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ON THE DEGENERATE BOUSSINESQ EQUATIONS ON SURFACES

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Abstract. In this paper we study the non-degenerate and partially degenerate Boussinesq equations on a closed surface Σ. When Σ has intrinsic curvature of finite Lipschitz norm, we prove the existence of global strong solutions to the Cauchy problem of the Boussinesq equations with full or partial dissipations. The issues of uniqueness and singular limits (vanishing viscosity/vanishing thermal diffusivity) are also addressed. In addition, we establish a breakdown criterion for the strong solutions for the case of zero viscosity and zero thermal diffusivity. These appear to be among the first results for Boussinesq systems on Riemannian manifolds.

1. **Introduction.** We consider the Cauchy problem for the Boussinesq equations on a smooth, closed (*i.e.*, compact and with no boundary) surface Σ:

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla P = \theta\mathbf{e} \qquad in \quad [0,T]\times\Sigma, \tag{1}$$

$$\partial_t \theta + u\cdot\nabla\theta - \kappa\Delta\theta = 0 \qquad in \quad [0,T]\times\Sigma, \tag{2}$$

$$\operatorname{div} u = 0 \qquad in \quad [0,T]\times\Sigma. \tag{3}$$

The initial condition is given by

$$(u,\theta)|_{t=0} = (u^\circ,\theta^\circ) \qquad on \quad \{0\}\times\Sigma. \tag{4}$$

Throughout this paper, (Σ, g) is a closed surface, *i.e.*, a 2-dimensional compact differentiable manifold without boundary. g is the Riemannian metric of Σ, *i.e.*, a positive definite symmetric 2×2 matrix field. At times we shall write $\langle\cdot,\cdot\rangle$ for the inner product given by g; thus the length of a vector field v is given by $|v| := \sqrt{\langle v, v\rangle}$. We denote by $\Gamma(T\Sigma)$ the space of tangential vector fields on Σ,

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and $\Gamma(T^*\Sigma)$ the space of 1-forms; $T\Sigma$ and $T^*\Sigma$ are the tangent and cotangent bundles of Σ, respectively. ∇ denotes the gradient operator on Σ; equivalently, it is the covariant derivative induced by the Levi-Civita connection on Σ. The divergence operator (div) corresponding to ∇ is obtained by taking the trace of ∇ with respect to g; it can be defined intrinsically on Σ. In addition, $\mathbf{e} \in \Gamma(T\Sigma)$ is a unit-length, Lipschitz vector field on Σ. For a vector field $v \in \Gamma(T\Sigma)$ we write its components in some local coordinates by v^i. The 1-form $v^\flat \in \Gamma(T^*\Sigma)$ dual to v^i has components $v_i := g_{ij}v^j$. Einstein's summation convention is adopted unless otherwise mentioned: the repeated upper and lower indices are summed over.

We denote by $Riem^i_{jkl}$ the components of the Riemann curvature tensor on Σ. For the 2-dimensional manifold Σ, there is only one intrinsic curvature: $Riem$ and the Gauss, Ricci, sectional and scalar curvatures are all equivalent. We shall simply refer to "the curvature of Σ". For the simplicity of presentation (*e.g.*, to state the Ricci identity in Lemma 2.1), in this paper we shall use $Riem$ for explicit computations.

We observe that Eqs. (1)–(3), as equations in $T\Sigma$, are formulated intrinsically, *i.e.*, independent of the choice of coordinates charts. Also, it is not necessary to assume that the fluid domain Σ is isometrically embedded into $\mathbb{R}^3$.

In the paper we impose one mild assumption on the geometry of Σ: the Lipschitz norm of the intrinsic curvature is bounded. That is,

$$\|\mathrm{Riem}\|_{W^{1,\infty}(\Sigma)} \leq R < \infty. \tag{5}$$

On the other hand, notice that Σ has a positive injectivity radius lower bound:

$$\mathrm{inj}_\Sigma \geq \iota_0 > 0,$$

and a finite volume:

$$\mathrm{Vol}\,\Sigma := \int_\Sigma 1\,\mathrm{d}x \leq V < \infty,$$

by the definition of the closed surface. Throughout this paper, a constant c is said to be "geometric" (or "depends on the geometry of Σ") if it depends on R, ι_0 and V. The bounds ι_0 and V will be needed for the Calderón–Zygmund estimates (Lemma 2.3).

The physical variables in Eqs. (1)–(4) are as follows: $u(t,\cdot) \in \Gamma(T\Sigma)$ is the velocity vector field, $\theta(t,\cdot) : \Sigma \to \mathbb{R}$ is the temperature function, and $P : \Sigma \to \mathbb{R}$ the pressure of an incompressible fluid; $\nu \geq 0$ is the viscosity and $\kappa \geq 0$ the thermal diffusivity of the fluid.

As an example, consider $\Sigma = \mathbb{S}^2$, the unit round sphere, with $\mathbf{e}$ equal to the natural vector field tangential to the geodesics from the north pole to the south pole (the latitudes). In the spherical coordinates, if

$$x = x(\underset{\sim}{\phi}, \underset{\sim}{\theta}) = \big(\sin\underset{\sim}{\phi}\cos\underset{\sim}{\theta},\, \sin\underset{\sim}{\phi}\sin\underset{\sim}{\theta},\, \cos\underset{\sim}{\phi}\big)^\top,$$

for the azimuthal angle $\underset{\sim}{\phi} \in [0,\pi[$ and the (x,y)-plane angle $\underset{\sim}{\theta} \in [0,2\pi[$, then

$$\mathbf{e}(x) = \big(\cos\underset{\sim}{\phi}\cos\underset{\sim}{\theta},\, \cos\underset{\sim}{\phi}\sin\underset{\sim}{\theta},\, -\sin\underset{\sim}{\phi}\big)^\top.$$

Clearly one has $\langle \mathbf{e}(x), x\rangle = 0$ and $|\mathbf{e}(x)| = 1$ for each $x \in \Sigma$. This has been studied by Saito [50].

To put things into perspective, we briefly survey the literature in connection with this work. When Σ is an Euclidean domain in $\mathbb{R}^2$ and $\mathbf{e} = \mathbf{e}_2 = (0,1)^\top$, the unit vertical vector, the qualitative behaviours of large-amplitude solutions to the 2D

Boussinesq equations, such as well-posedness, blowup criteria, regularity, explicit solutions, finite-time singularities, and long-time behaviour, subject to various initial and/or boundary conditions have been studied extensively in the literature. We refer the readers to

- [5, 17, 18, 19, 20, 22, 27, 37, 51, 53] for local well-posedness, blowup criteria, explicit solutions and finite-time singularities for the degenerate case (*i.e.*, $\nu = \kappa = 0$);
- [1, 2, 3, 4, 14, 15, 16, 21, 23, 24, 25, 32, 33, 34, 35, 36, 38, 39, 44, 45, 46, 48, 49, 62] for global well-posedness and regularity for the non-degenerate and partially degenerate cases;
- [41, 42, 47, 52, 59, 60, 61] for well-posedness and regularity with critical and supercritical dissipation; and
- [10, 26, 44, 54, 58, 62] for long-time behaviours.

There are also works dealing with the Boussinesq equations in the three-dimensional space; see, *e.g.*, [11] for long-time behaviour of small-amplitude solutions for the non-degenerate case.

On the other hand, comparing with the magnitude of research conducted on the Boussinesq equations on Euclidean domains, the qualitative behaviour of the model on Riemannian manifolds has been investigated relatively little. To the authors' knowledge, only the case of the two-dimensional round sphere has been studied, see [50], in which the convergence of the average of weak solutions of the 3D equations to a 2D problem is proved. The case of general Riemannian manifolds is widely open. This is the fact that primarily motivated the current work. In addition, the Boussinesq equations on surfaces may be potentially important in the modeling of geophysical fluids. The Boussinesq equations model buoyancy driven flows, which tend to become stratified. In fact, Earth's atmosphere is divided into a series of layers. The Boussinesq equations on surfaces become relevant for the dynamics of the layered flows.

In passing, we remark that despite the lack of literature on the analysis of Boussinesq equations on manifolds, various PDEs of hydrodynamic models have nevertheless been studied on manifolds, including the Navier–Stokes equations [40], the rotating Euler equations [55], and the SQG (surface quasi-geostrophic) equations [6, 7], even for critical cases without the smallness assumption on the initial data.

The goal of this work is to study the local/global well-posedness and blowup criteria (depending on the specific values of the dissipation parameters) of large-amplitude classical solutions to the Cauchy problem of the 2D Boussinesq equations on general closed surfaces, *i.e.*, 2-dimensional manifolds, with intrinsic curvature bounded in the Lipschitz norm. We reach the goal by combining approaches in Riemannian geometry and L^p-based energy methods. We remark that although the commonly utilized techniques, such as the Sobolev embeddings, interpolation inequalities (Gagliardo–Nirenberg, Ladyzhenskaya, *et al*), and special estimates (Brezis–Wainger, Calderón–Zygmund, *et al*) are still available in the Riemannian setting, the problem considered herein distinguishes itself from the problems on Euclidean domains significantly, mainly due to the Ricci identity for commuting covariant derivatives. In particular, the Ricci identity generates additional lower order terms when taking the spatial derivatives to the equations, which complicates the underlying analysis, especially for the global well-posedness of large-amplitude classical solutions. We overcome the difficulty by applying various interpolation inequalities and taking advantage of the dissipation mechanisms.

