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Journal of Differential Equations 271 (2021) 764–796

Journal of Differential Equations

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Stability and large-time behavior of the 2D Boussinesq equations with partial dissipation

Suhua Lai [a,b], Jiahong Wu [b], Yueyuan Zhong [c,b,*]

[a] *School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, 361005, PR China*
[b] *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States*
[c] *School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China*

Received 23 February 2020; revised 10 July 2020; accepted 16 September 2020

Abstract

This paper focuses on a special two-dimensional (2D) Boussinesq system modeling buoyancy driven fluids. It governs the motion of the velocity and temperature perturbations near the hydrostatic balance. This is a partially dissipated system with the velocity involving only the vertical dissipation. We are able to establish the global stability and the large-time behavior of the solutions. In particular, our results reveal that the buoyancy force actually stabilizes the fluids through the coupling and interaction. Without the coupling, the 2D Navier-Stokes equation with only vertical dissipation is not known to be stable. Mathematically the coupling allows us to deduce that both the velocity and the temperature obey degenerate damped wave equations, which generates the stabilization effect.

MSC: 35Q35; 35Q86; 76D03; 76D50

Keywords: Boussinesq equations; Large-time behavior; Partial dissipation; Stability

* Corresponding author.
E-mail addresses: ccnulaisuhua@163.com (S. Lai), jiahong.wu@okstate.edu (J. Wu), yueyuan.zhong@mail.bnu.edu.cn (Y. Zhong).

https://doi.org/10.1016/j.jde.2020.09.022
0022-0396/

1. Introduction

The goal of this paper is to understand the stability and large-time behavior of solutions to the following Boussinesq system

$$\begin{cases}\partial_t u + u\cdot\nabla u + \nabla P = \nu\partial_{22}u + \theta e_2, & x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 + \eta\theta = 0,\end{cases} \tag{1.1}$$

where $u=(u_1,u_2)$ denotes the velocity field, P the pressure, θ the temperature, $e_2=(0,1)$, and $\nu>0$ and $\eta>0$ are the viscosity and damping coefficients, respectively. Here we have written ∂_1 and ∂_2 for the partial derivatives ∂_{x_1} and ∂_{x_2}, respectively. The Boussinesq system focused here is slightly different from the standard ones. The dissipation in the velocity equation is degenerate and only in the vertical direction. In certain physical regimes and under suitable scaling, some partial differential equations (PDEs) modeling fluids involve only partial dissipation. A significant example is Prandtl's boundary layer equation. In addition to the standard transport term, the temperature equation also contains two extra terms. u_2 is generated from the convection when we write the equation of the perturbation near the hydrostatic equilibrium $\theta_{he}=x_2$. In fact, when we treat the temperature as a sum of the equilibrium x_2 and a perturbation θ, the standard convection term becomes $u\cdot\nabla(x_2+\theta)$, which is $u\cdot\nabla\theta+u_2$. The last term $\eta\theta$ represents thermal damping.

Due to their physical applications and mathematical significance, the Boussinesq equations have recently attracted considerable interests. The Boussinesq equations model buoyancy-driven fluids such as atmospheric and oceanographic flows, and the Rayleigh–Bénard convection (see, e.g., [13,17,37,40]). The Boussinesq equations are mathematically important. They share many crucial features with the Navier-Stokes equations. The 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. The inviscid 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [38]. In addition to the shared features with the Navier-Stokes and the Euler equations, the Boussinesq equations are also capable of modeling much richer phenomena such as degenerate waves. Furthermore, the Boussinesq system consists of an integrated and interactive system, and the coupling helps enhance the regularity and stability of the solutions. As we demonstrate in this paper, the temperature actually stabilizes the velocity field and the solutions to the system exhibit more regularity and decay property than those of each component equation in the system.

Two fundamental problems, the global (in time) regularity and stability, have been at the center of mathematical investigations on the Boussinesq equations. Due to the efforts of many researchers, significant progress has been made on the global regularity of the 2D Boussinesq equations, especially those with only partial or fractional dissipation or no dissipation at all (see, e.g., [1–4,6,8–12,14–16,20,21,23–31,33–36,39,41,44,50–58]). The study on the stability problem is relatively recent and has gained momentum in the last few years. Current investigations focused on the stability near the hydrostatic equilibrium or near the shear flow. These two steady states are very special and the stability problem on them are prominent topics in fluid dynamics, atmospherics and astrophysics. The investigations on the stability problem have so far been very fruitful ([7,18,19,46,47,49]).

The motivation for studying this particular Boussinesq system in (1.1) is two fold. The first is to reveal the phenomenon that the coupling and interaction between the velocity and the temperature actually stabilizes the fluid. The 2D Navier-Stokes equations with degenerate dissipation

(dissipation in only one direction, vertical or horizontal) are not known to be stable near the trivial solution in Sobolev spaces. When there is no dissipation at all, the Navier-Stokes equation reduces to the 2D Euler equation

$$\partial_t u + u \cdot \nabla u + \nabla P = 0, \quad x \in \mathbb{R}^2, \, t > 0$$

or, in terms of the vorticity $\omega = \nabla \times u$,

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{1.2}$$

As demonstrated in [32], $\nabla\omega$ of (1.2) allows double exponential growth in time. In particular, the velocity of the 2D Euler equations in the Sobolev space H^2 is not stable. In contrast, solution of the 2D Navier-Stokes equations with full dissipation

$$\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u, \quad x \in \mathbb{R}^2, \, t > 0$$

decays algebraically in time, as shown by Schonbek [42,43]. However, it is not clear how the solution of the Navier-Stokes equations with only vertical dissipation would behave. As we show in this paper, solution of the Boussinesq system in H^2 is actually stable in time. The coupling allows the temperature to stabilize the velocity field.

Our second motivation for this study is to develop efficient tools to extract the stabilization and regularization effects generated by the interaction of the velocity and temperature. More generally we would like to create a general framework that can handle the stability and large-time behavior problem on PDE systems with only partial dissipation. Our first step is to exploit the hidden structure due to the coupling in (1.1). Applying the Leray projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation leads to

$$\partial_t u = \nu \partial_{22} u + \mathbb{P}(\theta e_2) - \mathbb{P}(u \cdot \nabla u), \tag{1.3}$$

which separates the linear and nonlinear parts. More explicitly, in terms of the velocity components u_1 and u_2, (1.3) can be written as

$$\begin{cases} \partial_t u_1 = \nu \partial_{22} u_1 - \partial_1 \partial_2 \Delta^{-1}\theta - (u \cdot \nabla u_1 - \partial_1 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)), \\ \partial_t u_2 = \nu \partial_{22} u_2 + \theta - \partial_2 \partial_2 \Delta^{-1}\theta - (u \cdot \nabla u_2 - \partial_2 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)). \end{cases} \tag{1.4}$$

Differentiating (1.4) as well as the equation of θ in t and making several substitutions, we find (u_1, u_2, θ) satisfies

$$\begin{cases} \partial_{tt} u_1 + (\eta - \nu \partial_{22}) \partial_t u_1 - (\nu\eta \partial_{22} u_1 + \mathcal{R}_1^2 u_1) = N_1, \\ \partial_{tt} u_2 + (\eta - \nu \partial_{22}) \partial_t u_2 - (\nu\eta \partial_{22} u_2 + \mathcal{R}_1^2 u_2) = N_2, \\ \partial_{tt} \theta + (\eta - \nu \partial_{22}) \partial_t \theta - (\nu\eta \partial_{22} \theta + \mathcal{R}_1^2 \theta) = N_3, \end{cases} \tag{1.5}$$

where N_1, N_2 and N_3 are the nonlinear terms,

$$
\begin{aligned}
N_1 &= \partial_1\partial_2(\Delta)^{-1}(u\cdot\nabla\theta) - (\partial_t+\eta)(u\cdot\nabla u_1 - \partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u)),\\
N_2 &= -\partial_1\partial_1(\Delta)^{-1}(u\cdot\nabla\theta) - (\partial_t+\eta)(u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)),\\
N_3 &= (\nu\partial_{22}-\partial_t)(u\cdot\nabla\theta) + (u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)).
\end{aligned}
$$

Here $\mathcal{R}_1 = \partial_1(-\Delta)^{-\frac{1}{2}}$ denotes the standard Riesz transform (see, e.g., [22,45]). The fractional Laplacian operator is defined via the Fourier transform,

$$
\widehat{(-\Delta)^{\beta} f}(\xi) = |\xi|^{2\beta}\widehat{f}(\xi).
$$

The corresponding vorticity $\omega = \nabla\times u$ also satisfies

$$
\partial_{tt}\omega + (\eta - \nu\partial_{22})\partial_t\omega - (\nu\eta\partial_{22}\omega + \mathcal{R}_1^2\omega) = N_4, \tag{1.6}
$$

where

$$
N_4 = -(\partial_t+\eta)(u\cdot\nabla\omega) - \partial_1(u\cdot\nabla\theta).
$$

(1.6) can be obtained by taking the time derivative of the vorticity equation

$$
\partial_t\omega + u\cdot\nabla\omega = \nu\partial_{22}\omega + \partial_1\theta
$$

and making several substitutions. Amazingly u_1, u_2, θ and ω satisfy exactly the same damped and very degenerate wave type equation with different inhomogeneous terms. (1.5) and (1.6) exhibit much more stabilization and regularization properties than the original system in (1.1) can provide. By fully exploiting the damping and smoothing effects, we are able to prove the following stability and large-time behavior result.

Theorem 1.1. *Assume $(u_0,\theta_0)\in H^2$ with $\nabla\cdot u_0 = 0$. Then there exists a constant $\varepsilon > 0$ such that, if*

$$
\|(u_0,\theta_0)\|_{H^2} \le \varepsilon,
$$

then (1.1) has a unique global solution (u,θ) satisfying, for any $t > 0$,

$$
\|(u,\theta)(t)\|_{H^2}^2 + \int_0^t \left(\|(\partial_2 u,\theta)(\tau)\|_{H^2}^2 + \|(u_2,\partial_1 u_2)(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \le C\varepsilon^2,
$$

where $C > 0$ is a generic positive constant independent of ε and t.

Due to the lack of horizontal dissipation in the velocity equation, direct energy estimates involving only $\|u\|_{H^2}$ and $\|\theta\|_{H^2}$ will not lead to the proof of Theorem 1.1. When we estimate the L^2-norm of the second-order derivatives of u, or equivalently $\nabla\omega$, the immediate difficulty is how to bound the nonlinear term in the energy equality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t)\|_{L^2}^2+\nu\|\partial_2\nabla\omega\|_{L^2}^2=-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x+\int\omega\,\partial_1\theta\,\mathrm{d}x.$$

We can further write the nonlinear part as

$$\begin{aligned}&\int_{\mathbb{R}^2}\nabla w\cdot\nabla u\cdot\nabla w\,\mathrm{d}x\\&:=\int_{\mathbb{R}^2}\partial_1u_1\,(\partial_1w)^2\,\mathrm{d}x+\int_{\mathbb{R}^2}\partial_1u_2\partial_1w\,\partial_2w\,\mathrm{d}x+\int_{\mathbb{R}^2}\partial_2u\cdot\nabla w\partial_2w\,\mathrm{d}x.\end{aligned}$$

The first two terms resist suitable upper bounds when we do not have $\|\partial_1\nabla\omega\|_{L^2}^2$ on the left-hand side. What is really missing here is the control on ∂_1u_2.

