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Smoothing and stabilization effects of magnetic field on electrically conducting fluids

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Abstract

This paper solves the stability problem on a partially dissipated system of magnetohydrodynamic equations near a background magnetic field. Large-time behavior of the corresponding linearized system is also obtained. These results presented in this paper rigorously confirm a nonlinear phenomenon observed in physical experiments that the magnetic field actually stabilizes electrically conducting fluids.

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1. Introduction

The magnetohydrodynamic (MHD) equations are an interactive and integrated system which are composed of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The MHD system arises in geophysics, astrophysics, cosmology, and has been widely applied in engineering such as MHD generation and controlled thermonuclear reaction (see, e.g., [5,11]). The standard incompressible MHD equations can be written as

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$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B = \eta \Delta B + B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x,0) = u_0(x), \ B(x,0) = B_0(x), \end{cases}$$

where u and B are the velocity field and the magnetic field, respectively, and $\nu \geq 0$ kinematic viscosity and $\eta \geq 0$ the magnetic diffusivity.

Besides its wide physical applicability, the MHD system is also mathematically important. Numerous efforts have been devoted to understanding several fundamental issues on the MHD system including the global well-posedness and stability problems. Duvaut and Lions [16] established the local existence and uniqueness of Sobolev solutions and proved the global existence in the case of small initial data. Sermange and Temam [36] obtained the global existence and regularity of solutions to the 2D fully dissipative MHD equations. More recent studies focus on the MHD equations with only partial or fractional dissipation. Substantial progress has been made on the global existence and regularity problem concerning various partially or fractionally dissipated MHD systems. More information on these results can be found in a recent survey paper of Wu [45] and many other references (see, e.g., [7–10,13–15,17–19,24,27,28,34,35,39,40,42–44,49–54]).

During the last few years, there have been extensive interests on the stability problem concerning partially dissipated MHD systems near a background magnetic field. Some of these studies are partially motivated by significant nonlinear phenomena concerning electrically conducting fluids. It is observed in physical experiments and numerical simulations that the magnetic field can stabilize electrically conducting fluids and certain wave phenomena are associated with the stabilization process (see, e.g., [1,2,20,21]). The MHD systems are capable of modeling many different kind of wave phenomena. Most notably among them is the Alfvén wave, which was studied by the Nobel laureate Hannes Alfvén in 1942 [3]. The Alfvén wave or the electromagnetic-hydrodynamic wave is produced due to the interaction between the perturbations of the velocity field and the magnetic field near a constant background magnetic field. As stated in the work of Alfvén [3], if a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an electromagnetic field which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid. Thus a kind of combined electromagnetic-hydrodynamic wave is produced. There are substantial recent developments on the stability problem near a background magnetic field. The desired stability has been successfully established for various partially dissipated MHD systems (see, e.g., [4,6,12,22,23,25,26,29,31–33,37,41,46–48,55–57]).

This paper focuses on the following 2D MHD system with partial dissipation and magnetic diffusion

$$\begin{cases} \partial_t u + u \cdot \nabla u = \begin{bmatrix} 0 \\ \nu \Delta u_2 \end{bmatrix} - \nabla p + B \cdot \nabla B, \quad x \in \mathbb{R}^2, t > 0, \\ \partial_t B + u \cdot \nabla B = \begin{bmatrix} \eta \Delta B_1 \\ 0 \end{bmatrix} + B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x,0) = u_0(x), B(x,0) = B_0(x). \end{cases} \tag{1.1}$$

This MHD system arises in special physical circumstances when the dissipation is only present in the vertical component of the velocity and the magnetic diffusion only in the horizontal component of the magnetic field. Our aim here is to understand the stability problem on perturbations near a background magnetic field. Clearly,

$$u^{(0)}=0, \quad p^{(0)}=0, \quad B^{(0)}=(0,1)$$

is a stationary solution of (1.1). The perturbation (u, p, b) near this steady state with

$$b=B-B^{(0)}$$

satisfies

$$\begin{cases}\partial_t u+u\cdot\nabla u=\begin{bmatrix}0\\ \nu\Delta u_2\end{bmatrix}-\nabla p+b\cdot\nabla b+\partial_2 b,\\ \partial_t b+u\cdot\nabla b=\begin{bmatrix}\eta\Delta b_1\\ 0\end{bmatrix}+b\cdot\nabla u+\partial_2 u,\\ \nabla\cdot u=\nabla\cdot b=0,\\ u(x,0)=u_0(x), b(x,0)=b_0(x),\end{cases} \tag{1.2}$$

where ν, η are both positive constants. Our motivation for studying the stability problem is two-fold. The first is to verify that the magnetic field can actually stabilize the electrically conducting fluids, a phenomenon that has been observed in physical experiments and numerical simulations ([1,2,20,21]). The second is to uncover the mechanism on how the wave phenomenon helps with the stability problem on the MHD system with only partial dissipation.

We first point out that the system in (1.2) is globally well-posed. In fact, we can show that, for any initial data $(u_0, b_0)\in H^2(\mathbb{R}^2)$, (1.2) always has a unique global-in-time solution (u, b) that remains in $H^2(\mathbb{R}^2)$ for all time. The proof of this result is very similar to Theorem 4 of Cao and Wu [8], so we omit the details. For the sake of later references, we write this well-posedness result as a theorem.

Theorem 1.1. *Assume $(u_0, b_0)\in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0=\nabla\cdot b_0=0$. Then (1.2) has a unique global solution (u, b) that satisfies*

$$(u,b)\in L^\infty(0,\infty;H^2(\mathbb{R}^2)), \quad \nabla u_2\in L^2([0,\infty);H^2(\mathbb{R}^2)), \quad \nabla b_1\in L^2([0,\infty);H^2(\mathbb{R}^2)).$$

The focus of this paper is on the stability problem and the large-time behavior. Due to the lack of dissipation in the equations of u_1 and of b_2, the stability problem proposed for studying here does not appear to be trivial. One difficulty is that the velocity, especially u_1, can potentially grow in time. In fact, if we consider the equation of the vorticity $\omega=\nabla\times u$, given by

$$\partial_t\omega+u\cdot\nabla\omega=\nu\partial_{11}\omega+b\cdot\nabla j+\partial_2 j \tag{1.3}$$

where $j=\nabla\times b$ denotes the current density. (1.3) represents a forced Euler equation with only horizontal dissipation. Even when the force is not present or $b=0$, the stability problem on the reduced equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{11} \omega, \quad x \in \mathbb{R}^2,\ t > 0$$

remains open, the main difficulty is that $\nabla\omega$ could grow in time. When we estimate the L^2-norm of $\nabla\omega$ via the energy method,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega\, dx,$$

the term on the right-hand side can not be bounded suitably due to the lack of the vertical dissipation. In fact, if we further divide the term as

$$\begin{aligned} M := -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega\, dx &= -\int \partial_1 u_1\, (\partial_1\omega)^2\, dx - \int \partial_1 u_2 \partial_1\omega\, \partial_2\omega\, dx \\ &\quad - \int \partial_2 u_1\, \partial_1\omega\, \partial_2\omega\, dx - \int \partial_2 u_2\, (\partial_2\omega)^2\, dx, \end{aligned} \tag{1.4}$$

the last two terms in (1.4) resist any suitable control due to the lack of the vertical dissipation. This explains some of the difficulties we would encounter when we deal with the stability problem on (1.2).

How can it be possible to establish the stability for (1.2) if we can not handle the difficult term in (1.4)? The magic here is that the coupling and interaction between the velocity and the magnetic field actually stabilizes the fluid. The solution of the system in (1.2) behaves better than that of each individual equation alone. To explain this phenomenon precisely, we first separate the linear parts from the nonlinear parts in (1.2). Applying the Helmholtz-Leray projection operator

$$\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$$

to the equations in (1.2), we obtain

$$\begin{cases} \partial_t u - \nu\partial_{11} u - \partial_2 b = \mathbb{P}(b \cdot \nabla b - u \cdot \nabla u), \\ \partial_t b - \eta\partial_{22} b - \partial_2 u = \mathbb{P}(b \cdot \nabla u - u \cdot \nabla b), \end{cases} \tag{1.5}$$

where we have used the facts that

$$\mathbb{P}\begin{bmatrix} 0 \\ \nu\Delta u_2 \end{bmatrix} = \nu\,\partial_{11} u, \qquad \mathbb{P}\begin{bmatrix} \eta\Delta b_1 \\ 0 \end{bmatrix} = \eta\,\partial_{22} b. \tag{1.6}$$

(1.6) is obtained by some simple calculations. In fact, by $\nabla \cdot u = 0$,

$$\begin{aligned} \mathbb{P}\begin{bmatrix} 0 \\ \nu\Delta u_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ \nu\Delta u_2 \end{bmatrix} - \nabla\Delta^{-1}\nabla\cdot\begin{bmatrix} 0 \\ \nu\Delta u_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \nu\Delta u_2 \end{bmatrix} - \nu\nabla\partial_2 u_2 = \nu\begin{bmatrix} \partial_{11} u_1 \\ \partial_{11} u_2 \end{bmatrix} = \nu\partial_{11} u. \end{aligned}$$

We can further decouple the linear parts in (1.5) by differentiating (1.5) in t and making several substitutions to obtain

$$
\begin{cases}
\partial_{tt} u - (\nu\partial_{11} + \eta\partial_{22})\partial_t u + \nu\,\eta\,\partial_{1122} u - \partial_{22} u = N_1, \\
\partial_{tt} b - (\nu\partial_{11} + \eta\partial_{22})\partial_t b + \nu\,\eta\,\partial_{1122} b - \partial_{22} b = N_2,
\end{cases}
\tag{1.7}
$$

where N_1 and N_2 are the nonlinear terms,

$$
N_1 = (\partial_t - \eta\partial_{22})\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u) + \partial_2\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b),
$$
$$
N_2 = (\partial_t - \nu\partial_{11})\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b) + \partial_2\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u).
$$

In comparison with the original system in (1.2), the wave equations in (1.7) reveal more smoothing and stabilizing properties of the solution. These fine properties allow us to establish the desired stability by constructing suitable energy functionals. We now state our main results followed by the description of their proofs.

Statements of the main results. Our first result establishes the global existence and stability of solutions to (1.2) in the H^2-setting.