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results, such as geometric identities, Sobolev embeddings, interpolation inequalities and special estimates, which are frequently utilized in the proofs of the main results. Section 3 contains the global existence of weak solutions to the non-degenerate system ($\nu > 0$, $\kappa > 0$), uniqueness of strong solutions to the non-degenerate, partially degenerate and degenerate systems, and a Beale–Kato–Majda-type blowup criterion for all the cases. Sections 4 and 5 respectively deal with the global well-posedness of the partially degenerate system with either non-degenerate viscosity ($\nu > 0$ and $\kappa \geq 0$) or non-degenerate thermal diffusivity ($\nu \geq 0$ and $\kappa > 0$). In Section 6, we study the vanishing viscosity and diffusivity limits of the global solutions to the non-degenerate system, and establish the consistency between the non-degenerate and partially degenerate systems. The paper is finished with concluding remarks in Section 7 and a proof of the Beale–Kato–Majda-type blowup criterion in the Appendix.

2. **Preliminaries.**

Throughout this paper, the geometric constant R depends only on $\|Riem\|_{W^{1,\infty}(\Sigma)}$ (may change from line to line). The constants $K_i = K_i(t)$ depend only on the parameters of the fluid, and can be bounded uniformly in time by $K_i(t) \leq K_i(T)$ for any $t \in [0,T]$. We write c for geometric constants associated to various classical inequalities, *e.g.*, Gagliardo–Nirenberg–Sobolev, Poincaré, Brezis–Wainger and so on.

For 2×2 matrices M_1, M_2, we write $|M_1|$ for the Hilbert–Schmidt norm of M_1, and $M_1 : M_2 := tr(M_1 \cdot M_2)$. Given vector fields $a = (a^1, a^2)^\top$, $b = (b^1, b^2)^\top$ written in local coordinates, the tensor product $a \otimes b$ denotes the 2×2 matrix with the (i,j)-entry equal to $a^i b^j$. Let T_1, T_2 be two tensor fields on Σ; the schematic notation $T_1 \star T_2$ designates any bilinear combination of components of T_1, T_2. For a function f on Σ, $\nabla\nabla f$ denotes the Hessian matrix field $\{\nabla_i \nabla_j f\}_{1\leq i,j\leq 2}$. We shall always use Δ to denote *the Hodge Laplacian*

$$\Delta := -dd^* - d^* d,$$

where d is the exterior differential and d^* its L^2 formal adjoint. That is, Δ is the negative of the Laplace–Beltrami operator on Σ. For $\Sigma = \mathbb{R}^2$ we have the usual $\Delta = \partial^2/\partial(x^1)^2 + \partial^2/\partial(x^2)^2$. By elementary geometry, d is interpreted as the *curl* or *rot* and d^* as the *divergence* operator (modulo signs or obvious duality). Note that Δ maps differential r-forms to r-forms or, equivalently, from r-vector fields to r-vector fields. The arguments in this paper also apply to the Bochner Laplacian; see the end of §7 for discussions.

We define $\mathscr{J}$ as the space of smooth solenoidal vector fields on Σ:

$$\mathscr{J} := \Big\{ u \in C_0^\infty(\Sigma; T\Sigma) : \operatorname{div} u = 0 \Big\}, \tag{6}$$

and

$$\mathcal{H} := \overline{\mathscr{J}}^{L^2}, \tag{7}$$

the completion of $\mathscr{J}$ with respect to the L^2-topology.

It is well-known that the Sobolev spaces $W^{k,p}(\Sigma)$ can be defined globally on Σ via the Levi-Civita connection ∇, the metric g and the differentiable structure of Σ (see, *e.g.*, [31]). We write $W^{k,p}(\Sigma; T\Sigma)$ for the Sobolev space of $W^{k,p}$-vector fields on Σ, and similarly $W^{k,p}(\Sigma; T^*\Sigma)$ for the space of $W^{k,p}$-1-forms on Σ. As usual $H^k := W^{k,2}$. We use $\|\phi\|_{W^{k,p}(\Sigma)}$ to denote the $W^{k,p}$-norm of ϕ, where ϕ can be a function, a vector field, a 1-form or a tensor field of any type. Moreover, one denotes the homogeneous Sobolev norms by $\|\phi\|_{\dot{W}^{k,p}(\Sigma)}$, which consists of the L^p-norms of the higher-order covariant derivatives of ϕ. We shall also write $\int_\Sigma f(x)\,\mathrm{d}x$ as the integration with respect to the volume form on Σ; that is, $\mathrm{d}x$ denotes the volume (area) measure on (Σ, g).

In the sequel, we introduce several geometric identities and inequalities. They play a crucial role in our estimates.

First, we have the well-known Ricci identity on any Riemannian manifold (see [43]), which tells us how to commute covariant derivatives:

Lemma 2.1. *Let T be a covariant tensor field of rank m. Then*

$$\nabla_k\nabla_l T_{i_1 i_2 \ldots i_m} - \nabla_l\nabla_k T_{i_1 i_2 \ldots i_m} = \sum_{\alpha=1}^{m}\Big\{T_{i_1\ldots i_{\alpha-1} h i_{\alpha+1}\ldots l_m}\,\mathrm{Riem}^h_{i_\alpha k l}\Big\}.$$

For instance, for $f : \Sigma \to \mathbb{R}$ and $u \in \Gamma(T\Sigma)$ we have

$$\nabla_i\nabla_j\nabla_k f = \nabla_j\nabla_i\nabla_k f + \mathrm{Riem}^l_{ijk}\nabla_l f, \tag{8}$$

and by raising and lowering indices using the metric tensor we get

$$\begin{aligned}&\nabla_l\nabla_j\nabla_k u^i - \nabla_j\nabla_l\nabla_k u^i\\ &\qquad = g^{pi}\Big\{(\nabla_k u_h)\,\mathrm{Riem}^h_{plj} - (\nabla_h u_p)\,\mathrm{Riem}^h_{klj}\Big\},\end{aligned} \tag{9}$$

as well as

$$\begin{aligned}&\nabla_j\nabla_l\nabla_h\nabla_k u^i - \nabla_l\nabla_j\nabla_h\nabla_k u^i\\ &\qquad = g^{qi}\Big\{(\nabla_p\nabla_k u_q)\,\mathrm{Riem}^p_{hjl} + (\nabla_h\nabla_p u_q)\mathrm{Riem}^p_{kjl} + (\nabla_p\nabla_k u_h)\mathrm{Riem}^p_{hjl}\Big\}.\end{aligned} \tag{10}$$

Next, we remark that the usual Sobolev embedding theorems continue to hold on the surface Σ in our case. This follows from a more general result due to Varopoulos ([56]): Sobolev embedding theorems hold on complete Riemannian manifolds of arbitrary dimensions with Ricci curvature lower bound and a strictly positive lower bound for the volume of unit geodesic balls. As a consequence, any inequalities obtained from the Sobolev embeddings via interpolation continue to hold; *e.g.*, the Gagliardo–Nirenberg and the Ladyzhenskaya inequalities.

Let us also state an inequality due to Brezis–Wainger; see [12, 28] on Euclidean domains and [29] on closed manifolds. It is an end-point case of the classical Sobolev inequalities. We write $\log_+ s := \max\{\log s, 0\}$.

Lemma 2.2 (Brezis–Wainger)**.** *Let (Σ, g) be a closed Riemannian manifold. Assume $f \in L^2(\Sigma)\cap\dot{W}^{1,p}(\Sigma)$. Then there exists a constant c depending only on p and Σ such that*

$$\|f\|_{L^\infty(\Sigma)} \le c(1 + \|\nabla f\|_{L^2(\Sigma)})\sqrt{1 + \log_+(\|\nabla f\|_{L^p(\Sigma)})} + c\|f\|_{L^2(\Sigma)}. \tag{11}$$

In addition, we have the following Calderón–Zygmund estimate on Σ.

Lemma 2.3. *Let (Σ, g) be a closed surface with Lipschitz-bounded curvature. Then, for each differential s-form ψ on Σ and any $1 < p < \infty$, there exists a constant $c = c(p, s, \Sigma, \iota_0, V, \|\mathrm{Riem}\|_{L^\infty(\Sigma)})$ such that*

$$\|\nabla^2\psi\|_{L^p(\Sigma)} \le c\Big(\|\Delta\psi\|_{L^p(\Sigma)} + \|\psi\|_{L^p(\Sigma)}\Big).$$

The proof follows from Wang [57] with slight modifications (*e.g.*, using the Green's function on Σ; see [8]). In [57] it is proved that, in the above setting, if ψ is L^2-orthogonal to the space of harmonic s-forms on Σ, then there exists a constant $c' = c(p, s, \Sigma, \iota_0, V, \|\mathrm{Riem}\|_{L^\infty(\Sigma)})$ such that

$$\|\nabla^2\psi\|_{L^p(\Sigma)} \leq c'\|\Delta\psi\|_{L^p(\Sigma)}.$$

For the general case we include $\|\psi\|_{L^p(\Sigma)}$ on the right-hand side to control the non-trivial harmonic part of ψ. It follows immediately from the finite dimensionality of the space of harmonic forms on Σ, thanks to the theory of de Rham cohomology.

