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STABILIZING PHENOMENON FOR 2D ANISOTROPIC MAGNETOHYDRODYNAMIC SYSTEM NEAR A BACKGROUND MAGNETIC FIELD*

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Abstract. This paper intends to study an experimentally observed stabilizing phenomenon and to prove a mathematically rigorous stability result on the perturbations near a background magnetic field. Physical experiments and numerical simulations have observed a remarkable phenomenon that a background magnetic field can smooth and stabilize the electrically conducting turbulent fluids. To understand the mechanism of this phenomenon, we focus on a special 2D magnetohydrodynamic (MHD) system with anisotropic dissipation and partial damping and examine the stability near a background magnetic field. Due to the lack of full dissipation and damping, this stability problem is not trivial. Without the presence of a magnetic field, the fluid velocity is governed by the 2D Navier–Stokes equations in the whole space $\mathbb{R}^2$ with only vertical dissipation, and its stability (near the trivial solution) is still an open problem. However, when coupled to the magnetic field in such an MHD system, we are able to show that any perturbation near a background magnetic field is globally stable in Sobolev space H^2. This result reflects the observed stabilizing effect of the magnetic field. Mathematically, the MHD system obeyed by the perturbations can be converted to a system of wave equations which exhibits extra smoothing and stabilizing properties. These properties allow us to control the nonlinearity in the anisotropic Navier–Stokes equations and thus establish the desired stability result.

Key words. 2D MHD equations, background magnetic field, partial dissipation, stability

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1. Introduction.

An important issue in magnetohydrodynamic (MHD) turbulence is to understand the influence of magnetic field on the bulk turbulence involving various electrically conducting fluids such as liquid metals. Physical experiments and numerical simulations have observed a remarkable phenomenon that a background magnetic field can smooth and stabilize electrically conducting turbulent fluids (see, e.g., [2, 3, 7, 11, 12, 13, 16, 17]).

Our goal is to understand the mechanism and to establish the mathematically rigorous stability results for these observations in incompressible MHD flows. Attention here is focused on the following 2D MHD system with only vertical velocity

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dissipation,

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P = \nu\partial_2^2 U + B\cdot\nabla B, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t B + U\cdot\nabla B + \eta(0,B_2)^\top = B\cdot\nabla U,\\ \nabla\cdot U = \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

where $U=(U_1,U_2)^\top$ denotes the velocity field, and $B=(B_1,B_2)^\top$ the magnetic field, and P the total pressure and $\nu>0$ and $\eta>0$ are the viscosity and damping coefficients, respectively. Here $A^\top$ denotes the transpose of a matrix A.

The MHD system governs the motion of electrically conducting fluids such as plasmas, liquid, metals and electrolytes and have a very wide range of applications in astrophysics, geophysics, cosmology, and engineering (see, e.g., [5, 13, 28]). The MHD system is also mathematically important. It not only shares many crucial features with the Euler or the Navier–Stokes equations but also exhibits many more fascinating properties resulting from the coupling and interaction between the velocity and the magnetic field. In fact, the main result of this paper is about the smoothing and stabilizing effect of the magnetic field on the fluid motion. Without the coupling and interaction with the magnetic field, the wave structure and the stabilizing phenomenon associated with (1.1) seem not possible.

To understand the precise mechanism of the observed stabilizing phenomenon, we study the evolution of the perturbation near a background magnetic field. More precisely, we consider the perturbation near the steady-state solution $(U^{(0)},B^{(0)})$ associated with a background magnetic field in the x_1-direction, namely,

$$U^{(0)}\equiv 0,\quad B^{(0)}\equiv e_1 := (1,0).$$

The perturbation of (U,B) around $(U^{(0)},B^{(0)})$ is given by

$$u := U - U^{(0)},\qquad b := B - B^{(0)}.$$

Clearly,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = \nu\partial_2^2 u + b\cdot\nabla b + \partial_1 b, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t b + u\cdot\nabla b + \eta(0,b_2)^\top = b\cdot\nabla u + \partial_1 u,\\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{1.2}$$

Our attention will be focused on the initial-value problem of (1.2) supplemented with the Cauchy data:

$$u(x,0) = u_0(x),\ \ b(x,0) = b_0(x). \tag{1.3}$$

Compared with the original MHD system (1.1), the perturbation near the background magnetic field generates two extra terms in (1.2), $\partial_1 b$ and $\partial_1 u$ in the equations of u and b, respectively. These two terms play a crucial role in the stability theory considered in the present paper.

Due to the lack of full velocity dissipation and the lack of full damping of the magnetic field, the stability problem seems not trivial. Indeed, if there is an absence of a magnetic field, then (1.1) becomes the 2D anisotropic Navier–Stokes system,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = \nu\partial_2^2 u, \quad x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, \end{cases}$$

which can also be reformulated in terms of the vorticity $\omega := \nabla \times u$ as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_2^2 \omega, \quad x \in \mathbb{R}^2,\ t > 0, \\ u = \nabla^\perp \Delta^{-1} \omega := (-\partial_2, \partial_1)\Delta^{-1}\omega. \end{cases} \tag{1.4}$$

The vorticity ω of (1.4) is bounded a priori for all time, and Yudovich's approach for the 2D incompressible Euler equation can be applied to (1.4) to show the global regularity of the solutions. However, the vorticity may potentially grow rather rapidly in time since the one-directional dissipation is not sufficient to control the nonlinearity. The difficulty is essential when we attempt to estimate $\|\nabla\omega\|_{L^2}$. Indeed,

$$\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + 2\nu\|\partial_2\nabla\omega(t)\|_{L^2}^2 = -2\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx,$$

where the nonlinear part contains four component terms:

$$\begin{aligned} \text{Hard} := &-\int_{\mathbb{R}^2} \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx \\ = &-\int_{\mathbb{R}^2} \partial_1 u_1\,(\partial_1\omega)^2\,dx - \int_{\mathbb{R}^2} \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\ &-\int_{\mathbb{R}^2} \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx - \int_{\mathbb{R}^2} \partial_2 u_2\,(\partial_2\omega)^2\,dx. \end{aligned} \tag{1.5}$$

The first two terms in (1.5) do not admit a time-integrable upper bound. It appears that the best upper bound for $\|\nabla\omega(t)\|_{L^q}$ with $1 \le q \le \infty$ is double exponential in time,

$$\|\nabla\omega(t)\|_{L^q} \le (\|\nabla\omega_0\|_{L^q})^{e^{C\|\omega_0\|_{L^\infty} t}},$$

where ω_0 is the initial vorticity. The double exponential growth of the gradient of vorticity to the 2D incompressible Euler equation was confirmed by Kiselev and Sverak [22] for the unit disk domain. This particularly explains why we could not expect the solutions of the anisotropic 2D Navier–Stokes equations to be stable near the trivial solution. Since the classical approaches used to study the MHD well-posedness and stability problems generally treat the Lorentz forcing term as a bad term, the stability problem concerned here then appears to be nontrivial.

The new idea of the present paper is to treat the terms associated with the magnetic field as good terms and exploit the smoothing and stabilizing effects of the background magnetic field on the fluid motion. Due to the two extra terms generated by the background magnetic field and the coupling in (1.2), we are able to convert (1.2) into a system of wave equations,

$$\begin{cases} \partial_{tt}u_1 - \nu\partial_2^2\partial_t u_1 - \partial_1^2 u_1 = N_1 \\ \partial_{tt}u_2 + \left(\eta - \nu\partial_2^2\right)\partial_t u_2 - \left(\nu\eta\partial_2^2 u_2 + \partial_1^2 u_2\right) = N_2, \\ \partial_{tt}b_1 - \nu\partial_2^2\partial_t b_1 - \partial_1^2 b_1 = N_3, \\ \partial_{tt}b_2 + \left(\eta - \nu\partial_2^2\right)\partial_t b_2 - \left(\nu\eta\partial_2^2 b_2 + \partial_1^2 b_2\right) = N_4, \end{cases} \tag{1.6}$$

where N_1 through N_4 are nonlinear terms. In fact, by eliminating the pressure term and separating the linear and nonlinear parts in (1.2), we have

$$\begin{cases} \partial_t u - \nu\partial_2^2 u - \partial_1 b = \mathbb{P}(-u\cdot\nabla u + b\cdot\nabla b) \\ \partial_t b + \eta(0, b_2)^\top - \partial_1 u = -u\cdot\nabla b + b\cdot\nabla u. \end{cases} \tag{1.7}$$

One can easily obtain (1.6) from (1.7) after differentiating it in time and making several substitutions. The wave system (1.6) exhibits much more smoothing and stabilizing properties than (1.2). In particular, the velocity equation in (1.6) contains the term $-\partial_1^2 u$, which generates a weak dissipative effect in the horizontal direction. This is one of the main reasons that we can handle the Navier–Stokes nonlinearity in the MHD system (1.2), which cannot be bounded suitably without the coupling with the magnetic field.

We have to construct a suitable energy functional to incorporate the regularizing properties revealed by the wave structure in (1.6). The energy function consists of two layers. The first one is the natural H^2-energy functional, and the second one incorporates the extra regularization indicated by the wave structure. More precisely, we define

$$E(t) := E_1(t) + E_2(t), \tag{1.8}$$

where

$$E_1(t) := \sup_{0\le\tau\le t} \|(u,b)(\tau)\|_{H^2}^2 + 2\int_0^t \Big(\nu\|\partial_2 u(\tau)\|_{H^2}^2 + \eta\|b_2(\tau)\|_{H^2}^2\Big)d\tau, \tag{1.9}$$

$$E_2(t) := \int_0^t \|\partial_1 u(\tau)\|_{H^1}^2 d\tau. \tag{1.10}$$

We shall show that $E(t)$ satisfies a suitable energy inequality, from which we can derive a uniform global bound under the condition that the initial data are sufficiently small. This enables us to establish the following stability result.

