int_0^t \mathrm{e}^{-c(t-\tau)} \left\| \widehat{u\cdot\nabla\theta}(\tau) \right\|_{L^2} \mathrm{d}\tau + C\int_0^t \left\| |\xi|^2 \mathrm{e}^{-c|\xi|^2(t-\tau)} \widehat{u\theta}(\tau) \right\|_{L^2(S_{23})} \mathrm{d}\tau$$

$$
\begin{aligned}
&\leq C\int_0^t \mathrm{e}^{-c(t-\tau)}\,\|u\|_{L^\infty}\,\|\nabla\theta\|_{L^2}\mathrm{d}\tau + C\int_0^t (1+t-\tau)^{-\frac{3}{2}}(\|u\theta\|_{L^2}+\|u\theta\|_{L^1})\mathrm{d}\tau\\
&\leq CC_0^{\frac{1}{4}}\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-1}\mathrm{d}\tau + C\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}\mathrm{d}\tau\\
&\leq C\varepsilon^2(1+t)^{-1} + CC_0^{\frac{1}{4}}\varepsilon^2(1+t)^{-1}. \qquad (3.22)
\end{aligned}
$$

Inserting the upper bounds (3.13), (3.14), (3.20) and (3.22) in (3.12), we obtain

$$\|\partial_2 u\|_{L^2} \leq C_3\varepsilon(1+t)^{-1} + C_4(C_0^{\frac{1}{4}}+1)\varepsilon^2(1+t)^{-1}.$$

By choosing C_0 and ε satisfying

$$C_3 \leq \frac{C_0}{4}, \quad \varepsilon \leq \frac{C_0}{4C_4(C_0^{\frac{1}{4}}+1)}, \qquad (3.23)$$

we have

$$\|\partial_2 u\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1}. \qquad (3.24)$$

Next we estimate $\|\partial_1 u\|_{L^2}$. By the integral representation of $\widehat{\omega}$ in (2.3) and the Biot–Savart law in (3.5),

$$
\begin{aligned}
\|\partial_1 u\|_{L^2} = \|\widehat{\partial_1 u}\|_{L^2} &= \left\| |\xi_1||\xi|^{-1}\widehat{\omega}\right\|_{L^2}\\
&\leq \left\| |\xi_1||\xi|^{-1}\widehat{K_1}(t)\widehat{\omega_0}\right\|_{L^2} + \left\| |\xi_1||\xi|^{-1}\widehat{K_2}(t)\widehat{\partial_1\theta_0}\right\|_{L^2}\\
&\quad + \int_0^t \left\| |\xi_1||\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\quad + \int_0^t \left\| |\xi_1||\xi|^{-1}\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&:= L_1 + L_2 + L_3 + L_4. \qquad (3.25)
\end{aligned}
$$

L_1 and L_2 obey similar bounds as those for J_1 and J_2 in (3.13) and (3.14),

$$L_1 + L_2 \leq C(1+t)^{-1}\|(u_0,\theta_0)\|_{L^1\cap H^1}. \qquad (3.26)$$

By the ansatz in (3.1) and Lemma 3.3,

$$\|u\|_{L^\infty} \leq C\,\|u\|_{L^2}^{\frac{1}{4}}\,\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\,\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\,\|\partial_1\partial_2 u\|_{L^2}^{\frac{1}{4}} \leq CC_0^{\frac{1}{2}}\varepsilon(1+t)^{-\frac{3}{4}}. \qquad (3.27)$$

By (1.5), (3.3), (3.4), (3.27), and Lemma 3.6,

$$L_3 = \int_0^t \left\| |\xi_1||\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla\omega(\tau)}\right\|_{L^2}\mathrm{d}\tau$$

$$
\begin{aligned}
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\left\|\widehat{u\cdot\nabla\omega(\tau)}\right\|_{L^2}\mathrm{d}\tau + J_{33}\\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\mathrm{d}\tau + C\varepsilon^2(1+t)^{-1}\\
&\le CC_0^{\frac12}\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac34}\mathrm{d}\tau + C\varepsilon^2(1+t)^{-1}\\
&\le C(C_0^{\frac12}+1)\varepsilon^2(1+t)^{-\frac34}.
\end{aligned}
\tag{3.28}
$$