3. Existence and uniqueness of weak solutions.

Definition 3.1. *Let Σ be a closed surface with Lipschitz-bounded curvature. Let $(u^\circ, \theta^\circ) \in L^2(\Sigma; T\Sigma) \times L^2(\Sigma)$ be initial data satisfying $\operatorname{div} u^\circ = 0$. (u, θ) is a weak solution to the Boussinesq equations (1)–(4) if*

$$\begin{cases} u \in L^\infty\big(0, T; L^2(\Sigma; T\Sigma)\big) \cap L^2(0, T; \mathcal{H}), \\ \theta \in L^\infty\big(0, T; L^2(\Sigma)\big) \cap L^2\big(0, T; H^1(\Sigma)\big), \end{cases} \tag{12}$$

and they satisfy Eqs. (1)–(4) in the distributional sense:

$$\int_\Sigma \langle u^\circ(x), \Phi(x)\rangle \,\mathrm{d}x + \int_0^T \int_\Sigma \Big\{ \langle u(t,x), \partial_t\Phi(x)\rangle + \big[u(t,x) \otimes u(t,x)\big] : \nabla\Phi(x) \\ - \nu\nabla u(t,x) : \nabla\Phi(x) + \theta(t,x)\langle \Phi(x), \mathbf{e}(x)\rangle \Big\} \,\mathrm{d}x\mathrm{d}t = 0 \tag{13}$$

for every test vector field $\Phi \in C^\infty([0, T] \times \Sigma)$ with $\operatorname{div}\Phi = 0$ and $\Phi|_{t=T} = 0$, and

$$\int_\Sigma \theta^\circ(x)\varphi(x)\,\mathrm{d}x + \int_0^T \int_\Sigma \Big\{ \theta(t,x)\partial_t\varphi(x) \\ + \theta(t,x)\langle u(t,x), \nabla\varphi(x)\rangle - \kappa\langle\nabla\theta(t,x), \nabla\varphi(x)\rangle \Big\} \,\mathrm{d}x\mathrm{d}t = 0 \tag{14}$$

for each test function $\varphi \in C^\infty([0, T] \times \Sigma)$ with $\varphi|_{t=T} = 0$.

By standard methods of linearisation and Galerkin approximation, we can prove the existence of (distributional) weak solutions to Eqs.(1)–(4) under more stringent regularity assumptions on the initial data, *e.g.*, $(u^\circ, \theta^\circ) \in H^3(\Sigma, T\Sigma) \times H^3(\Sigma)$. Indeed, adapting the arguments in [30] by Guo–Yuan for 2D or 3D bounded Euclidean domains, which in turn are based on the classical work [13] by Caffarelli–Kohn–Nirenberg on the incompressible Navier–Stokes equations, it is possible to establish the existence of "suitable weak solutions"; see the appendix in [13] for the proof on compact surfaces-with-boundary.

It remains a major open problem regarding the uniqueness of weak solutions to the Boussinesq equations, even in $\mathbb{R}^2$ or bounded Euclidean domains. Nevertheless, the uniqueness of strong solutions in the following sense can be easily established. For the non-degenerate case, we have:

Theorem 3.2. *Let Σ be a closed surface with Lipschitz-bounded curvature. Let $u^\circ \in \mathcal{H}$, $\theta^\circ \in L^2(\Sigma)$, and $T > 0$. Suppose $\kappa > 0$ and $\nu > 0$. Then solutions (u, θ) to the system (1)–(4) on $[0, T]$ with initial data (u°, θ°) is unique in the following space:*

$$\begin{cases} u \in L^2(0, T; \mathcal{J}) \cap C^0(0, T; \mathcal{H}); \\ \theta \in L^2\big(0, T; H^1_0(\Sigma)\big) \cap C^0\big(0, T; L^2(\Sigma)\big). \end{cases} \tag{15}$$

Proof. Let (u_i, θ_i, P_i), $i \in \{1, 2\}$, be two solutions in the indicated function spaces. Denote $\hat{v} := u_1 - u_2$, $\hat{\theta} := \theta_1 - \theta_2$ and $\hat{P} := P_1 - P_2$. The equations for the hat-variables are

$$\begin{cases} \partial_t \hat{v} - \nu\Delta\hat{v} + \hat{v}\cdot\nabla u_1 + u_2\cdot\nabla\hat{v} + \nabla\hat{P} = \hat{\theta}\mathbf{e}; \\ \partial_t\hat{\theta} - \kappa\Delta\hat{\theta} + \hat{v}\cdot\nabla\theta_1 + u_2\cdot\nabla\hat{\theta} = 0; \\ \nabla\cdot\hat{v} = 0, \end{cases} \tag{16}$$

together with the initial data

$$(\hat{v}, \hat{\theta})|_{t=0} = (0, 0) \qquad \text{on } \{0\}\times\Sigma. \tag{17}$$

Now, standard energy estimates, Cauchy–Schwarz and the Ladyzhenskaya's inequality

$$\|f\|_{L^4(\Sigma)} \le k^{1/2}\big\{\|f\|^{1/2}_{L^2(\Sigma)}\|\nabla f\|^{1/2}_{L^2(\Sigma)} + \|f\|_{L^2(\Sigma)}\big\}$$

give us

$$\begin{aligned} &\int_\Sigma |\hat{v}(t,x)|^2\,\mathrm{d}x + 2\nu\int_0^t\int_\Sigma|\nabla\hat{v}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad\le 2\int_0^t \Big\{\int_\Sigma |\nabla u_1(\tau,x)||\hat{v}(\tau,x)|^2\,\mathrm{d}x + \int_\Sigma|\hat{\theta}(\tau,x)||\hat{v}(\tau,x)|\,\mathrm{d}x\Big\}\,\mathrm{d}\tau \\ &\quad\le 2\int_0^t\Big\{\|\hat{v}(\tau,\cdot)\|^2_{L^4(\Sigma)}\|\nabla u_1(\tau,\cdot)\|_{L^2(\Sigma)} + \|\hat{\theta}(\tau,\cdot)\|_{L^2(\Sigma)}\|\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\Big\}\,\mathrm{d}\tau \\ &\quad\le \int_0^t\Big\{4k\|\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla u_1(\tau,\cdot)\|_{L^2(\Sigma)} \\ &\qquad\quad + 4k\|\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\|\nabla u_1(\tau,\cdot)\|_{L^2(\Sigma)} + 2\|\hat{\theta}(\tau,\cdot)\|_{L^2(\Sigma)}\|\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\Big\}\,\mathrm{d}\tau, \end{aligned}$$

where we used the fundamental inequality: $(a+b)^2 \le 2(a^2+b^2)$. Here k is a geometric constant depending only on the geometry of Σ. By the Cauchy inequalities $kabc \le \nu a^2/4 + (kbc)^2/\nu$ and $2ab \le a^2 + b^2$, we deduce that

$$\begin{aligned} &\int_\Sigma|\hat{v}(t,x)|^2\,\mathrm{d}x + \nu\int_0^t\int_\Sigma|\nabla\hat{v}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad\le \frac{4k^2}{\nu}\int_0^t\|\nabla u_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\int_\Sigma|\hat{v}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Sigma|\hat{\theta}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\qquad + \int_0^t\Big(1 + 4k\|\nabla u_1(\tau,\cdot)\|_{L^2(\Sigma)}\Big)\int_\Sigma|\hat{v}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{18}$$

On the other hand, similar arguments for the $\hat{\theta}$-equation lead to

$$\begin{aligned} &\int_\Sigma|\hat{\theta}(t,x)|^2\,\mathrm{d}x + 2\kappa\int_0^t\int_\Sigma|\nabla\hat{\theta}(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad\le 2\int_0^t\int_\Sigma|\theta_1\langle\hat{v},\nabla\hat{\theta}\rangle|(\tau,x)\,\mathrm{d}x\mathrm{d}\tau \\ &\quad\le\int_0^t\Big\{\kappa\|\nabla\hat{\theta}(\tau,\cdot)\|^2_{L^2(\Sigma)} + \kappa^{-1}\|\theta_1(\tau,\cdot)\|^2_{L^4(\Sigma)}\|\hat{v}(\tau,\cdot)\|^2_{L^4(\Sigma)}\Big\}\,\mathrm{d}\tau \\ &\quad\le\int_0^t\Big\{\kappa\|\nabla\hat{\theta}(\tau,\cdot)\|^2_{L^2(\Sigma)} + \frac{4k}{\kappa}\Big(\|\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)} + \|\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\times \end{aligned}$$

$$\times \Big(\|\hat{\theta}_1(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla\hat{\theta}_1(\tau,\cdot)\|_{L^2(\Sigma)} + \|\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\Big\}\,\mathrm{d}\tau.$$

Applying the Cauchy's inequality, one may infer

$$\begin{aligned}
&\int_\Sigma |\hat{\theta}(t,x)|^2\,\mathrm{d}x + \kappa\int_0^t\int_\Sigma |\nabla\theta(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau\\
&\leq 4k\kappa^{-1}\int_0^t \|\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla\hat{v}(\tau,\cdot)\|_{L^2(\Sigma)}\times\\
&\quad\times\Big(\|\hat{\theta}_1(\tau,\cdot)\|_{L^2(\Sigma)}\|\nabla\hat{\theta}_1(\tau,\cdot)\|_{L^2(\Sigma)} + \|\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\,\mathrm{d}\tau\\
&\quad+ 2k\kappa^{-1}\int_0^t \|\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big(\|\nabla\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)} + 3\|\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\,\mathrm{d}\tau\\
&\leq \frac{\nu}{2}\int_0^t \|\nabla\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\,\mathrm{d}\tau\\
&\quad+\frac{16k^2}{\nu\kappa^2}\int_0^t \|\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big(\|\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\|\nabla\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)} + \|\hat{\theta}_1(\tau,\cdot)\|^4_{L^2(\Sigma)}\Big)\,\mathrm{d}\tau\\
&\quad+ 2k\kappa^{-1}\int_0^t \|\hat{v}(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big(\|\nabla\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)} + 3\|\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\,\mathrm{d}\tau. \qquad (19)
\end{aligned}$$