THEOREM 1.1. *Assume initial data* $(u_0, b_0) \in H^2$ *with* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. *Then there exists a constant* $\varepsilon > 0$, *depending only on* ν *and* η, *such that if*

$$\|(u_0,b_0)\|_{H^2} \le \varepsilon,$$

then the problem (1.2)–(1.3) *has a unique global solution* $(u,b)(x,t)$ *on* $\mathbb{R}^2\times[0,\infty)$, *satisfying*

$$\|(u,b)(t)\|_{H^2}^2 + \int_0^t \big(\|(\partial_2 u, b_2)(\tau)\|_{H^2}^2 + \|\partial_1 u(\tau)\|_{H^1}^2\big)d\tau \le C\,\varepsilon^2 \quad \forall\ t\ge 0,$$

where C *is an absolute positive constant.*

Remark 1.1. The stability result stated in Theorem 1.1 depends mathematically on the anisotropic dissipation, the vertical damping, and the background magnetic field. It is easy to check that if (1.1) is with the horizontal dissipation, the horizontal damping, and a background magnetic field in the x_2-direction, then the same stability result still holds.

Remark 1.2. It is an interesting problem to study the explicit decay rates of the solutions obtained in Theorem 1.1. Mathematically, this is not a trivial problem. A natural idea is to convert (1.2) into a system of wave equations and understand the spectral properties of the linearized system. Indeed, if we apply the Leray projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the equations in (1.2), then

$$\begin{cases} \partial_t u = \nu\partial_2^2 u + \partial_1 b + \mathbb{P}(b\cdot\nabla b - u\cdot\nabla u), \\ \partial_t b = -\eta\partial_1^2\Delta^{-1} b + \partial_1 u + \mathbb{P}(b\cdot\nabla u - u\cdot\nabla b), \\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{1.11}$$

Differentiating (1.11) in t and making several substitutions, we have

$$\begin{cases} \partial_{tt} u - (\nu\partial_2^2 - \eta\partial_1^2\Delta^{-1})\partial_t u - \partial_1^2 u - \nu\eta\partial_1^2\partial_2^2\Delta^{-1}u = M_1, \\ \partial_{tt} b - (\nu\partial_2^2 - \eta\partial_1^2\Delta^{-1})\partial_t b - \partial_1^2 b - \nu\eta\partial_1^2\partial_2^2\Delta^{-1}b = M_2, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1.12}$$

where M_1 and M_2 are the nonlinear terms. The large-time behavior of the solutions depends actually on the eigenvalues of the linear parts of (1.12), which are determined through the following characteristic polynomial:

$$\lambda^2 + \left(\nu\xi_2^2 + \frac{\eta\xi_1^2}{|\xi|^2}\right)\lambda + \frac{\nu\eta\xi_1^2\xi_2^2}{|\xi|^2} + \xi_1^2 = 0.$$

Its roots λ_1 and λ_2 are given by

$$\lambda_1 := \frac{-\left(\nu\xi_2^2 + \frac{\eta\xi_1^2}{|\xi|^2}\right) - \sqrt{\Gamma}}{2}, \qquad \lambda_2 := \frac{-\left(\nu\xi_2^2 + \frac{\eta\xi_1^2}{|\xi|^2}\right) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma := \left(\nu\xi_2^2 + \frac{\eta\xi_1^2}{|\xi|^2}\right)^2 - 4\left(\frac{\nu\eta\xi_1^2\xi_2^2}{|\xi|^2} + \xi_1^2\right).$$

One can also obtain the corresponding eigenvectors and the kernel functions. In view of the definitions of λ_1, λ_2, and Γ, we see that λ_2 may be degenerate and vanish for some of the frequencies ξ with $\xi_1 = 0$. In order to understand the large-time behavior, we need to decompose the frequency space into suitable subdomains to deal with the degeneracy and the anisotropicity encountered here. This is in our future research plan.

Theorem 1.1 asserts the H^2-stability of the perturbation near a background magnetic field and rigorously confirms the experimentally observed stabilizing phenomenon. Mathematically, the approach used herein may provide some hints for the stability problem of other anisotropic PDEs systems with partial damping since the classical methods usually require full dissipation or full damping (see, e.g., [31, 32]).

Next, we briefly recall some of the closely related works to place our result in a suitable context. The stability problem and related issues on the MHD system near a background magnetic field have recently attracted considerable interest. [4, 8, 18] studied the stability problem near a background magnetic field of either the ideal MHD system or the fully dissipative MHD system with identical viscosity and magnetic diffusivity, based on the use of the Elsässer variables. [35] allows these two coefficients to be slightly different. The stability problem of the nonresistive MHD system was pioneered by [23] by using the Lagrangian approach. [29] revisited this stability problem by resorting to Eulerian energy estimates in the anisotropic Besov setting and obtained the large-time behavior of the solutions. [38] established the stability of the nonresistive MHD system with velocity damping (without dissipation) via the approach of wave equations. The stability and large-time behavior of the solutions of the 2D/3D MHD system with anisotropic dissipation and magnetic diffusion was considered in [6, 15, 25, 39]. There is a very large literature on the global regularity of the MHD system and related issues; see, e.g., [1, 10, 14, 19, 20, 21, 23, 27, 30, 33, 36, 37, 40, 41, 42]. This list is by no means exhaustive.

We now comment on the proof of Theorem 1.1. Since the local (in time) existence theorem can be shown by following a standard approach (see, e.g., [26]), our attention will be focused on the global (in time) bounds of the H^2-norm. The bootstrapping argument serves this strategy well (see, e.g., [34, p. 21]). Let $E(t)$ be the energy functional defined in (1.8). We aim to show that

$$E(t) \le C_1(E(0) + E(0)^{\frac{3}{2}} + E(0)^2) + C_2 E(t)^{\frac{3}{2}} + C_3 E(t)^2 + C_4 E(t)^{\frac{5}{2}}, \tag{1.13}$$

where C_i $(i = 1, \ldots, 4)$ are generic positive constants, depending only on ν and η. Once (1.13) is established, the bootstrapping argument implies that if

$$\|(u_0, b_0)\|_{H^2} \le \varepsilon \quad \text{or} \quad E(0) \le \varepsilon^2,$$

then there exists a generic positive constant $C > 0$ such that

$$E(t) \le C\varepsilon^2 \quad \forall\, t \ge 0,$$

which leads to the desired uniformly global bounds.

Due to the anisotropic dissipation and partial damping, the verification of (1.13) is not trivial. For the sake of clarity, we divide the proof of (1.13) into two parts, which concern the estimates of $E_1(t)$ and $E_2(t)$, respectively.

PROPOSITION 1.1. *Let $E_1(t)$ and $E_2(t)$ be the ones as in* (1.9) *and* (1.10)*. Then there exists a generic positive constant $C > 0$, depending only on ν and η, such that*

$$\begin{aligned} E_1(t) \le\; & CE_1(0) + CE_1(0)^{\frac{3}{2}} + CE_1(0)^2 + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}} \\ & + CE_1(t)^2 + CE_2(t)^2 + CE_1(t)^{\frac{5}{2}} + CE_2(t)^{\frac{5}{2}}. \end{aligned}$$

PROPOSITION 1.2. *Let $E_1(t)$ and $E_2(t)$ be the ones as in* (1.9) *and* (1.10)*. Then there exists a generic positive constant $C > 0$, depending only on ν and η, such that*

$$E_2(t) \le CE_1(0) + CE_1(t) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}.$$

The proofs of Propositions 1.1 and 1.2 involve the estimates of many terms. Some of the terms cannot be bounded directly in terms of $E_1(t)$ and $E_2(t)$, and new ideas have to be developed. As mentioned, the first two terms in (1.5) simply do not admit any suitable upper bound when the velocity equation is not coupled to the magnetic field. But they can now be bounded suitably due to the inclusion of the energy functional $E_2(t)$ (see section 3 for details). However, there are also some of the nonlinear terms that cannot be bounded by $E_1(t)$ and $E_2(t)$ directly. Two of the most difficult ones are

$$\int \partial_1 u_1 |\partial_2^2 b_1|^2 \, dx \quad \text{and} \quad \int b_1 \partial_1 u_1 |\partial_2^2 b_1|^2 \, dx.$$

The strategy here is to replace $\partial_1 u_1$ by using the equation of the magnetic field:

$$\partial_1 u_1 = \partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1. \tag{1.14}$$

For example, by (1.14), we have

$$\begin{aligned} \int \partial_1 u_1 |\partial_2^2 b_1|^2 \, dx &= \int (\partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1) |\partial_2^2 b_1|^2 \, dx \\ &= \frac{d}{dt} \int b_1 |\partial_2^2 b_1|^2 \, dx - 2 \int b_1 \partial_2^2 b_1 \partial_2^2 \partial_t b_1 \, dx \\ &\quad + \int u \cdot \nabla b_1 |\partial_2^2 b_1|^2 \, dx - \int b \cdot \nabla u_1 |\partial_2^2 b_1|^2 \, dx. \end{aligned}$$