The wave structure in (1.5) reveals that we may be able to control $\|u_2\|_{L^2}$ and $\|\partial_1u_2\|_{L^2}$ if we design suitable Lyapunov functional. We set the Lyapunov functional to be

$$L(t)=\|(u(t),\theta(t))\|_{H^2}^2+\lambda(\theta,u_2)+\lambda(\partial_1\theta,\partial_1u_2),$$

where the parameter λ is to be specified later and (f,g) denotes the L^2-inner product. For the sake of conciseness, we have written $\|(u(t),\theta(t))\|_{H^2}^2:=\|u(t)\|_{H^2}^2+\|\theta(t)\|_{H^2}^2$. By selecting suitable $\lambda>0$, we are able to show that, for any $t\geq0$,

$$E(t):=\|(u,\theta)(t)\|_{H^2}^2+2\int_0^t\Big(\nu\|\partial_2u(\tau)\|_{H^2}^2+\eta\|\theta(\tau)\|_{H^2}^2+\lambda\|(u_2,\partial_1u_2)(\tau)\|_{L^2}^2\Big)\mathrm{d}\tau$$

obeys

$$E(t)\leq CE(0)+CE^{\frac{3}{2}}(t).\tag{1.7}$$

Applying the bootstrapping argument (see, e.g., [48]) to (1.7), we find that, if $E(0)$ is sufficiently small, namely

$$E(0)\leq\varepsilon^2\quad\text{or}\quad\|(u_0,\theta_0)\|_{H^2}\leq\varepsilon$$

for some suitable $\varepsilon>0$, then for a constant $C>0$ and for all time $t>0$,

$$E(t)\leq C\varepsilon^2.$$

More details are provided in Section 2.

We are also able to establish certain large-time behavior on the solutions obtained in Theorem 1.1. More precisely, the following theorem on the large-time behavior holds.

Theorem 1.2. *Assume the initial data* $(u_0,\theta_0)\in H^2$, $\partial_{22}\nabla u_0\in L^2$ *satisfies* $\nabla\cdot u_0=0$ *and*

$$\|(u_0,\theta_0)\|_{H^2}\leq\varepsilon$$

for some sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution of (1.1) obtained in Theorem 1.1. Then (u, θ) obeys the following large-time behavior.

(a) *As $t \to \infty$,*

$$\|u_2(t)\|_{L^2} \to 0, \quad \|\partial_2 \nabla u(t)\|_{L^2} \to 0, \quad \|\partial_t u(t)\|_{L^2} \to 0;$$
$$\|\theta(t)\|_{L^2} \to 0, \quad \|\nabla^2 \theta(t)\|_{L^2} \to 0, \quad \|\partial_t \theta(t)\|_{H^1} \to 0.$$

(b) *For some constant $C > 0$,*

$$\|\nabla u(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}, \quad \|\nabla \theta(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}.$$

In order to prove Part (a) in Theorem 1.2, we first refine the Lemma 3.1 of [19] and then combine it with suitable energy estimates. The explicit decay rates provided in Part (b) are shown by obtaining a self-contained estimate for $\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2$.

Efforts are also devoted to exploiting the full regularization and stabilization effects that the wave structure can bring. First, we solve the linearized system

$$\begin{cases} \partial_{tt} u_1 + (\eta - \nu\partial_{22})\partial_t u_1 - (\nu\eta\partial_{22} u_1 + \mathcal{R}_1^2 u_1) = 0, \\ \partial_{tt} u_2 + (\eta - \nu\partial_{22})\partial_t u_2 - (\nu\eta\partial_{22} u_2 + \mathcal{R}_1^2 u_2) = 0, \\ \partial_{tt}\theta + (\eta - \nu\partial_{22})\partial_t \theta - (\nu\eta\partial_{22}\theta + \mathcal{R}_1^2 \theta) = 0 \end{cases} \tag{1.8}$$

via the method of operator splitting. We analyze the large-time behavior by performing the L^2 estimates on the explicit representation. Second, we combine two energy inequalities obtained from (1.8) to induce precise decay rates on several quantities of the solution. Finally we establish that any frequency away from the origin decays exponentially in time. These findings are stated in the following theorems.

Theorem 1.3. *Consider (1.8) with the initial data (u_0, θ_0). The solution of (1.8) can be explicitly represented in terms of u_0 and θ_0 as*

$$u = G_1\left(\theta_0 e_2 - \nabla\Delta^{-1}\partial_2\theta_0 + \frac{1}{2}(\eta + \nu\partial_{22})\, u_0\right) + G_2 u_0,$$
$$\theta = G_1\left(-u_{20} - \frac{1}{2}(\eta + \nu\partial_{22})\,\theta_0\right) + G_2\theta_0,$$

where G_1 and G_2 are Fourier multiplier operators,

$$\widehat{G_1}(\xi, t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad \widehat{G_2}(\xi, t) = \frac{1}{2}\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right) \tag{1.9}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2 + \left(\eta + \nu\xi_2^2\right)\lambda + \left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right) = 0$$

or

$$\lambda_1 = -\frac{1}{2}\left(\eta + \nu\xi_2^2\right)\left(1 + \sqrt{1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2}}\right),$$

$$\lambda_2 = -\frac{1}{2}\left(\eta + \nu\xi_2^2\right)\left(1 - \sqrt{1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta + \nu\xi_2^2\right)^2}}\right).$$

As a consequence, if $(u_0, \theta_0) \in L^1 \cap L^2$ *with* $\nabla \cdot u_0 = 0$*, then* (u, θ) *satisfies, for any* $0 < a < 1$*,*

$$\|(u, \theta)\|_{L^2} \le C(a)\ (1+t)^{-\frac{a}{2}}\ \|(u_0, \theta_0)\|_{L^1 \cap L^2},$$

where $C = C(a)$ *is a constant depending on* a*.*

The decay rate for the first component u_1 appears to be optimal and can not be improved to be $(1+t)^{-\frac{1}{2}}$. The decay rates for u_2 and θ can actually be improved to $(1+t)^{-\frac{1}{2}}$.

Theorem 1.4. *Let the initial data* $(u_0, \theta_0) \in L^2$*,* $\partial_2\theta_0 \in L^2$ *and* $\nabla \cdot u_0 = 0$*. Let* (u, θ) *be the corresponding solution of (1.8). Then* (u_2, θ) *satisfies,*

$$\|(u_2, \theta)(t)\|_{L^2} \le C(u_0, \theta_0)\ (1+t)^{-\frac{1}{2}}, \qquad \|\partial_t\theta(t)\|_{L^2} \le C(u_0, \theta_0)\ (1+t)^{-\frac{1}{2}},$$

where C *are constants depending on the initial data only.*

When the Fourier frequency is away from the origin, the solution actually decays exponentially in time. To precisely state our result, we define $\widehat{\chi}(\xi)$ to be the following cutoff function in the frequency space

$$\widehat{\chi}(\xi) = \begin{cases} 0, & \text{if} \quad \xi \in D, \\ 1, & \text{if} \quad \xi \in D^c, \end{cases}$$

where D is given by

$$D := \left\{\xi \in \mathbb{R}^2 :\ |\xi| < \varrho \text{ and } \ |\xi_2| > |\xi_1|\right\} \quad \text{with } \varrho > 0 \text{ being fixed.} \tag{1.10}$$

Then

$$\widehat{\chi * f}(\xi) = \widehat{\chi}(\xi)\,\widehat{f}(\xi). \tag{1.11}$$

Theorem 1.5. *Assume the initial data* (u_0, θ_0) *satisfies*

$$\nabla \cdot u_0 = 0, \quad \chi * u_0,\ \chi * \theta_0,\ \chi * \partial_2 u_0,\ \chi * \partial_2 \theta_0,\ \chi * \partial_{22} u_0 \in L^2.$$

Then the corresponding solution (u, θ) *of* (1.8) *obeys the following exponential decay estimates, for two constants* $C = C(u_0, \theta_0, \varrho) > 0$ *and* $c = c(\nu, \eta, \varrho) > 0$*,*

$$\|(\chi * u, \chi * \theta)\|_{L^2},\ \|(\chi * \partial_2 u, \chi * \partial_2 \theta)\|_{L^2},\ \ \|(\chi * \partial_t u, \chi * \partial_t \theta)\|_{L^2} \le C\, e^{-c(\eta,\nu,\varrho)t}.$$

The rest of this paper is devoted to the proofs of the results described above. Section 2 proves Theorem 1.1 and Theorem 1.2. Section 3 solves the linearized system in (1.8) and proves Theorem 1.3. The last section proves both Theorem 1.4 and Theorem 1.5.

2. Proofs the Theorem 1.1 and Theorem 1.2

This section proves Theorem 1.1 and Theorem 1.2. As we have explained in the introduction, due to the lack of horizontal dissipation, the stability result stated in Theorem 1.1 can not be shown by classical energy estimates. The approach here is to construct a suitable Lyapunov functional by taking into account of the wave structure in (1.5). This Lyapunov functional recovers the damping and the horizontal dissipative effect on u_2. We can then close the estimates and obtain the desired inequality in (1.7). The bootstrapping argument then allows us to conclude the desired stability.

Theorem 1.2 contains two types of large-time behavior results. Part (a) assesses that six quantities including $\|u_2(t)\|_{L^2}$, $\|\partial_2 \nabla u(t)\|_{L^2}$, $\|\partial_t u(t)\|_{L^2}$, $\|\theta(t)\|_{L^2}$, $\|\nabla^2 \theta(t)\|_{L^2}$ and $\|\partial_t \theta(t)\|_{H^1}$ approach zero as $t \to \infty$. Part (b) states that $\|\nabla u(t)\|_{L^2}$ and $\|\nabla \theta(t)\|_{L^2}$ decays at the rate $(1+t)^{-\frac{1}{2}}$. To prove Part (a), we first refine Lemma 3.1 of [19] and then apply it to suitable bounds involving the six quantities. The explicit decay rates for $\|\nabla u(t)\|_{L^2}$ and $\|\nabla \theta(t)\|_{L^2}$ are obtained by applying a simple fact (see Lemma 2.5 below) to suitable upper bounds for $\|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2$.

We use several anisotropic inequalities extensively. They are given in the following two lemmas. Lemma 2.1 can be found in [5] while Lemma 2.2 contains several classical anisotropic inequalities, which can be shown by making use of the following basic one-dimensional inequality,

$$\|g\|_{L^\infty(\mathbb{R})} \le \sqrt{2} \|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

Lemma 2.1. *Assume* f*,* g*,* h*,* $\partial_2 g$ *and* $\partial_1 h$ *are all in* $L^2(\mathbb{R}^2)$*. Then, for a pure constant* C*,*

$$\int_{\mathbb{R}^2} |fgh|\, \mathrm{d}x \le C\, \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_1 h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.$$

Lemma 2.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

Consequently,

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}},$$
$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}.$$

The following lemma is a refined version of Lemma 3.1 in [19], which is more suitable for the purpose of applications.