Now we add up Eqs. (18) and (19) together. Denoting by

$$\mathbf{Y}(t) := \int_\Sigma \Big(|\hat{v}(t,x)|^2 + |\hat{\theta}(t,x)|^2\Big)\,\mathrm{d}x, \qquad (20)$$

one has

$$\mathbf{Y}(t) \lesssim \int_0^t \Big(1 + \|\nabla u_1(\tau,\cdot)\|^2_{L^2(\Sigma)} + \|\nabla\hat{\theta}_1(\tau,\cdot)\|^2_{L^2(\Sigma)}\Big)\mathbf{Y}(\tau)\,\mathrm{d}\tau, \qquad (21)$$

where $X \lesssim Y$ stands for $X \leq cY$ for some constant depending on ν, k, κ and the uniform (in time) estimate of $\|\hat{\theta}_1(\tau,\cdot)\|_{L^2(\Sigma)}$. Therefore, in view of the integrability assumptions for u_1, θ_1, we conclude from Grönwall's inequality $\mathbf{Y}(t) \equiv 0$. This completes the proof. □

Theorem 3.2remains valid on surfaces-with-boundaries under the following boundary conditions:

$$\begin{cases}\langle\hat{v},\mathbf{n}\rangle = 0 \text{ or } \partial_{\mathbf{n}}\hat{v} = 0 \qquad \text{on } \partial\Sigma,\\ \hat{\theta} = 0 \qquad \text{on } \partial\Sigma.\end{cases}$$

In contrast, to prove the uniqueness in the degenerate case, one needs more stringent regularity assumptions on at least one of the solutions:

Theorem 3.3. *Let Σ be a closed surface with Lipschitz-bounded curvature. Let (u_i,θ_i,P_i), $i\in\{1,2\}$ be two solutions to the system (1)–(4) on $[0,T]$ with the same initial data. Assume as in Theorem 3.2 that*

$$\begin{cases}u_i \in L^2(0,T;\mathcal{J})\cap C^0(0,T;\mathcal{H}),\\ \theta_i \in L^2\big(0,T;H^1_0(\Sigma)\big)\cap C^0\big(0,T;L^2(\Sigma)\big),\end{cases} \qquad (22)$$

for $i\in\{1,2\}$. Then $(u_1,\theta_1,P_1)\equiv(u_2,\theta_2,P_2)$ under the additional hypotheses below:

1. *When $\nu = 0, \kappa > 0$, assume $\theta_1 \in L^2(0,T;L^\infty(\Sigma))$ and $u_1 \in L^1(0,T;W^{1,\infty}(\Sigma;T\Sigma))$;*
2. *When $\nu > 0, \kappa = 0$, assume $\theta_1 \in L^1(0,T;W^{1,\infty}(\Sigma))$ and $u_1 \in L^2(0,T;L^\infty(\Sigma;T\Sigma))$;*
3. *When $\nu = \kappa = 0$, assume $\theta_1 \in L^1(0,T;W^{1,\infty}(\Sigma))$ and $u_1 \in L^1(0,T;W^{1,\infty}(\Sigma;T\Sigma))$.*

Proof. Let us show (1) in detail and sketch the proof for (2) and (3).

Define $(\hat{v}, \hat{\theta}, \hat{P})$ as in the proof of Theorem 3.2. The equations (16) and (17) for the hat-variables remain valid. For $\nu = 0$, $\kappa > 0$, standard energy estimate gives us

$$\begin{aligned}\int_\Sigma |\hat{v}(t,x)|^2 \, \mathrm{d}x \\ \leq 2\int_0^t \left\{ \int_\Sigma |\hat{v}(\tau,x)|^2 |\nabla u_1(\tau,x)| \, \mathrm{d}x + \int_\Sigma \big|\langle \hat{v}(\tau,x), \mathbf{e}(x)\rangle\big| |\hat{\theta}(\tau,x)| \right\} \mathrm{d}\tau, \end{aligned} \tag{23}$$

and via integration by parts,

$$\begin{aligned}\int_\Sigma |\hat{\theta}(t,x)|^2 \, \mathrm{d}x + 2\kappa \int_0^t \int_\Sigma |\nabla\hat{\theta}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau \\ \leq 2\int_0^t \int_\Sigma |\nabla\hat{\theta}(\tau,x)||\hat{v}(\tau,x)||\theta_1(\tau,x)| \, \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{24}$$

Eq. (23) is controlled by

$$\begin{aligned}\int_\Sigma |\hat{v}(t,x)|^2 \, \mathrm{d}x \\ \leq \int_0^t \Big(1 + \|u_1(\tau,\cdot)\|_{W^{1,\infty}(\Sigma)}\Big) \int_\Sigma |\hat{v}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Sigma |\hat{\theta}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

For Eq. (24), thanks to $\kappa > 0$, one utilises Cauchy's inequality to get

$$\int_\Sigma |\hat{\theta}(t,x)|^2 \, \mathrm{d}x + \kappa \int_0^t \int_\Sigma |\nabla\hat{\theta}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau \leq \frac{1}{\kappa}\int_0^t \|\theta_1(\tau,\cdot)\|^2_{L^\infty(\Sigma)} \int_\Sigma |\hat{v}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau.$$

We can now deduce (1) from Grönwall's inequality, by considering the quantity $\mathbf{Y}$ as in (20).

When $\nu > 0$, $\kappa = 0$, in place of Eq. (23) there holds

$$\begin{aligned}\int_\Sigma |\hat{v}(t,x)|^2 \, \mathrm{d}x + 2\nu \int_0^t \int_\Sigma |\nabla\hat{v}(\tau,x)|^2 \, \mathrm{d}x\mathrm{d}\tau \\ \leq 2\int_0^t \Big\{ \int_\Sigma |\nabla\hat{v}(\tau,x)||\hat{v}(\tau,x)||u_1(\tau,x)| \, \mathrm{d}x \\ + \int_\Sigma \big|\langle \hat{v}(\tau,x), \mathbf{e}(x)\rangle\big| |\hat{\theta}(\tau,x)| \, \mathrm{d}x \Big\} \mathrm{d}\tau, \end{aligned} \tag{25}$$

and, in place of Eq. (24),

$$\int_\Sigma |\hat{\theta}(t,x)|^2 \, \mathrm{d}x \leq 2\int_0^t \int_\Sigma |\hat{\theta}(\tau,x)||\hat{v}(\tau,x)||\nabla\theta_1(\tau,x)| \, \mathrm{d}x\mathrm{d}\tau. \tag{26}$$

We apply Cauchy's inequality to Eq. (25) and argue by Grönwall as before. This proves (2).

Finally, if $\nu = \kappa = 0$, then only Eqs. (23) and (26) are available. We thus need the $L^1_t W^{1,\infty}_x$ bounds on both u_1 and θ_1 for the Grönwall inequality. This proves (3). □

We also consider the *strong solution* (u, θ) in the space:

$$\begin{cases} u \in C^0\big(0,T; H^3(\Sigma; T\Sigma) \cap \mathcal{H}\big), \\ \theta \in C^0\big(0,T; H^3(\Sigma)\big). \end{cases}$$

The subsequent two sections will be based on a Beale–Kato–Majda-type *breakdown criterion*:

Theorem 3.4. *Let Σ be a closed surface with Lipschitz curvature. Let $(u^\circ, \theta^\circ) \in H^3(\Sigma; T\Sigma) \times H^3(\Sigma)$ be initial data satisfying $\operatorname{div} u^\circ = 0$. Assume that (u, θ) is a strong solution to the Boussinesq equations (1)–(4) on $[0,T] \times \Sigma$. If*

$$\int_0^T \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\,\mathrm{d}t < \infty, \tag{27}$$

then the strong solution can be continued to $[0, T+\epsilon[$ for some $\epsilon > 0$.

The proof is based on energy estimates and an end-point case of the Sobolev–Morrey embeddings (Lemma 2.2). To avoid repetitions with the following sections, we postpone the proof to the Appendix.

4. **Strong solutions: Non-degenerate viscosity.** In this section, we establish the existence of global strong solutions of the Boussinesq equations on a closed surface Σ with non-degenerate viscosity. More precisely, let us prove:

Theorem 4.1. *Let Σ be a closed surface with Lipschitz-bounded curvature, and let $T > 0$ be arbitrary. Suppose that $\nu > 0$, $\kappa \geq 0$, and $u^\circ, \theta^\circ \in H^3$ with $\operatorname{div}(u^\circ) = 0$. Then, there exists a unique solution (u, θ) to the Boussinesq equations (1)–(4) on $[0,T] \times \Sigma$ in the following space:*

$$\begin{cases} u \in C^0\big(0,T; H^3(\Sigma; T\Sigma) \cap \mathcal{H}\big) \cap L^2\big(0,T; H^4(\Sigma; T\Sigma)\big), \\ \theta \in C^0\big(0,T; H^3(\Sigma)\big). \end{cases}$$

Proof. The strategy of the proof is largely based on [16] by D. Chae, which made use of energy estimates and the Brezis–Wainger inequality in Lemma 2.2. We divide the arguments into nine steps.