We have to replace $\partial_t b_1$ by using (1.14) again. This process generates many more terms. Fortunately, all the terms can be bounded by $E_1(t)$ and $E_2(t)$. This idea comes from the work of Lin and Zhang [24]. Collecting these estimates together, we are able to establish the energy inequality in Proposition 1.1. To prove Proposition 1.2, we also need to make use of the connection between u and b via the equation of the magnetic field:

$$\partial_1 u = \partial_t b + u \cdot \nabla b + \eta(0, b_2)^\top - b \cdot \nabla u.$$

This strategy allows us to convert the time integral of $\|\partial_1 u\|_{H^1}$ into other terms which potentially have better time integrability:

$$\begin{aligned}\int_0^t \|\partial_1 u\|_{H^1}^2 d\tau = &\int_0^t \int \partial_1 u \cdot \partial_t b \; dx d\tau + \int_0^t \int u \cdot \nabla b \cdot \partial_1 u \; dx d\tau \\ &+ \eta \int_0^t \int b_2 \partial_1 u_2 \; dx d\tau - \int_0^t \int b \cdot \nabla u \cdot \partial_1 u \; dx d\tau \\ &+ \int_0^t \int \nabla \partial_1 u \cdot \partial_t \nabla b \; dx d\tau + \int_0^t \int \nabla (u \cdot \nabla b) \cdot \nabla \partial_1 u \; dx d\tau \\ &+ \eta \int_0^t \int \nabla \partial_1 u_2 \cdot \nabla b_2 \; dx d\tau - \int_0^t \int \nabla (b \cdot \nabla u) \cdot \nabla \partial_1 u \; dx d\tau.\end{aligned}$$

By further shifting the two terms associated with time derivative $\partial_t b$ and $\partial_t \nabla b$ to $\partial_1 u$ and $\nabla \partial_1 u$, respectively, and replacing $\partial_t u$ by other terms in the velocity equation, we are able to establish the desired energy inequality in Proposition 1.2. More technical details are left to section 4.

The rest of this paper is organized as follows. Theorem 1.1 is shown in section 2 by assuming Propositions 1.1 and 1.2 hold. Sections 3 and 4 are devoted to the proofs of Propositions 1.1 and 1.2, respectively.

2. Proof of Theorem 1.1. This section aims to prove Theorem 1.1, based on Propositions 1.1 and 1.2. We show that an application of the bootstrapping argument to the energy inequalities in Propositions 1.1 and 1.2 will lead to the desired global well-posedness and stability result of Theorem 1.1.

Proof of Theorem 1.1. Since the local well-posedness on (1.2) can be established by the standard approach (see, e.g., [26]), we omit the details for the conciseness of the proof. Attention here is focused on the global H^2-bound of (u, b). Adding the energy inequality in Propositions 1.1 to the inequality in Proposition 1.2 multiplied by a suitable number yields

$$\begin{aligned}E_1(t) + \frac{1}{2C} E_2(t) \le\; & C\left(E_1(0) + E_1(0)^{\frac{3}{2}} + E_1(0)^2\right) + C\left(E_1(t)^{\frac{3}{2}} + E_2(t)^{\frac{3}{2}}\right) \\ &+ \frac{1}{2} E_1(t) + C\left(E_1(t)^2 + E_2(t)^2\right) + C\left(E_1(t)^{\frac{5}{2}} + E_2(t)^{\frac{5}{2}}\right).\end{aligned}$$

Thus, writing $E(t) := E_1(t) + E_2(t)$ and noting that $E_1(0) = E(0) = \|(u_0, b_0)\|_{H^2}^2$, we have

$$E(t) \le C_1\left(E(0) + E(0)^{\frac{3}{2}} + E(0)^2\right) + C_2\, E(t)^{\frac{3}{2}} + C_3\, E(t)^2 + C_4\, E(t)^{\frac{5}{2}}, \tag{2.1}$$

where $C_i (i = 1, 2, 3, 4)$ are positive constants depending only on ν and η.

Now an application of the bootstrapping argument to (2.1) leads to the desired upper bound in Theorem 1.1. In fact, if we take the initial H^2-norm $\|(u_0, b_0)\|_{H^2}$ to be sufficiently small such that $E(0)$ satisfies

$$C_1\left(E(0)+E(0)^{\frac{3}{2}}+E(0)^2\right) \le \frac{1}{4}\min\left\{\frac{1}{36C_2^2},\ \frac{1}{6C_3},\ \left(\frac{1}{6C_4}\right)^{\frac{2}{3}}\right\} \tag{2.2}$$

and make the ansatz that, for $t > 0$,

$$E(t) \le \min\left\{\frac{1}{36C_2^2},\ \frac{1}{6C_3},\ \left(\frac{1}{6C_4}\right)^{\frac{2}{3}}\right\}, \tag{2.3}$$

then we infer from (2.1) that

$$E(t) \le C_1\left(E(0)+E(0)^{\frac{3}{2}}+E(0)^2\right)+\frac{1}{2}E(t)$$

or

$$E(t) \le 2C_1\left(E_1(0)+E_1(0)^{\frac{3}{2}}+E_1(0)^2\right),$$

which, combined with (2.2), gives

$$E(t) \le \frac{1}{2}\min\left\{\frac{1}{36C_2^2},\ \frac{1}{6C_3},\ \left(\frac{1}{6C_4}\right)^{\frac{2}{3}}\right\}. \tag{2.4}$$

Since the upper bound in (2.4) is just half of the bound in the ansatz (2.3), the bootstrapping argument then asserts that (2.4) indeed holds for all time $t > 0$, provided the initial data satisfy (2.2). Thus, we obtain the desired global H^2-bound, which, together with the local well-posedness theory on (1.2), leads to the global well-posedness and stability theory stated in Theorem 1.1. This completes the proof of Theorem 1.1. ∎

3. Proof of Proposition 1.1. This section is devoted to the proof of Proposition 1.1. We will use several anisotropic inequalities extensively. They are listed in the following two lemmas. Lemma 3.1 can be found in [9], while Lemma 3.2 was established in [25].

LEMMA 3.1. *Assume that f, g, h, $\partial_2 g$, and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then there exists a generic constant $C > 0$ such that*

$$\int_{\mathbb{R}^2}|fgh|\,dx \le C\,\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\partial_1 h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.$$

LEMMA 3.2. *The following estimates hold when the right-hand sides are all bounded:*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

Consequently,

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}},$$

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}.$$

Here C in the bounds are all generic constants.

We are now in a position of proving Proposition 1.1. For simplicity, in the rest of this paper, we denote by C the various positive constants, which may depend on ν and η but not on t.

Proof of Proposition 1.1. Due to the equivalence of the norm $\|(u,b)\|_{H^2}$ with the norm $\|(u,b)\|_{L^2} + \|(u,b)\|_{\dot{H}^2}$, it suffices to bound the L^2 and the homogeneous $\dot{H}^2$-norm of (u,b). We start with the L^2-estimate. First, by the basic energy estimates and the facts that $\nabla\cdot u = \nabla\cdot b = 0$, we obtain

$$\|(u,b)\|_{L^2}^2 + 2\int_0^t \left(\nu\|\partial_2 u\|_{L^2}^2 + \eta\|b_2\|_{L^2}^2\right) d\tau = \|(u_0,b_0)\|_{L^2}^2. \tag{3.1}$$

To estimate the $\dot{H}^2$-norm, applying $\partial_i^2 (i=1,2)$ to (1.2) and then dotting them by $(\partial_i^2 u, \partial_i^2 b)$ in L^2, we deduce

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\|(\partial_i^2 u, \partial_i^2 b)\|_{L^2}^2 + \nu\sum_{i=1}^{2}\|\partial_i^2\partial_2 u\|_{L^2}^2 + \eta\sum_{i=1}^{2}\|\partial_i^2 b_2\|_{L^2}^2 = \sum_{i=1}^{5} I_i, \tag{3.2}$$

where

$$\begin{aligned}
I_1 &:= \sum_{i=1}^{2}\int \left(\partial_i^2\partial_1 b\cdot\partial_i^2 u + \partial_i^2\partial_1 u\cdot\partial_i^2 b\right) dx,\\
I_2 &:= -\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla u)\cdot\partial_i^2 u\, dx,\\
I_3 &:= \sum_{i=1}^{2}\int \left(\partial_i^2(b\cdot\nabla b) - b\cdot\nabla\partial_i^2 b\right)\cdot\partial_i^2 u\, dx,\\
I_4 &:= -\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla b)\cdot\partial_i^2 b\, dx,\\
I_5 &:= \sum_{i=1}^{2}\int \left(\partial_i^2(b\cdot\nabla u) - b\cdot\nabla\partial_i^2 u\right)\cdot\partial_i^2 b\, dx.
\end{aligned}$$

By integration by parts,

$$I_1 = 0. \tag{3.3}$$

To bound I_2, we divide it into two parts,

$$I_2 = -\int \partial_1^2(u\cdot\nabla u)\cdot\partial_1^2 u\, dx - \int \partial_2^2(u\cdot\nabla u)\cdot\partial_2^2 u\, dx := I_{21} + I_{22},$$

where the terms on right-hand side can be bounded as follows. First, using the divergence-free condition $\nabla\cdot u = 0$ and Lemma 3.1, we obtain

$$\begin{aligned}
I_{21} &= -\int (\partial_1^2 u\cdot\nabla u + 2\partial_1 u\cdot\nabla\partial_1 u)\cdot\partial_1^2 u\, dx\\
&\le C\|\partial_1^2 u\|_{L^2}\Big(\|\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\\
&\quad + \|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\partial_1 u\|_{L^2}^{\frac{1}{2}}\Big)\\
&\le C\|u\|_{H^2}\left(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned} \tag{3.4}$$