Theorem 1.2. *Consider (1.2) with $\nu > 0$ and $\eta > 0$. Assume $(u_0, b_0) \in H^2(\mathbb{R}^2)$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then there exists a constant $\varepsilon > 0$ depending on ν and η such that if*

$$
\|(u_0, b_0)\|_{H^2} \le \varepsilon,
$$

then (1.2) has a unique global solution (u, b) satisfying

$$
\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \int_0^t (\nu\|\partial_1 u(\tau)\|_{H^2}^2 + \eta\|\partial_2 b(\tau)\|_{H^2}^2)\,d\tau + \int_0^t \|\partial_2 u_1\|_{H^1}^2\,d\tau \le C\,\varepsilon^2
$$

for a uniform constant $C > 0$ and for all $t > 0$.

Our second main result explores the large-time behavior of solutions to the linearized system

$$
\begin{cases}
\partial_t u - \nu\partial_{11} u - \partial_2 b = 0, \\
\partial_t b - \eta\partial_{22} b - \partial_2 u = 0, \\
\nabla\cdot u = \nabla\cdot b = 0, \\
u(x,0) = u_0(x), \quad b(x,0) = b_0(x),
\end{cases}
\tag{1.8}
$$

which can be converted to the following system of wave equations

$$
\begin{cases}
\partial_{tt} u - (\nu\partial_{11} + \eta\partial_{22})\partial_t u + \nu\,\eta\,\partial_{1122} u - \partial_{22} u = 0, \\
\partial_{tt} b - (\nu\partial_{11} + \eta\partial_{22})\partial_t b + \nu\,\eta\,\partial_{1122} b - \partial_{22} b = 0, \\
\nabla\cdot u = \nabla\cdot b = 0, \\
u(x,0) = u_0(x), \quad b(x,0) = b_0(x).
\end{cases}
\tag{1.9}
$$

To obtain the optimal decay rates, we make use of the symmetric structure of (1.8) and exploit the regularization of the wave equations in (1.9). As we know, the initial data needs to be a Sobolev space of negative index or in the Lebesgue space L^q with $1 \le q < 2$ in order to attain the L^2-decay rate for the solution itself. Sobolev spaces of negative indices will be employed here. To give a precise statement of our result, we provide the definition of the fractional Laplacian operator. For any $\gamma \in \mathbb{R}$,

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma}\, \widehat{f}(\xi),$$

where $\widehat{f}$ denotes the standard Fourier transform

$$\widehat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}\, f(x)\, dx.$$

For notational convenience, we sometimes write $\Lambda = (-\Delta)^{\frac{1}{2}}$. Similarly, the fractional partial derivative operator Λ_i^γ with $i = 1, 2$ and $\gamma \in \mathbb{R}$ can be defined as

$$\widehat{\Lambda_i^\gamma f}(\xi) = |\xi_i|^\gamma\, \widehat{f}(\xi).$$

With these definitions at our disposal, we can now state our second main result.

Theorem 1.3. *Consider the linearized system (1.8) with $\nu > 0$ and $\eta > 0$. Its solution has the following decay properties.*

(a) Let $\sigma > 0$. Assume (u_0, b_0) satisfies

$$(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})u_0 \in H^{1+\sigma},\ (\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})b_0 \in H^{1+\sigma}, \quad \nabla\cdot u_0 = \nabla\cdot b_0 = 0.$$

Then the corresponding solution (u, b) of (1.8) satisfies

$$u,\ b \in L^\infty(0,\infty; H^1), \quad \partial_1 u,\ \partial_2 b \in L^2(0,\infty; H^1)$$

and, for a constant C depending on u_0 and b_0 only and for any $t > 0$,

$$\|u(t)\|_{H^1} + \|b(t)\|_{H^1} \le C\,(1+t)^{-\frac{\sigma}{2}}. \tag{1.10}$$

(b) Assume (u_0, b_0) satisfies

$$\Lambda_2^{-1}u_0 \in H^2,\ \Lambda_2^{-1}b_0 \in H^2, \quad \nabla\cdot u_0 = \nabla\cdot b_0 = 0.$$

Then the corresponding solution (u, b) of (1.8) satisfies

$$\|\partial_t u(t)\|_{L^2}^2 + \|\partial_2 u(t)\|_{L^2}^2 + \nu\eta\|\partial_{12}u(t)\|_{L^2}^2 \le C\,(1+t)^{-1}, \tag{1.11}$$

$$\|\partial_t b(t)\|_{L^2}^2 + \|\partial_2 b(t)\|_{L^2}^2 + \nu\eta\|\partial_{12}b(t)\|_{L^2}^2 \le C\,(1+t)^{-1}. \tag{1.12}$$

We remark that it is much more difficult to establish the large-time behavior for the nonlinear problem. There appears to be essential difficulties. The large-time behavior of solutions to the nonlinear system relies crucially on the eigenvalues associated with the wave equation in (1.9). Due to the partial dissipation, the wave equation is degenerate and this degeneracy makes one of the eigenvalues really close to zero. More precisely, the characteristic polynomial associated with (1.9) (in Fourier space) is given by

$$\lambda^2 + (\nu\xi_1^2 + \eta\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \xi_2^2 = 0,$$

and the two eigenvalues are

$$\lambda_1 = \frac{-(\nu\xi_1^2 + \eta\xi_2^2) - \sqrt{(\nu\xi_1^2 + \eta\xi_2^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + \xi_2^2)}}{2},$$
$$\lambda_2 = \frac{-(\nu\xi_1^2 + \eta\xi_2^2) + \sqrt{(\nu\xi_1^2 + \eta\xi_2^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + \xi_2^2)}}{2}.$$

λ_2 could be quite close to 0 when ξ_2 is close to 0. In fact,

$$\lambda_2 = -\frac{2(\nu\eta\xi_1^2\xi_2^2 + \xi_2^2)}{(\nu\xi_1^2 + \eta\xi_2^2) + \sqrt{(\nu\xi_1^2 + \eta\xi_2^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + \xi_2^2)}} \approx 0 \quad \text{for } \xi_2 \approx 0.$$

When we represent the nonlinear system in an integral form, this degeneracy of λ_2 makes the decay evaluation of the nonlinear terms extremely difficult. We will explore the large-time behavior of the nonlinear problem in the future.

A brief outline of the proofs. We present the main ideas in the proofs of Theorem 1.2 and Theorem 1.3. The framework in the proof of Theorem 1.2 is the bootstrapping argument (see, e.g., [38]). We need to construct suitable energy functionals in order to derive self-contained inequalities. Since the solution sought is in H^2, a natural part of the energy functional is

$$E_1(t) = \sup_{0\le\tau\le t} (\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2) + \nu\int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau + \eta\int_0^t \|\partial_2 b(\tau)\|_{H^2}^2 d\tau. \quad (1.13)$$

As aforementioned, this part is not sufficient since the difficult term (1.4) emerged in the H^2-estimate can not be bounded by $E_1(t)$. We seek a second part of the energy functional. By examining the regularization provided by the wave equations (1.7), we are led to the definition of the second part of the energy functional,

$$E_2(t) = \int_0^t \|\partial_2 u(\tau)\|_{H^1}^2 d\tau. \quad (1.14)$$

The inclusion of E_2 helps the control of (1.4). Of course, we also need to verify that E_2 can be bounded by a combination of E_1 and E_2. More precisely, we set

$$E(t) := E_1(t) + \delta\, E_2(t) \tag{1.15}$$

for a suitable $\delta > 0$, and verify that, for a constant $C > 0$ and for all $t > 0$,

$$E(t) \le C_0\, E(0) + C_0\, E(t)^{\frac{3}{2}}. \tag{1.16}$$

Once (1.16) is confirmed, a bootstrapping argument applied to (1.16) would lead to the desired global existence and stability of Theorem 1.2. Our main efforts are devoted to check (1.16). This is a very lengthy process and we divided into two parts. The first part proves

$$E_1(t) \le E_1(0) + C_1\, E_1^{\frac{3}{2}}(t) + C_2\, E_2^{\frac{3}{2}}(t),$$

while the second part verifies

$$E_2(t) \le E_1(0) + C_3\, E_1(t) + C_4\, E_1^{\frac{3}{2}}(t) + C_5\, E_2^{\frac{3}{2}}(t). \tag{1.17}$$

The proof of (1.17) makes use of the wave structure of (1.7). More details can be found in Section 2.

We also briefly explain the proof of Theorem 1.3. The proof for Part (a) makes use of two energy inequalities, one for the H^1-norm of (u, b) and one for $(\Lambda_1^{-\sigma} u, \Lambda_2^{-\sigma} b)$ in $H^{1+\sigma}$. By developing anisotropic interpolation inequalities, we are able to convert the H^1-energy inequality into an ODE type inequality, which leads to the desired decay estimate. The proof of Part (b) relies on the following lemma providing a precise decay rate for a nonnegative integrable function when it decreases in a generalized sense.

Lemma 1.1. *Let $f = f(t)$ be a nonnegative continuous function satisfying, for two constants $a_0 > 0$ and $a_1 > 0$,*

$$\int_0^\infty f(\tau)\, d\tau \le a_0 < \infty \quad \textit{and} \quad f(t) \le a_1\, f(s) \quad \textit{for any } 0 \le s < t. \tag{1.18}$$

Then, for $a_2 = \max\{2a_1 f(0), 2a_0 a_1\}$ and for any $t > 0$,

$$f(t) \le a_2 (1+t)^{-1}.$$

With the help of this lemma, the proof of Part (b) is then reduced to verify that

$$\int_0^\infty \left(\|\partial_t u(t)\|_{L^2}^2 + \|\partial_2 u(t)\|_{L^2}^2 + \nu\eta \|\partial_{12} u(t)\|_{L^2}^2 \right) dt \le C$$

and

$$\begin{aligned} &\|\partial_t u(t)\|_{L^2}^2 + \|\partial_2 u(t)\|_{L^2}^2 + \nu\eta \|\partial_{12} u(t)\|_{L^2}^2 \\ &\quad \le \|\partial_t u(s)\|_{L^2}^2 + \|\partial_2 u(s)\|_{L^2}^2 + \nu\eta \|\partial_{12} u(s)\|_{L^2}^2 \quad \text{for any } t \ge s \ge 0. \end{aligned}$$

These two properties are proven via the wave equations in (1.9). The equations in (1.9) are very versatile and allow us to perform and combine different types of energy estimates. These two properties follow as a consequence of suitable combination of three energy inequalities derived from the wave equations in (1.9).