By the ansatz in (3.1), (1.5) and Lemma 3.2,

$$
\|\partial_1 u\cdot\nabla\theta\|_{L^2}\le C\,\|\partial_1 u\|_{L^2}^{\frac12}\,\|\partial_2\partial_1 u\|_{L^2}^{\frac12}\,\|\nabla\theta\|_{L^2}^{\frac12}\,\|\partial_1\nabla\theta\|_{L^2}^{\frac12}\le CC_0\varepsilon^2(1+t)^{-\frac34}.
\tag{3.29}
$$

By the ansatz in (3.1), (1.5), (1.6), (3.3), (3.4), (3.27), (3.29), Lemmas 3.5 and 3.6, we have

$$
\begin{aligned}
L_4 &= \int_0^t \left\||\xi_1||\xi|^{-1}\widehat{K_2}(t-\tau)\partial_1\widehat{(u\cdot\nabla\theta)}(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\left\|\partial_1\widehat{(u\cdot\nabla\theta)}(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\quad + C\int_0^t \left\||\mathrm{e}^{-c|\xi|^2(t-\tau)}\partial_1\widehat{\nabla\cdot(u\theta)}(\tau)\right\|_{L^2(S_{23})}\mathrm{d}\tau\\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\left(\|u\|_{L^\infty}\|\partial_1\nabla\theta\|_{L^2}+\|\partial_1 u\cdot\nabla\theta\|_{L^2}\right)\mathrm{d}\tau\\
&\quad + C\int_0^t \left\||\xi|^2|\mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\theta}(\tau)\right\|_{L^2(S_{23})}\mathrm{d}\tau\\
&\le C(C_0^{\frac12}+C_0)\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac34}\mathrm{d}\tau\\
&\quad + C\int_0^t (1+t-\tau)^{-\frac32}(\|u\theta\|_{L^1}+\|u\theta\|_{L^2})\mathrm{d}\tau\\
&\le C(C_0^{\frac12}+C_0)\varepsilon^2(1+t)^{-\frac34} + C\varepsilon^2\int_0^t (1+t-\tau)^{-\frac32}(1+\tau)^{-1}\mathrm{d}\tau\\
&\le C(C_0^{\frac12}+C_0+1)\varepsilon^2(1+t)^{-\frac34}.
\end{aligned}
\tag{3.30}
$$

Inserting (3.26), (3.28) and (3.30) in (3.25) yields

$$
\|\partial_1 u\|_{L^2}\le C\varepsilon(1+t)^{-\frac34}+C(C_0^{\frac12}+C_0+1)\varepsilon^2(1+t)^{-\frac34}.
\tag{3.31}
$$

We now tend to $\|\partial_2\nabla u\|_{L^2}$. By the integral representation of $\widehat{\omega}$ in (2.3),

$$
\|\partial_2\nabla u\|_{L^2} = \|\widehat{\partial_2\omega}\|_{L^2} = \||\xi_2|\widehat{\omega}\|_{L^2}
$$

$$
\begin{aligned}
&\le \left\| |\xi_2| \widehat{K_1}(t) \widehat{\omega_0} \right\|_{L^2} + \left\| |\xi_2| \widehat{K_2}(t) \widehat{\partial_1 \theta_0} \right\|_{L^2} \\
&\quad + \int_0^t \left\| |\xi_2| \widehat{K_1}(t-\tau) \widehat{N_1}(\tau) \right\|_{L^2} \mathrm{d}\tau + \int_0^t \left\| |\xi_2| \widehat{K_2}(t-\tau) \widehat{N_2}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned} \tag{3.32}
$$

According to Proposition 2.1,

$$
\begin{aligned}
M_1 &= \left\| |\xi_2| \widehat{K_1}(t) \widehat{\omega_0} \right\|_{L^2} \\
&\le C \mathrm{e}^{-ct} \| \widehat{\partial_2 \omega_0} \|_{L^2} + C \left\| |\xi| \mathrm{e}^{-c|\xi|^2 t} |\widehat{\omega_0}| \right\|_{L^2} \\
&\le C \mathrm{e}^{-ct} \| \partial_2 \omega_0 \|_{L^2} + C (1+t)^{-\frac{3}{2}} \| u_0 \|_{L^1 \cap H^2} \\
&\le C (1+t)^{-\frac{3}{2}} \| u_0 \|_{L^1 \cap H^2}.
\end{aligned} \tag{3.33}
$$