1. First, we multiply $p|\theta|^{p-2}\theta$ to Eq. (2) for any $p \geq 1$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t,\cdot)\|^p_{L^p(\Sigma)} + \kappa p(p-1)\int_\Sigma |\theta(t,\cdot)|^{p-2}|\nabla\theta(t,\cdot)|^2\,\mathrm{d}x = 0. \tag{28}$$

As a result,

$$\|\theta(t,\cdot)\|_{L^p(\Sigma)} \leq \|\theta^\circ\|_{L^p(\Sigma)}, \tag{29}$$

and

$$\kappa p(p-1)\int_0^t\int_\Sigma |\theta(t,\cdot)|^{p-2}|\nabla\theta(\tau,x)|^2\,\mathrm{d}x\mathrm{d}\tau \leq \|\theta^\circ\|^p_{L^p(\Sigma)}. \tag{30}$$

2. Next, multiplying u to Eq. (1), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Sigma |u(t,x)|^2\,\mathrm{d}x\Big) + 2\nu\int_\Sigma |\nabla u(t,x)|^2\,\mathrm{d}x = 2\int_\Sigma \big\langle \theta(t,x)u(t,x), \mathbf{e}(x)\big\rangle\,\mathrm{d}x.$$

As $|\mathbf{e}| = 1$, the right-hand side can be bounded by

$$2\Big|\int_\Sigma \big\langle \theta(t,x)u(t,x), \mathbf{e}(x)\big\rangle\,\mathrm{d}x\Big| \leq \|\theta^\circ\|^2_{L^2(\Sigma)} + \|u\|^2_{L^2(\Sigma)}.$$

Hence, the Grönwall's inequality implies

$$\|u(t,\cdot)\|^2_{L^2(\Sigma)} \leq \|u^\circ\|^2_{L^2(\Sigma)}e^t + \|\theta^\circ\|^2_{L^2(\Sigma)}(e^t - 1) =: K_0(t), \tag{31}$$

for any $t \in]0,T]$.

3. Now we consider the *vorticity*

$$\omega = \operatorname{rot} u. \tag{32}$$

In any local coordinate $\{e_1, e_2\}$ of $T\Sigma$, we define

$$\mathrm{rot}\, u := -\nabla_2 u^1 + \nabla_1 u^2 \equiv \nabla^\perp \cdot u, \tag{33}$$

where $u = u^1 e_1 + u^2 e_2$. Let us emphasize that ω is a scalar function on Σ.

We *claim* that ω satisfies the *vorticity equation* below, which shall be used frequently in the subsequent developments:

$$\partial_t \omega + u \cdot \nabla\omega - \nu\Delta\omega + \mathrm{Riem} \star |u|^2 = \mathrm{rot}\,(\theta \mathbf{e}). \tag{34}$$

To see this, we take *rot* to Eq. (1); thus

$$\begin{aligned} 0 &= \partial_t \omega + \nabla^\perp \cdot (u \cdot \nabla u) - \nu \nabla^\perp \cdot \Delta u - \mathrm{rot}\,(\theta\mathbf{e}) \\ &= \partial_t \omega + u \cdot \nabla \omega - \nu\Delta\omega + \nu[\Delta, \nabla^\perp \cdot] u - [\nabla^\perp \cdot, u \cdot \nabla] u - \mathrm{rot}\,(\theta\mathbf{e}), \end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutator of two differential operators. The first commutator vanishes: recall that the Hodge Laplacian is $\Delta = -dd^* - d^*d$ where d^* is the L^2-adjoint of d, namely the co-differential operator. Also, let $u^\flat$ be the 1-form canonically dual to $u \in \Gamma(T\Sigma)$ via the metric; then

$$\mathrm{rot}\, u = \star_H [d(u^\flat)],$$

where $\star_H$ is the Hodge star operator on differential forms. Thus, the 2-form $\star_H[\Delta, \nabla^\perp \cdot]u$ equals

$$\star_H \big\{ [\Delta, \nabla^\perp \cdot] u \big\} = d(d^*d + dd^*)u - (d^*d + dd^*)du = 0,$$

as $dd = 0$ and $d^*d^* = 0$. For the second commutator term one may compute directly:

$$\begin{aligned} \nabla^\perp \cdot (u \cdot \nabla u) &= \nabla_2 \Big(u^1 \nabla_1 u^1 + u^2 \nabla_2 u^1 \Big) - \nabla_1 \Big(u^1 \nabla_1 u^2 + u^2 \nabla_2 u^2 \Big) \\ &= (\nabla_2 u^1)(\nabla_1 u^1) - (\nabla_1 u^1)(\nabla_1 u^2) + (\nabla_2 u^2)(\nabla_2 u^1) - (\nabla_1 u^2)(\nabla_2 u^2) \\ &\quad + u^1 \nabla_1 \Big(\nabla_2 u^1 - \nabla_1 u^2 \Big) + u^2 \nabla_2 \Big(\nabla_2 u^1 - \nabla_1 u^2 \Big) \\ &\quad + u^1 \nabla_2 \nabla_1 u^1 - u^1 \nabla_1 \nabla_2 u^1 + u^2 \nabla_2 \nabla_1 u^2 - u^2 \nabla_1 \nabla_2 u^2 \\ &= (\mathrm{div}\, u)(\mathrm{rot}\, u) + u \cdot \nabla(\mathrm{rot}\, u) + u^i \mathrm{Riem} \star u^i, \end{aligned}$$

where the last line follows from the Ricci identity (Lemma 2.1). Hence the *claim* follows.

4. To resume, multiplying $p|\omega|^{p-2}\omega$ to Eq. (34), one gets

$$\begin{aligned} 0 &= \partial_t(|\omega|^p) + u \cdot \nabla(|\omega|^p) - \nu\Delta(|\omega|^p) \\ &\quad + p(p-1)\nu|\omega|^{p-2}|\nabla\omega|^2 \\ &\quad + p\nu\langle \omega, \mathrm{Riem} \star \omega \rangle |\omega|^{p-2} - p|\omega|^{p-2}\langle \omega, \mathrm{rot}\,(\theta\mathbf{e})\rangle + p\omega|\omega|^{p-2} \mathrm{Riem} \star u \star u. \end{aligned} \tag{35}$$

Integrating over Σ, we thus get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + p(p-1)\nu \int_\Sigma |\nabla\omega(t,x)|^2 |\omega(t,x)|^{p-2}\,\mathrm{d}x \\ &\quad \leq Rp\nu \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + p(p-1) \int_\Sigma |\omega(t,x)|^{p-2} |\nabla\omega(t,x)||\theta(t,x)|\,\mathrm{d}x \\ &\qquad + pR \int_\Sigma |\omega(t,x)|^{p-1} |u(t,x)|^2\,\mathrm{d}x, \end{aligned}$$

where R only depends on $\|Riem\|_{L^\infty(\Sigma)}$.

By Hölder's inequality, we have

$$\int_{\Sigma} |\omega(t,x)|^{p-1}|u(t,x)|^2\,\mathrm{d}x \leq \|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)}\|u(t,\cdot)\|^2_{L^{2p}(\Sigma)}. \tag{36}$$

Then, by the Gargliardo–Nirenberg interpolation inequality, there is a constant $c = c(p,\Sigma)$ such that

$$\|u(t,\cdot)\|^2_{L^{2p}(\Sigma)} \leq c\left(\|\nabla u(t,\cdot)\|_{L^p}\|u(t,\cdot)\|_{L^2(\Sigma)} + \|u(t,\cdot)\|^2_{L^2(\Sigma)}\right).$$

But by Calderón–Zygmund (Lemma 2.3) and an obvious interpolation there holds

$$\|\nabla u(t,\cdot)\|_{L^p(\Sigma)} \leq c\|\omega(t,\cdot)\|_{L^p(\Sigma)} + c\|u\|_{L^\infty(\Sigma)}.$$

To estimate the L^∞-norm of u, we notice that by the Calderón–Zygmund estimates (Lemma 2.3), the Gagliardo–Nirenberg interpolation inequality and the Young inequality, there are constants $c = C(p,\Sigma)$ for $2 < p < \infty$ and $\delta > 0$ (to be determined) such that

$$\begin{aligned}
\|u(t,\cdot)\|_{L^\infty(\Sigma)} &\leq c\|u(t,\cdot)\|^{\frac{p-2}{2p-2}}_{L^2(\Sigma)}\Big\{\|\omega(t,\cdot)\|_{L^p(\Sigma)} + \|u(t,\cdot)\|_{L^p(\Sigma)}\Big\}^{\frac{p}{2p-2}}\\
&\leq c\|u(t,\cdot)\|^{\frac{p-2}{2p-2}}_{L^2(\Sigma)}\|\omega(t,\cdot)\|^{\frac{p}{2p-2}}_{L^p(\Sigma)} + c\|u(t,\cdot)\|^{\frac{p-2}{2p-2}}_{L^\infty(\Sigma)}\|u(t,\cdot)\|^{\frac{p}{2p-2}}_{L^2(\Sigma)}\\
&\leq c\|u(t,\cdot)\|^{\frac{p-2}{2p-2}}_{L^2(\Sigma)}\|\omega(t,\cdot)\|^{\frac{p}{2p-2}}_{L^p(\Sigma)} + \frac{p}{2p-2}\left(\frac{c}{\delta}\right)^{\frac{2p-2}{p}}\\
&\quad + \frac{p-2}{2p-2}\delta^{\frac{2p-2}{p-2}}\|u(t,\cdot)\|_{L^\infty(\Sigma)}.
\end{aligned}$$