The rest of this paper is naturally divided into two sections. Section 2 proves Theorem 1.2 while Section 3 proves Theorem 1.3.

2. Proof of Theorem 1.2

This section proves Theorem 1.2. Our main efforts are devoted to verifying (1.16), namely

$$E(t) \le C_0\, E(0) + C_0\, E(t)^{\frac{3}{2}}, \tag{2.1}$$

where $E(t)$ is a combination of $E_1(t)$ and $E_2(t)$ (see (1.15)) with $E_1(t)$ and $E_2(t)$ given by (1.13) and (1.14), respectively. (2.1) is accomplished by the following proposition.

Proposition 2.1. *Let E_1 and E_2 be defined as in (1.13) and (1.14), respectively. Then*

$$E_1(t) \le E_1(0) + C_1\, E_1^{\frac{3}{2}}(t) + C_2\, E_2^{\frac{3}{2}}(t), \tag{2.2}$$

$$E_2(t) \le E_1(0) + C_3\, E_1(t) + C_4\, E_1^{\frac{3}{2}}(t) + C_5\, E_2^{\frac{3}{2}}(t). \tag{2.3}$$

As a special consequence, by choosing δ such that $C_3\,\delta \le \frac{1}{2}$ and define $E(t)$ as in (1.15), or

$$E(t) = E_1(t) + \delta E_2(t),$$

we obtain (2.1) for some suitable constant C_0.

The proof of Theorem 1.2 follows as a consequence of (2.1).

Proof of Theorem 1.2. First of all, for any $(u_0, b_0) \in H^2$, (1.2) always has a unique local-in-time solution. This can be shown by following a standard process involving the contraction mapping principle. One example of this process can be found in the book of Majda and Bertozzi [30]. An application of the bootstrapping argument to (2.1) would yield the global uniform bound that would guarantee the global existence and the stability of the solution.

To apply the bootstrapping argument, we take

$$\varepsilon := \frac{1}{4C_0^{3/2}} \quad \text{and} \quad \|u_0\|_{H^2} + \|b_0\|_{H^2} \le \varepsilon.$$

Then

$$E(0) := \|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 \le \frac{1}{16C_0^3}.$$

The argument starts with the ansatz that

$$E(t) \leq A := \frac{1}{4C_0^2}.$$

Then, by (2.1),

$$E(t) \leq C_0\, E(0) + C_0\, E(t)^{\frac{3}{2}} \leq C_0\, E(0) + \frac{1}{2} E(t)$$

or

$$E(t) \leq 2C_0 E(0) \leq \frac{1}{8C_0^2} = \frac{1}{2} A.$$

The bootstrapping argument then concludes that, for all $t > 0$,

$$E(t) \leq \frac{1}{8C_0^2} \leq 2C_0\, \varepsilon^2.$$

This concludes the proof of Theorem 1.2. □

It remains to prove Proposition 2.1. To deal with nonlinear terms, we need the following anisotropic inequality (see [8]).

Lemma 2.1. *Assume that $f, g, \partial_2 g, h$ and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then,*

$$\int_{\mathbb{R}^2} |fgh| dx \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

Proof of Proposition 2.1. We first prove (2.2). Using $\nabla \cdot u = \nabla \cdot b = 0$, the L^2-estimate yields the uniform global L^2-bound,

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_2(\tau)\|_{L^2}^2 \mathrm{d}\tau + 2\eta \int_0^t \|\nabla b_1(\tau)\|_{L^2}^2 \mathrm{d}\tau \\ &\leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned} \tag{2.4}$$

To estimate $\|\nabla u\|_{L^2}$ and $\|\nabla b\|_{L^2}$, we write the system of (ω, j) with $\omega = \nabla \times u$, $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_1 \Delta u_2 + b \cdot \nabla j + \partial_2 j, \\ \partial_t j + u \cdot \nabla j = -\eta \partial_2 \Delta b_1 + b \cdot \nabla \omega + Q + \partial_2 \omega, \end{cases} \tag{2.5}$$

where

$$Q = 2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2).$$

Multiplying the first and the second equation in (2.5) by ω and j, respectively, adding the results and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2)+\nu\|\partial_1\omega\|_{L^2}^2+\eta\|\partial_2 j\|_{L^2}^2\\
&=\int Qj\,\mathrm{d}x\\
&=\int 2\partial_1 b_1\partial_2 u_1 j\,\mathrm{d}x+\int 2\partial_1 b_1\partial_1 u_2 j\,\mathrm{d}x-\int 2\partial_1 u_1\partial_2 b_1 j\,\mathrm{d}x-\int 2\partial_1 u_1\partial_1 b_2 j\,\mathrm{d}x\\
&\doteq I_1+I_2+I_3+I_4,
\end{aligned}\tag{2.6}$$

where we used $\partial_1\Delta u_2=\partial_{11}\omega$ and $-\partial_2\Delta b_1=\partial_{22}j$. By Lemma 2.1 and Young's inequality,

$$\begin{aligned}
I_1&\le C\|\partial_1 b_1\|_{L^2}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_1\partial_2 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\\
&\le C\|\partial_1 b_1\|_{L^2}\|\omega\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12},\\
I_2&\le C\|\partial_1 b_1\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_1 u_2\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12},\\
I_3&\le C\|j\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_1\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 b_1\|_{L^2}^{\frac12}\\
&=C\|j\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 b_1\|_{L^2}^{\frac12},
\end{aligned}$$

where we used the simple fact $\|\partial_1\nabla u\|_{L^2}=\|\partial_1\omega\|_{L^2}$. All these upper bounds are suitable since the sum of the powers of the time-integrable parts in each upper bound is at least 2. For example, the upper bound for I_1 contains three time-integrable parts, $\|\partial_1 b_1\|_{L^2}$, $\|\partial_1\omega\|_{L^2}^{\frac12}$ and $\|\partial_2 j\|_{L^2}^{\frac12}$ and the sum of these powers is 2. To estimate I_4, we write $j=\partial_1 b_2-\partial_2 b_1$,

$$\begin{aligned}
I_4&=-2\int\partial_1 u_1\partial_1 b_2 j\,\mathrm{d}x\\
&=-2\int\partial_1 u_1\partial_1 b_2\partial_1 b_2\,\mathrm{d}x+2\int\partial_1 u_1\partial_1 b_2\partial_2 b_1\,\mathrm{d}x\\
&=I_{41}+I_{42}.
\end{aligned}$$

By integration by parts and Lemma 2.1,

$$\begin{aligned}
I_{41}&=2\int\partial_2 u_2\partial_1 b_2\partial_1 b_2\,\mathrm{d}x=-4\int u_2\partial_2\partial_1 b_2\partial_1 b_2\,\mathrm{d}x\\
&\le C\|\partial_2\partial_1 b_2\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_2\|_{L^2}^{\frac12}\\
&\le C\|u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_2\|_{L^2}^{\frac32}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}
\end{aligned}$$

and

$$\begin{aligned}
I_{42}&\le C\|\partial_1 b_2\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_{11} u_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 b_1\|_{L^2}^{\frac12}\\
&\le C\|\partial_1 b_2\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}.
\end{aligned}$$

Inserting the bounds above in (2.6) and integrating in time, we find

$$\begin{aligned}
&\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 \omega\|_{L^2}^2 \mathrm{d}\tau + 2\eta \int_0^t \|\partial_2 j\|_{L^2}^2 \mathrm{d}\tau \\
&\le E_1(0) + C\, E_1^{\frac{3}{2}}(t), \qquad (2.7)
\end{aligned}$$

where we have used Hölder's inequality to show the time integral of the bounds are all controlled by $E_1^{\frac{3}{2}}(t)$. For example,

$$\begin{aligned}
&\int_0^t \|\partial_1 b_1\|_{L^2} \|\omega\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 j\|_{L^2}^{\frac{1}{2}} \, d\tau \\
&\le \sup_{0\le\tau\le t} \|\omega\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \int_0^t \|\partial_1 b_1\|_{L^2} \, \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 j\|_{L^2}^{\frac{1}{2}} \, d\tau \\
&\le E_1^{\frac{1}{2}} \, E_1(t).
\end{aligned}$$

Next we estimate the H^2-norm. Multiplying the first equation in (2.5) by $(-\Delta\omega)$ and the second by $(-\Delta j)$, and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\big) + \nu\|\partial_1\nabla\omega\|_{L^2}^2 + \eta\|\partial_2\nabla j\|_{L^2}^2 \\
&= -\int \nabla u\cdot\nabla\omega\cdot\nabla\omega \mathrm{d}x + \int \nabla b\cdot\nabla j\cdot\nabla\omega \mathrm{d}x - \int \nabla u\cdot\nabla j\cdot\nabla j \mathrm{d}x \\
&\quad + \int \nabla b\cdot\nabla\omega\cdot\nabla j \mathrm{d}x + \int \nabla Q\cdot\nabla j \mathrm{d}x, \qquad (2.8)
\end{aligned}$$

where we used the facts, as $\nabla\cdot u = \nabla\cdot b = 0$,

$$\int u\cdot\nabla\nabla\omega\cdot\nabla\omega \mathrm{d}x = 0, \quad \int b\cdot\nabla\nabla j\cdot\nabla\omega \mathrm{d}x + \int b\cdot\nabla\nabla\omega\cdot\nabla j \mathrm{d}x = 0.$$

Integrating (2.8) in time, we have

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + 2\int_0^t (\nu\|\partial_1\nabla\omega\|_{L^2}^2 + \eta\|\partial_2\nabla j\|_{L^2}^2) \,\mathrm{d}\tau \\
&\le E_1(0) - C\int_0^t \int \nabla u\cdot\nabla\omega\cdot\nabla\omega \mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t\int \nabla b\cdot\nabla j\cdot\nabla\omega\,\mathrm{d}x\mathrm{d}\tau - C\int_0^t\int \nabla u\cdot\nabla j\cdot\nabla j\,\mathrm{d}x\mathrm{d}\tau\\
&+C\int_0^t\int \nabla b\cdot\nabla \omega\cdot\nabla j\,\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int \nabla Q\cdot\nabla j\,\mathrm{d}x\mathrm{d}\tau\\
&= E_1(0)+\sum_{i=1}^{5} J_i.
\end{aligned}\tag{2.9}
$$