Similarly,

$$
M_2 \le C (1+t)^{-\frac{3}{2}} \| \theta_0 \|_{L^1 \cap H^2}. \tag{3.34}
$$

By Proposition 2.1,

$$
\begin{aligned}
M_3 &= \int_0^t \left\| |\xi_2| \widehat{K_1}(t-\tau) \widehat{u \cdot \nabla \omega}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\le C \int_0^t \left\| |\xi_2| \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)} \widehat{u \cdot \nabla \omega}(\tau) \right\|_{L^2} \mathrm{d}\tau + C \int_0^t \left\| \mathrm{e}^{-c(t-\tau)} \widehat{u \cdot \nabla \omega}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\quad + C \int_0^t \left\| |\xi_2| \mathrm{e}^{-c|\xi|^2 (t-\tau)} \widehat{u \cdot \nabla \omega} \right\|_{L^2(S_{23})} \mathrm{d}\tau := M_{31} + M_{32} + M_{33}.
\end{aligned}
$$

As in the estimate of J_{31},

$$
\begin{aligned}
M_{31} &= C \int_0^t \left\| |\xi_2| \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)} \widehat{u \cdot \nabla \omega}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} \| u \|_{L^\infty} \| \nabla \omega \|_{L^2} \mathrm{d}\tau \\
&\le C \varepsilon^2 \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} \mathrm{d}\tau + C C_0^{\frac{1}{2}} \varepsilon^2 \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-\frac{3}{4}} d\tau \\
&\le C \left(\frac{t}{2} \right)^{\frac{1}{2}} \mathrm{e}^{-\frac{c}{2} t} \varepsilon^2 + C C_0^{\frac{1}{2}} \varepsilon^2 (1 + \frac{t}{2})^{-\frac{3}{4}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \mathrm{e}^{-c(t-\tau)} \mathrm{d}\tau \\
&\le C \varepsilon^2 (1+t)^{-\frac{3}{4}} + C C_0^{\frac{1}{2}} \varepsilon^2 (1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{3.35}
$$

For M_{32}, it is easy to conclude from (1.5) and (3.27) that

$$\begin{aligned}M_{32} &= C\int_0^t \left\|\mathrm{e}^{-c(t-\tau)}\widehat{u\cdot\nabla\omega}(\tau)\right\|_{L^2}\mathrm{d}\tau \le C\int_0^t \mathrm{e}^{-c(t-\tau)}\left\|u\cdot\nabla\omega\right\|_{L^2}\mathrm{d}\tau\\ &\le CC_0^{\frac{1}{2}}\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \le CC_0^{\frac{1}{2}}\varepsilon^2(1+t)^{-\frac{3}{4}}. \end{aligned}\tag{3.36}$$

By (1.6), (3.7) and (3.9), and the ansatz (3.1),

$$\begin{aligned}M_{33} &= C\int_0^t \left\||\xi_2|\mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\cdot\nabla\omega}\right\|_{L^2(S_{23})}\mathrm{d}\tau\\ &\le C\int_0^t \left\||\xi|^2\mathrm{e}^{-c|\xi|^2(t-\tau)}|\widehat{u\omega}|\right\|_{L^2(S_{23})}\mathrm{d}\tau\\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}(\|u\omega\|_{L^1}+\|u\omega\|_{L^2})\mathrm{d}\tau\\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}(\|u\|_{L^2}\|\omega\|_{L^2}+\|u\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}})\mathrm{d}\tau\\ &\le C\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \le C\varepsilon^2(1+t)^{-\frac{3}{4}}. \end{aligned}\tag{3.37}$$