By choosing $\delta > 0$ sufficiently small (depending on p), the final term can be absorbed to the left-hand side. Thus, for another constant $c = c(p,\Sigma)$, it holds that

$$\|u(t,\cdot)\|_{L^\infty(\Sigma)} \leq c\Big\{1 + \|u(t,\cdot)\|^{\frac{p-2}{2p-2}}_{L^2(\Sigma)}\|\omega(t,\cdot)\|^{\frac{p}{2p-2}}_{L^p(\Sigma)}\Big\}. \tag{37}$$

Hence, we can continue Eq. (36) as follows:

$$\begin{aligned}
&\int_{\Sigma} |\omega(t,x)|^{p-1}|u(t,x)|^2\,\mathrm{d}x\\
&\quad\leq \|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)} \times c\sqrt{K_0}\Big\{\|\omega(t,\cdot)\|_{L^p(\Sigma)} + 1 + (K_0)^{\frac{p-2}{4p-4}}\|\omega(t,\cdot)\|^{\frac{p}{2p-2}}_{L^p(\Sigma)} + \sqrt{K_0}\Big\}\\
&\quad\leq c\left(\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1\right).
\end{aligned} \tag{38}$$

The constant c depends on Σ, p, T and K_0 (hence on $\|u^\circ\|_{L^2(\Sigma)}$ and $\|\theta^\circ\|_{L^2(\Sigma)}$). Here it is crucial that $p \geq 2$, so that $\frac{p}{2p-2} \leq 1$ and we may apply interpolation to the term in the parenthesis on the second line.

Using (38) we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + p(p-1)\nu\int_{\Sigma}|\nabla\omega(t,x)|^2|\omega(t,x)|^{p-2}\,\mathrm{d}x\\
&\quad\leq Rp\nu\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + p\int_{\Sigma}|\omega(t,x)|^{p-2}|\nabla\omega(t,x)||\theta(t,x)|\,\mathrm{d}x\\
&\qquad + c\left(\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1\right).
\end{aligned} \tag{39}$$

As a special case, when $p = 2$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + 2\nu\|\nabla\omega(t,\cdot)\|^2_{L^2(\Sigma)}$$

$$\leq 2R\nu\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + 2\int_\Sigma |\nabla\omega(t,x)||\theta(t,x)|\,\mathrm{d}x + c\left(\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + 1\right).$$

For the second term on the right-hand side, we apply $ab \leq \frac{\nu}{2}a^2 + \frac{1}{2\nu}b^2$ and the conservation of $\|\theta(t,\cdot)\|_{L^2(\Sigma)}$ to infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \nu\|\nabla\omega(t,\cdot)\|^2_{L^2(\Sigma)} &\leq 2R\nu\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \frac{1}{\nu}\|\theta^\circ\|^2_{L^2(\Sigma)} \\ &\quad + c\left(\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + 1\right).\end{aligned}$$

By the Grönwall inequality, we get

$$\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \nu\int_0^t \|\nabla\omega(\tau,\cdot)\|^2_{L^2(\Sigma)}\,\mathrm{d}\tau \leq K_1(t), \tag{40}$$

where $K_1(t) = C(R,\nu,\nu^{-1},\|\theta^\circ\|_{L^2(\Sigma)},\|\omega^\circ\|_{L^2(\Sigma)},\Sigma,K_0(t))$.

5. To proceed, let us take the gradient of the temperature equation (2). Note that

$$\nabla(u\cdot\nabla\theta) = u\cdot\nabla(\nabla\theta) + \langle\nabla u,\nabla\theta\rangle,$$

as well as

$$\nabla\Delta\theta = \Delta\nabla\theta - \mathrm{Riem}\star\nabla\theta.$$

Hence

$$\partial_t(\nabla\theta) + u\cdot\nabla(\nabla\theta) + \langle\nabla u,\nabla\theta\rangle - \kappa\Delta(\nabla\theta) + \kappa\mathrm{Riem}\star\nabla\theta = 0. \tag{41}$$

Now, taking the inner product with $p|\nabla\theta|^{p-2}\nabla\theta$ to (41), we get

$$\begin{aligned}0 &= \partial_t(|\nabla\theta|^p) + p\,\mathrm{div}\left((u\cdot\nabla\theta)|\nabla\theta|^{p-2}\nabla\theta\right) \\ &\quad + u\cdot\nabla(|\nabla\theta|^p) + p|\nabla\theta|^{p-2}\nabla u : \nabla\theta\otimes\nabla\theta \\ &\quad - \frac{1}{2}p\kappa\,\mathrm{div}\left(|\nabla\theta|^{p-2}\nabla(|\nabla\theta|^2)\right) + p\kappa|\nabla\theta|^{p-2}|\nabla\nabla\theta|^2 \\ &\quad + p(p-2)\kappa|\nabla\theta|^{p-4}|\langle\nabla\nabla\theta,\nabla\theta\rangle|^2 \\ &\quad + p\kappa|\nabla\theta|^{p-2}\langle\nabla\theta,\mathrm{Riem}\star\nabla\theta\rangle,\end{aligned} \tag{42}$$

where $\nabla\nabla\theta$ is the Hessian matrix of θ. Thus, for any $\kappa \geq 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa\int_\Sigma |\nabla\nabla\theta(t,x)|^2|\nabla\theta(t,x)|^{p-2}\,\mathrm{d}x \\ &\quad + p(p-2)\kappa\int_\Sigma |\nabla\theta(t,x)|^{p-4}|\langle\nabla\nabla\theta(t,x),\nabla\theta(t,x)\rangle|^2\,\mathrm{d}x \\ &\quad \leq p\int_\Sigma |\nabla\theta(t,x)|^{p-2}|\nabla u(t,x):\nabla\theta(t,x)\otimes\nabla\theta(t,x)|\,\mathrm{d}x + p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\ &\quad \leq p\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}.\end{aligned}$$

For $\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)}$, by the Brezis–Wainger inequality (Lemma 2.2) and Calderón–Zygmund estimate (Lemma 2.3), it holds that

$$\|\nabla u\|_{L^\infty(\Sigma)} \leq c(1+\|\nabla\omega\|_{L^2(\Sigma)})\sqrt{1+\log_+(\|\nabla\omega\|_{L^p(\Sigma)})} + c\|\omega\|_{L^2(\Sigma)}. \tag{43}$$

By plugging (43) into its preceding estimate, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa\int_\Sigma |\nabla\nabla\theta(t,x)|^2|\nabla\theta(t,x)|^{p-2}\,\mathrm{d}x \\ &\quad + p(p-2)\kappa\int_\Sigma |\nabla\theta(t,x)|^{p-4}|\langle\nabla\nabla\theta(t,x),\nabla\theta(t,x)\rangle|^2\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + pc\|\omega(t,\cdot)\|_{L^2(\Sigma)}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\
&\quad + pc(1+\|\nabla\omega(t,\cdot)\|_{L^2(\Sigma)})\sqrt{1+\log_+(\|\nabla\omega(t,\cdot)\|_{L^p(\Sigma)})}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\
&\leq p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + pc\sqrt{K_2(t)}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\
&\quad + pc(1+\|\nabla\omega(t,\cdot)\|_{L^2(\Sigma)})\sqrt{1+\log_+(\|\nabla\omega(t,\cdot)\|_{L^p(\Sigma)})}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}, \qquad (44)
\end{aligned}
$$

where the estimate (40) was applied.

6. Now let us bound $\|\nabla\omega(t,\cdot)\|_{L^p(\Sigma)}$ and $\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}$. To this end, we take the gradient of the vorticity equation (34) to get

$$
\partial_t(\nabla\omega) + \nabla(u\cdot\nabla\omega) - \nu\nabla\Delta\omega + \nabla(\mathrm{Riem}\star\{\omega+|u|^2\}) - \nabla\mathrm{rot}\,(\theta\mathbf{e}) = 0. \qquad (45)
$$

Commuting ∇ and Δ yields a curvature term as before; hence,

$$
\begin{aligned}
&\partial_t(\nabla\omega) + \nabla(u\cdot\nabla\omega) - \nu\Delta(\nabla\omega) \\
&\qquad + \nu\mathrm{Riem}\star\omega + \nu(\nabla\mathrm{Riem})\star(\omega+|u|^2) \\
&\qquad + \nu\mathrm{Riem}\star(\nabla\omega + u\cdot\nabla u) - \nabla\mathrm{rot}\,(\theta\mathbf{e}) = 0. \qquad (46)
\end{aligned}
$$

In the above we have utilized the Ricci identities (Lemma 2.1).