We write J_1 into the following four terms explicitly,

$$
\begin{aligned}
J_1 = &-C\int_0^t\int \partial_1 u_1(\partial_1\omega)^2\mathrm{d}x\mathrm{d}\tau - C\int_0^t\int \partial_2 u_1\partial_1\omega\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau\\
&-C\int_0^t\int \partial_1 u_2\partial_2\omega\partial_1\omega\,\mathrm{d}x\mathrm{d}\tau - C\int_0^t\int \partial_2 u_2(\partial_2\omega)^2\mathrm{d}x\mathrm{d}\tau\\
= &\; J_{11}+J_{12}+J_{13}+J_{14}.
\end{aligned}
$$

By Lemma 2.1 and Young's inequality,

$$
\begin{aligned}
J_{11}+J_{13} \le\; & C\int_0^t \|\partial_1 u_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_1\partial_1\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&+C\int_0^t \|\partial_1 u_2\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
\le\; & C\sup_{0\le\tau\le t}\|\nabla\omega\|_{L^2}\int_0^t(\|\nabla u_2\|_{L^2}^2+\|\partial_1\nabla\omega\|_{L^2}^2)\mathrm{d}\tau\\
\le\; & CE_1^{\frac12}(t)E_1(t),\\
J_{12}\le\; & C\int_0^t \|\partial_2 u_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
\le\; & C\sup_{0\le\tau\le t}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\int_0^t\big(\|\partial_2 u_1\|_{L^2}^2+\|\partial_1\partial_2\omega\|_{L^2}^2\big)\mathrm{d}\tau\\
\le\; & CE_1^{\frac12}(t)(E_2(t)+E_1(t)),
\end{aligned}
$$

and by $\nabla\cdot u=0$,

$$J_{14} \leq C\int_0^t \|\partial_2\omega\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau$$

$$\leq C\sup_{0\leq\tau\leq t}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\int_0^t \big(\|\partial_2\omega\|_{L^2}^2+\|\partial_1\partial_2\omega\|_{L^2}^2\big)\mathrm{d}\tau$$

$$\leq CE_1^{\frac{1}{2}}(t)(E_1(t)+E_2(t)).$$

Therefore,

$$J_1 \leq CE_1^{\frac{3}{2}}(t)+E_1^{\frac{1}{2}}(t)E_2(t). \tag{2.10}$$

J_2 can be handled similarly as J_1,

$$J_2 = C\int_0^t\int \partial_1 b_1\partial_1 j\partial_1\omega\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int \partial_2 b_1\partial_1 j\partial_2\omega\mathrm{d}x\mathrm{d}\tau$$

$$+C\int_0^t\int \partial_1 b_2\partial_2 j\partial_1\omega\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int \partial_2 b_2\partial_2 j\partial_2\omega\mathrm{d}x\mathrm{d}\tau.$$

By $\nabla\cdot b = 0$ and Young's inequality,

$$J_2 \leq C\int_0^t \|\partial_1 j\|_{L^2}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau$$

$$+C\int_0^t \|\partial_2 b_1\|_{L^2}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau$$

$$+C\int_0^t \|\partial_2 j\|_{L^2}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau$$

$$+C\int_0^t \|\partial_2\omega\|_{L^2}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_2 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 j\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau$$

$$\leq C\sup_{0\leq\tau\leq t}\big(\|b\|_{H^2}+\|u\|_{H^2}\big)\int_0^t \big(\|\partial_2 b\|_{H^2}^2+\|\partial_1\omega\|_{H^1}^2\big)\mathrm{d}\tau$$

$$\leq CE_1^{\frac{3}{2}}(t). \tag{2.11}$$

Next we estimate J_3,

$$
\begin{aligned}
J_3 = &-C\int_0^t\int \partial_1 u_1 \partial_1 j \partial_1 j \mathrm{d}x\mathrm{d}\tau + \big(-C\int_0^t\int \partial_1 u_2 \partial_2 j \partial_1 j \mathrm{d}x\mathrm{d}\tau \\
&-C\int_0^t\int \partial_2 u_1 \partial_1 j \partial_2 j \mathrm{d}x\mathrm{d}\tau - C\int_0^t\int \partial_2 u_2 \partial_2 j \partial_2 j \mathrm{d}x\mathrm{d}\tau\big) \\
= &J_{31} + J_{32}.
\end{aligned}
$$

By integration by parts,

$$
\begin{aligned}
J_{31} &= C\int_0^t\int \partial_2 u_2 \partial_1 j \partial_1 j \mathrm{d}x\mathrm{d}\tau = -2C\int_0^t\int u_2 \partial_2 \partial_1 j \partial_1 j \mathrm{d}x\mathrm{d}\tau \\
&\le C\int_0^t \|\partial_1\partial_2 j\|_{L^2} \|u_2\|_{L^2}^{\frac12} \|\partial_1 u_2\|_{L^2}^{\frac12} \|\partial_1 j\|_{L^2}^{\frac12} \|\partial_2\partial_1 j\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t} \|u_2\|_{L^2}^{\frac12} \|\partial_1 j\|_{L^2}^{\frac12} \int_0^t \big(\|\partial_2\partial_1 j\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2\big)\mathrm{d}\tau \\
&\le C E_1^{\frac12}(t) E_1(t), \\
J_{32} &\le C\int_0^t \|\partial_1 u_2\|_{L^2} \|\partial_2 j\|_{L^2}^{\frac12} \|\partial_1\partial_2 j\|_{L^2}^{\frac12} \|\partial_1 j\|_{L^2}^{\frac12} \|\partial_2\partial_1 j\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\quad + C\int_0^t \|\partial_2 j\|_{L^2} \|\partial_1 j\|_{L^2}^{\frac12} \|\partial_2\partial_1 j\|_{L^2}^{\frac12} \|\partial_2 u_1\|_{L^2}^{\frac12} \|\partial_1\partial_2 u_1\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\quad + C\int_0^t \|\partial_2 u_2\|_{L^2} \|\partial_2 j\|_{L^2}^{\frac12} \|\partial_1\partial_2 j\|_{L^2}^{\frac12} \|\partial_2 j\|_{L^2}^{\frac12} \|\partial_2\partial_2 j\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t} \big(\|b\|_{H^2} + \|u\|_{H^1}\big) \int_0^t \big(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 b\|_{H^2}^2\big)\mathrm{d}\tau \\
&\le C E_1^{\frac12}(t) E_1(t).
\end{aligned}
$$

Summing them up leads to

$$
J_3 \le E_1^{\frac32}(t). \tag{2.12}
$$

Next we bound J_4,

$$\begin{aligned}
J_4 &= C\int_0^t\int \partial_1 b_1\partial_1\omega\partial_1 j\,\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int \partial_1 b_2\partial_2\omega\partial_1 j\,\mathrm{d}x\mathrm{d}\tau \\
&\quad + C\int_0^t\int \partial_2 b_1\partial_1\omega\partial_2 j\,\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int \partial_2 b_2\partial_2\omega\partial_2 j\,\mathrm{d}x\mathrm{d}\tau \\
&= J_{41} + J_{42} + J_{43} + J_{44}.
\end{aligned}$$

By $\nabla\cdot b = 0$,

$$\begin{aligned}
J_{41} &= -C\int_0^t\int \partial_2 b_2\partial_1\omega\partial_1 j\,\mathrm{d}x\mathrm{d}\tau \\
&\le C\int_0^t \|\partial_1 j\|_{L^2}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_1\partial_1\omega\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\|\partial_1 j\|_{L^2}\int_0^t \big(\|\partial_2 b\|_{H^1}^2 + \|\partial_1 u\|_{H^2}^2\big)\mathrm{d}\tau \\
&\le C E_1^{\frac12}(t)E_1(t).
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
J_{42} &= -C\int_0^t\int \partial_2\partial_1 b_2\omega\partial_1 j\,\mathrm{d}x\mathrm{d}\tau - C\int_0^t\int \partial_1 b_2\omega\partial_2\partial_1 j\,\mathrm{d}x\mathrm{d}\tau \\
&\le C\int_0^t \big(\|\omega\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12}\big)\big(\|\partial_2\partial_1 b_2\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1 j\|_{L^2}^{\frac12}\big)\mathrm{d}\tau \\
&\quad + C\int_0^t \big(\|\omega\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\big)\big(\|\partial_2\partial_1 j\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_2\|_{L^2}^{\frac12}\big)\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\big(\|\omega\|_{L^2} + \|\partial_1 b\|_{H^1}\big)\int_0^t \big(\|\partial_2 b\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2\big)\mathrm{d}\tau \\
&\le C E_1^{\frac12}(t)E_1(t),
\end{aligned}$$

$$
\begin{aligned}
J_{43}+J_{44} &\le C\int_0^t \|\nabla\omega\|_{L^2}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1 b\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_2\partial_2 j\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\|\nabla\omega\|_{L^2}\int_0^t \|\partial_2 b\|_{H^2}^2\mathrm{d}\tau \\
&\le CE_1^{\frac12}(t)E_1(t).
\end{aligned}
$$

Therefore,

$$
J_4 \le CE_1^{\frac32}(t). \tag{2.13}
$$

We write J_5 into four terms,

$$
\begin{aligned}
J_5 &= \int_0^t\int \nabla Q\cdot\nabla j\,\mathrm{d}x\mathrm{d}\tau = -\int_0^t\int Q\Delta j\,\mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t\int 2\big[\partial_1 b_1(\partial_2 u_1+\partial_1 u_2)-\partial_1 u_1(\partial_2 b_1+\partial_1 b_2)\big]\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau \\
&\quad -\int_0^t\int 2\big[\partial_1 b_1(\partial_2 u_1+\partial_1 u_2)-\partial_1 u_1(\partial_2 b_1+\partial_1 b_2)\big]\partial_{22} j\,\mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t\int 2\partial_1 b_1(\partial_2 u_1+\partial_1 u_2)\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau + \int_0^t\int 2\partial_1 u_1(\partial_2 b_1+\partial_1 b_2)\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau \\
&\quad -\int_0^t\int 2\big[\partial_1 b_1(\partial_2 u_1+\partial_1 u_2)-\partial_1 u_1(\partial_2 b_1+\partial_1 b_2)\big]\partial_{22} j\,\mathrm{d}x\mathrm{d}\tau \\
&\doteq \sum_{i=1}^3 J_{5i}.
\end{aligned}
$$