To bound I_{22}, we infer from Hölder's inequality that

$$\begin{aligned} I_{22} &= -\int (\partial_2^2 u \cdot \nabla u + 2\partial_2 u \cdot \nabla \partial_2 u) \cdot \partial_2^2 u \, dx \\ &\le C\|\partial_2^2 u\|_{L^2} \left(\|\nabla u\|_{L^4} \|\partial_2^2 u\|_{L^4} + \|\partial_2 u\|_{L^4} \|\nabla \partial_2 u\|_{L^4} \right) \\ &\le C\|u\|_{H^2} \|\partial_2 u\|_{H^2}^2, \end{aligned}$$

which, together with (3.4), yields

$$I_2 \le C\|u\|_{H^2} \left(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 \right). \tag{3.5}$$

Similarly to estimate I_3, we separate it into two items:

$$I_3 = \sum_{i=1}^{2} \int 2\partial_i b \cdot \nabla \partial_i b \cdot \partial_i^2 u \, dx + \sum_{i=1}^{2} \int \partial_i^2 b \cdot \nabla b \cdot \partial_i^2 u \, dx := I_{31} + I_{32}.$$

For I_{31}, by the divergence-free condition $\nabla \cdot b = 0$, Lemma 3.1, and Hölder's inequality, we find

$$\begin{aligned} I_{31} &= 2\int \left(\partial_1 b \cdot \nabla \partial_1 b \cdot \partial_1^2 u + \partial_2 b \cdot \nabla \partial_2 b \cdot \partial_2^2 u \right) dx \\ &= 2\int \left(\partial_1 b \cdot \nabla \partial_1 b \cdot \partial_1^2 u + \partial_2 b_1 \partial_1 \partial_2 b \cdot \partial_2^2 u - \partial_1 b_1 \partial_2^2 b \cdot \partial_2^2 u \right) dx \\ &\le C\|\partial_1 \nabla b\|_{L^2} \|\partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\left(\|\partial_2 b_1\|_{L^4} \|\partial_1 \partial_2 b\|_{L^2} + \|\partial_1 b_1\|_{L^4} \|\partial_2^2 b\|_{L^2} \right) \|\partial_2^2 u\|_{L^4} \\ &\le C\|(u,b)\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 \right) \end{aligned} \tag{3.6}$$

and analogously

$$\begin{aligned} I_{32} &= \int \left(\partial_1^2 b \cdot \nabla b \cdot \partial_1^2 u + \partial_2^2 b_1 \partial_1 b \cdot \partial_2^2 u - \partial_1 \partial_2 b_1 \partial_2 b \cdot \partial_2^2 u \right) dx \\ &\le C\|\partial_1^2 b\|_{L^2} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\left(\|\partial_2^2 b_1\|_{L^2} \|\partial_1 b\|_{L^4} + \|\partial_1 \partial_2 b_1\|_{L^2} \|\partial_2 b\|_{L^4} \right) \|\partial_2^2 u\|_{L^4} \\ &\le C\|(u,b)\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 \right). \end{aligned} \tag{3.7}$$

Thus, combining (3.6) and (3.7) gives

$$I_3 \le C\|(u,b)\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 \right). \tag{3.8}$$

We proceed to estimate I_4, which is more troublesome. Note that

$$I_4 = -\sum_{i=1}^{2} \int 2\partial_i u \cdot \nabla \partial_i b \cdot \partial_i^2 b \, dx - \sum_{i=1}^{2} \int \partial_i^2 u \cdot \nabla b \cdot \partial_i^2 b \, dx := I_{41} + I_{42}. \tag{3.9}$$

To estimate I_{41}, we split it into four parts,

$$\begin{aligned} I_{41} &= -2\int \left(\partial_1 u \cdot \nabla \partial_1 b \cdot \partial_1^2 b + \partial_2 u \cdot \nabla \partial_2 b \cdot \partial_2^2 b \right) dx \\ &= -2\int \Big(\partial_1 u \cdot \nabla \partial_1 b \cdot \partial_1^2 b + \partial_2 u_1 \partial_1 \partial_2 b \cdot \partial_2^2 b \\ &\qquad + \partial_2 u_2 \partial_1 \partial_2 b_1 \partial_1 \partial_2 b_1 - \partial_1 u_1 |\partial_2^2 b_1|^2 \Big) dx \\ &:= I_{411} + I_{412} + I_{413} + I_{414}, \end{aligned} \tag{3.10}$$

where the first three terms can be bounded by Lemma 3.2 and Hölder's inequality:

$$
\begin{aligned}
&I_{411}+I_{412}+I_{413}\\
&\le C\left(\|\partial_1 u\|_{L^\infty}\|\partial_1\nabla b\|_{L^2}\|\partial_1^2 b\|_{L^2}+\|\partial_2 u\|_{L^\infty}\|\partial_1\partial_2 b\|_{L^2}\|\nabla^2 b\|_{L^2}\right)\\
&\le C\left(\|\partial_1 u\|_{H^1}^{\frac12}\|\partial_2\partial_1 u\|_{H^1}^{\frac12}\|\partial_1\nabla b\|_{L^2}\|\partial_1^2 b\|_{L^2}+\|\partial_2 u\|_{H^2}\|\partial_1\partial_2 b\|_{L^2}\|\nabla^2 b\|_{L^2}\right)\\
&\le C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right). \qquad (3.11)
\end{aligned}
$$

Next we turn to estimate I_{414}, which is the most difficult term. As mentioned, the strategy here is to replace $\partial_1 u_1$ by other terms in the equation of the magnetic field:

$$\partial_1 u_1=\partial_t b_1+u\cdot\nabla b_1-b\cdot\nabla u_1. \qquad (3.12)$$

Indeed, substituting (3.12) into I_{414}, we have

$$
\begin{aligned}
I_{414}&=2\int\partial_1 u_1|\partial_2^2 b_1|^2\,dx=2\int\left(\partial_t b_1+u\cdot\nabla b_1-b\cdot\nabla u_1\right)|\partial_2^2 b_1|^2\,dx\\
&=2\frac{d}{dt}\int b_1|\partial_2^2 b_1|^2\,dx-4\int b_1\partial_2^2 b_1\partial_2^2\partial_t b_1\,dx\\
&\quad+2\int\left(u\cdot\nabla b_1\right)|\partial_2^2 b_1|^2\,dx-2\int\left(b\cdot\nabla u_1\right)|\partial_2^2 b_1|^2\,dx. \qquad (3.13)
\end{aligned}
$$

To deal with the second term on the right-hand side of (3.13), we use (3.12) again to write it as

$$
\begin{aligned}
-4\int b_1\partial_2^2 b_1\partial_2^2\partial_t b_1\,dx&=-4\int b_1\partial_2^2 b_1\partial_2^2(\partial_1 u_1-u\cdot\nabla b_1+b\cdot\nabla u_1)\,dx\\
&=-4\int b_1\partial_2^2 b_1\partial_2^2\partial_1 u_1\,dx+4\int b_1\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1\,dx\\
&\quad+8\int b_1\partial_2^2 b_1\partial_2 u\cdot\nabla\partial_2 b_1\,dx+2\int b_1 u\cdot\nabla|\partial_2^2 b_1|^2\,dx\\
&\quad-4\int b_1\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1)\,dx. \qquad (3.14)
\end{aligned}
$$

Inserting (3.14) into (3.13) and using the fact that

$$\int b_1 u\cdot\nabla|\partial_2^2 b_1|^2\,dx+\int u\cdot\nabla b_1|\partial_2^2 b_1|^2\,dx=0,$$

we obtain

$$
\begin{aligned}
I_{414}&=2\frac{d}{dt}\int b_1|\partial_2^2 b_1|^2\,dx-4\int b_1\partial_2^2 b_1\partial_2^2\partial_1 u_1\,dx+4\int b_1\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1\,dx\\
&\quad+8\int b_1\partial_2^2 b_1\partial_2 u\cdot\nabla\partial_2 b_1\,dx-4\int b_1\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1)\,dx\\
&\quad-2\int b\cdot\nabla u_1|\partial_2^2 b_1|^2\,dx:=J_1+J_2+J_3+J_4+J_5+J_6. \qquad (3.15)
\end{aligned}
$$

We now have to bound $J_2, \dots, J_6$ one by one. To bound J_2, by integration by parts twice and Hölder's inequality, we have

$$\begin{aligned}
J_2 &= -4\int b_1\partial_2^2 b_1\partial_2^2\partial_1 u_1 \; dx\\
&= 4\int \partial_1 b_1\partial_2^2 b_1\partial_2^2 u_1 \; dx - 4\int \partial_2 b_1\partial_1\partial_2 b_1\partial_2^2 u_1\; dx - 4\int b_1\partial_1\partial_2 b_1\partial_2^3 u_1 \; dx\\
&\le C\left(\|\partial_1 b_1\|_{L^4}\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u_1\|_{L^4} + \|\partial_2 b_1\|_{L^4}\|\partial_1\partial_2 b\|_{L^2}\|\partial_2^2 u_1\|_{L^4}\right)\\
&\quad + C\|b_1\|_{L^\infty}\|\partial_1\partial_2 b_1\|_{L^2}\|\partial_2^3 u_1\|_{L^2}\\
&\le C\|b\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right). \qquad (3.16)
\end{aligned}$$