Lemma 2.3. *Let $f = f(t)$ with $t \in [0, \infty)$ be nonnegative continuous function. Assume f is integrable on $[0, \infty)$,*

$$\int_0^\infty f(t)\,\mathrm{d}t < \infty. \tag{2.1}$$

Assume that for any $\delta > 0$, there is $\rho > 0$ such that, for any $0 \le t_1 < t_2$ with $t_2 - t_1 \le \rho$,

$$\textit{either } f(t_2) \le f(t_1) \quad \textit{or} \quad f(t_2) \ge f(t_1) \textit{ and } f(t_2) - f(t_1) \le \delta.$$

Then

$$f(t) \to 0 \quad \textit{as } t \to \infty.$$

Proof. Let $\varepsilon > 0$ be a given small number. By the assumption of this lemma, for $\delta = \varepsilon/2$, there is $\rho > 0$ such that, for any $0 < t_2 - t_1 \le \rho$,

$$\text{either } f(t_2) \le f(t_1) \quad \text{or} \quad f(t_2) \ge f(t_1) \text{ and } f(t_2) - f(t_1) \le \delta. \tag{2.2}$$

It follows from (2.1) that there exists $M_1 > 0$ such that

$$\int_{M_1}^\infty f(t)\,\mathrm{d}t < \frac{\varepsilon\rho}{2}. \tag{2.3}$$

We can then conclude that, for any $t > M := M_1 + \rho$,

$$f(t) < \varepsilon. \tag{2.4}$$

To show (2.4), we consider the interval $[t - \rho, t]$. Since $t - \rho > M_1$, (2.3) implies

$$\int_{t-\rho}^{t} f(\tau)\,\mathrm{d}\tau < \frac{\varepsilon\rho}{2}. \tag{2.5}$$

Then there must be at least one point $t_1 \in [t - \rho, t]$ such that $f(t_1) < \frac{\varepsilon}{2}$. Otherwise, (2.5) would be false. By (2.2),

$$\text{either } f(t) \le f(t_1) \quad \text{or} \quad f(t) \ge f(t_1) \text{ and } f(t) - f(t_1) \le \delta.$$

Either case would lead to (2.4). This completes the proof of Lemma 2.3. $\square$

A special consequence of Lemma 2.3 is the following large-time behavior result for functions in $W^{1,1}([0,\infty))$.

Lemma 2.4. *Assume $f \in W^{1,1}([0,\infty))$, namely*

$$\int_0^\infty |f(t)|\,\mathrm{d}t < \infty \quad \textit{and} \quad \int_0^\infty |f'(t)|\,\mathrm{d}t < \infty.$$

Then $f(t) \to 0$ as $t \to \infty$.

Another useful fact to be used frequently is about the decay rate of an integrable function.

Lemma 2.5. *Let $f = f(t)$ be a nonnegative function satisfying, for two constants $C_0 > 0$ and $C_1 > 0$,*

$$\int_0^\infty f(\tau)\,\mathrm{d}\tau \le C_0 < \infty \quad \textit{and} \quad f(t) \le C_1 f(s) \quad \textit{for any } 0 \le s < t. \tag{2.6}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$ and for any $t > 0$,

$$f(t) \le C_2(1+t)^{-1}. \tag{2.7}$$

Lemma 2.5 remains valid even if (2.6) only holds for large $s < t$. The proof of Lemma 2.5 is really simple, but we provide it for the convenience of readers.

Proof. For any $0 \le t \le 1$, we have by (2.6),

$$f(t) \le C_1 f(0). \tag{2.8}$$

By (2.6), for any $t \ge 1$,

$$C_0 \ge \int_{\frac{t}{2}}^{t} f(\tau)\,\mathrm{d}\tau \ge \int_{\frac{t}{2}}^{t} C_1^{-1} f(t)\,\mathrm{d}\tau = C_1^{-1} f(t)\frac{t}{2}$$

or

$$f(t) \le 2C_0 C_1 t^{-1}. \tag{2.9}$$

(2.8) and (2.9) then imply (2.7) for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$. This completes the proof of Lemma 2.5. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We work with the Lyapunov functional

$$L(t) = \|(u(t), \theta(t))\|_{H^2}^2 + \lambda(\theta, u_2) + \lambda(\partial_1\theta, \partial_1 u_2)$$

and shows that, for suitably chosen $\lambda_0 > 0$ (specified in (2.20)) and for any $t \geq 0$,

$$\begin{aligned} E(t) :=& \|(u, \theta)(t)\|_{H^2}^2 \\ &+2\int_0^t \left(\nu\|\partial_2 u(\tau)\|_{H^2}^2 + \eta\|\theta(\tau)\|_{H^2}^2 + \lambda_0\|(u_2, \partial_1 u_2)(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \end{aligned}$$

obeys

$$E(t) \leq CE(0) + CE^{\frac{3}{2}}(t). \tag{2.10}$$

We start with the L^2 bound. Dotting (1.1) by (u, θ) in L^2, integrating by parts and using $\nabla \cdot u = 0$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u, \theta)\|_{L^2}^2 + \nu\|\partial_2 u\|_{L^2}^2 + \eta\|\theta\|_{L^2}^2 = 0. \tag{2.11}$$

To estimate the $\dot{H}^2$-norm, we resort to the vorticity equation

$$\partial_t\omega + u \cdot \nabla\omega = \nu\partial_{22}\omega + \partial_1\theta. \tag{2.12}$$

Applying ∇ to (2.12), dotting with $\nabla\omega$, and applying Δ to the θ-equation in (1.1) and multiplying by $\Delta\theta$, we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\omega, \Delta\theta)\|_{L^2}^2 + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\Delta\theta\|_{L^2}^2 \\ &\quad = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \mathrm{d}x - \int \Delta(u \cdot \nabla\theta)\Delta\theta \mathrm{d}x. \end{aligned} \tag{2.13}$$

According to (1.4), the equation of u_2 can be written as, after eliminating the pressure term,

$$\partial_t u_2 + u \cdot \nabla u_2 - \partial_2\Delta^{-1}\nabla \cdot \nabla \cdot (u \otimes u) = \nu\partial_{22}u_2 + \Delta^{-1}\partial_{11}\theta. \tag{2.14}$$

Combining with the θ-equation of (1.1), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\theta, u_2) + \|u_2\|_{L^2}^2 =& -(u \cdot \nabla\theta, u_2) - \eta(\theta, u_2) - (\theta, u \cdot \nabla u_2) \\ &+ \left(\theta, \partial_2\Delta^{-1}\nabla \cdot \nabla \cdot (u \otimes u)\right) + (\theta, \nu\partial_{22}u_2) + \left(\theta, \Delta^{-1}\partial_{11}\theta\right). \end{aligned} \tag{2.15}$$

Similarly,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\partial_1\theta, \partial_1 u_2) + \|\partial_1 u_2\|_{L^2}^2 \\
=& -(\partial_1(u\cdot\nabla\theta), \partial_1 u_2) - \eta\,(\partial_1\theta, \partial_1 u_2) - (\partial_1\theta, \partial_1(u\cdot\nabla u_2)) \\
&+ \left(\partial_1\theta, \partial_{12}\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\right) + (\partial_1\theta, \nu\partial_{122}u_2) + \left(\partial_1\theta, \Delta^{-1}\partial_{111}\theta\right).
\end{aligned} \tag{2.16}
$$

For a constant $\lambda > 0$, (2.11)+(2.13)+ λ(2.15)+λ(2.16) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(u,\theta)\|_{H^2}^2 + 2\lambda(\theta, u_2) + 2\lambda(\partial_1\theta, \partial_1 u_2)\right) + \nu\|\partial_2 u\|_{H^2}^2 \\
&\quad + \eta\|\theta\|_{H^2}^2 + \lambda\|u_2\|_{L^2}^2 + \lambda\|\partial_1 u_2\|_{L^2}^2 := I + J + \sum_{i=1}^{6} K_i,
\end{aligned} \tag{2.17}
$$

where

$$
\begin{aligned}
&I = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x, \\
&J = -\int \Delta(u\cdot\nabla\theta)\Delta\theta\,\mathrm{d}x, \\
&K_1 = -\lambda\,(u\cdot\nabla\theta, u_2) - \lambda\,(\partial_1(u\cdot\nabla\theta), \partial_1 u_2), \\
&K_2 = -\lambda\eta\,(\theta, u_2) - \lambda\eta\,(\partial_1\theta, \partial_1 u_2), \\
&K_3 = -\lambda\,(\theta, u\cdot\nabla u_2) - \lambda\,(\partial_1\theta, \partial_1(u\cdot\nabla u_2)), \\
&K_4 = \lambda\left(\theta, \partial_2\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\right) + \lambda\left(\partial_1\theta, \partial_{12}\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\right), \\
&K_5 = \lambda\,(\theta, \nu\partial_{22}u_2) + \lambda\,(\partial_1\theta, \nu\partial_{122}u_2), \\
&K_6 = \lambda\left(\theta, \Delta^{-1}\partial_{11}\theta\right) + \lambda\left(\partial_1\theta, \Delta^{-1}\partial_{111}\theta\right).
\end{aligned}
$$

Thanks to Lemma 2.1 and by Hölder's inequality, Sobolev's embedding and Young's inequality,

$$
\begin{aligned}
I &= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x \\
&= -2\int \partial_{21}\omega u_2\partial_1\omega\,\mathrm{d}x - \int \partial_1\omega\partial_1 u_2\partial_2\omega\,\mathrm{d}x \\
&\quad - \int \partial_2\omega\partial_2 u_1\partial_1\omega\,\mathrm{d}x - \int \partial_2\omega\partial_2 u_2\partial_2\omega\,\mathrm{d}x \\
&\le C\|\partial_{21}\omega\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{21}\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1\omega\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_{12}u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{21}\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_2 u\|_{L^\infty}\|\partial_2\omega\|_{L^2}\big(\|\partial_1\omega\|_{L^2} + \|\partial_2\omega\|_{L^2}\big) \\
&\le C\|u\|_{H^2}\big(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2\big).
\end{aligned}
$$

Similarly, due to Lemma 2.2,

$$
\begin{aligned}
J &= -\int (\Delta u \cdot \nabla\theta)\, \Delta\theta \mathrm{d}x - 2\int \left(\nabla u \cdot \nabla^2\theta\right) \Delta\theta \mathrm{d}x \\
&\le C\|\Delta\theta\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u\|_{L^2}^{\frac{1}{2}} + C\|\nabla u\|_{L^\infty}\|\theta\|_{H^2}^2 \\
&\le C\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^2\|\partial_2 u\|_{H^2}^{\frac{1}{2}} + C\|\nabla u\|_{H^1}^{\frac{1}{2}}\|\partial_2\nabla u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^2}^2 \\
&\le C\big(\|u\|_{H^2} + \|\theta\|_{H^2}\big)\big(\|\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\big).
\end{aligned}
$$