First we estimate J_{53},

$$
\begin{aligned}
J_{53} &\le C\int_0^t \|\partial_{22} j\|_{L^2}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\big(\|\nabla b\|_{L^2}+\|\nabla u\|_{L^2}\big)\int_0^t\big(\|\partial_2\nabla b\|_{H^1}^2+\|\partial_1\nabla u\|_{L^2}^2\big)\mathrm{d}\tau
\end{aligned}
$$

$$\leq CE_1^{\frac{1}{2}}(t)E_1(t).$$

By $\nabla\cdot b=0$ and integration by parts,

$$\begin{aligned}
J_{51} &= \int_0^t\int 2\partial_2 b_2(\partial_2 u_1+\partial_1 u_2)\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau\\
&= -\int_0^t\int 2\partial_1\partial_2 b_2(\partial_2 u_1+\partial_1 u_2)\partial_1 j\,\mathrm{d}x\mathrm{d}\tau - \int_0^t\int 2\partial_2 b_2\partial_1(\partial_2 u_1+\partial_1 u_2)\partial_1 j\,\mathrm{d}x\mathrm{d}\tau\\
&\leq C\int_0^t \|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau\\
&\quad +C\int_0^t \|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau\\
&\leq C\sup_{0\leq\tau\leq t}\big(\|\partial_1 j\|_{L^2}+\|u\|_{H^1}\big)\int_0^t\big(\|\partial_2 b\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\big)\mathrm{d}\tau\\
&\leq CE_1^{\frac{1}{2}}(t)E_1(t).
\end{aligned}$$

$$\begin{aligned}
J_{52} &= \int_0^t\int 2\partial_1 u_1\partial_2 b_1\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau + \int_0^t\int 2\partial_1 u_1\partial_1 b_2\partial_{11} j\,\mathrm{d}x\mathrm{d}\tau\\
&\doteq J_{521}+J_{522}.
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
J_{521} &= -\int_0^t\int 2\partial_{11} u_1\partial_2 b_1\partial_1 j\,\mathrm{d}x\mathrm{d}\tau - \int_0^t\int 2\partial_1 u_1\partial_1\partial_2 b_1\partial_1 j\,\mathrm{d}x\mathrm{d}\tau\\
&\leq C\int_0^t \|\partial_1 j\|_{L^2}\|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{111} u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_{22} b_1\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau\\
&\quad +\int_0^t \|\partial_1 j\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{22} b_1\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau\\
&\leq C\sup_{0\leq\tau\leq t}\|\partial_1 j\|_{L^2}\int_0^t\big(\|\partial_1 u_1\|_{H^2}^2+\|\partial_2 b_1\|_{H^2}^2\big)\mathrm{d}\tau
\end{aligned}$$

$$\leq CE_1^{\frac{1}{2}}(t)E_1(t).$$

By $\nabla \cdot u = 0$ and integration by parts,

$$\begin{aligned}
J_{522} &= -\int_0^t \int 2\partial_2 u_2 \partial_1 b_2 \partial_{11} j \,\mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int 2u_2 \partial_2 \partial_1 b_2 \partial_{11} j \,\mathrm{d}x\mathrm{d}\tau + \int_0^t \int 2u_2 \partial_1 b_2 \partial_2 \partial_{11} j \,\mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t \int 2\partial_1 u_2 \partial_2 \partial_1 b_2 \partial_1 j \,\mathrm{d}x\mathrm{d}\tau - \int_0^t \int 2u_2 \partial_2 \partial_{11} b_2 \partial_1 j \,\mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int 2\partial_1 u_2 \partial_1 b_2 \partial_2 \partial_1 j \,\mathrm{d}x\mathrm{d}\tau - \int_0^t \int 2u_2 \partial_{11} b_2 \partial_2 \partial_1 j \,\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
J_{522} &\leq C\int_0^t \|\partial_1 j\|_{L^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_{22} \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad + C\int_0^t \|\partial_2 \partial_{11} b_2\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 j\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad + C\int_0^t \|\partial_2 \partial_1 j\|_{L^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad + C\int_0^t \|\partial_2 \partial_1 j\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_{11} b_2\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\
&\leq C \sup_{0\leq\tau\leq t} \big(\|b\|_{H^2} + \|u\|_{H^1}\big) \int_0^t \big(\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 b\|_{H^2}^2\big)\mathrm{d}\tau \\
&\leq CE_1^{\frac{1}{2}}(t)E_1(t).
\end{aligned}$$

Therefore,

$$J_5 \leq E_1^{\frac{3}{2}}(t). \tag{2.14}$$

Inserting (2.10), (2.11), (2.12), (2.13) and (2.14) into (2.9), and combining with (2.4) and (2.7), we obtain

$$E_1(t) \le E_1(0) + C_1 E_1^{\frac{3}{2}}(t) + C_2 E_1^{\frac{1}{2}}(t) E_2(t).$$

This verifies (2.2). Next we check (2.3). We rewrite

$$\begin{aligned}
E_2(t) &= \int_0^t \|\partial_2 u\|_{H^1}^2 \mathrm{d}\tau = \int_0^t \left(\|\partial_2 u\|_{L^2}^2 + \|\partial_2 \nabla u\|_{L^2}^2\right)\mathrm{d}\tau \\
&= \int_0^t \left(\|\partial_2 u\|_{L^2}^2 + \|\partial_2 \omega\|_{L^2}^2\right)\mathrm{d}\tau. \qquad (2.15)
\end{aligned}$$

Recalling the perturbations equations in (1.2), multiplying the second equation in (1.2) by $\partial_2 u$ and integrating over $\mathbb{R}^2$ and also in time lead to

$$\begin{aligned}
\int_0^t \|\partial_2 u\|_{L^2}^2 \mathrm{d}\tau &= \int_0^t \int \partial_\tau b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau + \int_0^t \int u \cdot \nabla b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&\quad - \eta \int_0^t \int \Delta b_1 \partial_2 u_1 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int b \cdot \nabla u \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int \left(\partial_\tau (b \cdot \partial_2 u) - b \cdot \partial_\tau \partial_2 u\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t \int u \cdot \nabla b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&\quad - \eta \int_0^t \int \Delta b_1 \partial_2 u_1 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int b \cdot \nabla u \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \int b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau + \int_0^t \int \partial_\tau u \cdot \partial_2 b \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t \int u \cdot \nabla b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau - \eta \int_0^t \int \Delta b_1 \partial_2 u_1 \mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int b \cdot \nabla u \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \int b \cdot \partial_2 u \mathrm{d}x\mathrm{d}\tau - \int_0^t \int u \cdot \nabla u \cdot \partial_2 b \mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int \nabla p \cdot \partial_2 b \mathrm{d}x\mathrm{d}\tau + \int_0^t \int \nu \Delta u_2 \partial_2 b_2 \mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$+\int_0^t\int b\cdot\nabla b\cdot\partial_2 b\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_2 b\cdot\partial_2 b\mathrm{d}x\mathrm{d}\tau$$
$$+\int_0^t\int u\cdot\nabla b\cdot\partial_2 u\mathrm{d}x\mathrm{d}\tau-\eta\int_0^t\int \Delta b_1\partial_2 u_1\mathrm{d}x\mathrm{d}\tau$$
$$-\int_0^t\int b\cdot\nabla u\cdot\partial_2 u\mathrm{d}x\mathrm{d}\tau$$
$$\doteq\sum_{i=1}^9 L_i, \tag{2.16}$$

where we have used the first equation in (1.2) as well.

$$\begin{aligned}L_1&=\int b\cdot\partial_2 u\mathrm{d}x-\int b_0\cdot\partial_2 u_0\mathrm{d}x\\&\le C\|b\|_{L^2}\|\partial_2 u\|_{L^2}+C\|b_0\|_{L^2}\|\partial_2 u_0\|_{L^2}\\&\le CE_1(t)+CE_1(0).\end{aligned} \tag{2.17}$$

By integration by parts,

$$\begin{aligned}L_2&=-\int_0^t\int u_1\partial_1 u\partial_2 b\mathrm{d}x\mathrm{d}\tau-\int_0^t\int u_2\partial_2 u\partial_2 b\mathrm{d}x\mathrm{d}\tau\\&=-\int_0^t\int u_1\partial_1 u\partial_2 b\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_2 u_2 u\partial_2 b\mathrm{d}x\mathrm{d}\tau+\int_0^t\int u_2 u\partial_{22} b\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

By Lemma 2.1 and $\nabla\cdot u=0$,

$$\begin{aligned}L_2&\le C\int_0^t \|u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_{11} u\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_{22} b\|_{L^2}^{\frac12}\mathrm{d}\tau\\&\quad+C\int_0^t \|\partial_{22} b\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\partial_2 u_2\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\mathrm{d}\tau\\&\le C\sup_{0\le\tau\le t}\|u\|_{L^2}\int_0^t\big(\|\partial_1 u\|_{H^1}^2+\|\partial_2 b\|_{H^1}^2\big)\mathrm{d}\tau\\&\le CE_1^{\frac32}(t).\end{aligned} \tag{2.18}$$

By $\nabla \cdot b = 0$,

$$L_3 = \int_0^t \int p \partial_2 \nabla \cdot b \mathrm{d}x \mathrm{d}\tau = 0. \tag{2.19}$$

Clearly,

$$\begin{aligned} L_4 &= \int_0^t \int \nu \Delta u_2 \partial_2 b_2 \mathrm{d}x \mathrm{d}\tau = \int_0^t \int \nu \partial_1 \omega \partial_2 b_2 \mathrm{d}x \mathrm{d}\tau \\ &\le C \int_0^t \|\partial_1 \omega\|_{L^2} \|\partial_2 b_2\|_{L^2} \mathrm{d}\tau \\ &\le C E_1(t), \end{aligned} \tag{2.20}$$

where we used $\Delta u_2 = \partial_1 \omega$. By integration by parts,

$$\begin{aligned} L_5 &= \int_0^t \int b_1 \partial_1 b \partial_2 b \mathrm{d}x \mathrm{d}\tau + \int_0^t \int b_2 \partial_2 b \partial_2 b \mathrm{d}x \mathrm{d}\tau \\ &= -\int_0^t \int \partial_1 b_1 b \partial_2 b \mathrm{d}x \mathrm{d}\tau - \int_0^t \int b_1 b \partial_1 \partial_2 b \mathrm{d}x \mathrm{d}\tau + \int_0^t \int b_2 \partial_2 b \partial_2 b \mathrm{d}x \mathrm{d}\tau \\ &= \int_0^t \int \partial_2 b_2 b \partial_2 b \mathrm{d}x \mathrm{d}\tau - \int_0^t \int b_1 b \partial_1 \partial_2 b \mathrm{d}x \mathrm{d}\tau + \int_0^t \int b_2 \partial_2 b \partial_2 b \mathrm{d}x \mathrm{d}\tau. \end{aligned}$$