Collecting (3.35), (3.36) and (3.37) yields

$$M_3 \le C(1+C_0^{\frac{1}{2}})\varepsilon^2(1+t)^{-\frac{3}{4}}. \tag{3.38}$$

Invoking (1.5), (1.6), (3.1), (3.9), (3.21), (3.27) and (3.29), we have

$$\begin{aligned}M_4 &= \int_0^t \left\||\xi_2|\widehat{K_2}(t-\tau)\partial_1\widehat{(u\cdot\nabla\theta)}(\tau)\right\|_{L^2}\mathrm{d}\tau\\ &\le C\int_0^t \left\|\mathrm{e}^{-c(t-\tau)}\partial_1\widehat{(u\cdot\nabla\theta)}\right\|_{L^2}\mathrm{d}\tau + C\int_0^t \left\||\xi|^2\mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\cdot\nabla\theta}\right\|_{L^2(S_{23})}\mathrm{d}\tau\\ &\le C\int_0^t \mathrm{e}^{-c(t-\tau)}(\|u\|_{L^\infty}\|\partial_1\nabla\theta\|_{L^2}+\|\partial_1 u\cdot\nabla\theta\|_{L^2})\mathrm{d}\tau\\ &\quad + C\int_0^t (1+t-\tau)^{-\frac{3}{2}}(\|u\cdot\nabla\theta\|_{L^1}+\|u\cdot\nabla\theta\|_{L^2})\mathrm{d}\tau\\ &\le C(C_0+C_0^{\frac{1}{2}})\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\quad + C\int_0^t (1+t-\tau)^{-\frac{3}{2}}(\|u\|_{L^2}\|\nabla\theta\|_{L^2}+\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}})\mathrm{d}\tau\\ &\le C(C_0+C_0^{\frac{1}{2}})\varepsilon^2(1+t)^{-\frac{3}{4}} + C\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\le C(C_0+C_0^{\frac{1}{2}}+1)\varepsilon^2(1+t)^{-\frac{3}{4}}. \end{aligned}\tag{3.39}$$

Inserting the uppers (3.33), (3.34), (3.38) and (3.39) in (3.32) leads to

$$\|\partial_2 \nabla u\|_{L^2} \le C\varepsilon(1+t)^{-\frac{3}{4}} + C(C_0 + C_0^{\frac{1}{2}} + 1)\varepsilon^2(1+t)^{-\frac{3}{4}}. \tag{3.40}$$

By (3.31) and (3.40), we get

$$\|\partial_1 u\|_{L^2} + \|\partial_2 \nabla u\|_{L^2} \le C_5\varepsilon(1+t)^{-\frac{3}{4}} + C_6(C_0 + C_0^{\frac{1}{2}} + 1)\varepsilon^2(1+t)^{-\frac{3}{4}}.$$

If C_0 and ε satisfy

$$C_5 \le \frac{C_0}{4}, \quad \varepsilon \le \frac{C_0}{4C_6(C_0 + C_0^{\frac{1}{2}} + 1)}, \tag{3.41}$$

then

$$\|\partial_1 u\|_{L^2} + \|\partial_2 \nabla u\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{3}{4}}. \tag{3.42}$$

In summary, if C_0 and ε satisfy (3.10), (3.23) and (3.41), then (u, θ) obeys (3.11), (3.24) and (3.42), namely (3.2). The bootstrapping argument then assesses the desired decay estimates.

3.2 Faster Decay Rate for $\|\nabla\theta\|_{L^2}$

By making use of (3.2), we are able to derive a faster decay rate for $\|\nabla\theta\|_{L^2}$ than the one in Theorem 1.1. Using the integral form for $\widehat{\partial_1\theta}$ in (2.4) and the fact that $\widehat{K_3} = -\frac{\xi_1^2}{|\xi|^2}\widehat{K_2}$ in (2.5), we obtain

$$\begin{aligned}
\|\nabla\theta\|_{L^2} &= \||\xi|\widehat{\theta}\|_{L^2} = \||\xi||\xi_1|^{-1}\widehat{\partial_1\theta}\|_{L^2} \\
&\le \||\xi_1||\xi|^{-1}\widehat{K_2}\widehat{\omega_0}\|_{L^2} + \||\xi||\xi_1|^{-1}\widehat{K_4}\widehat{\partial_1\theta_0}\|_{L^2} \\
&\quad + \int_0^t \left\| |\xi_1||\xi|^{-1}\widehat{K_2}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&\quad + \int_0^t \left\| |\xi||\xi_1|^{-1}\widehat{K_4}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&:= Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned} \tag{3.43}$$