To bound K_1, we again apply Lemma 2.1 to get

$$
\begin{aligned}
K_1 &= -\lambda\int (u\cdot\nabla\theta)\, u_2 \mathrm{d}x - \lambda\int (u\cdot\nabla\partial_1\theta)\,\partial_1 u_2 \mathrm{d}x - \lambda\int (\partial_1 u\cdot\nabla\theta)\,\partial_1 u_2 \mathrm{d}x \\
&\le C\lambda\,\|u\|_{L^\infty}\|u_2\|_{L^2}\|\nabla\theta\|_{L^2} + C\lambda\,\|u\|_{L^\infty}\|\nabla\partial_1\theta\|_{L^2}\|\partial_1 u_2\|_{L^2} \\
&\quad + C\lambda\,\|\partial_1 u_2\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_{12} u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}} \\
&\le C\lambda\left(\|u\|_{H^2} + \|\theta\|_{H^2}\right)\big(\|u_2\|_{L^2}^2 + \|\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2\big).
\end{aligned}
$$

By Hölder's inequality,

$$
K_2 \le \lambda\eta\|\theta\|_{H^2}(\|u_2\|_{L^2} + \|\partial_1 u_2\|_{L^2}) \le \frac{\eta}{8}\|\theta\|_{H^2}^2 + 2\lambda^2\eta(\|u_2\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2).
$$

By Hölder's and Sobolev's inequalities,

$$
K_3 \le C\lambda\,\|u\|_{L^\infty}\|\theta\|_{H^2}\|\nabla u_2\|_{L^2} \le C\lambda\,\|u\|_{H^2}\big(\|\theta\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2\big).
$$

Recalling that the singular integral operators $\mathcal{R}_{ij} = \partial_i\partial_j(-\Delta)^{-1}$ with $i, j = 1, 2$ are bounded on L^p for $1 < p < \infty$ (see [45]), namely

$$
\|\mathcal{R}_{ij} f\|_{L^p} \le C\|f\|_{L^p}, \quad 1 < p < \infty, \tag{2.18}
$$

we have

$$
K_4 \le C\lambda\,\|\theta\|_{H^2}\|u\|_{H^2}\|\partial_2 u\|_{H^2} \le C\lambda\,\|u\|_{H^2}\big(\|\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\big).
$$

By Hölder's inequality,

$$
K_5 \le \lambda\nu\|\theta\|_{H^2}\|\partial_2 u\|_{H^2} \le \frac{\eta}{8}\|\theta\|_{H^2}^2 + \frac{2\lambda^2\nu^2}{\eta}\|\partial_2 u\|_{H^2}^2.
$$

By (2.18), K_6 is bounded by

$$
K_6 \le c_0\lambda\|\theta\|_{H^2}^2.
$$

Collecting the bounds for I, J and K_1 through K_6 and inserting them in (2.17), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|(u,\theta)\|_{H^2}^2+2\lambda(\theta,u_2)+2\lambda(\partial_1\theta,\partial_1 u_2)\big)+\big(\nu-\frac{2\lambda^2\nu^2}{\eta}\big)\|\partial_2 u\|_{H^2}^2$$
$$+\big(\frac{3\eta}{4}-c_0\lambda\big)\|\theta\|_{H^2}^2+\big(\lambda-2\lambda^2\eta\big)\big(\|u_2\|_{L^2}^2+\|\partial_1 u_2\|_{L^2}^2\big)$$
$$\leq C(1+\lambda)\big(\|u\|_{H^2}+\|\theta\|_{H^2}\big)\big(\|\theta\|_{H^2}^2+\|u_2\|_{L^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u\|_{H^2}^2\big). \tag{2.19}$$

We now specify λ and require that

$$0<\lambda\leq\lambda_0=\min\left\{\frac{1}{2},\frac{1}{2}\sqrt{\frac{\eta}{\nu}},\frac{\eta}{4c_0},\frac{1}{4\eta}\right\}. \tag{2.20}$$

When (2.20) is satisfied, we have

$$\nu-\frac{2\lambda^2\nu^2}{\eta}\geq\frac{1}{2}\nu,\qquad \frac{3\eta}{4}-c_0\lambda\geq\frac{1}{2}\eta,\qquad \lambda-2\lambda^2\eta\geq\frac{1}{2}\lambda$$

and

$$\|(u,\theta)\|_{H^2}^2+2\lambda(\theta,u_2)+2\lambda(\partial_1\theta,\partial_1 u_2)$$
$$\geq\|(u,\theta)\|_{H^2}^2-\lambda\,(\|\theta\|_{H^1}^2+\|u_2\|_{H^1}^2)\geq\frac{1}{2}\|u\|_{H^2}^2+\frac{1}{2}\|\theta\|_{H^2}^2.$$

Choosing $\lambda=\lambda_0$ and integrating (2.19) in t, we obtain

$$\sup_{0\leq\tau\leq t}\|(u,\theta)\|_{H^2}^2+2\int_0^t\big(\nu\|\partial_2 u\|_{H^2}^2+\eta\|\theta\|_{H^2}^2+\lambda_0\|(u_2,\partial_1 u_2)\|_{L^2}^2\big)\mathrm{d}\tau$$
$$\leq C\|(u_0,\theta_0)\|_{H^2}^2+C\sup_{0\leq\tau\leq t}\|(u,\theta)\|_{H^2}\int_0^t\big(\nu\|\partial_2 u\|_{H^2}^2+\eta\|\theta\|_{H^2}^2+\lambda_0\|(u_2,\partial_1 u_2)\|_{L^2}^2\big)\mathrm{d}\tau,$$

which implies (2.10), namely

$$E(t)\leq C_3E(0)+C_4\,E^{\frac{3}{2}}(t). \tag{2.21}$$

The bootstrapping argument then allows us to establish the stability of Theorem 1.1 if the initial data is sufficiently small,

$$E(0)\leq\frac{1}{16C_3C_4^2}\quad\text{or}\quad\|(u_0,\theta_0)\|_{H^2}\leq\varepsilon\leq\frac{1}{4\sqrt{C_3}C_4}. \tag{2.22}$$

In fact, if we make the ansatz that, for $0<T\leq\infty$,

$$E(T)\leq\frac{1}{4C_4^2},$$

then (2.21) implies

$$E(T) \le C_3 E(0) + C_4 E^{\frac{3}{2}}(T) \le C_3 E(0) + C_4 \frac{1}{2C_4} E(T) \quad \text{or} \quad \frac{1}{2} E(T) \le C_3 E(0),$$

which, according to the smallness assumption on the initial data (2.22), leads to

$$E(T) \le 2C_3 E(0) \le \frac{1}{8C_4^2}. \tag{2.23}$$

The bootstrapping argument then concludes that $T = \infty$ and (2.23) holds for all time,

$$\sup_{0\le\tau<\infty} \|(u,\theta)\|_{H^2}^2 + 2\int_0^\infty \big(\nu\|\partial_2 u\|_{H^2}^2 + \eta\|\theta\|_{H^2}^2 + \lambda_0\|(u_2, \partial_1 u_2)\|_{L^2}^2\big)\mathrm{d}\tau \le 2C_3\varepsilon^2.$$

This finishes the proof for the global stability. It is very easy to check that any two solutions $(u^{(1)}, P^{(1)}, \theta^{(1)})$ and $(u^{(2)}, P^{(2)}, \theta^{(2)})$ of (1.1) with one of them in the regularity class, say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T; H^2)$ must coincide. In fact, the difference between the two solutions $(\overline{u}, \overline{P}, \overline{\theta})$ with

$$\overline{u} = u^{(2)} - u^{(1)}, \quad \overline{P} = P^{(2)} - P^{(1)}, \quad \overline{\theta} = \theta^{(2)} - \theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t \overline{u} + u^{(2)} \cdot \nabla\overline{u} + \overline{u} \cdot \nabla u^{(1)} + \nabla\overline{P} = \nu\partial_{22}\overline{u} + \overline{\theta} e_2, \\ \partial_t \overline{\theta} + u^{(2)} \cdot \nabla\overline{\theta} + \overline{u} \cdot \nabla\theta^{(1)} + \overline{u}_2 + \eta\overline{\theta} = 0, \\ \nabla\cdot\overline{u} = 0, \\ \overline{u}(x,0) = 0, \ \ \overline{\theta}(x,0) = 0. \end{cases} \tag{2.24}$$

Taking the L^2-inner product of (2.24) with $(\overline{u}, \overline{\theta})$, by Lemma 2.1, Young's inequality and the uniformly global bounds for $\|(\theta^{(1)}, u^{(1)})\|_{H^2}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\overline{u},\overline{\theta})\|_{L^2}^2 + \nu\|\partial_2\overline{u}\|_{L^2}^2 + \eta\|\overline{\theta}\|_{L^2}^2 \\
=& -\int \overline{u}\cdot\nabla u^{(1)}\cdot\overline{u}\mathrm{d}x - \int \overline{u}\cdot\nabla\theta^{(1)}\overline{\theta}\mathrm{d}x \\
\le& C\|\overline{u}\|_{L^2}\|\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\overline{\theta}\|_{L^2}\|\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\theta^{(1)}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta^{(1)}\|_{L^2}^{\frac{1}{2}} \\
\le& C\|\overline{u}\|_{L^2}^{\frac{3}{2}}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}} + C\|\overline{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{u}\|_{L^2}^{\frac{1}{2}}\|\overline{\theta}\|_{L^2} \\
\le& \frac{\nu}{2}\|\partial_2\overline{u}\|_{L^2}^2 + C\|(\overline{u},\overline{\theta})\|_{L^2}^2.
\end{aligned}$$

Grönwall's inequality then yields the desired uniqueness.

$$\|\overline{u}\|_{L^2}^2 = \|\overline{\theta}\|_{L^2}^2 = 0.$$

This completes the proof of Theorem 1.1. $\square$

Next we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. This proof assesses the large time behavior of the solution (u, θ) to (1.1). The solution (u, θ) satisfies

$$\sup_{0\le\tau<\infty} \|(u,\theta)\|_{H^2}^2 + 2\int_0^\infty \left(\nu\|\partial_2 u\|_{H^2}^2 + \eta\|\theta\|_{H^2}^2 + \lambda_0\|(u_2, \partial_1 u_2)\|_{L^2}^2\right) d\tau \le 2C_3\varepsilon^2. \tag{2.25}$$

We now prove the decay results and start by showing that, as $t \to \infty$,

$$\|u_2(t)\|_{L^2} \to 0.$$

This is shown by applying Lemma 2.3. We verify the conditions of Lemma 2.3. Due to (2.25),

$$\int_0^\infty \|u_2(t)\|_{L^2}^2 \, dt < \infty.$$

We perform the energy estimates to verify the uniform continuity in Lemma 2.3. Taking the inner product of (2.14) with u_2, integrating by parts and invoking (2.18), we obtain

$$\begin{aligned}\frac{d}{dt}\|u_2\|_{L^2}^2 + 2\nu\|\partial_2 u_2\|_{L^2}^2 &= 2\int \Delta^{-1}\partial_{11}\theta u_2 dx - 2\int \Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\partial_2 u_2 dx\\ &\le C\|\theta\|_{L^2}\|u_2\|_{L^2} + C\|\partial_2 u_2\|_{L^2}\|u\|_{L^4}^2\\ &\le C\|\theta\|_{L^2}\|u\|_{L^2} + C\|u\|_{H^2}^3.\end{aligned}$$