For J_3, by Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
J_3 &= 4\int b_1\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1\; dx\\
&\le C\|b_1\|_{H^1}^{\frac12}\|\partial_1 b_1\|_{H^1}^{\frac12}\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u\|_{L^2}^{\frac12}\|\partial_2^3 u\|_{L^2}^{\frac12}\|\nabla b_1\|_{L^2}^{\frac12}\|\partial_1\nabla b_1\|_{L^2}^{\frac12}\\
&\le C\|b\|_{H^2}^2\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right). \qquad (3.17)
\end{aligned}$$

To bound J_4, we first decompose it into two pieces,

$$J_4 = 8\int\left(b_1\partial_2^2 b_1\partial_2 u_1\partial_1\partial_2 b_1 + b_1\partial_2^2 b_1\partial_2 u_2\partial_2^2 b_1\right)\; dx := J_{41} + J_{42},$$

where the first term on the right-hand side can be bounded as follows, using Hölder's and Sobolev's inequalities:

$$\begin{aligned}
J_{41} &\le C\|b_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_1\partial_2 b_1\|_{L^2}\\
&\le C\|b\|_{H^2}^2\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2\right). \qquad (3.18)
\end{aligned}$$

To estimate J_{42}, the second difficult term, we use the special structure of the magnetic field equation (3.12) to rewrite it as

$$\begin{aligned}
J_{42} &= -8\int b_1\partial_1 u_1|\partial_2^2 b_1|^2\; dx\\
&= -8\int b_1|\partial_2^2 b_1|^2\left(\partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1\right)\; dx\\
&= -4\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\; dx + 8\int b_1^2\partial_2^2 b_1\partial_2^2\partial_t b_1\; dx\\
&\quad - 4\int(u\cdot\nabla|b_1|^2)|\partial_2^2 b_1|^2\; dx + 8\int b_1(b\cdot\nabla u_1)|\partial_2^2 b_1|^2\; dx. \qquad (3.19)
\end{aligned}$$

For the second term on the right-hand side of (3.19), we apply (3.12) again to obtain

$$\begin{aligned}
8\int b_1^2\partial_2^2 b_1\partial_2^2\partial_t b_1\; dx &= 8\int b_1^2\partial_2^2 b_1\partial_2^2(\partial_1 u_1 - u\cdot\nabla b_1 + b\cdot\nabla u_1)\; dx\\
&= 8\int b_1^2\partial_2^2 b_1\partial_2^2\partial_1 u_1\; dx - 8\int b_1^2\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1\; dx\\
&\quad - 16\int b_1^2\partial_2^2 b_1\partial_2 u\cdot\nabla\partial_2 b_1\; dx - 4\int b_1^2 u\cdot\nabla|\partial_2^2 b_1|^2\; dx\\
&\quad + 8\int b_1^2\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1)\; dx. \qquad (3.20)
\end{aligned}$$

Thus, substituting (3.20) into (3.19) and using the fact that

$$\int (u\cdot\nabla b_1^2)|\partial_2^2 b_1|^2\,dx + \int b_1^2 u\cdot\nabla|\partial_2^2 b_1|^2\,dx = 0,$$

we immediately obtain

$$\begin{aligned} J_{42} &= -4\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\,dx + 8\int b_1^2\partial_2^2 b_1\partial_2^2\partial_1 u_1\,dx \\ &\quad - 8\int b_1^2\partial_2^2 b_1\partial_2^2 u\cdot\nabla b_1\,dx - 16\int b_1^2\partial_2^2 b_1\partial_2 u\cdot\nabla\partial_2 b_1\,dx \\ &\quad + 8\int b_1^2\partial_2^2 b_1\partial_2^2(b\cdot\nabla u_1)\,dx + 8\int b_1(b\cdot\nabla u_1)|\partial_2^2 b_1|^2\,dx \\ &:= J_{421}+J_{422}+J_{423}+J_{424}+J_{425}+J_{426}. \end{aligned} \tag{3.21}$$

The three terms J_{422}, J_{423}, J_{424} can be bounded by

$$\begin{aligned} &J_{422}+J_{423}+J_{424} \\ &\le C\|b_1\|_{L^\infty}^2\left(\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2\partial_1 u_1\|_{L^2} + \|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u\|_{L^4}\|\nabla b_1\|_{L^4}\right) \\ &\quad + C\|b_1\|_{L^\infty}^2\|\partial_2^2 b_1\|_{L^2}\|\partial_2 u\|_{L^\infty}\|\nabla\partial_2 b_1\|_{L^2} \\ &\le C\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\left(\|\partial_2^2 b_1\|_{L^2}\|\partial_2^2\partial_1 u_1\|_{L^2} + \|\partial_2^2 b_1\|_{L^2}\|\partial_2^2 u\|_{H^1}\|\nabla b_1\|_{H^1}\right) \\ &\quad + C\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\|\partial_2^2 b_1\|_{L^2}\|\partial_2 u\|_{H^2}\|\nabla\partial_2 b_1\|_{L^2} \\ &\le C\left(\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{3.22}$$

For J_{425}, noting that $\|\nabla u_1\|_{H^2} = \|\partial_2 u\|_{H^2}$, we infer from Lemma 3.2 and Sobolev's inequalities that

$$\begin{aligned} &J_{425} = 8\int b_1^2\partial_2^2 b_1(\partial_2^2 b\cdot\nabla u_1 + 2\partial_2 b\cdot\nabla\partial_2 u_1 + b\cdot\nabla\partial_2^2 u_1)\,dx \\ &\le C\|b_1\|_{L^\infty}^2\|\partial_2^2 b_1\|_{L^2}\left(\|\partial_2^2 b\|_{L^2}\|\nabla u_1\|_{L^\infty} + \|\partial_2 b\|_{L^4}\|\nabla\partial_2 u_1\|_{L^4} + \|b\|_{L^\infty}\|\nabla\partial_2^2 u_1\|_{L^2}\right) \\ &\le C\|b_1\|_{H^1}\|\partial_1 b_1\|_{H^1}\|\partial_2^2 b_1\|_{L^2}\|b\|_{H^2}\|\partial_2 u\|_{H^2} \le C\|b\|_{H^2}^3\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned} J_{426} &\le C\|b\|_{L^\infty}^2\|\nabla u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}^2 \le C\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|\partial_2 u\|_{H^2}\|\partial_2^2 b_1\|_{L^2}^2 \\ &\le C\|b\|_{H^2}^3\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{3.24}$$

Plugging (3.22), (3.23), (3.24) into (3.21), we have

$$J_{42} \le -4\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\,dx + C\left(\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right), \tag{3.25}$$

which, together with (3.18), leads to

$$J_4 \le -4\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\,dx + C\left(\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right). \tag{3.26}$$

For the last two terms J_5 and J_6, we use Lemmas 3.1 and 3.2 and (3.25) to get

$$\begin{aligned}
J_5+J_6 &= -4\int b_1\partial_2^2 b_1\left(\partial_2^2 b\cdot\nabla u_1+2\partial_2 b\cdot\nabla\partial_2 u_1+b\cdot\nabla\partial_2^2 u_1\right)dx\\
&\quad -2\int b\cdot\nabla u_1|\partial_2^2 b_1|^2 dx\\
&= \frac{3}{4}J_{42}-4\int b_1\partial_2 u_1\partial_2^2 b_1\partial_2^2 b_2\ dx-8\int b_1\partial_2^2 b_1\partial_2 b\cdot\nabla\partial_2 u_1\ dx\\
&\quad -4\int b_1\partial_2^2 b_1 b\cdot\nabla\partial_2^2 u_1\ dx-2\int b_2\partial_2 u_1|\partial_2^2 b_1|^2 dx\\
&\le \frac{3}{4}J_{42}+C\|b_1\|_{L^\infty}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_1\partial_2 b_1\|_{L^2}\\
&\quad +C\|b_1\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2 b\|_{L^2}^{\frac12}\|\nabla\partial_2 u_1\|_{L^2}^{\frac12}\|\nabla\partial_2^2 u_1\|_{L^2}^{\frac12}\\
&\quad +C\|b\|_{L^\infty}^2\|\partial_2^2 b_1\|_{L^2}\|\nabla\partial_2^2 u_1\|_{L^2}+C\|b_2\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}^2\|\partial_2 u_1\|_{L^\infty}\\
&\le \frac{3}{4}J_{42}+C\|b\|_{H^2}^2\|\partial_2 u\|_{H^2}\|\partial_1 b\|_{H^1}+C\|b\|_{H^2}^2\|\partial_2 u\|_{H^2}\|b_2\|_{H^2}\\
&\le \frac{3}{4}J_{42}+C\|b\|_{H^2}^2\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right)\\
&\le -3\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\ dx+C\left(\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right),
\end{aligned}\tag{3.27}$$

where we have used the simple fact that $\|\partial_1 b\|_{H^1}=\|\nabla b_2\|_{H^1}$. Substituting (3.16), (3.17), (3.26), (3.27) into (3.15), we find

$$\begin{aligned}
I_{414} &\le 2\frac{d}{dt}\int b_1|\partial_2^2 b_1|^2\ dx-7\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\ dx\\
&\quad +C\left(\|b\|_{H^2}+\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}\tag{3.28}$$

This, combining with (3.11) and (3.10), shows that

$$\begin{aligned}
I_{41} &\le 2\frac{d}{dt}\int b_1|\partial_2^2 b_1|^2\ dx-7\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2\ dx\\
&\quad +C\left(\|(u,b)\|_{H^2}+\|b\|_{H^2}^2+\|b\|_{H^2}^3\right)\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}\tag{3.29}$$