We now estimate Q_1 through Q_4. Q_1 and Q_2 can be estimated similarly as J_1 and J_2 in (3.13) and (3.14),

$$Q_1 + Q_2 \le C(1+t)^{-1}\|(u_0, \theta_0)\|_{L^1 \cap H^1} \le C\varepsilon(1+t)^{-1}. \tag{3.44}$$

Thanks to (3.2), (3.3), (3.4), (3.8) and (3.27), we obtain

$$
\begin{aligned}
Q_3 &= \int_0^t \left\| |\xi_1||\xi|^{-1}\widehat{K_2}(t-\tau)(\widehat{u\cdot\nabla\omega})(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\|u\cdot\nabla\omega\|_{L^2}\mathrm{d}\tau \\
&\quad + C\int_0^t \left\|\mathrm{e}^{-c|\xi|^2(t-\tau)}\nabla\times(\widehat{\nabla\cdot(u}\otimes u))\right\|_{L^2(S_{23})}\mathrm{d}\tau \\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\mathrm{d}\tau + J_{33} \\
&\le C\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau + +C\varepsilon^2(1+t)^{-1} \le C\varepsilon^2(1+t)^{-\frac{3}{4}}. \qquad (3.45)
\end{aligned}
$$

For Q_4, applying the upper bound for $\widehat{K_4}$ in (3.3) and (3.4), we have

$$
\begin{aligned}
Q_4 &\le \int_0^t \left\| |\xi|\widehat{K_4}(t-\tau)\widehat{u\cdot\nabla\theta}(\tau)\right\|_{L^2}\mathrm{d}\tau \\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}\||\xi|\widehat{u\cdot\nabla\theta}\|_{L^2}\mathrm{d}\tau + C\int_0^t \||\xi|\mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\cdot\nabla\theta}\|_{L^2(S_{23})}\mathrm{d}\tau \\
&:= Q_{41} + Q_{42}.
\end{aligned}
$$

By the decay rate in (3.2), the uniform bound in (1.5), Lemmas 3.2 and 3.3 , we have

$$
\begin{aligned}
\||\xi|\widehat{u\cdot\nabla\theta}\|_{L^2} = \|\nabla(u\cdot\nabla\theta)\|_{L^2} &\le \|\nabla u\cdot\nabla\theta\|_{L^2} + \|u\cdot\nabla^2\theta\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_{12}u\|_{L^2}^{\frac{1}{4}}\|\nabla^2\theta\|_{L^2} \le C\varepsilon^2(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

which, together with Lemma 3.6, implies

$$
Q_{41} \le C\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \le C\varepsilon^2(1+t)^{-\frac{3}{4}}.
$$

By Lemma 3.4, (3.2) and (3.9),

$$
\begin{aligned}
Q_{42} &\le C\int_0^t \||\xi|^2\mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\theta}(\tau)\|_{L^2(S_{23})}\mathrm{d}\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\left(\|u\theta\|_{L^1} + \|u\theta\|_{L^2}\right)\mathrm{d}\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\left(\|u\|_{L^2}\|\theta\|_{L^2} + \|u\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\right)\mathrm{d}\tau \\
&\le C\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}\mathrm{d}\tau \le C\varepsilon^2(1+t)^{-1}.
\end{aligned}
$$

Therefore,

$$Q_4 \leq C\varepsilon^2(1+t)^{-\frac{3}{4}}. \tag{3.46}$$

Inserting (3.44), (3.45) and (3.46) in (3.43) yields

$$\|\nabla\theta\|_{L^2} \leq C_7\varepsilon(1+t)^{-\frac{3}{4}} + C_8\varepsilon^2(1+t)^{-\frac{3}{4}} \leq 2C_7\varepsilon(1+t)^{-\frac{3}{4}} \tag{3.47}$$

if ε is small. This completes the decay estimate for $\|\nabla\theta\|_{L^2}$.