The terms on the right can be bounded as follows,

$$\begin{aligned} L_5 &\le C \int_0^t \|b\|_{L^2} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_{22} b\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\ &\quad + C \int_0^t \|\partial_2 \partial_1 b\|_{L^2} \|b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\ &\le C \sup_{0 \le \tau \le t} \|b\|_{L^2} \int_0^t \|\partial_2 b\|_{H^1}^2 \mathrm{d}\tau \\ &\le C E_1^{\frac{3}{2}}(t). \end{aligned} \tag{2.21}$$

Obviously,

$$L_6 = \int_0^t \|\partial_2 b\|_{L^2}^2 \mathrm{d}\tau \le E_1(t). \tag{2.22}$$

By integration by parts,

$$\begin{aligned}
L_7 &= \int_0^t \int u_1 \partial_1 b \partial_2 u \mathrm{d}x\mathrm{d}\tau + \int_0^t \int u_2 \partial_2 b \partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t \int \partial_1 u_1 b \partial_2 u \mathrm{d}x\mathrm{d}\tau - \int_0^t \int u_1 b \partial_1\partial_2 u \mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int \partial_2 u_2 \partial_2 b u \mathrm{d}x\mathrm{d}\tau - \int_0^t \int u_2 \partial_{22} b u \mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

the terms on the right can be bounded as follows,

$$\begin{aligned}
L_7 &\le C \int_0^t \|\partial_1 u_1\|_{L^2} \|b\|_{L^2}^{\frac12} \|\partial_2 b\|_{L^2}^{\frac12} \|\partial_2 u\|_{L^2}^{\frac12} \|\partial_1\partial_2 u\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\quad + C \int_0^t \|\partial_1\partial_2 u\|_{L^2} \|u_1\|_{L^2}^{\frac12} \|\partial_1 u_1\|_{L^2}^{\frac12} \|b\|_{L^2}^{\frac12} \|\partial_2 b\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\quad + C \int_0^t \|u\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac12} \|\partial_{11} u_1\|_{L^2}^{\frac12} \|\partial_2 b\|_{L^2}^{\frac12} \|\partial_{22} b\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\quad + C \int_0^t \|\partial_{22} b\|_{L^2} \|u_2\|_{L^2}^{\frac12} \|\partial_2 u_2\|_{L^2}^{\frac12} \|u\|_{L^2}^{\frac12} \|\partial_1 u\|_{L^2}^{\frac12} \mathrm{d}\tau \\
&\le C \sup_{0\le\tau\le t} \big(\|b\|_{L^2} + \|u\|_{H^1}\big) \int_0^t \big(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 b\|_{H^1}^2\big) \mathrm{d}\tau \\
&\le C E_1^{\frac32}(t),
\end{aligned} \tag{2.23}$$

where we used $\nabla \cdot u = 0$. By Hölder's inequality,

$$L_8 = -\eta \int_0^t \int \big(\partial_{11} b_1 \partial_2 u_1 + \partial_{22} b_1 \partial_2 u_1\big) \mathrm{d}x\mathrm{d}\tau$$

$$
\begin{aligned}
&\le C\int_0^t \big(\|\partial_1\partial_2 b_2\|_{L^2}\|\partial_2 u_1\|_{L^2} + \|\partial_{22} b_1\|_{L^2}\|\partial_2 u_1\|_{L^2}\big)\mathrm{d}\tau \\
&\le \frac{1}{8}E_2(t) + C\,E_1(t). \qquad (2.24)
\end{aligned}
$$

By integration by parts,

$$
\begin{aligned}
L_9 &= -\int_0^t\int b_1\partial_1 u\partial_2 u\mathrm{d}x\mathrm{d}\tau - \int_0^t\int b_2\partial_2 u\partial_2 u\mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t\int \partial_1 b_1 u\partial_2 u\mathrm{d}x\mathrm{d}\tau + \int_0^t\int b_1 u\partial_1\partial_2 u\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int \partial_2 b_2 u\partial_2 u\mathrm{d}x\mathrm{d}\tau + \int_0^t\int b_2 u\partial_{22} u\mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t\int b_1 u\partial_1\partial_2 u\mathrm{d}x\mathrm{d}\tau + \int_0^t\int b_2 u\partial_{22} u\mathrm{d}x\mathrm{d}\tau \\
&\doteq L_{91} + L_{92},
\end{aligned}
$$

where we used $\nabla\cdot b = 0$. The terms on the right can be bounded as follows,

$$
\begin{aligned}
L_{91} &\le C\int_0^t \|\partial_1\partial_2 u\|_{L^2}\|b_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\|b_1\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\int_0^t \big(\|\partial_1 u\|_{H^1}^2 + \|\partial_2 b_1\|_{L^2}^2\big)\mathrm{d}\tau \\
&\le C E_1^{\frac12}(t)E_1(t).
\end{aligned}
$$

By Young's inequality,

$$
\begin{aligned}
L_{92} &\le C\int_0^t \|\partial_{22} u\|_{L^2}\|b_2\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\int_0^t \frac{1}{8}\|\partial_{22} u\|_{L^2}^2\mathrm{d}\tau + C\int_0^t \|b_2\|_{L^2}\|\partial_2 b_2\|_{L^2}\|u\|_{L^2}\|\partial_1 u\|_{L^2}\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^t \frac{1}{8}\|\partial_{22}u\|_{L^2}^2 d\tau + C\int_0^t \left(\|\partial_2 b_2\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2\right) d\tau \\
&\le \frac{1}{8}E_2(t) + CE_1(t),
\end{aligned}
$$

where the uniform L^2-bound (2.4) is used. Therefore,

$$
L_9 \le \frac{1}{8}E_2(t) + CE_1(t) + CE_1^{\frac{3}{2}}(t). \tag{2.25}
$$

Inserting (2.17) through (2.25) in (2.16) leads to

$$
\int_0^t \|\partial_2 u\|_{L^2}^2 d\tau \le CE_1(0) + CE_1(t) + \frac{1}{4}E_2(t) + CE_1^{\frac{3}{2}}(t). \tag{2.26}
$$

Now we bound the second part in (2.15). Multiplying the second equation in (2.5) by $\partial_2\omega$ and integrating over $\mathbb{R}^2$ and also in time lead to

$$
\begin{aligned}
\int_0^t \|\partial_2\omega\|_{L^2}^2 d\tau = &\int_0^t\int \partial_\tau j\partial_2\omega dx d\tau + \int_0^t\int u\cdot\nabla j\partial_2\omega dx d\tau \\
&+\eta\int_0^t\int \partial_2\Delta b_1\partial_2\omega dx d\tau - \int_0^t\int b\cdot\nabla\omega\partial_2\omega dx d\tau \\
&-\int_0^t\int Q\partial_2\omega dx d\tau \\
=&\int_0^t\int \left(\partial_\tau(j\partial_2\omega) - j\partial_2\partial_\tau\omega\right) dx d\tau + \int_0^t\int u\cdot\nabla j\partial_2\omega dx d\tau \\
&+\eta\int_0^t\int \partial_2\Delta b_1\partial_2\omega dx d\tau - \int_0^t\int b\cdot\nabla\omega\partial_2\omega dx d\tau \\
&-\int_0^t\int Q\partial_2\omega dx d\tau \\
=&\int_0^t\int \partial_\tau(j\partial_2\omega) dx d\tau + \int_0^t\int j\partial_2(u\cdot\nabla\omega) dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
& -\nu\int_0^t\int j\partial_2(\partial_1\Delta u_2)\mathrm{d}x\mathrm{d}\tau-\int_0^t\int j\partial_2(b\cdot\nabla j)\mathrm{d}x\mathrm{d}\tau \\
& -\int_0^t\int j\partial_2\partial_2 j\mathrm{d}x\mathrm{d}\tau+\int_0^t\int u\cdot\nabla j\partial_2\omega\mathrm{d}x\mathrm{d}\tau \\
& +\eta\int_0^t\int \partial_2\Delta b_1\partial_2\omega\mathrm{d}x\mathrm{d}\tau-\int_0^t\int b\cdot\nabla\omega\partial_2\omega\mathrm{d}x\mathrm{d}\tau \\
& -\int_0^t\int Q\partial_2\omega\mathrm{d}x\mathrm{d}\tau \\
\doteq & \sum_{i=1}^9 H_i, \qquad (2.27)
\end{aligned}
$$

where we have used the first equation in (2.5). Clearly,

$$
\begin{aligned}
H_1 &= \int j(x,t)\partial_2\omega(x,t)\mathrm{d}x-\int j_0\partial_2\omega_0\mathrm{d}x \\
&\le \|j\|_{L^2}\|\partial_2\omega\|_{L^2}+\|j_0\|_{L^2}\|\partial_2\omega_0\|_{L^2} \\
&\le E_1(t)+E_1(0). \qquad (2.28)
\end{aligned}
$$

By integration by parts,

$$
\begin{aligned}
H_2 &= -\int_0^t\int \partial_2 j(u\cdot\nabla\omega)\mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t\int \partial_2 ju_1\partial_1\omega\mathrm{d}x\mathrm{d}\tau-\int_0^t\int \partial_2 ju_2\partial_2\omega\mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t\int \partial_2 ju_1\partial_1\omega\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_{22} ju_2\omega\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_2 j\partial_2 u_2\omega\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