Next we return to estimate I_{42}. By the divergence-free conditions $\nabla\cdot u=\nabla\cdot b=0$,

$$\begin{aligned}
I_{42} &= -\int\partial_1^2 u\cdot\nabla b\cdot\partial_1^2 b\ dx-\int\partial_2^2 u\cdot\nabla b\cdot\partial_2^2 b\ dx\\
&= -\int\partial_1^2 u\cdot\nabla b\cdot\partial_1^2 b\ dx-\int\partial_2^2 u_1\partial_1 b\cdot\partial_2^2 b\ dx+\int\partial_2^2 u_2\partial_1 b_1\partial_2^2 b_2\ dx\\
&\quad +\int\partial_1\partial_2 u_1\partial_2 b_1\partial_2^2 b_1\ dx := I_{421}+I_{422}+I_{423}+I_{424}.
\end{aligned}$$

For I_{421}, I_{422}, and I_{423}, by Lemma 3.1 and Sobolev's inequalities, we obtain

$$\begin{aligned}
I_{421}+I_{422}+I_{423} &\le C\|\partial_1^2 b\|_{L^2}\|\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\\
&\quad +C\|\partial_2^2 b\|_{L^2}\|\partial_2^2 u\|_{L^4}\|\partial_1 b\|_{L^4}\\
&\le C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}$$

For I_{424}, after integrating by parts twice, we have from Hölder's and Sobolev's inequalities that

$$\begin{aligned} I_{424} &= \int \partial_1\partial_2 u_1 \partial_2 b_1 \partial_2^2 b_1 \ dx = \int \partial_2^2 u_1 \partial_2 b_1 \partial_1\partial_2 b_1 \ dx \\ &\leq C\|\partial_2^2 u_1\|_{L^4}\|\partial_2 b_1\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^2} \\ &\leq C\|b\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right), \end{aligned}$$

and hence

$$I_{42} \leq C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right). \tag{3.30}$$

Thus, inserting (3.29) and (3.30) into (3.9), we find

$$\begin{aligned} I_4 &\leq 2\frac{d}{dt}\int b_1|\partial_2^2 b_1|^2 \ dx - 7\frac{d}{dt}\int b_1^2|\partial_2^2 b_1|^2 \ dx \\ &\quad + C\left(\|(u,b)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3\right)\left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{3.31}$$

Finally, it remains to estimate I_5:

$$I_5 = \sum_{i=1}^{2}\int 2\partial_i b\cdot\nabla\partial_i u\cdot\partial_i^2 b \ dx + \sum_{i=1}^{2}\int \partial_i^2 b\cdot\nabla u\cdot\partial_i^2 b \ dx := I_{51} + I_{52}.$$

To estimate I_{51}, we first use $\nabla\cdot b = 0$ to get

$$\begin{aligned} I_{51} &= 2\int \partial_1 b\cdot\nabla\partial_1 u\cdot\partial_1^2 b \ dx + 2\int \partial_2 b\cdot\nabla\partial_2 u\cdot\partial_2^2 b \ dx \\ &= 2\int \partial_1 b\cdot\nabla\partial_1 u\cdot\partial_1^2 b \ dx - 2\int \partial_1 b_1\partial_2^2 u\cdot\partial_2^2 b \ dx \\ &\quad - 2\int \partial_2 b_1\partial_1\partial_2 u_2\partial_2\partial_1 b_1 \ dx + 2\int \partial_2 b_1\partial_1\partial_2 u_1\partial_2^2 b_1 \ dx \\ &:= I_{511} + I_{512} + I_{513} + I_{514}, \end{aligned}$$

where the first three terms I_{511}, I_{512}, I_{513} on the right-hand side are bounded by

$$\begin{aligned} &I_{511} + I_{512} + I_{513} \\ &\leq C\|\partial_1^2 b\|_{L^2}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1^2 b\|_{L^2}^{\frac12}\|\nabla\partial_1 u\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2 u\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_2^2 b\|_{L^2}\|\partial_2^2 u\|_{L^4}\|\partial_1 b_1\|_{L^4} + C\|\partial_2 b_1\|_{L^4}\|\partial_2\partial_1 u_2\|_{L^4}\|\partial_1\partial_2 b_1\|_{L^2} \\ &\leq C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned}$$

Similarly to the derivation of I_{424}, we have

$$I_{514} \leq C\|b\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right),$$

and consequently,

$$I_{51} \leq C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\right). \tag{3.32}$$

Due to $\nabla \cdot b = 0$, we can rewrite I_{52} as

$$
\begin{aligned}
I_{52} &= \int \partial_1^2 b \cdot \nabla u \cdot \partial_1^2 b \, dx + \int \partial_2^2 b \cdot \nabla u \cdot \partial_2^2 b \, dx \\
&= \int \partial_1^2 b \cdot \nabla u \cdot \partial_1^2 b \, dx - \int \partial_1 \partial_2 b_1 \partial_2 u \cdot \partial_2^2 b \, dx \\
&\quad - \int \partial_2^2 b_1 \partial_1 u_2 \partial_1 \partial_2 b_1 \, dx + \int \partial_1 u_1 |\partial_2^2 b_1|^2 \, dx \\
&:= I_{521} + I_{522} + I_{523} + \frac{1}{2} I_{414}.
\end{aligned}
$$

For I_{521}, I_{522}, and I_{523}, by Lemma 3.2 and Sobolev's inequalities, we obtain

$$
\begin{aligned}
I_{521} + I_{522} + I_{523} &\le C\|\nabla u\|_{L^\infty} \|\partial_1^2 b\|_{L^2}^2 + C\|\partial_2 u\|_{L^\infty} \|\partial_1 \partial_2 b_1\|_{L^2} \|\partial_2^2 b\|_{L^2} \\
&\quad + C\|\partial_1 u_2\|_{L^\infty} \|\partial_1 \partial_2 b_1\|_{L^2} \|\partial_2^2 b_1\|_{L^2} \\
&\le C\|\nabla u\|_{H^1}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{H^1}^{\frac{1}{2}} \|\partial_1^2 b\|_{L^2}^2 + C\|\partial_2 u\|_{H^2} \|\partial_1 \partial_2 b_1\|_{L^2} \|\partial_2^2 b\|_{L^2} \\
&\quad + C\|\partial_1 u_2\|_{H^1}^{\frac{1}{2}} \|\partial_2 \partial_1 u_2\|_{H^1}^{\frac{1}{2}} \|\partial_1 \partial_2 b_1\|_{L^2} \|\partial_2^2 b_1\|_{L^2} \\
&\le C\|(u,b)\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right).
\end{aligned} \tag{3.33}
$$

Collecting (3.32), (3.33), and the estimate of I_{414} in (3.28) together, we find

$$
\begin{aligned}
I_5 &\le \frac{d}{dt} \int b_1 |\partial_2^2 b_1|^2 \, dx - \frac{7}{2} \frac{d}{dt} \int b_1^2 |\partial_2^2 b_1|^2 \, dx \\
&\quad + C \left(\|(u,b)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3 \right) \left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right).
\end{aligned} \tag{3.34}
$$

Now, plugging (3.3), (3.5), (3.8), (3.31), and (3.34) into (3.2), we arrive at

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 + \nu \|\partial_2 \nabla^2 u\|_{L^2}^2 + \eta \|\nabla^2 b_2\|_{L^2}^2 \\
&\le 3 \frac{d}{dt} \int b_1 |\partial_2^2 b_1|^2 \, dx - \frac{21}{2} \frac{d}{dt} \int b_1^2 |\partial_2^2 b_1|^2 \, dx \\
&\quad + C \left(\|(u,b)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3 \right) \left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right).
\end{aligned} \tag{3.35}
$$

Thus, integrating (3.35) over $[0,t]$, we obtain

$$
\begin{aligned}
&\|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 + 2 \int_0^t \left(\nu \|\partial_2 \nabla^2 u\|_{L^2}^2 + \eta \|\nabla^2 b_2\|_{L^2}^2 \right) d\tau \\
&\le \|(u_0, b_0)\|_{H^2}^2 + 6 \int b_1 |\partial_2^2 b_1|^2 \, dx - 21 \int b_1^2 |\partial_2^2 b_1|^2 \, dx \\
&\quad - 6 \int b_1(x,0) |\partial_2^2 b_1|^2(x,0) \, dx + 21 \int b_1^2(x,0) |\partial_2^2 b_1|^2(x,0) \, dx \\
&\quad + C \int_0^t \left(\|(u,b)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3 \right) \left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right) d\tau \\
&\le \|(u_0, b_0)\|_{H^2}^2 + C \left(\|b_1(0)\|_{L^\infty} + \|b_1(0)\|_{L^\infty}^2 \right) \|b(0)\|_{H^2}^2 \\
&\quad + C \left(\|b_1(t)\|_{L^\infty} + \|b_1(t)\|_{L^\infty}^2 \right) \|b(t)\|_{H^2}^2 \\
&\quad + C \sup_{0 \le \tau \le t} \left(\|(u,b)\|_{H^2} + \|b\|_{H^2}^2 + \|b\|_{H^2}^3 \right) \int_0^t \left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right) d\tau,
\end{aligned}
$$

which, combined with (3.1) and Cauchy–Schwarz's inequality, gives

$$E_1(t) \le CE_1(0) + CE_1(0)^{\frac{3}{2}} + CE_1(0)^2 + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}} + CE_1(t)^2 + CE_2(t)^2 + CE_1(t)^{\frac{5}{2}} + CE_2(t)^{\frac{5}{2}}.$$