Let us now take the inner product with $p\nabla\omega|\nabla\omega|^{p-2}$ for $p\geq 2$. Then

$$
\begin{aligned}
0 &= \partial_t(|\nabla\omega|^p) + u\cdot\nabla(|\nabla\omega|^p) + p|\nabla\omega|^{p-2}\nabla u : \nabla\omega\otimes\nabla\omega \\
&\quad - \frac{1}{2}p\nu\,\mathrm{div}\left(|\nabla\omega|^{p-2}\nabla(|\nabla\omega|^2)\right) \\
&\quad + \nu p|\nabla\omega|^{p-2}|\nabla\nabla\omega|^2 \\
&\quad + p(p-2)\nu|\nabla\omega|^{p-4}|\langle\nabla\nabla\omega,\nabla\omega\rangle|^2 \\
&\quad + \nu p|\nabla\omega|^{p-2}\big\langle\nabla\omega, \mathrm{Riem}\star\omega + \nabla\mathrm{Riem}\star\omega + \mathrm{Riem}\star\nabla\omega\big\rangle \\
&\quad + p\,\mathrm{div}\left(\nabla\omega|\nabla\omega|^{p-2}\mathrm{rot}\,(\theta\mathbf{e})\right) \\
&\quad + p\,\mathrm{rot}\,(\theta\mathbf{e})|\nabla\omega|^{p-2}\Delta\omega + p(p-2)\,\mathrm{rot}\,(\theta\mathbf{e})|\nabla\omega|^{p-4}\nabla\nabla\omega : \nabla\omega\otimes\nabla\omega \\
&\quad + p\nu|\nabla\omega|^{p-2}\nabla\omega\mathrm{Riem}\star(u\cdot\nabla u). \qquad (47)
\end{aligned}
$$

Integration over Σ gives us

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \nu p\int_\Sigma |\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|^2\,\mathrm{d}x \\
&\quad + p(p-2)\nu\int_\Sigma |\nabla\omega(t,x)|^{p-4}|\langle\nabla\nabla\omega(t,x),\nabla\omega(t,x)\rangle|^2\,\mathrm{d}x \\
&\qquad \leq p\left|\int_\Sigma |\nabla\omega(t,x)|^{p-2}\nabla u(t,x) : \nabla\omega(t,x)\otimes\nabla\omega(t,x)\,\mathrm{d}x\right| \\
&\qquad\quad + \nu p\left\{R\int_\Sigma |\nabla\omega(t,x)|^{p-1}|\omega(t,x)|\,\mathrm{d}x + R\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)}\right\} \\
&\qquad\quad + p(p-1)\int_\Sigma \left|\mathrm{rot}\left(\theta(t,x)\mathbf{e}(x)\right)\right||\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|\,\mathrm{d}x \\
&\qquad\quad + p\nu R\int_\Sigma |\nabla\omega(t,x)|^{p-1}|u(t,x)||\nabla u(t,x)|\,\mathrm{d}x. \qquad (48)
\end{aligned}
$$

Estimates for (48), Step 1. Let us first bound the third line in Eq. (48). It is less than or equal to

$$p^2 \int_\Sigma \Big(|\nabla\omega(t,x)|^{\frac{p-2}{2}} |\nabla\nabla\omega(t,x)| \Big) \Big(|u(t,x)| |\nabla\omega(t,x)|^{\frac{p}{2}} \Big) \,\mathrm{d}x.$$

As $\nu > 0$, we utilise Cauchy–Schwarz and $ab \le \frac{\nu a^2}{p} + \frac{pb^2}{2\nu}$ to bound it by

$$\frac{p\nu}{2} \int_\Sigma |\nabla\omega(t,x)|^{p-2} |\nabla\nabla\omega(t,x)|^2 \,\mathrm{d}x + \frac{p^3}{2\nu} \|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} \|u(t,\cdot)\|^2_{L^\infty(\Sigma)}.$$

To continue, we need Eq. (39) to bound the L^p-norm of ω. The second term on the right-hand side of (39) can be estimated as follows, via Hölder inequality:

$$\begin{aligned}
&\int_\Sigma |\omega(t,x)|^{p-2} |\nabla\omega(t,x)| |\theta(t,x)| \,\mathrm{d}x \\
&\le \Big(\int_\Sigma |\omega(t,x)|^{p-2} |\nabla\omega(t,x)|^2 \,\mathrm{d}x \Big)^{1/2} \Big(\int_\Sigma |\theta(t,x)|^p \,\mathrm{d}x \Big)^{1/p} \Big(\int_\Sigma |\omega(t,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{2p}} \\
&\le \frac{(p-1)\nu}{2} \int_\Sigma |\omega(t,x)|^{p-2} |\nabla\omega(t,x)|^2 \,\mathrm{d}x + \frac{1}{2(p-1)\nu} \|\theta(t,\cdot)\|^2_{L^p(\Sigma)} \|\omega(t,\cdot)\|^{p-2}_{L^p(\Sigma)}.
\end{aligned}$$

As $\|\theta(t,\cdot)\|_{L^p(\Sigma)} \le \|\theta^\circ\|_{L^p(\Sigma)}$, we may infer that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \frac{p(p-1)\nu}{2} \int_\Sigma |\nabla\omega(t,x)|^2 |\omega(t,x)|^{p-2} \,\mathrm{d}x \\
&\quad \le Rp\nu \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \frac{c}{\nu} \|\theta^\circ\|^2_{L^p(\Sigma)} \Big(1 + \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} \Big) \\
&\qquad + c \Big(\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1 \Big).
\end{aligned}$$

Now, the Grönwall inequality implies

$$\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} \le K_2(t), \tag{49}$$

where K_2 depends on $K_0(t), p, \nu, \operatorname{Vol}\Sigma, \nu^{-1}, \|\omega^\circ\|_{\mathrm{L}^p(\Sigma)}$ and $\|\theta^\circ\|_{L^p(\Sigma)}$. It grows exponentially in t. Substituting into Eq. (37), we get

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^\infty(\Sigma)} \\
&\quad \le c \Big(1 + K_2(t)^{\frac{1}{2p-2}} \Big\{ e^t \|u^\circ\|^2_{L^2(\Sigma)} + (e^t - 1) \|\theta^\circ\|^2_{L^2(\Sigma)} \Big\}^{\frac{p-2}{4(p-1)}} \Big) =: K_3(t),
\end{aligned} \tag{50}$$

where we applied (31) for the estimate of $\|u(t,\cdot)\|_{L^2(\Sigma)}$.

To sum up, the third line in Eq. (48) can be bounded by

$$\begin{aligned}
&p \Big| \int_\Sigma |\nabla\omega(t,x)|^{p-2} \nabla u(t,x) : \nabla\omega(t,x) \otimes \nabla\omega(t,x) \,\mathrm{d}x \Big| \\
&\quad \le \frac{p\nu}{2} \int_\Sigma |\nabla\omega(t,x)|^{p-2} |\nabla\nabla\omega(t,x)|^2 \,\mathrm{d}x + \frac{p^3}{2\nu} \big(K_3(t) \big)^2 \|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)}.
\end{aligned} \tag{51}$$

Estimates for (48), Step 2. We can bound the fourth line in Eq. (48) using the Hölder inequality, $a^{\frac{p-1}{p}} b^{\frac{1}{p}} \le \frac{p-1}{p} a + \frac{b}{p}$ and Eq. (49):

$$\begin{aligned}
&p\nu \Big\{ R \int_\Sigma |\nabla\omega(t,x)|^{p-1} |\omega(t,x)| \,\mathrm{d}x + R \|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} \Big\} \\
&\quad \le \nu R K_2(t) + p\nu \Big(\frac{p-1}{p} R + R \Big) \|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)}.
\end{aligned} \tag{52}$$

Estimates for (48), Step 3. For the fifth line in Eq. (48) we estimate as follows:

$$\begin{aligned}
&\frac{p\nu}{4}\int_\Sigma |\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|^2\,\mathrm{d}x \\
&\quad + \frac{2p(p-1)^2}{\nu}\int_\Sigma |\nabla\omega(t,x)|^{p-2}\Big(|\nabla\theta(t,x)|^2 + S|\theta(t,x)|^2\Big)\,\mathrm{d}x \\
&\leq \frac{p\nu}{4}\int_\Sigma |\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|^2\,\mathrm{d}x \\
&\quad + \frac{2p(p-1)^2}{\nu}\Big\{\|\nabla\omega(t,\cdot)\|^{p-2}_{L^p(\Sigma)}\|\nabla\theta(t,\cdot)\|^2_{L^p(\Sigma)} + S\|\nabla\omega(t,\cdot)\|^{p-2}_{L^p(\Sigma)}\|\theta(t,\cdot)\|^2_{L^p(\Sigma)}\Big\} \\
&\leq \frac{p\nu}{4}\int_\Sigma |\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|^2\,\mathrm{d}x + \frac{2(p-1)^2(p-2)(1+S)}{\nu}\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} \\
&\quad + \frac{4(p-1)^2}{\nu}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + \frac{4(p-1)^2S}{\nu}\|\theta^\circ\|^p_{L^p(\Sigma)},
\end{aligned} \tag{53}$$

where $S = \|\mathbf{e}\|_{W^{1,\infty}(\Sigma)}$. In the first inequality we utilised the Hölder inequality and $ab \leq \frac{\nu a^2}{4(p-1)} + \frac{b^2(p-1)}{\nu}$, and in the second the Young's inequality $ab \leq \frac{p-2}{p}a^{\frac{p}{p-2}} + \frac{2}{p}b^{\frac{p}{2}}$.