3.3 Optimal Decay Rate for $\|\partial_{22}u\|_{L^2}$

We now turn to the estimate for $\|\partial_{22}u(t)\|_{L^2}$. Applying ∂_{22} to (3.5) yields

$$\partial_{22}u = (-\partial_2^3\Delta^{-1}\omega, \partial_2^2\partial_1\Delta^{-1}\omega).$$

By the integral representation for $\widehat{\omega}$ in (2.3),

$$\begin{aligned}
\|\partial_{22}u\|_{L^2} = \|\widehat{\partial_{22}u}\|_{L^2} &= \left\||\xi_2|^2|\xi|^{-1}\widehat{\omega}\right\|_{L^2} \\
&\leq \left\||\xi_2|^2|\xi|^{-1}\widehat{K_1}\widehat{\omega_0}\right\|_{L^2} + \left\||\xi_2|^2|\xi|^{-1}\widehat{K_2}\widehat{\partial_1\theta_0}\right\|_{L^2} \\
&\quad + \int_0^t \left\||\xi_2|^2|\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&\quad + \int_0^t \left\||\xi_2|^2|\xi|^{-1}\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} \mathrm{d}\tau \\
&:= R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

Invoking the bounds for $\widehat{K_1}$ in (2.6), (2.7), (2.8) and (2.9), and Lemma 3.4, we have

$$\begin{aligned}
R_1 &= \||\xi_2|^2|\xi|^{-1}\widehat{K_1}\widehat{\omega_0}\|_{L^2} \\
&\leq C\mathrm{e}^{-ct}\|\xi_2^2\widehat{u_0}\|_{L^2} + C\||\xi|^2\mathrm{e}^{-c|\xi|^2t}\widehat{u_0}\|_{L^2} \\
&\leq C\mathrm{e}^{-ct}\|u_0\|_{H^2} + (1+t)^{-\frac{3}{2}}\|u_0\|_{L^1\cap H^2} \leq C\varepsilon(1+t)^{-\frac{5}{4}}.
\end{aligned}$$

Similarly

$$R_2 \leq C\varepsilon(1+t)^{-\frac{5}{4}}.$$

By (3.7),

$$R_3 = \int_0^t \left\||\xi_2|^2|\xi|^{-1}\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla\omega}(\tau)\right\|_{L^2} \mathrm{d}\tau$$

$$\le C\int_0^t \||\xi_2|^2 \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)}\widehat{u\omega}(\tau)\|_{L^2}\mathrm{d}\tau + C\int_0^t \|\mathrm{e}^{-c(t-\tau)}\widehat{u\omega}(\tau)\|_{L^2(S_{21})}\mathrm{d}\tau$$
$$+ C\int_0^t \||\xi|^2 \mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\omega}(\tau)\|_{L^2(S_{23})}\mathrm{d}\tau$$
$$:= R_{31} + R_{32} + R_{33}.$$

Using the bounds (3.2), we have

$$\begin{aligned}
\|\partial_2(u\omega)(t)\|_{L^2} &\le \|\partial_2 u\|_{L^4}\|\omega\|_{L^4} + \|u\|_{L^\infty}\|\partial_2\omega\|_{L^2}\\
&\le C\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_{12} u\|_{L^2}^{\frac{1}{4}}\|\partial_2\omega\|_{L^2}\\
&\le C\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{3.48}$$

In addition, due to the uniform bound in (1.5),

$$\|\partial_2(u\omega)(t)\|_{L^2} \le C\varepsilon^2. \tag{3.49}$$

(3.48), (3.49) and the simple fact that $x^{\frac{1}{2}}\mathrm{e}^{-x} \le C$ imply

$$\begin{aligned}
R_{31} &= C\int_0^t \||\xi_2|^2 \mathrm{e}^{-c(1+\xi_2^2)(t-\tau)}\widehat{u\omega}(\tau)\|_{L^2}\mathrm{d}\tau\\
&\le C\int_0^t \mathrm{e}^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_2(u\omega)(\tau)\|_{L^2}\mathrm{d}\tau\\
&\le C\int_0^{\frac{t}{2}} \mathrm{e}^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\varepsilon^2 d\tau + C\varepsilon^2\int_{\frac{t}{2}}^t \mathrm{e}^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}d\tau\\
&\le C\varepsilon^2\left(\frac{t}{2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{c}{2}t} + C\varepsilon^2\left(1+\frac{t}{2}\right)^{-\frac{5}{4}}\int_{\frac{t}{2}}^t \mathrm{e}^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}d\tau\\
&\le C\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{3.50}$$