The proof of Proposition 1.1 is therefore complete. ∎

4. Proof of Proposition 1.2. This section aims to prove Proposition 1.2. It relies on the special structure of the magnetic field equation in $(1.2)_2$, namely,

$$\partial_1 u = \partial_t b + u \cdot \nabla b + \eta (0, b_2)^\top - b \cdot \nabla u. \tag{4.1}$$

Proof of Proposition 1.2. Multiplying (4.1) with $\partial_1 u$ in L^2 and integrating it over $\mathbb{R}^2$ yields

$$\|\partial_1 u\|_{L^2}^2 = \int \partial_1 u \cdot \partial_t b \, dx + \int u \cdot \nabla b \cdot \partial_1 u \, dx + \eta \int b_2 \partial_1 u_2 \, dx - \int b \cdot \nabla u \cdot \partial_1 u \, dx := K_1 + K_2 + K_3 + K_4. \tag{4.2}$$

To estimate K_1, we use the velocity equation in $(1.2)_1$ to get

$$K_1 = \frac{d}{dt} \int \partial_1 u \cdot b \, dx - \int b \cdot \partial_1 (\nu \partial_2^2 u + b \cdot \nabla b + \partial_1 b - u \cdot \nabla u) \, dx := K_{11} + K_{12} + K_{13} + K_{14} + K_{15},$$

where we have eliminated the pressure term due to $\nabla \cdot b = 0$. By integration by parts and Hölder's inequality, K_{12}, K_{13}, K_{14} can be easily bounded by

$$K_{12} = -\nu \int b \cdot \partial_1 \partial_2^2 u \, dx = \nu \int \partial_1 b \cdot \partial_2^2 u \, dx \le C \|\partial_1 b\|_{L^2} \|\partial_2^2 u\|_{L^2},$$

$$\begin{aligned} K_{13} &= -\int b \cdot \partial_1 (b \cdot \nabla b) \, dx = \int \partial_1 b \cdot (b \cdot \nabla b) \, dx \\ &= \int b_1 \partial_1 b \cdot \partial_1 b \, dx + \int b_2 \partial_2 b \cdot \partial_1 b \, dx \\ &\le C \|b_1\|_{L^\infty} \|\partial_1 b\|_{L^2}^2 + C \|b_2\|_{L^\infty} \|\partial_2 b\|_{L^2} \|\partial_1 b\|_{L^2} \\ &\le C \|b\|_{H^2} \|b_2\|_{H^2}^2, \end{aligned}$$

and

$$K_{14} = -\int b \cdot \partial_1^2 b \, dx = \int \partial_1 b \cdot \partial_1 b \, dx \le C \|\partial_1 b\|_{H^1}^2.$$

To bound K_{15}, by Lemma 3.1 and integration by parts, we have

$$\begin{aligned} K_{15} &= \int b \cdot \partial_1 (u \cdot \nabla u) \, dx = -\int \partial_1 b \cdot (u \cdot \nabla u) \, dx \\ &\le C \|\partial_1 b\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\le C \|u\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right). \end{aligned}$$

In a similar manner,

$$\begin{aligned}K_2 &= \int u\cdot\nabla b\cdot\partial_1 u\ dx\\ &\le C\|\partial_1 u\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\\ &\le C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\right),\end{aligned}$$

$$K_3=\eta\int b_2\partial_1 u_2\ dx\le C\|b_2\|_{L^2}\|\partial_1 u_2\|_{L^2}\le\frac12\|\partial_1 u\|_{L^2}^2+C\|b_2\|_{L^2}^2,$$

and

$$\begin{aligned}K_4 &= -\int b\cdot\nabla u\cdot\partial_1 u\ dx\\ &\le C\|\partial_1 u\|_{L^2}\|b\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\\ &\le C\|(u,b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\right).\end{aligned}$$

This completes the L^2-estimate of $\partial_1 u$.

Next we proceed to bound the $\dot{H}^1$-norm. Applying ∇ to (4.1) and then multiplying it by $\nabla\partial_1 u$ in L^2, we find

$$\begin{aligned}\|\nabla\partial_1 u\|_{L^2}^2 &= \int\nabla\partial_1 u\cdot\partial_t\nabla b\ dx+\int\nabla(u\cdot\nabla b)\cdot\nabla\partial_1 u\ dx\\ &\quad+\eta\int\nabla\partial_1 u_2\cdot\nabla b_2\ dx-\int\nabla(b\cdot\nabla u)\cdot\nabla\partial_1 u\ dx\\ &:= L_1+L_2+L_3+L_4.\end{aligned}\tag{4.3}$$

To bound L_1, we integrate by parts and use the velocity equation in $(1.2)_1$ to get

$$\begin{aligned}L_1 &= \frac{d}{dt}\int\nabla\partial_1 u\cdot\nabla b\ dx-\int\nabla b\cdot\nabla\partial_1(\nu\partial_2^2 u+b\cdot\nabla b+\partial_1 b-u\cdot\nabla u)\ dx\\ &:= L_{11}+L_{12}+L_{13}+L_{14}+L_{15}.\end{aligned}$$

The bounds of L_{12}, L_{13}, L_{14} are directly obtained by integrating by parts and using Hölder's inequalities,

$$L_{12}=-\nu\int\nabla b\cdot\nabla\partial_1\partial_2^2 u\ dx=\nu\int\partial_1\nabla b\cdot\partial_2^2\nabla u\ dx\le C\|\partial_1 b\|_{H^1}\|\partial_2 u\|_{H^2},$$

$$\begin{aligned}L_{13} &= -\int\nabla b\cdot\nabla\partial_1(b\cdot\nabla b)\ dx=\int\partial_1\nabla b\cdot\nabla(b\cdot\nabla b)\ dx\\ &= \sum_{i,j=1}^2\int\partial_1\partial_i b\,(\partial_i b_j\partial_j b+b_j\partial_j\partial_i b)\,dx\\ &= \int\partial_1\nabla b\cdot(\nabla b_1\partial_1 b+\nabla b_2\partial_2 b)\,dx+\int\left(b_1|\partial_1\nabla b|^2+b_2\partial_2\nabla b\cdot\partial_1\nabla b\right)dx\\ &\le C\|\partial_1\nabla b\|_{L^2}\left(\|\nabla b_1\|_{L^4}\|\partial_1 b\|_{L^4}+\|\nabla b_2\|_{L^4}\|\partial_2 b\|_{L^4}\right)\\ &\quad+C\left(\|b_1\|_{L^\infty}\|\partial_1\nabla b\|_{L^2}^2+\|b_2\|_{L^\infty}\|\partial_2\nabla b\|_{L^2}\|\partial_1\nabla b\|_{L^2}\right)\\ &\le C\|b\|_{H^2}\|b_2\|_{H^2}^2,\end{aligned}$$

and

$$L_{14} = -\int \nabla b \cdot \nabla \partial_1^2 b \, dx = \int \nabla \partial_1 b \cdot \nabla \partial_1 b \, dx \leq C \|\partial_1 b\|_{H^1}^2,$$

where we have used the fact that $\|\nabla b_2\|_{H^1} = \|\partial_1 b\|_{H^1}$ in the estimate of L_{13}. Based on Lemma 3.1 and integration by parts, we deduce

$$\begin{aligned} L_{15} &= \int \nabla b \cdot \nabla \partial_1 (u \cdot \nabla u) \, dx = -\int \partial_1 \nabla b \cdot \nabla (u \cdot \nabla u) \, dx \\ &= -\int \partial_1 \nabla b \cdot \left(\nabla u \cdot \nabla u + u \cdot \nabla^2 u \right) \, dx \\ &\leq C \|\partial_1 \nabla b\|_{L^2} \|\nabla u\|_{L^2} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_1 \nabla b\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_1 u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|u\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right). \end{aligned}$$

For L_2, we infer from integration by parts, Lemma 3.1, and Hölder's inequalities that

$$\begin{aligned} L_2 &= \int \nabla (u \cdot \nabla b) \cdot \nabla \partial_1 u \, dx \\ &= \int \nabla u \cdot \nabla b \cdot \nabla \partial_1 u \, dx + \int u_1 \partial_1 \nabla b \cdot \partial_1 \nabla u \, dx + \int u_2 \partial_2 \nabla b \cdot \partial_1 \nabla u \, dx \\ &= \int \nabla u \cdot \nabla b \cdot \nabla \partial_1 u \, dx + \int u_1 \partial_1 \nabla b \cdot \partial_1 \nabla u \, dx - \int \partial_1 u_2 \partial_2 \nabla b \cdot \nabla u \, dx \\ &\quad + \int \partial_2 u_2 \partial_1 \nabla b \cdot \nabla u \, dx + \int u_2 \partial_1 \nabla b \cdot \partial_2 \nabla u \, dx \\ &\leq C \|\partial_1 \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_1 \nabla u\|_{L^2} \|u_1\|_{L^\infty} \|\partial_1 \nabla b\|_{L^2} + C \|\partial_1 u_2\|_{L^4} \|\partial_2 \nabla b\|_{L^2} \|\nabla u\|_{L^4} \\ &\quad + C \|\partial_2 u_2\|_{L^4} \|\partial_1 \nabla b\|_{L^2} \|\nabla u\|_{L^4} + C \|u_2\|_{L^\infty} \|\partial_1 \nabla b\|_{L^2} \|\partial_2 \nabla u\|_{L^2} \\ &\leq C \|(u, b)\|_{H^2} \left(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2 \right). \end{aligned}$$