For any $0 \le s \le t < \infty$, we integrate the inequality above in time and use the upper bound in (2.25) to obtain

$$\|u_2(t)\|_{L^2}^2 - \|u_2(s)\|_{L^2}^2 \le C\left(\varepsilon^2 + \varepsilon^3\right)(t-s).$$

Applying Lemma 2.3 leads to

$$\|u_2\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

We now turn to the proof that $\|\theta\|_{H^2} \to 0$ as $t \to \infty$. The idea is still to apply Lemma 2.3. (2.25) provides the time integrability,

$$\int_0^\infty \|\theta(t)\|_{H^2}^2 \, \mathrm{d}t < \infty. \tag{2.26}$$

The verification of the uniform continuity condition in Lemma 2.3 relies on the energy estimate on $\|\theta(t)\|_{H^2}$. Taking the H^2-inner product of the θ-equation in (1.1) with θ, making use of Lemma 2.1, Lemma 2.2 and Young's inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^2}^2 + \eta\|\theta\|_{H^2}^2 &= -\int u_2\theta \mathrm{d}x - \int \Delta u_2 \Delta\theta \mathrm{d}x \\
&\quad - 2\int \left(\nabla u \cdot \nabla^2\theta\right)\Delta\theta \mathrm{d}x - \int (\Delta u \cdot \nabla\theta)\,\Delta\theta \mathrm{d}x \\
&\le \|u_2\|_{H^2}\|\theta\|_{H^2} + C\|\nabla u\|_{H^1}^{\frac{1}{2}}\|\partial_2\nabla u\|_{H^1}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^2 \\
&\quad + C\|\Delta\theta\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\eta}{2}\|\theta\|_{H^2}^2 + C\|\partial_2 u\|_{H^2}^2\|\theta\|_{H^2}^2 + C\|u\|_{H^2}^2(\|\theta\|_{H^2}^2 + 1).
\end{aligned}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^2}^2 + \eta\|\theta\|_{H^2}^2 \le C\|\partial_2 u\|_{H^2}^2\|\theta\|_{H^2}^2 + C\|u\|_{H^2}^2(\|\theta\|_{H^2}^2 + 1).$$

Ignoring the term $\eta\|\theta\|_{H^2}^2$ and combining the first term and the third term, we find that

$$A(t) := \exp\left\{-C\int_0^t \|\partial_2 u\|_{H^2}^2 \mathrm{d}\tau\right\}\|\theta(t)\|_{H^2}^2$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) \le C\exp\left\{-C\int_0^t \|\partial_2 u\|_{H^2}^2 \mathrm{d}\tau\right\}\|u\|_{H^2}^2(\|\theta\|_{H^2}^2 + 1).$$

Integrating in time and using (2.25), we obtain, for any $0 \le s \le t$,

$$A(t) - A(s) \le C\,\varepsilon^2\,(\varepsilon^2 + 1)(t - s).$$

(2.26) also implies

$$\int_0^\infty A(\tau)\,\mathrm{d}\tau < \infty.$$

By Lemma 2.3, $A(t) \to 0$ as $t \to \infty$. Therefore,

$$\|\theta(t)\|_{H^2}^2 = \exp\left\{C\int_0^t \|\partial_2 u\|_{H^2}^2 \mathrm{d}\tau\right\} A(t) \le e^{C\varepsilon^2} A(t) \to 0.$$

We now prove the decay rate

$$\|(\omega(t), \nabla\theta(t))\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.$$

It follows from the equation of ω in (2.12) and the equation of θ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega, \nabla\theta)(t)\|_{L^2}^2 + \eta\|\nabla\theta\|_{L^2}^2 + \nu\|\partial_2\omega\|_{L^2}^2 \\
&= -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta \mathrm{d}x \\
&\le C\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}} \\
&\le C\big(\|u\|_{H^2} + \|\theta\|_{H^2}\big)\big(\|\nabla\theta\|_{L^2}^2 + \|\partial_2\omega\|_{L^2}^2\big) \\
&\le c_1\varepsilon\big(\|\nabla\theta\|_{L^2}^2 + \|\partial_2\omega\|_{L^2}^2\big),
\end{aligned}$$

where we have invoked the bound in (2.25). Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2\big) + 2\big(\min\{\eta, \nu\} - c_1\varepsilon\big)\big(\|\nabla\theta\|_{L^2}^2 + \|\partial_2\omega\|_{L^2}^2\big) \le 0,$$

where ε is taken to be sufficiently small such that $\min\{\eta, \nu\} - c_1\varepsilon \ge 0$. Therefore, $Y(t) := \|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2$ is a non-increasing function for $t \in [0, \infty)$. (2.25) also implies that $\|\theta(t)\|_{H^2}^2$, $\|\partial_2 u\|_{L^2}^2$ and $\|\partial_1 u_2\|_{L^2}^2$ are all time integrable. That is

$$\int_0^\infty Y(t)\,\mathrm{d}t < \infty.$$

By Lemma 2.5,

$$\|(\omega, \nabla\theta)\|_{L^2} \le C(1+t)^{-\frac{1}{2}} \quad \text{as} \quad t \to \infty.$$

Next we show that, as $t \to \infty$,

$$\|\partial_2\omega(t)\|_{L^2} \to 0, \quad \|\partial_t u(t)\|_{L^2} \to 0 \quad \text{and} \quad \|\partial_t\theta(t)\|_{H^1} \to 0.$$

In order to obtain the large time behavior of $\|\partial_2\omega\|_{L^2}$, we make use of the wave equation in (1.6), namely

$$\partial_{tt}\omega + (\eta - \nu\partial_{22})\partial_t\omega - (\eta\nu\partial_{22}\omega + \mathcal{R}_1^2\omega) = N_4. \tag{2.27}$$

Taking the L^2-inner product of (2.27) with $\partial_t\omega$, integrating by parts, and bounding the nonlinear terms by Lemma 2.1, Sobolev's embedding and Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_t\omega\|_{L^2}^2+\eta\nu\|\partial_2\omega\|_{L^2}^2+\|\mathcal{R}_1\omega\|_{L^2}^2\big)+\eta\|\partial_t\omega\|_{L^2}^2+\nu\|\partial_2\partial_t\omega\|_{L^2}^2\\
&=-\int\partial_1(u\cdot\nabla\theta)\partial_t\omega\mathrm{d}x-\eta\int(u\cdot\nabla\omega)\partial_t\omega\mathrm{d}x-\int\partial_t(u\cdot\nabla\omega)\partial_t\omega\mathrm{d}x\\
&=-\int\partial_1 u\cdot\nabla\theta\partial_t\omega\mathrm{d}x-\int u\cdot\nabla\partial_1\theta\partial_t\omega\mathrm{d}x-\eta\int u\cdot\nabla\omega\partial_t\omega\mathrm{d}x\\
&\quad-\int\partial_t u\cdot\nabla\omega\partial_t\omega\mathrm{d}x\\
&\le C\|\partial_1 u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac12}\|\partial_1\nabla\theta\|_{L^2}^{\frac12}\|\partial_t\omega\|_{L^2}^{\frac12}\|\partial_2\partial_t\omega\|_{L^2}^{\frac12}\\
&\quad+C\|\partial_1\nabla\theta\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_t\omega\|_{L^2}^{\frac12}\|\partial_2\partial_t\omega\|_{L^2}^{\frac12}\\
&\quad+C\eta\|\partial_t\omega\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla\omega\|_{L^2}^{\frac12}\|\partial_2\nabla\omega\|_{L^2}^{\frac12}\\
&\quad+C\|\partial_t\omega\|_{L^2}\|\partial_t u\|_{L^2}^{\frac12}\|\partial_1\partial_t u\|_{L^2}^{\frac12}\|\nabla\omega\|_{L^2}^{\frac12}\|\partial_2\nabla\omega\|_{L^2}^{\frac12}\\
&\le\frac{\nu}{2}\|\partial_2\partial_t\omega\|_{L^2}^2+\frac{\eta}{2}\|\partial_t\omega\|_{L^2}^2\\
&\quad+C\big(\|u\|_{H^2}^2+\|u\|_{H^2}^2\|\partial_t u\|_{L^2}^2\big)\big(\|\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{L^2}^2\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_t\omega\|_{L^2}^2+\eta\nu\|\partial_2\omega\|_{L^2}^2+\|\mathcal{R}_1\omega\|_{L^2}^2\big)+\eta\|\partial_t\omega\|_{L^2}^2+\nu\|\partial_2\partial_t\omega\|_{L^2}^2\\
&\le C\big(\|u\|_{H^2}^2+\|u\|_{H^2}^2\|\partial_t u\|_{L^2}^2\big)\big(\|\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{L^2}^2\big).
\end{aligned}
\tag{2.28}
$$

Next we establish a bound for $\|\partial_t u\|_{L^\infty(0,\infty;L^2)}$. According to (1.3),

$$
\partial_t u+\mathbb{P}(u\cdot\nabla u)=\nu\partial_{22}u+\mathbb{P}(\theta e_2),
$$

where $\mathbb{P}=I-\nabla\Delta^{-1}\nabla\cdot$ denotes the Leray projection onto divergence-free vector fields. By the fact that $\|\mathbb{P}f\|_{L^2}\le\|f\|_{L^2}$,

$$
\begin{aligned}
\|\partial_t u\|_{L^2}&\le\|\mathbb{P}(u\cdot\nabla u)\|_{L^2}+\nu\|\partial_{22}u\|_{L^2}+\|\mathbb{P}(\theta e_2)\|_{L^2}\\
&\le C\|u\|_{H^2}\|\omega\|_{L^2}+C\|\partial_2\omega\|_{L^2}+\|\theta\|_{L^2}.
\end{aligned}
\tag{2.29}
$$

The global uniform bound in (2.25) implies

$$
\partial_t u\in L^\infty(0,\infty;\ L^2).
\tag{2.30}
$$

Integrating (2.28) from 0 to T and invoking the uniform bounds in (2.25) and (2.30), we have

$$
\begin{aligned}
&\|\partial_t\omega\|_{L^2}^2+\eta\nu\|\partial_2\omega\|_{L^2}^2+\|\mathcal{R}_1\omega\|_{L^2}^2+\eta\int_0^T\|\partial_t\omega\|_{L^2}^2\mathrm{d}\tau+\nu\int_0^T\|\partial_2\partial_t\omega\|_{L^2}^2\mathrm{d}\tau\\
&\le\|\partial_t\omega_0\|_{L^2}^2+\eta\nu\|\partial_2\omega_0\|_{L^2}^2+\|\mathcal{R}_1\omega_0\|_{L^2}^2\\
&\quad+C\sup_{0\le t\le T}\big(\|u\|_{H^2}^2+\|u\|_{H^2}^2\|\partial_t u\|_{L^2}^2\big)\int_0^T\big(\|\theta\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{L^2}^2\big)\mathrm{d}\tau.
\end{aligned}
$$

As a consequence,

$$
\int_0^\infty\|\partial_t\omega\|_{L^2}^2\mathrm{d}\tau<\infty \quad\text{and}\quad \int_0^\infty\|\partial_2\partial_t\omega\|_{L^2}^2\mathrm{d}\tau<\infty.
$$

It then follows from Lemma 2.4 that

$$
\|\partial_2\omega(t)\|_{L^2}\to 0 \quad\text{as}\quad t\to\infty.
$$

Since $\|\omega\|_{L^2}\to 0$, $\|\partial_2\omega\|_{L^2}\to 0$ and $\|\theta\|_{L^2}\to 0$, (2.29) yields

$$
\|\partial_t u\|_{L^2}\to 0 \quad\text{as}\quad t\to\infty.
$$

Similarly, the equation of θ implies

$$
\begin{aligned}
\|\partial_t\theta\|_{H^1}&\le\|u\cdot\nabla\theta\|_{H^1}+\|u_2\|_{H^1}+\eta\|\theta\|_{H^1}\\
&\le\|u\|_{H^2}\|\theta\|_{H^2}+\|u_2\|_{H^1}+\eta\|\theta\|_{H^1}
\end{aligned}
$$

and thus

$$
\|\partial_t\theta\|_{H^1}\to 0 \quad\text{as}\quad t\to\infty.
$$

This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

This section presents the proof of Theorem 1.3. It relies on two crucial lemmas. The first lemma represents the solution of a degenerate wave equation. The second lemma provides upper bounds for the Fourier multiplier operators G_1 and G_2 defined in (1.9).