By (3.2) and (3.27),

$$\begin{aligned}
R_{32} &= C\int_0^t \mathrm{e}^{-c(t-\tau)}\|u\omega(\tau)\|_{L^2}\mathrm{d}\tau \le C\int_0^t \mathrm{e}^{-c(t-\tau)}\|u\|_{L^\infty}\|\omega\|_{L^2}\mathrm{d}\tau\\
&\le C\varepsilon^2\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau \le C\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{3.51}$$

By (3.2), (3.27), Lemmas 3.4 and 3.5,

$$R_{33} = C\int_0^t \||\xi|^2 \mathrm{e}^{-c|\xi|^2(t-\tau)}\widehat{u\omega}(\tau)\|_{L^2(S_{23})}\mathrm{d}\tau$$

$$
\begin{aligned}
&\le C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left(\|u\omega\|_{L^1} + \|u\omega\|_{L^2}\right) \mathrm{d}\tau \\
&\le C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left(\|u\|_{L^2}\|\omega\|_{L^2} + \|u\|_{L^\infty}\|\omega\|_{L^2}\right) \mathrm{d}\tau \\
&\le C\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left((1+\tau)^{-\frac{5}{4}} + (1+\tau)^{-\frac{3}{2}}\right) \mathrm{d}\tau \\
&\le C\varepsilon^2 (1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{3.52}
$$

Combining (3.50), (3.51) and (3.52) yields

$$
R_3 \le C\varepsilon^2(1+t)^{-\frac{5}{4}}.
$$

By (3.21), the upper bounds for $\widehat{K_2}$ in (2.6), (2.7), (2.8) and (2.9), and (3.2), (3.27) and (3.47), we obtain

$$
\begin{aligned}
R_4 &= \int_0^t \left\| |\xi_2|^2 |\xi|^{-1} \widehat{K_2}(t-\tau) \partial_1 \widehat{(u\cdot\nabla\theta)}(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\le C \int_0^t \|\mathrm{e}^{-c(t-\tau)} \widehat{u\cdot\nabla\theta}(\tau)\|_{L^2} \mathrm{d}\tau \\
&\quad + C \int_0^t \||\xi|^2 \mathrm{e}^{-c|\xi|^2(t-\tau)} \widehat{u\cdot\nabla\theta}(\tau)\|_{L^2(S_{23})} \mathrm{d}\tau \\
&\le C \int_0^t \mathrm{e}^{-c(t-\tau)} \|u\|_{L^\infty} \|\nabla\theta\|_{L^2} \mathrm{d}\tau \\
&\quad + C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left(\|u\cdot\nabla\theta\|_{L^1} + \|u\cdot\nabla\theta\|_{L^2}\right) \mathrm{d}\tau \\
&\le C\varepsilon^2 \int_0^t \mathrm{e}^{-c(t-\tau)} (1+\tau)^{-\frac{3}{2}} \mathrm{d}\tau \\
&\quad + C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left(\|u\|_{L^2}\|\nabla\theta\|_{L^2} + \|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\right) \mathrm{d}\tau \\
&\le C\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} \left((1+\tau)^{-\frac{5}{4}} + (1+\tau)^{-\frac{3}{2}}\right) \mathrm{d}\tau \\
&\le C\varepsilon^2 (1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Collecting the upper bounds for R_1 through R_4, we obtain

$$
\|\partial_{22}u\|_{L^2} \le C_9\varepsilon(1+t)^{-\frac{5}{4}} + C_{10}\varepsilon^2(1+t)^{-\frac{5}{4}} \le 2C_9\varepsilon(1+t)^{-\frac{5}{4}}
$$

if ε is sufficiently small. This finishes the decay estimate for $\|\partial_{22}u\|_{L^2}$ and thus, the proof of Theorem 1.2.

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