Obviously, L_3, L_4 can be bounded as follows:

$$L_3 = \eta \int \nabla \partial_1 u_2 \cdot \nabla b_2 \, dx \leq C \|\nabla b_2\|_{L^2} \|\nabla \partial_1 u_2\|_{L^2} \leq \frac{1}{2} \|\partial_1 \nabla u\|_{L^2}^2 + C \|\nabla b_2\|_{L^2}^2$$

and

$$\begin{aligned} L_4 &= -\int \nabla (b \cdot \nabla u) \cdot \nabla \partial_1 u \, dx = -\int \nabla \partial_1 u \cdot \left(\nabla b \cdot \nabla u + b \cdot \nabla^2 u \right) \, dx \\ &\leq C \|\partial_1 \nabla u\|_{L^2} \Big(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\qquad + \|b\|_{L^2}^{\frac{1}{2}} \|\partial_1 b\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \Big) \\ &\leq C \|(u, b)\|_{H^2} \left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right). \end{aligned}$$

Now, adding up (4.2) and (4.3), then combining all the estimates for K_1 through K_4 and L_1 through L_4, we obtain

$$\begin{aligned} \frac{1}{2} \|\partial_1 u\|_{H^1}^2 &\leq \frac{d}{dt} \int \partial_1 u \cdot b \, dx + \frac{d}{dt} \int \nabla \partial_1 u \cdot \nabla b \, dx + C \left(\|\partial_2 u\|_{H^2}^2 + \|b_2\|_{H^2}^2 \right) \\ &\quad + C \|(u, b)\|_{H^2} \left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2 \right). \end{aligned} \tag{4.4}$$

Integrating (4.4) over $[0,t]$ leads to

$$\begin{aligned}\int_0^t \|\partial_1 u\|_{H^1}^2 d\tau &\leq 2\int \partial_1 u\cdot b\ dx - 2\int \partial_1 u(x,0)\cdot b(x,0)\ dx + 2\int \nabla\partial_1 u\cdot\nabla b\ dx\\ &\quad - 2\int \nabla\partial_1 u(x,0)\cdot\nabla b(x,0)\ dx + C\int_0^t \left(\|\partial_2 u\|_{H^2}^2+\|b_2\|_{H^2}^2\right) d\tau\\ &\quad + C\int_0^t \|(u,b)\|_{H^2}\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\right) d\tau\\ &\leq C\|(u_0,b_0)\|_{H^2}^2 + C\|(u,b)\|_{H^2}^2 + C\int_0^t \|(\partial_2 u, b_2)\|_{H^2}^2 d\tau\\ &\quad + C\sup_{0\leq\tau\leq t}\|(u,b)\|_{H^2}\int_0^t \left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\right) d\tau,\end{aligned}$$

which particularly implies that

$$E_2(t) \leq CE_1(0) + CE_1(t) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}.$$

The proof of Proposition 1.2 is thus complete. ∎

REFERENCES

[1] H. Abidi and P. Zhang, *On the global solution of 3D MHD system with initial data near equilibrium*, Commun. Pure Appl. Math., 70 (2017), pp. 1509–1561.

[2] A. Alexakis, *Two-dimensional behavior of three-dimensional magnetohydrodynamic flow with a strong guiding field*, Phys. Rev. E, 84 (2011), 056330.

[3] H. Alfvén, *Existence of electromagnetic-hydrodynamic waves*, Nature, 150 (1942), pp. 405–406.

[4] C. Bardos, C. Sulem, and P. L. Sulem, *Longtime dynamics of a conductive fluid in the presence of a strong magnetic field*, Trans. Amer. Math. Soc., 305 (1988), pp. 175–191.

[5] D. Biskamp, *Nonlinear Magnetohydrodynamics*, Cambridge University Press, Cambridge, 1993.

[6] N. Boardman, H. Lin, and J. Wu, *Stabilization of a background magnetic field on a 2D magnetohydrodynamic flow*, SIAM J. Math. Anal., 52 (2020), pp. 5001–5035.

[7] P. Burattini, O. Zikanov, and B. Knaepen, *Decay of magnetohydrodynamic turbulence at low magnetic Reynolds number*, J. Fluid Mech., 657 (2010), pp. 502–538.

[8] Y. Cai and Z. Lei, *Global well-posedness of the incompressible magnetohydrodynamics*, Arch. Ration. Mech. Anal., 228 (2018), pp. 969–993.

[9] C. Cao and J. Wu, *Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion*, Adv. Math., 226 (2011), pp. 1803–1822.

[10] Q. Chen and X. Ren, *Global well-posedness for the 2D non-resistive MHD equations in two kinds of periodic domains*, Z. Angew. Math. Phys., 70 (2019), 18.

[11] P. A. Davidson, *Magnetic damping of jets and vortices*, J. Fluid Mech., 299 (1995), pp. 153–186.

[12] P. A. Davidson, *The role of angular momentum in the magnetic damping of turbulence*, J. Fluid Mech., 336 (1997), pp. 123–150.

[13] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge University Press, Cambridge, 2001.

[14] W. Deng and P. Zhang, *Large time behavior of solutions to 3-D MHD system with initial data near equilibrium*, Arch. Ration. Mech. Anal., 230 (2018), pp. 1017–1102.

[15] W. Feng, F. Hafeez, and J. Wu, *Influence of a background magnetic field on a 2D magnetohydrodynamic flow*, Nonlinearity, 34 (2021), pp. 2527–2562.

[16] B. Gallet, M. Berhanu, and N. Mordant, *Influence of an external magnetic field on forced turbulence in a swirling flow of liquid metal*, Phys. Fluids, 21 (2009), 085107.

[17] B. Gallet and C. R. Doering, *Exact two-dimensionalization of low-magnetic-Reynolds-number flows subject to a strong magnetic field*, J. Fluid Mech., 773 (2015), pp. 154–177.

[18] L. He, L. Xu, and P. Yu, *On global dynamics of three dimensional magnetohydrodynamics: Nonlinear stability of Alfvén waves*, Ann. PDE, 4 (2018), 5.

[19] X. Hu, *Global Existence for Two Dimensional Compressible Magnetohydrodynamic Flows with Zero Magnetic Diffusivity*, preprint, arXiv:1405.0274v1 [math.AP], 2014.
[20] X. Hu and F. Lin, *Global Existence for Two Dimensional Incompressible Magnetohydrodynamic Flows with Zero Magnetic Diffusivity*, preprint, arXiv:1405.0082v1 [math.AP], 2014.
[21] R. Ji and J. Wu, *The resistive magnetohydrodynamic equation near an equilibrium*, J. Differential Equations, 268 (2020), pp. 1854–1871.
[22] A. Kiselev and V. Sverak, *Small scale creation for solutions of the incompressible two-dimensional Euler equation*, Ann. Math., 180 (2014), pp. 1205–1220.
[23] F. Lin, L. Xu, and P. Zhang, *Global small solutions to 2-D incompressible MHD system*, J. Differential Equations, 259 (2015), pp. 5440–5485.
[24] F. Lin and T. Zhang, *Global small solutions to a complex fluid model in three dimensional*, Arch. Ration. Mech. Anal., 216 (2015), pp. 905–920.
[25] H. Lin, R. Ji, J. Wu, and L. Yan, *Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation*, J. Funct. Anal., 279 (2020), 108519.
[26] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2002.
[27] R. Pan, Y. Zhou, and Y. Zhu, *Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes*, Arch. Ration. Mech. Anal., 227 (2018), pp. 637–662.
[28] E. Priest and T. Forbes, *Magnetic Reconnection, MHD Theory and Applications*, Cambridge University Press, Cambridge, 2000.
[29] X. Ren, J. Wu, Z. Xiang, and Z. Zhang, *Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion*, J. Funct. Anal., 267 (2014), pp. 503–541.
[30] X. Ren, Z. Xiang, and Z. Zhang, *Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain*, Nonlinearity, 29 (2016), pp. 1257–1291.
[31] M. Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Ration. Mech. Anal., 88 (1985), pp. 209–222.
[32] M. Schonbek and M. Wiegner, *On the decay of higher-order norms of the solutions of Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), pp. 677–685.
[33] Z. Tan and Y. Wang, *Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems*, SIAM J. Math. Anal., 50 (2018), pp. 1432–1470.
[34] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Reg. Conf. Ser. Math. 106, American Mathematical Society, Providence, RI, 2006.
[35] D. Wei and Z. Zhang, *Global well-posedness of the MHD equations in a homogeneous magnetic field*, Anal. PDE, 10 (2017), pp. 1361–1406.
[36] J. Wu, *The 2D magnetohydrodynamic equations with partial or fractional dissipation*, in Lectures on the Analysis of Nonlinear Partial Differential Equations, Morningside Lectures on Mathematics, Part 5, MLM5, International Press, Somerville, MA, 2018, pp. 283–332.
[37] J. Wu and Y. Wu, *Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion*, Adv. Math., 310 (2017), pp. 759–888.
[38] J. Wu, Y. Wu, and X. Xu, *Global small solution to the 2D MHD system with a velocity damping term*, SIAM J. Math. Anal., 47 (2015), pp. 2630–2656.
[39] J. Wu and Y. Zhu, *Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium*, Adv. Math., 377 (2021), 107466.
[40] T. Zhang, *An Elementary Proof of the Global Existence and Uniqueness Theorem to 2-D Incompressible Non-Resistive MHD System*, preprint, arXiv:1404.5681v1 [math.AP], 2014.
[41] T. Zhang, *Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field*, J. Differential Equations, 260 (2016), pp. 5450–5480.
[42] Y. Zhou and Y. Zhu, *Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain*, J. Math. Phys., 59 (2018), 081505.