The terms on the right can be bounded as follows,

$$
\begin{aligned}
H_2 \le\ & C\int_0^t \|u_1\|_{L^2}\|\partial_2 j\|_{L^2}^{\frac{1}{2}}\|\partial_{22} j\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_{11}\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau \\
& +C\int_0^t \|\partial_{22} j\|_{L^2}\|u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t \|\omega\|_{L^2}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_{22} j\|_{L^2}^{\frac12}\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_{11} u_1\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\le C\sup_{0\le\tau\le t}\|u\|_{H^1}\int_0^t \big(\|\partial_2 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\big)\mathrm{d}\tau\\
&\le CE_1^{\frac32}(t),
\end{aligned}
\tag{2.29}
$$

where we used $\nabla\cdot u=0$. H_3 is bounded by

$$
\begin{aligned}
H_3&=-\nu\int_0^t\int \partial_1\partial_2 j\,\partial_{11}u_2\,\mathrm{d}x\mathrm{d}\tau-\nu\int_0^t\int \partial_1\partial_2 j\,\partial_{22}u_2\,\mathrm{d}x\mathrm{d}\tau\\
&=-\nu\int_0^t\int \partial_1\partial_2 j\,\partial_{11}u_2\,\mathrm{d}x\mathrm{d}\tau-\nu\int_0^t\int \partial_1\partial_2 j\,\partial_2\partial_1 u_1\,\mathrm{d}x\mathrm{d}\tau\\
&\le C\int_0^t \|\partial_1\partial_2 j\|_{L^2}\|\partial_1 u\|_{H^1}\mathrm{d}\tau\\
&\le CE_1(t).
\end{aligned}
\tag{2.30}
$$

By integration by parts,

$$
\begin{aligned}
H_4&=\int_0^t\int \partial_2 j\,(b\cdot\nabla j)\,\mathrm{d}x\mathrm{d}\tau\\
&=\int_0^t\int \partial_2 j\, b_1\partial_1 j\,\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_2 j\, b_2\partial_2 j\,\mathrm{d}x\mathrm{d}\tau\\
&=-\int_0^t\int \partial_1\partial_2 j\, b_1 j\,\mathrm{d}x\mathrm{d}\tau-\int_0^t\int \partial_2 j\,\partial_1 b_1 j\,\mathrm{d}x\mathrm{d}\tau+\int_0^t\int \partial_2 j\, b_2\partial_2 j\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

The terms on the right are bounded by

$$
\begin{aligned}
H_4&\le C\int_0^t \|\partial_1\partial_2 j\|_{L^2}\|b_1\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\quad+C\int_0^t \|j\|_{L^2}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_{22} j\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_{11} b_1\|_{L^2}^{\frac12}\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t \|b_2\|_{L^2}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_1\partial_2 j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_{22} j\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\le C\sup_{0\le\tau\le t}\big(\|b\|_{L^2}+\|j\|_{L^2}\big)\int_0^t \|\partial_2 b\|_{H^2}^2\mathrm{d}\tau\\
&\le CE_1^{\frac12}(t)E_1(t),
\end{aligned}
\tag{2.31}
$$

where we used $\nabla\cdot b=0$. Obviously,

$$
H_5=\int_0^t\int \partial_2 j\,\partial_2 j\,\mathrm{d}x\mathrm{d}\tau\le CE_1(t). \tag{2.32}
$$

By integration by parts,

$$
\begin{aligned}
H_6&=\int_0^t\int u_1\partial_1 j\,\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau+\int_0^t\int u_2\partial_2 j\,\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau\\
&=-\int_0^t\int \partial_1 u_1\, j\,\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau-\int_0^t\int u_1 j\,\partial_1\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau+\int_0^t\int u_2\partial_2 j\,\partial_2\omega\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

We estimate the terms on the right as follows.

$$
\begin{aligned}
H_6&\le C\int_0^t \|\partial_1 u_1\|_{L^2}\|j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\quad+C\int_0^t \|\partial_1\partial_2\omega\|_{L^2}\|u_1\|_{L^2}^{\frac12}\|\partial_1 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_2 j\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\quad+C\int_0^t \|\partial_2 j\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\le C\sup_{0\le\tau\le t}\big(\|b\|_{H^1}+\|u\|_{H^2}\big)\int_0^t\big(\|\partial_1 u\|_{H^2}^2+\|\partial_2 b\|_{H^1}^2\big)\mathrm{d}\tau\\
&\le CE_1^{\frac12}(t)E_1(t).
\end{aligned}
\tag{2.33}
$$

Clearly,

$$H_7 \le \int_0^t \|\partial_2 b_1\|_{H^2}\|\partial_2\omega\|_{L^2}\mathrm{d}\tau \le \frac{1}{4}E_2(t) + CE_1(t).$$

We write

$$H_8 = -\int_0^t\int b_1\partial_1\omega\partial_2\omega\mathrm{d}x\mathrm{d}\tau - \int_0^t\int b_2\partial_2\omega\partial_2\omega\mathrm{d}x\mathrm{d}\tau \doteq H_{81} + H_{82}.$$

The bounds for H_{81} and H_{82} are

$$\begin{aligned}
H_{81} &\le C\int_0^t \|\partial_1\omega\|_{L^2}\|b_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\big(\|b_1\|_{L^2} + \|\partial_2\omega\|_{L^2}\big)\int_0^t\big(\|\partial_1\omega\|_{H^1}^2 + \|\partial_2 b_1\|_{L^2}^2\big)\mathrm{d}\tau \\
&\le CE_1^{\frac12}(t)E_1(t), \\
H_{82} &\le C\int_0^t \|\partial_2\omega\|_{L^2}\|b_2\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\|b_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\int_0^t\|\partial_2\omega\|_{L^2}\big(\|\partial_2 b_2\|_{L^2} + \|\partial_1\partial_2\omega\|_{L^2}\big)\mathrm{d}\tau \\
&\le C\sup_{0\le\tau\le t}\|b_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\Big(\int_0^t\|\partial_2\omega\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac12}\Big(\int_0^t\|\partial_2 b_2\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac12} \\
&\quad + C\sup_{0\le\tau\le t}\|b_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\Big(\int_0^t\|\partial_2\omega\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac12}\Big(\int_0^t\|\partial_1\partial_2\omega\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac12} \\
&\le CE_1(t)E_2^{\frac12}(t).
\end{aligned}$$

Therefore,

$$H_8 \le CE_1^{\frac32}(t) + CE_1(t)\,E_2^{\frac12}(t). \tag{2.34}$$

Finally,

$$H_9 = 2\int_0^t\int \partial_1 b_1\partial_2 u_1\partial_2\omega\mathrm{d}x\mathrm{d}\tau + \big(2\int_0^t\int \partial_1 b_1\partial_1 u_2\partial_2\omega\mathrm{d}x\mathrm{d}\tau$$

$$
\begin{aligned}
&-2\int_0^t\int \partial_1 u_1\partial_2 b_1\partial_2\omega \mathrm{d}x\mathrm{d}\tau - 2\int_0^t\int \partial_1 u_1\partial_1 b_2\partial_2\omega \mathrm{d}x\mathrm{d}\tau\big)\\
&\doteq H_{91}+H_{92}.
\end{aligned}
$$

The terms on the right can be bounded as follows,

$$
\begin{aligned}
H_{91} &\le \int_0^t \|\partial_2\omega\|_{L^2}\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_1\partial_2 u_1\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\le \sup_{0\le\tau\le t}\|\partial_2\omega\|_{L^2}\int_0^t \|\partial_2 u_1\|_{L^2}^{\frac12}\big(\|\partial_2 b\|_{H^1}^{\frac32}+\|\partial_1\partial_2 u_1\|_{L^2}^{\frac32}\big)\mathrm{d}\tau\\
&\le \sup_{0\le\tau\le t}\|\partial_2\omega\|_{L^2}\big(\int_0^t \|\partial_2 u_1\|_{L^2}^2\mathrm{d}\tau\big)^{\frac14}\big(\int_0^t \|\partial_2 b\|_{H^1}^2\mathrm{d}\tau\big)^{\frac34}\\
&\quad+\sup_{0\le\tau\le t}\|\partial_2\omega\|_{L^2}\big(\int_0^t \|\partial_2 u_1\|_{L^2}^2\mathrm{d}\tau\big)^{\frac14}\big(\int_0^t \|\partial_1\partial_2 u_1\|_{L^2}^2\mathrm{d}\tau\big)^{\frac34}\\
&\le E_1^{\frac12}(t)E_2^{\frac14}(t)E_1^{\frac34}(t),\\
H_{92} &\le C\int_0^t \|\partial_2\omega\|_{L^2}\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_{11} u_2\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\quad+C\int_0^t \|\partial_2\omega\|_{L^2}\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_{11} u_1\|_{L^2}^{\frac12}\|\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_{22} b_1\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\quad+C\int_0^t \|\partial_1 u_1\|_{L^2}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1\partial_2\omega\|_{L^2}^{\frac12}\mathrm{d}\tau\\
&\le C\sup_{0\le\tau\le t}\big(\|\partial_2\omega\|_{L^2}+\|\partial_1 b\|_{L^2}\big)\int_0^t \big(\|\partial_2 b\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2\big)\mathrm{d}\tau\\
&\le C E_1^{\frac12}(t)E_1(t).
\end{aligned}
$$

Therefore,

$$
H_9 \le C E_1^{\frac54}(t)E_2^{\frac14}(t)+E_1^{\frac32}(t). \tag{2.35}
$$

Inserting the upper bounds (2.28) through (2.35) in (2.27), we obtain, after applying a simple Young's inequality,

$$\int_0^t \|\partial_2\omega\|_{L^2}^2 d\tau \le E_1(0) + \frac{1}{4}E_2(t) + C\,E_1(t) + E_1^{\frac{3}{2}}(t) + E_2^{\frac{3}{2}}(t). \tag{2.36}$$

Adding (2.26) and (2.36) leads to

$$E_2(t) \le C\,E_1(0) + \frac{1}{2}E_2(t) + C\,E_1(t) + C\,E_1^{\frac{3}{2}}(t) + C\,E_2^{\frac{3}{2}}(t).$$

Eliminating $1/2E_2(t)$ from both sides yields the desired inequality in (2.3). This completes the proof of Proposition 2.1. □

3. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. The proof of Part (a) in Theorem 1.3 makes use of the symmetric structure of the linearized system (1.8). Anisotropic interpolation inequalities are employed. The proof of Part (b) takes advantage of the wave structure in (1.9). These wave equations are very versatile. They allow us to perform different types of energy estimates and make various combinations. Lemma 1.1 will be used to facilitate the proof.