Lemma 3.1. *Assume that g satisfies the degenerate wave type equation*

$$
\begin{cases}
\partial_{tt}g+(\eta-\nu\partial_{22})\partial_t g-(\nu\eta\partial_{22}g+\mathcal{R}_1^2 g)=0,\\
g(x,0)=g_0(x),\ \ \partial_t g(x,0)=g_1(x).
\end{cases}
$$

Then g can be explicitly represented as

$$g = G_1 \left(g_1 + \frac{1}{2} (\eta - \nu \partial_{22})\, g_0 \right) + G_2 g_0, \tag{3.1}$$

where G_1 and G_2 are defined as in (1.9), namely

$$\widehat{G_1}(\xi, t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad \widehat{G_2}(\xi, t) = \frac{1}{2} \left(e^{\lambda_1 t} + e^{\lambda_2 t} \right) \tag{3.2}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2 + \left(\eta + \nu \xi_2^2 \right) \lambda + \left(\nu \eta \xi_2^2 + \frac{\xi_1^2}{|\xi|^2} \right) = 0$$

or

$$\lambda_1 = -\frac{1}{2} \left(\eta + \nu \xi_2^2 \right) \left(1 + \sqrt{1 - \frac{4 \left(\nu \eta \xi_2^2 + \frac{\xi_1^2}{|\xi|^2} \right)}{\left(\eta + \nu \xi_2^2 \right)^2}} \right), \tag{3.3}$$

$$\lambda_2 = -\frac{1}{2} \left(\eta + \nu \xi_2^2 \right) \left(1 - \sqrt{1 - \frac{4 \left(\nu \eta \xi_2^2 + \frac{\xi_1^2}{|\xi|^2} \right)}{\left(\eta + \nu \xi_2^2 \right)^2}} \right). \tag{3.4}$$

When $\lambda_1 = \lambda_2$, (3.1) remains valid if we replace $\widehat{G_1}$ and $\widehat{G_2}$ in (3.2) by their corresponding limit form, namely

$$\widehat{G_1} = \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = t e^{\lambda_1 t}, \quad \widehat{G_2}(\xi, t) = \lim_{\lambda_2 \to \lambda_1} \frac{1}{2} \left(e^{\lambda_1 t} + e^{\lambda_2 t} \right) = e^{\lambda_1 t}.$$

Proof. To obtain the solution representation, we split the wave operator and write

$$\left(\partial_t + \frac{1}{2} (\eta - \nu \partial_{22}) - \frac{1}{2} \sqrt{(\eta + \nu \partial_{22})^2 + 4 \mathcal{R}_1^2} \right) \cdot \left(\partial_t + \frac{1}{2} (\eta - \nu \partial_{22}) + \frac{1}{2} \sqrt{(\eta + \nu \partial_{22})^2 + 4 \mathcal{R}_1^2} \right) g = 0.$$

It is then clear that we can rewrite the wave equation into two different systems,

$$(\partial_t - \Gamma_1)\, g = h, \tag{3.5}$$

$$(\partial_t - \Gamma_2)\, h = 0, \tag{3.6}$$

and

$$(\partial_t - \Gamma_2)\, g = f, \tag{3.7}$$

$$(\partial_t - \Gamma_1)\, f = 0, \tag{3.8}$$

where

$$\Gamma_1 = -\frac{1}{2}(\eta - \nu\partial_{22}) - \frac{1}{2}\sqrt{(\eta + \nu\partial_{22})^2 + 4\mathcal{R}_1^2},$$
$$\Gamma_2 = -\frac{1}{2}(\eta - \nu\partial_{22}) + \frac{1}{2}\sqrt{(\eta + \nu\partial_{22})^2 + 4\mathcal{R}_1^2}.$$

Taking the difference of (3.7) and (3.5), we obtain

$$g(x,t) = (\Gamma_2 - \Gamma_1)^{-1}(h(x,t) - f(x,t)). \tag{3.9}$$

(3.6) and (3.8) yield

$$f(x,t) = e^{\Gamma_1 t} f(x,0), \quad h(x,t) = e^{\Gamma_2 t} h(x,0). \tag{3.10}$$

The initial data $f(x,0)$ and $h(x,0)$ can be obtained by (3.5) and (3.7), respectively, that is

$$f(x,0) = g_1 - \Gamma_2 g_0, \quad h(x,0) = g_1 - \Gamma_1 g_0. \tag{3.11}$$

Plugging (3.10), (3.11) into (3.9) yields (3.1). This completes the proof of Lemma 3.1. □

We now analyze the behavior of $\widehat{G_1}(\xi,t)$ and $\widehat{G_2}(\xi,t)$, which clearly relies on the Fourier frequencies ξ. The following lemma provides upper bounds for $\widehat{G_1}(\xi,t)$ and $\widehat{G_2}(\xi,t)$ in different subdomains of the frequency space. The notation $Re\,\rho$ denotes the real part of a complex number ρ.

Lemma 3.2. *Let S_1, S_2, S_3 and A be the following subsets of $\mathbb{R}^2$,*

$$S_1 := \left\{ \xi \in \mathbb{R}^2 : \ 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{(\eta + \nu\xi_2^2)^2} \le \frac{1}{4} \right\},$$

$$S_2 := \left\{ \xi \in \mathbb{R}^2 : \ 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{(\eta + \nu\xi_2^2)^2} > \frac{1}{4} \quad \text{and} \quad \xi \in A^c \right\},$$

$$S_3 := \left\{ \xi \in \mathbb{R}^2 : \ 1 - \frac{4\left(\nu\eta\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}{(\eta + \nu\xi_2^2)^2} > \frac{1}{4} \quad \text{and} \quad \xi \in A \right\},$$

$$A := \left\{ \xi \in \mathbb{R}^2 : \ \xi_1^2 \le \frac{\eta\delta(t)}{\nu\eta - \nu\delta(t)}, \quad \xi_2^2 \le \frac{\eta\delta(t)}{\nu\eta - \nu\delta(t)} \right\},$$

where A^c denotes the complement of A and $0 < \delta(t) < \min\left\{\eta, \frac{1}{\eta}\right\}$ is specified in (3.15). Then $\widehat{G_1}(\xi,t)$ and $\widehat{G_2}(\xi,t)$ admit the following upper bounds:
(a) For any $\xi \in S_1$,

$$Re\lambda_1 \le -\frac{1}{2}\left(\eta+\nu\xi_2^2\right), \quad Re\lambda_2 \le -\frac{1}{4}\left(\eta+\nu\xi_2^2\right),$$
$$|\widehat{G_1}(\xi,t)| \le te^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}, \quad |\widehat{G_2}(\xi,t)| \le Ce^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}.$$

(b) For any $\xi \in S_2$,

$$\lambda_1 \le -\frac{3}{4}\left(\eta+\nu\xi_2^2\right), \quad \lambda_2 \le -\delta(t),$$
$$|\widehat{G_1}(\xi,t)| \le \frac{2}{\eta+\nu\xi_2^2}\left(e^{-\frac{3}{4}\eta t}+e^{-\delta(t)t}\right), \quad |\widehat{G_2}(\xi,t)| \le C\left(e^{-\frac{3}{4}\eta t}+e^{-\delta(t)t}\right).$$

(c) For any $\xi \in S_3$,

$$\lambda_1 \le -\frac{3}{4}\left(\eta+\nu\xi_2^2\right), \quad \lambda_2 \le 0,$$
$$|\widehat{G_1}(\xi,t)| \le \frac{2}{\eta+\nu\xi_2^2}\left(e^{-\frac{3}{4}\eta t}+1\right), \quad |\widehat{G_2}(\xi,t)| \le C\left(e^{-\frac{3}{4}\eta t}+1\right).$$

Proof. (a) For $\xi \in S_1$, the upper bounds for the real parts of λ_1 and λ_2, and for $\widehat{G_2}(\xi,t)$ follow directly from (3.3), (3.4), (3.2) and the definition of S_1. The upper bound for $\widehat{G_1}(\xi,t)$ is a consequence of the mean-value theorem. To be completely clear, we consider two cases:

$$1-\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2} \ge 0 \quad \text{and} \quad 1-\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2} < 0.$$

Both λ_1 and λ_2 are real in the first case. It then follows from the mean-value theorem that there is $\zeta \in (\lambda_1, \lambda_2)$ such that

$$\widehat{G_1}(\xi,t) = te^{\zeta t} \le te^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}.$$

In the second case λ_1 and λ_2 are a pair of complex conjugates and

$$\widehat{G_1} = \frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1} = e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t}\,\frac{2\sin(\frac{1}{2}Qt)}{Q}, \quad Q := (\eta+\nu\xi_2^2)\sqrt{\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2}-1},$$

which clearly implies $|\widehat{G_1}| \le t\,e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t}$.

(b) For $\xi \in S_2$, the bound for λ_1 is obvious. To estimate λ_2, we rewrite λ_2 as

$$\begin{aligned}
\lambda_2 &= -\frac{1}{2}\left(\eta+\nu\xi_2^2\right)\left(1-\sqrt{1-\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2}}\right) \\
&= -\frac{1}{2}\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)+\sqrt{\left(\eta+\nu\xi_2^2\right)^2-4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}} \\
&\le -\frac{\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}}{\eta+\nu\xi_2^2} \le -\delta(t),
\end{aligned}$$

where we have used $\xi \in A^c$ in the last inequality. The upper bound for $\widehat{G_2}(\xi,t)$ follows directly from the bounds for λ_1 and λ_2. Noticing that the lower bound

$$\lambda_2-\lambda_1=\left(\eta+\nu\xi_2^2\right)\sqrt{1-\frac{4\left(\nu\eta\xi_2^2+\frac{\xi_1^2}{|\xi|^2}\right)}{\left(\eta+\nu\xi_2^2\right)^2}} \ge \frac{1}{2}\left(\eta+\nu\xi_2^2\right),$$

we easily obtain the upper bound for $\widehat{G_1}(\xi,t)$.

(c) For $\xi \in S_3$, the upper bounds for λ_1 and λ_2 follows directly from (3.3) and (3.4). These upper bounds yield the desired bounds for $\widehat{G_1}(\xi,t)$ and $\widehat{G_2}(\xi,t)$. The proof is then complete. □

Lemma 3.1 and Lemma 3.2 allow us to prove Theorem 1.3.