Proof of Theorem 1.3. We start with the proof of Part (a). Taking the inner product of (1.8) with (u, b) in H^1, we find

$$\frac{d}{dt}L(t) + M(t) = 0, \tag{3.1}$$

where

$$\begin{aligned}
L(t) &= \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2,\\
M(t) &= 2\nu\|\partial_1 u(t)\|_{L^2}^2 + 2\eta\|\partial_2 b(t)\|_{L^2}^2 + 2\nu\|\partial_1\nabla u(t)\|_{L^2}^2 + 2\eta\|\partial_2\nabla b\|_{L^2}^2.
\end{aligned}$$

We also compute the norm of (u, b) in anisotropic Sobolev spaces of negative indices. Applying $\Lambda_1^{-\sigma}$ to (1.8) and then dotting with $(\Lambda_1^{-\sigma}u, \Lambda_1^{-\sigma}b)$ in $H^{1+\sigma}$, and applying $\Lambda_2^{-\sigma}$ to (1.8) and dotting with $(\Lambda_2^{-\sigma}u, \Lambda_2^{-\sigma}b)$ in $H^{1+\sigma}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}N(t) + 2\nu\|\partial_1(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + 2\eta\|(\partial_2(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})b(t)\|_{L^2}^2\\
&\quad + 2\nu\|\partial_1\Lambda^{1+\sigma}(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + 2\eta\|\partial_2\Lambda^{1+\sigma}(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})b(t)\|_{L^2}^2 = 0,
\end{aligned}$$

where

$$\begin{aligned}
N(t) &= \|(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + \|(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})b(t)\|_{L^2}^2\\
&\quad + \|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + \|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma})b(t)\|_{L^2}^2.
\end{aligned}$$

Clearly,

$$N(t) \le N(0).$$

We now bound L in terms of M and N and show that, for a constant $C > 0$,

$$L(t) \le C M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}, \tag{3.2}$$

(3.2) follows from the interpolation inequalities below,

$$\|u(t)\|_{L^2}^2 \le \|\partial_1 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_1^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}, \tag{3.3}$$

$$\|\nabla u(t)\|_{L^2}^2 \le \|\partial_1 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}, \tag{3.4}$$

$$\|b(t)\|_{L^2}^2 \le C\|\partial_2 b(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_2^{-\sigma} b(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le C M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}, \tag{3.5}$$

$$\|\nabla b(t)\|_{L^2}^2 \le C\|\partial_2 b(t)(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_2^{-\sigma} b(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le C M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}. \tag{3.6}$$

We provide some details on proving (3.3) and (3.4). The proofs of (3.5) and (3.6) are similar. To prove (3.3), we use Plancherel theorem and Hölder's inequality to obtain

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &= \int |\widehat{u}(\xi,t)|^2\, d\xi = \int (|\xi_1|^2|\widehat{u}|^2)^{\frac{\sigma}{1+\sigma}} (|\xi_1|^{-2\sigma}|\widehat{u}|^2)^{\frac{1}{1+\sigma}}\, d\xi \\
&\le \left(\int \xi_1^2|\widehat{u}|^2 d\xi\right)^{\frac{\sigma}{1+\sigma}} \left(\int |\xi_1|^{-2\sigma}|\widehat{u}|^2 d\xi\right)^{\frac{1}{1+\sigma}} \\
&= \|\partial_1 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_1^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \\
&\le M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla u(t)\|_{L^2}^2 &= \int_{\xi\in\mathbb{R}^2} |\xi|^2|\widehat{u}(\xi)|^2 \mathrm{d}\xi \\
&\le \left(\int_{\xi\in\mathbb{R}^2} |\xi_1|^2|\widehat{u}(\xi)|^2\mathrm{d}\xi\right)^{\frac{\sigma}{1+\sigma}} \left(\int_{\xi\in\mathbb{R}^2} |\xi|^{2+2\sigma}|\xi_1|^{-2\sigma}|\widehat{u}(\xi)|^2\mathrm{d}\xi\right)^{\frac{1}{1+\sigma}} \\
&= \|\partial_1 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \\
&\le M(t)^{\frac{\sigma}{1+\sigma}} N(t)^{\frac{1}{1+\sigma}}.
\end{aligned}$$

Since $N(t) \le N(0)$,

$$L(t) \le C M(t)^{\frac{\sigma}{1+\sigma}} N(0)^{\frac{1}{1+\sigma}} \quad \text{or} \quad M(t) \ge C N(0)^{-\frac{1}{\sigma}} L(t)^{1+\frac{1}{\sigma}}.$$

Substituting the inequality above in (3.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t)+CN(0)^{-\frac{1}{\sigma}}L(t)^{1+\frac{1}{\sigma}}\le 0.$$

This inequality leads to

$$L(t)\le\left(L(0)^{-\frac{1}{\sigma}}+\frac{C}{\sigma}N_0^{-\frac{1}{\sigma}}t\right)^{-\sigma},$$

which is (1.10). We now turn to the proof of Part (b). According to Lemma 1.1 stated in the introduction, it suffices to verify the two conditions in (1.18) in order to prove (1.11). That is, we show that

$$\int_0^\infty\left(\|\partial_t u(t)\|_{L^2}^2+\|\partial_2 u(t)\|_{L^2}^2+\nu\eta\|\partial_{12}u(t)\|_{L^2}^2\right)dt\le C \tag{3.7}$$

and

$$\begin{aligned}&\|\partial_t u(t)\|_{L^2}^2+\|\partial_2 u(t)\|_{L^2}^2+\nu\eta\|\partial_{12}u(t)\|_{L^2}^2\\ &\le\|\partial_t u(s)\|_{L^2}^2+\|\partial_2 u(s)\|_{L^2}^2+\nu\eta\|\partial_{12}u(s)\|_{L^2}^2\quad\text{for any } t\ge s\ge 0.\end{aligned} \tag{3.8}$$

We take advantage of the wave equations in (1.9). Dotting the equation of u in (1.9) by $\partial_t u$ and integrating by parts, we have

$$\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2+\|\partial_2 u\|_{L^2}^2+\nu\eta\|\partial_{12}u\|_{L^2}^2\right)+2\nu\|\partial_1\partial_t u\|_{L^2}^2+2\eta\|\partial_2\partial_t u\|_{L^2}^2=0. \tag{3.9}$$

Since the equations in (1.9) are linear, $\Lambda_2^{-1}u$ satisfies the same equation

$$\frac{d}{dt}\left(\|\partial_t\Lambda_2^{-1}u\|_{L^2}^2+\|u\|_{L^2}^2+\nu\eta\|\partial_1 u\|_{L^2}^2\right)+2\nu\|\partial_1\partial_t\Lambda_2^{-1}u\|_{L^2}^2+2\eta\|\partial_t u\|_{L^2}^2=0. \tag{3.10}$$

It follows from (3.9) that

$$\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2+\|\partial_2 u\|_{L^2}^2+\nu\eta\|\partial_{12}u\|_{L^2}^2\right)\le 0,$$

which implies (3.8). To obtain (3.7), we need to establish one more energy estimate. We dot the equation of u in (1.9) by u and integrate by parts,

$$\frac{d}{dt}\left(\nu\|\partial_1 u\|_{L^2}^2+\eta\|\partial_2 u\|_{L^2}^2\right)+2\|\partial_2 u\|_{L^2}^2+2\nu\eta\,\|\partial_{12}u\|_{L^2}^2+2\int\partial_{tt}u\cdot u\,dx=0.$$

Rewriting the last term in the equation above, we find

$$\begin{aligned}&\frac{d}{dt}\left(\nu\|\partial_1 u\|_{L^2}^2+\eta\|\partial_2 u\|_{L^2}^2+2(\partial_t u,u)\right)\\ &+2\|\partial_2 u\|_{L^2}^2+2\nu\eta\,\|\partial_{12}u\|_{L^2}^2-2\|\partial_t u\|_{L^2}^2=0,\end{aligned} \tag{3.11}$$

where $(\partial_t u, u)$ denotes the L^2-inner product. Let $\rho = \min\left\{\frac{1}{2}, \eta\right\}$. Then the combination (3.9) + (3.10) + $\rho \times$ (3.11) yields

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\partial_t u\|_{L^2}^2 + \|u\|_{L^2}^2 + 2\rho(\partial_t u, u) + (\nu\eta + \rho\nu)\|\partial_1 u\|_{L^2}^2 \\
&\quad + (1+\rho\eta)\|\partial_2 u\|_{L^2}^2 + \|\partial_t \Lambda_2^{-1} u\|_{L^2}^2 + \nu\eta\|\partial_{12} u\|_{L^2}^2\Big) \\
&\quad + 2\nu\|\partial_1\partial_t u\|_{L^2}^2 + 2\eta\|\partial_2\partial_t u\|_{L^2}^2 + 2\nu\|\partial_1\partial_t\Lambda_2^{-1} u\|_{L^2}^2 \\
&\quad + 2(\eta-\rho)\|\partial_t u\|_{L^2}^2 + 2\rho\|\partial_2 u\|_{L^2}^2 + 2\nu\eta\rho\,\|\partial_{12} u\|_{L^2}^2 = 0.
\end{aligned}$$

Integrating the inequality above in time and using the fact that $\rho < \eta$ and $\rho < 1/2$, we obtain

$$\begin{aligned}
&\int_0^t \Big(2\nu\|\partial_1\partial_t u\|_{L^2}^2 + 2\eta\|\partial_2\partial_t u\|_{L^2}^2 + 2\nu\|\partial_1\partial_t\Lambda_2^{-1} u\|_{L^2}^2 \\
&\quad + 2(\eta-\rho)\|\partial_t u\|_{L^2}^2 + 2\rho\|\partial_2 u\|_{L^2}^2 + 2\nu\eta\rho\,\|\partial_{12} u\|_{L^2}^2\Big)d\tau \\
&\le C\,\|\Lambda_2^{-1} u_0\|_{H^2}^2 + \|\Lambda_2^{-1} b_0\|_{H^2}^2,
\end{aligned}$$

which, in particular, implies (3.7). We have thus verified (3.7) and (3.8). Lemma 1.1 gives the desired rate. Since b satisfies the same equation, the proof for (1.12) is the same. This completes the Theorem 1.3. □

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