Proof of Theorem 1.3. Applying Lemma 3.1 to (1.8) leads to

$$u = G_1\left((\partial_t u)(x,0)+\frac{1}{2}(\eta-\nu\partial_{22})\,u_0\right)+G_2u_0, \tag{3.12}$$

$$\theta = G_1\left((\partial_t \theta)(x,0)+\frac{1}{2}(\eta-\nu\partial_{22})\,\theta_0\right)+G_2\theta_0. \tag{3.13}$$

Setting $t=0$ in the linearized system

$$\partial_t u=\nu\partial_{22}u+(I-\nabla\Delta^{-1}\nabla\cdot)(\theta e_2), \qquad \partial_t\theta+u_2+\eta\theta=0$$

and inserting them in (3.12) and (3.13), we obtain

$$u = G_1\left(\theta_0 e_2-\nabla\Delta^{-1}\partial_2\theta_0+\frac{1}{2}(\eta+\nu\partial_{22})\,u_0\right)+G_2u_0,$$

$$\theta = G_1\left(-u_{20} - \frac{1}{2}(\eta + \nu\partial_{22})\theta_0\right) + G_2\theta_0.$$

To estimate the L^2-norm of u, we consider the Fourier transform of u, which satisfies

$$\widehat{u}(\xi, t) = \frac{1}{2}\left(\eta - \nu\xi_2^2\right)\widehat{G_1}(\xi, t)\widehat{u_0} + \begin{pmatrix} -\frac{\xi_1\xi_2}{|\xi|^2} \\ \\ \frac{\xi_1^2}{|\xi|^2} \end{pmatrix} \widehat{G_1}(\xi, t)\widehat{\theta_0} + \widehat{G_2}(\xi, t)\widehat{u_0}.$$

Therefore, by Plancherel's Theorem,

$$\begin{aligned} \|u\|_{L^2}^2 \le C \int \left|\eta - \nu\xi_2^2\right|^2 \left|\widehat{G_1}(\xi, t)\right|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi + C \int \left|\widehat{G_1}(\xi, t)\right|^2 \left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi \\ + C \int \left|\widehat{G_2}(\xi, t)\right|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi := M_1 + M_2 + M_3. \end{aligned} \tag{3.14}$$

To bound M_1, we estimate it on the subdomains S_1, S_2 and S_3 and use the upper bounds obtained in Lemma 3.2. We also use the simple fact that $x^n e^{-x} \le C(n)$ for any $n \ge 0$ and $x \ge 0$ repeatedly.

$$\begin{aligned} M_1 &= \int\limits_{\xi\in S_1} \left|\eta - \nu\xi_2^2\right|^2 \left|\widehat{G_1}(\xi, t)\right|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi + \int\limits_{\xi\in S_2} \left|\eta - \nu\xi_2^2\right|^2 \left|\widehat{G_1}(\xi, t)\right|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\quad + \int\limits_{\xi\in S_3} \left|\eta - \nu\xi_2^2\right|^2 \left|\widehat{G_1}(\xi, t)\right|^2 |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\le C \int \left|\eta - \nu\xi_2^2\right|^2 t^2 e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\quad + C \int \left|\frac{\eta - \nu\xi_2^2}{\eta + \nu\xi_2^2}\right|^2 \left(e^{-\frac{3}{2}\eta t} + e^{-2\delta(t)t}\right) |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\quad + C \int\limits_A \left|\frac{\eta - \nu\xi_2^2}{\eta + \nu\xi_2^2}\right|^2 \left(e^{-\frac{3}{2}\eta t} + 1\right) |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\le C \int \left(\eta + \nu\xi_2^2\right)^2 t^2 e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\quad + C \int \left(e^{-\frac{3}{2}\eta t} + e^{-2\delta(t)t}\right) |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi + C \int\limits_A |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\le C \int e^{-C\eta t} |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi + C \int e^{-2\delta(t)t} |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi + C \int\limits_A |\widehat{u_0}(\xi)|^2 \,\mathrm{d}\xi \\ &\le C\left(e^{-C\eta t} + e^{-2\delta(t)t}\right) \|u_0\|_{L^2}^2 + \frac{C\,\eta\delta(t)}{\nu\eta - \nu\delta(t)} \|u_0\|_{L^1}^2. \end{aligned}$$

Similarly,

$$
\begin{aligned}
M_2 \le\, & C \int\limits_{\xi\in S_1} t^2 e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} \left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi + C \int\limits_{\xi\in S_2} \frac{1}{\left|\eta+\nu\xi_2^2\right|^2}\left(e^{-\frac{3}{2}\eta t}+e^{-2\delta(t)t}\right)\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi \\
& + C \int\limits_{\xi\in S_3} \frac{1}{\left|\eta+\nu\xi_2^2\right|^2}\left(e^{-\frac{3}{2}\eta t}+1\right)\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi \\
\le\, & C \int t^2 e^{-\frac{1}{2}\eta t}\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi + C\int \left(e^{-\frac{3}{2}\eta t}+e^{-2\delta(t)t}\right)\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi + C\int\limits_A \left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi \\
\le\, & C\int e^{-C\eta t}\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi + C\int e^{-2\delta(t)t}\left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi + C\int\limits_A \left|\widehat{\theta_0}(\xi)\right|^2 \mathrm{d}\xi \\
\le\, & C\left(e^{-C\eta t}+e^{-2\delta(t)t}\right)\|\theta_0\|_{L^2}^2 + \frac{C\,\eta\delta(t)}{\nu\eta-\nu\delta(t)}\|\theta_0\|_{L^1}^2.
\end{aligned}
$$

The estimates for M_3 are similar to those for M_2 and the bound is

$$
M_3 \le C\left(e^{-C\eta t}+e^{-2\delta(t)t}\right)\|u_0\|_{L^2}^2 + \frac{C\,\eta\delta(t)}{\nu\eta-\nu\delta(t)}\|u_0\|_{L^1}^2.
$$

Inserting the bounds for M_1, M_2 and M_3 in (3.14) yields

$$
\|u\|_{L^2}^2 \le C\left(e^{-C\eta t}+e^{-2\delta(t)t}\right)\|(u_0,\theta_0)\|_{L^2}^2 + \frac{C\,\eta\delta(t)}{\nu\eta-\nu\delta(t)}\|(u_0,\theta_0)\|_{L^1}^2.
$$

Since the attention is focused on the large time behavior, we can assume, without loss of generality, that $t \ge 1$. Now we choose, for any $t \ge 1$,

$$
\delta(t) = \min\left\{\eta,\ \frac{1}{\eta}\right\}(1+t)^{-a}, \quad 0<a<1. \tag{3.15}
$$

The selection of this special form of δ is to balance the two parts in the upper bound. Clearly,

$$
e^{-2\delta(t)t} = e^{-\frac{2\min\left\{\eta,\ \frac{1}{\eta}\right\}t}{(1+t)^a}} \le e^{-\min\left\{\eta,\ \frac{1}{\eta}\right\}(1+t)^{1-a}} \le C(1+t)^{-\beta} \quad \text{for any } \beta>0
$$

and

$$
\frac{\eta\delta(t)}{\nu\eta-\nu\delta(t)} \le \frac{\eta}{\nu(1-(1+t)^{-a})}(1+t)^{-a}.
$$

Therefore,

$$
\|u(t)\|_{L^2}^2 \le C(a)\,(1+t)^{-a}\,\|(u_0,\theta_0)\|_{L^1\cap L^2}^2.
$$

The bound for $\|\theta(t)\|_{L^2}^2$ is similar. The proof of Theorem 1.3 is now complete. □

4. Proofs of Theorem 1.4 and Theorem 1.5

This section proves Theorem 1.4 and Theorem 1.5. The proofs make use of the special wave structure in the linearized system (1.8).

Proof of Theorem 1.4. Taking the inner product of the θ-equation in (1.8) with $\partial_t\theta$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_t\theta\|_{L^2}^2+\nu\eta\|\partial_2\theta\|_{L^2}^2+\|\mathcal{R}_1\theta\|_{L^2}^2\right)+\eta\|\partial_t\theta\|_{L^2}^2+\nu\|\partial_2\partial_t\theta\|_{L^2}^2=0. \tag{4.1}$$

Taking the inner product of the θ-equation with θ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta\|\theta\|_{L^2}^2+\nu\|\partial_2\theta\|_{L^2}^2+2(\partial_t\theta,\theta)\right)+\nu\eta\|\partial_2\theta\|_{L^2}^2+\|\mathcal{R}_1\theta\|_{L^2}^2-\|\partial_t\theta\|_{L^2}^2=0. \tag{4.2}$$

For a constant μ to be specified later, (4.1)+μ(4.2) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t\theta\|_{L^2}^2+\mu\eta\|\theta\|_{L^2}^2+\|\mathcal{R}_1\theta\|_{L^2}^2+(\nu\eta+\mu\nu)\,\|\partial_2\theta\|_{L^2}^2+2\mu(\partial_t\theta,\theta)\Big)\\&\qquad+(\eta-\mu)\|\partial_t\theta\|_{L^2}^2+\nu\|\partial_2\partial_t\theta\|_{L^2}^2+\mu\nu\eta\|\partial_2\theta\|_{L^2}^2+\mu\|\mathcal{R}_1\theta\|_{L^2}^2=0.\end{aligned} \tag{4.3}$$

For $0\le s<t<\infty$, we integrate (4.3) over $[s,t]$ to obtain

$$F(t)+2\int_s^t\left((\eta-\mu)\|\partial_t\theta\|_{L^2}^2+\nu\|\partial_2\partial_t\theta\|_{L^2}^2+\mu\nu\eta\|\partial_2\theta\|_{L^2}^2+\mu\|\mathcal{R}_1\theta\|_{L^2}^2\right)\mathrm{d}\tau=F(s),$$

where we have set

$$F(t):=\|\partial_t\theta\|_{L^2}^2+\mu\eta\|\theta\|_{L^2}^2+\|\mathcal{R}_1\theta\|_{L^2}^2+(\nu\eta+\mu\nu)\,\|\partial_2\theta\|_{L^2}^2+2\mu(\partial_t\theta,\theta).$$

If we take $\mu<\eta$, then

$$F(t)\le F(s). \tag{4.4}$$

Now we take $\mu=\frac{\eta}{4}$. By the Cauchy-Schwartz inequality

$$\frac{1}{2}\|\partial_t\theta\|_{L^2}^2+\frac{1}{8}\eta^2\|\theta\|_{L^2}^2\le\|\partial_t\theta\|_{L^2}^2+\mu\eta\|\theta\|_{L^2}^2+2\mu(\partial_t\theta,\theta),$$

we obtain a lower bound for $F(t)$,

$$\frac{1}{2}\|\partial_t\theta(t)\|_{L^2}^2+\frac{1}{8}\eta^2\|\theta(t)\|_{L^2}^2+\|\mathcal{R}_1\theta(t)\|_{L^2}^2+\frac{5}{4}\nu\eta\|\partial_2\theta(t)\|_{L^2}^2\le F(t). \tag{4.5}$$

Clearly, $F(s)$ admits the following upper bound

$$F(s)\le\frac{3}{2}\|\partial_s\theta(s)\|_{L^2}^2+\frac{3}{8}\eta^2\|\theta(s)\|_{L^2}^2+\|\mathcal{R}_1\theta(s)\|_{L^2}^2+\frac{5}{4}\nu\eta\|\partial_2\theta(s)\|_{L^2}^2 \tag{4.6}$$

It then follows from (4.4), (4.5) and (4.6) that, for any $0 \le s < t$,

$$
\begin{aligned}
&\frac{1}{2}\|\partial_t\theta(t)\|_{L^2}^2 + \frac{1}{8}\eta^2\|\theta(t)\|_{L^2}^2 + \|\mathcal{R}_1\theta(t)\|_{L^2}^2 + \frac{5}{4}\nu\eta\|\partial_2\theta(t)\|_{L^2}^2 \\
&\le \frac{3}{2}\|\partial_s\theta(s)\|_{L^2}^2 + \frac{3}{8}\eta^2\|\theta(s)\|_{L^2}^2 + \|\mathcal{R}_1\theta(s)\|_{L^2}^2 + \frac{5}{4}\nu\eta\|\partial_2\theta(s)\|_{L^2}^2 \\
&\le 3\left(\frac{1}{2}\|\partial_s\theta(s)\|_{L^2}^2 + \frac{1}{8}\eta^2\|\theta(s)\|_{L^2}^2 + \|\mathcal{R}_1\theta(s)\|_{L^2}^2 + \frac{5}{4}\nu\eta\|\partial_2\theta(s)\|_{L^2}^2\right)
\end{aligned} \tag{4.7}
$$

We intend to apply Lemma 2.5. (4.7) fulfills the decrease condition. We need to verify the time integrability condition. For $\mu = \frac{\eta}{4}$, we have

$$
\int_0^\infty \|\partial_t\theta\|_{L^2}^2 \mathrm{d}t < \infty, \quad \int_0^\infty \|\partial_2\theta\|_{L^2}^2 \mathrm{d}t < \infty, \quad \int_0^\infty \|\mathcal{R}_1\theta\|_{L^2}^2 \mathrm{d}t < \infty.
$$

In addition, the L^2-estimate

$$
\|(u,\theta)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_2 u\|_{L^2}^2 \mathrm{d}\tau + 2\eta\int_0^t \|\theta\|_{L^2}^2 \mathrm{d}\tau = \|(u_0,\theta_0)\|_{L^2}^2
$$

ensures that

$$
\int_0^\infty \|\theta\|_{L^2}^2 \mathrm{d}t < \infty.
$$

It then follows from Lemma 2.5 that

$$
\|\partial_t\theta(t)\|_{L^2}^2, \ \|\theta(t)\|_{L^2}^2, \ \|\partial_2\theta(t)\|_{L^2}^2 \le C(1+t)^{-1}.
$$

As a consequence, using the linearized temperature equation, we have

$$
\|u_2\|_{L^2} \le \|\partial_t\theta\|_{L^2} + \eta\|\theta\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.
$$

This finishes the proof of Theorem 1.4. □

We now turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let χ be the Fourier cutoff operator defined in (1.11). Taking the convolution of χ with the velocity equation in (1.8) leads to

$$
\partial_{tt}(\chi * u) + (\eta - \nu\partial_{22})\partial_t(\chi * u) - (\eta\nu\partial_{22}(\chi * u) + \mathcal{R}_1^2(\chi * u)) = 0. \tag{4.8}
$$

Taking the L^2-inner product of (4.8) with $\partial_t(\chi * u)$ and integrating by parts yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t(\chi*u)\|_{L^2}^2+\eta\nu\|\partial_2(\chi*u)\|_{L^2}^2+\|\mathcal{R}_1(\chi*u)\|_{L^2}^2\Big)$$
$$+2\eta\|\partial_t(\chi*u)\|_{L^2}^2+2\nu\|\partial_2\partial_t(\chi*u)\|_{L^2}^2=0. \tag{4.9}$$

Similarly, taking the L^2-inner product of (4.8) with $\chi*u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\eta\|\chi*u\|_{L^2}^2+\nu\|\partial_2(\chi*u)\|_{L^2}^2+2\left(\partial_t(\chi*u),\chi*u\right)\Big)$$
$$+2\eta\nu\|\partial_2(\chi*u)\|_{L^2}^2+2\|\mathcal{R}_1(\chi*u)\|_{L^2}^2-2\|\partial_t(\chi*u)\|_{L^2}^2=0. \tag{4.10}$$

Next we show that we can bound $\|\mathcal{R}_2(\chi*u)\|_{L^2}^2$ in terms of $\|\partial_2(\chi*u)\|_{L^2}^2$ and $\|\mathcal{R}_1(\chi*u)\|_{L^2}^2$. Let D be defined as in (1.10) and D^c be its complement. We further divide D^c into two regions

$$Q_1=\Big\{\xi\in\mathbb{R}^2:\ |\xi|\ge\varrho\Big\},\qquad Q_2=\Big\{\xi\in\mathbb{R}^2:\ |\xi|<\varrho\ \text{ and }\ |\xi_2|\le|\xi_1|\Big\}.$$

Then

$$\begin{aligned}\|\mathcal{R}_2(\chi*u)\|_{L^2}^2&=\left\|\frac{\xi_2}{|\xi|}\widehat{\chi u}\right\|_{L^2}^2=\int_{Q_1}\frac{\xi_2^2}{|\xi|^2}|\widehat{\chi u}|^2\mathrm{d}\xi+\int_{Q_2}\frac{\xi_2^2}{|\xi|^2}|\widehat{\chi u}|^2\mathrm{d}\xi\\&\le\varrho^{-2}\int_{Q_1}\xi_2^2|\widehat{\chi u}|^2\mathrm{d}\xi+\int_{Q_2}\frac{\xi_1^2}{|\xi|^2}|\widehat{\chi u}|^2\mathrm{d}\xi\\&\le\varrho^{-2}\|\partial_2(\chi*u)\|_{L^2}^2+\|\mathcal{R}_1(\chi*u)\|_{L^2}^2.\end{aligned}\tag{4.11}$$

For a constant $\kappa>0$, (4.9)+κ(4.10)+κ^2(4.11) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t(\chi*u)\|_{L^2}^2+\eta\kappa\|\chi*u\|_{L^2}^2+\nu(\eta+\kappa)\|\partial_2(\chi*u)\|_{L^2}^2\\&\quad+\|R_1(\chi*u)\|_{L^2}^2+2\kappa\left(\partial_t(\chi*u),\chi*u\right)\Big)+(2\eta-2\kappa)\|\partial_t(\chi*u)\|_{L^2}^2\\&\quad+(2\eta\nu\kappa-\varrho^{-2}\kappa^2)\|\partial_2(\chi*u)\|_{L^2}^2+(2\kappa-2\kappa^2)\|R_1(\chi*u)\|_{L^2}^2+\kappa^2\|\chi*u\|_{L^2}^2\\&\quad+2\nu\|\partial_2\partial_t(\chi*u)\|_{L^2}^2\le0.\end{aligned}\tag{4.12}$$

If we take

$$\kappa\le\min\left\{\frac{1}{2},\frac{\eta}{2},\eta\nu\varrho^2\right\},$$

(4.12) then implies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)+\eta\|\partial_t(\chi*u)\|_{L^2}^2+\eta\nu\kappa\|\partial_2(\chi*u)\|_{L^2}^2$$
$$+\kappa\|\mathcal{R}_1(\chi*u)\|_{L^2}^2+\kappa^2\|\chi*u\|_{L^2}^2\le0, \tag{4.13}$$

where, for notational convenience, we have set

$$G(t) := \|\partial_t(\chi * u)\|_{L^2}^2 + \eta\kappa\|\chi * u\|_{L^2}^2 + \nu(\eta+\kappa)\|\partial_2(\chi * u)\|_{L^2}^2 \\ + \|\mathcal{R}_1(\chi * u)\|_{L^2}^2 + 2\kappa\,(\partial_t(\chi * u), \chi * u)\,.$$

We invoke the basic inequality,

$$2\kappa\,(\partial_t(\chi * u), \chi * u) \le \kappa\|\partial_t(\chi * u)\|_{L^2}^2 + \kappa\|\chi * u\|_{L^2}^2. \tag{4.14}$$

(4.13)+κ^2(4.14) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}G + \kappa^2 G + (\eta - \kappa^2 - \kappa^3)\|\partial_t(\chi * u)\|_{L^2}^2 + \Big(\eta\nu\kappa - \kappa^2\nu(\eta+\kappa)\Big)\|\partial_2(\chi * u)\|_{L^2}^2 \\ + \Big(\kappa - \kappa^2\Big)\|R_1(\chi * u)\|_{L^2}^2 + \kappa^2(1 - \eta\kappa - \kappa)\|\chi * u\|_{L^2}^2 \le 0. \tag{4.15}$$

If we take κ to be sufficiently small, say

$$\kappa \le \min\left\{\frac{1}{2}, \frac{\eta}{2}, \eta\nu\varrho^2, \frac{1}{\eta+1}, \frac{\sqrt{\eta^2+4\eta}-\eta}{2}\right\}, \tag{4.16}$$

then (4.15) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}G + \kappa^2 G \le 0 \quad \text{or} \quad G(t) \le G(0)e^{-\kappa^2 t}.$$

Using the simple inequality

$$2\kappa\,(\partial_t(\chi * u), \chi * u) \le \frac{1}{2}\|\partial_t(\chi * u)\|_{L^2}^2 + 2\kappa^2\|\chi * u\|_{L^2}^2,$$

we have

$$\frac{1}{2}\|\partial_t(\chi * u)\|_{L^2}^2 + \frac{1}{2}\eta\kappa\|\chi * u\|_{L^2}^2 + \nu(\eta+\kappa)\|\partial_2(\chi * u)\|_{L^2}^2 + \|R_1(\chi * u)\|_{L^2}^2 \le G(t).$$

Therefore, for κ satisfying (4.16),

$$\|\chi * u\|_{L^2}^2,\ \|\partial_t(\chi * u)\|_{L^2}^2,\ \|\partial_2(\chi * u)\|_{L^2}^2\ \le Ce^{-c(\eta,\nu)t}$$

with

$$c(\eta,\nu) = \left\{\min\Big\{\frac{1}{2}, \frac{\eta}{4}, \eta\nu\varrho^2, \frac{1}{\eta+1}, \frac{\sqrt{\eta^2+4\eta}-\eta}{2}\Big\}\right\}^2.$$

The estimates for θ are very similar. This finishes the proof of Theorem 1.5. $\square$

Acknowledgments

This manuscript was completed during the visits of Suhua Lai and Yueyuan Zhong at Oklahoma State University. They would like to thank the warm hospitality and support of Department of Mathematics at Oklahoma State University. S. Lai was partially supported by the National Natural Science Foundation of China (No. 11871407), J. Wu was partially supported by The National Science Foundation of USA under grant DMS 1624146 and the AT&T Foundation at Oklahoma State University. Y. Zhong was partially supported by the National Natural Science Foundation of China (Nos. 11771043, 11771045).

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