Estimates for (48), Step 4. The estimate for the last line in Eq. (48) is as follows. By Hölder inequality

$$\int_\Sigma |\nabla\omega(t,x)|^{p-1}|u(t,x)||\nabla u(t,x)|\,\mathrm{d}x \leq \|u(t,\cdot)\|_{L^\infty(\Sigma)}\|\nabla\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)}\|\nabla u(t,\cdot)\|_{L^p(\Sigma)},$$

where $\|u(t,\cdot)\|_{L^\infty(\Sigma)}$ is bounded by K_3 in Eq. (50). Moreover, by Lemma 2.3 we have

$$\|\nabla u(t,\cdot)\|_{L^p(\Sigma)} \leq c\|\omega(t,\cdot)\|_{L^p(\Sigma)} + c\|u(t,\cdot)\|_{L^\infty(\Sigma)}$$

for some $c = c(p,\Sigma)$. But the L^p-norm of ω is bounded in Eq. (49). Therefore, by the above estimates and the Young's inequality $ab \leq \frac{(p-1)a^{\frac{p}{p-1}}}{p} + \frac{b^p}{p}$, we can get

$$p\nu R\int_\Sigma |\nabla\omega(t,x)|^{p-1}|u(t,x)||\nabla u(t,x)|\,\mathrm{d}x \leq \frac{p-1}{p}\Big(\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)}\Big) + K_4(t),$$

where

$$K_4(t) := \frac{\Big(cp\nu RK_3(t)\Big\{\big[K_2(t)\big]^{\frac{1}{p}} + K_3(t)\Big\}\Big)^p}{p}.$$

Estimates for (48), Step 5. Thus, combining the estimates in (51), (52) and (53), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \frac{\nu p}{4}\int_\Sigma |\nabla\omega(t,x)|^{p-2}|\nabla\nabla\omega(t,x)|^2\,\mathrm{d}x \\
&\quad \leq K_5(t) + K_6\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + K_7(t)\|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)},
\end{aligned} \tag{54}$$

where for notational convenience we defined the constants:

$$\begin{aligned}
K_5(t) &:= \frac{4(p-1)S}{p\nu}\|\theta^\circ\|^p_{L^p(\Sigma)} + \nu RK_2(t) + K_4(t), \\
K_6 &:= \frac{4(p-1)^2}{\nu}, \\
K_7(t) &:= \Big\{\frac{p-1}{p} + \frac{2(p-1)^2(p-2)(1+S)}{\nu} + \nu(2p-1)R + \frac{p^3K_3(t)}{2\nu}\Big\}.
\end{aligned}$$

7. By combining the estimates (44) and (54), we see that the quantity

$$\mathbf{X}_1(t) := \|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + \|\nabla\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1 \tag{55}$$

satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_1(t) \le \left(K_8(t) + pc(1+\|\nabla\omega(t,\cdot)\|_{L^2(\Sigma)})\sqrt{1+\frac{1}{p}\log\,\mathbf{X}_1(t)}\right)\mathbf{X}_1(t), \tag{56}$$

where

$$K_8(t) := \max\left\{K_5(t) + K_6 + K_7(t) + pc\sqrt{K_2(t)} + p\kappa R\right\}.$$

It is crucial that the bound is still valid for $\kappa = 0$. Moreover, by letting

$$\mathbf{X}_2(t) := 1 + \frac{1}{p}\log\,\mathbf{X}_1(t),$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_2(t) \le \frac{1}{p}\left(K_8(t) + pc(1+\|\nabla\omega(t,\cdot)\|_{L^2(\Sigma)})\mathbf{X}_2(t)\right).$$

Then Grönwall's inequality implies for any $0 < T < \infty$ and $t \in [0,T]$,

$$\mathbf{X}_2(t) \le \exp\left\{c\int_0^t (1+\|\nabla\omega(\tau,\cdot)\|_{L^2(\Sigma)})\mathrm{d}\tau\right\}\left(\frac{1}{p}\int_0^t K_8(\tau)\mathrm{d}\tau + \mathbf{X}_2(0)\right) =: K_9(t),$$

where $K_9(t)$ is finite for any $t \in [0,T]$, due to the Hölder inequality and (40). Hence,

$$\mathbf{X}_1(t) \le e^{pK_9(t)} =: K_{10}(t). \tag{57}$$

Notice that $K_{10}(t) \le K_{10}(T)$, where $K_{10}(T)$ depends on p, T, ν, κ, $\|\theta^\circ\|_{W^{1,p}(\Sigma)}$, $\|u^\circ\|_{W^{1,p}(\Sigma)}$, $\|\mathbf{e}\|_{W^{1,\infty}(\Sigma)}$ and the geometry of Σ, and it does not blow up when $\kappa = 0$. In particular, Eqs. (43), (55) and (57) imply that

$$\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)} \le K_{11}(t), \tag{58}$$

where $K_{11}(t)$ depends on the previous time-dependent constants and all of which are evaluated at some fixed $p > 2$, e.g. $p = 3$. Moreover, $K_{11}(t)$ is finite for any finite $t > 0$.

8. To proceed, notice that the estimate right before Eq. (43) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \le p\Big(K_{11}(t) + \kappa R\Big)\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)},$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)} \le \Big(K_{11}(t) + \kappa R\Big)\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}.$$

Thus, by Grönwall inequality,

$$\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)} \le \|\nabla\theta^\circ\|_{L^p(\Sigma)}\exp\left\{\kappa Rt + \int_0^t K_{11}(\tau)\,\mathrm{d}\tau\right\}. \tag{59}$$

Since $K_{11}(t)$ does not depend on p, we may send p to ∞ to get

$$\|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)} \le \|\nabla\theta^\circ\|_{L^\infty(\Sigma)}\exp\left\{\kappa Rt + \int_0^t K_{11}(\tau)\,\mathrm{d}\tau\right\}. \tag{60}$$

Thus, for all $t \in]0,T]$, there holds

$$\|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)} \le \|\nabla\theta^\circ\|_{L^\infty(\Sigma)}e^{[\kappa R + K_{11}(T)]T} < \infty. \tag{61}$$

9. We are now ready to conclude. First, in view of the remark ensuing Definition 3.1, the strong solution exists on $[0,T_1]$ for some $T_1 > 0$. Then, suppose that the maximal lifespan T of the solution is finite. In view of the breakdown criterion in Theorem 3.4, we can extend the solution up to time $(T+\epsilon)$ for some $\epsilon > 0$, thus contradicting the maximality of T; hence, the strong solution exists globally. The uniqueness of solutions follows from Theorems 3.2 and 3.3. Finally, all the above estimates are valid for both $\kappa > 0$ and $\kappa = 0$. So the proof is complete. □

5. **Strong solutions: Non-degenerate thermal diffusivity.** In this section we consider the case $\nu \geq 0$ but $\kappa > 0$ (non-degenerate thermal diffusivity).

Theorem 5.1. *Let Σ be a closed surface with Lipschitz curvature, and let $T > 0$ be arbitrary. Suppose that $\kappa > 0$, $\nu \geq 0$, and $u^\circ, \theta^\circ \in H^3$ with* $\mathrm{div}(u^\circ) = 0$. *Then, there exists a unique solution (u, θ) on $[0, T]$ in the following space:*

$$\begin{cases} u \in C^0\big(0, T; H^3(\Sigma; T\Sigma) \cap \mathcal{H}\big), \\ \theta \in C^0\big(0, T; H^3(\Sigma)\big) \cap L^2\big(0, T; H^4(\Sigma)\big). \end{cases} \tag{62}$$

Proof. We divide the arguments into eight steps.

1. First let us summarise some estimates from the previous section which carry over to this case. By setting $\nu = 0$ in Eqs. (31), (28) and (44), we obtain

$$\|u(t,\cdot)\|^2_{L^2(\Sigma)} \leq \|u^\circ\|^2_{L^2(\Sigma)} e^t + \|\theta^\circ\|^2_{L^2(\Sigma)}(e^t - 1) = K_0(t), \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t,\cdot)\|^p_{L^p(\Sigma)} = -p(p-1)\kappa \int_\Sigma |\theta(t,x)|^{p-2}|\nabla\theta(t,x)|^2 \,\mathrm{d}x, \tag{64}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa \int_\Sigma |\nabla\nabla\theta(t,x)|^2|\nabla\theta(t,x)|^{p-2}\,\mathrm{d}x \\ &\quad + p(p-2)\kappa \int_\Sigma |\nabla\theta(t,x)|^{p-4}|\langle\nabla\nabla\theta(t,x), \nabla\theta(t,x)\rangle|^2\,\mathrm{d}x \\ &\qquad \leq p\Big|\int_\Sigma |\nabla\theta(t,x)|^{p-2}\nabla u(t,x) : \nabla\theta(t,x)\otimes\nabla\theta(t,x)\,\mathrm{d}x\Big| \\ &\qquad\quad + p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}, \end{aligned} \tag{65}$$

for each $p \geq 2$ and $t \in]0, T]$.

In addition, taking $\nu = 0$ in the vorticity equation (34) and applying the L^p-energy estimate as before, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} &\leq pR\int_\Sigma |u(t,x)|^2|\omega(t,x)|^{p-1}\,\mathrm{d}x \\ &\quad + p\int_\Sigma |\omega(t,x)|^{p-1}\Big(|\nabla\theta(t,x)| + S|\theta(t,x)|\Big)\,\mathrm{d}x. \end{aligned} \tag{66}$$

Again, R depends only on the Lipschitz norm of the curvature of Σ, and S depends only on the Lipschitz norm of the vector field $\mathbf{e}$.

2. When $p = 2$, we derive from (66) by using the Hölder, Ladyzhenskaya and Cauchy inequalities and the Calderón–Zygmund estimates (Lemma 2.3) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} &\leq 2R\|\omega(t,\cdot)\|_{L^2(\Sigma)}\|u(t,\cdot)\|^2_{L^4(\Sigma)} \\ &\quad + 2\|\omega(t,\cdot)\|_{L^2(\Sigma)}\left(\|\nabla\theta(t,\cdot)\|_{L^2(\Sigma)} + S\|\theta(t,\cdot)\|_{L^2(\Sigma)}\right) \\ &\leq 2Rc\|\omega(t,\cdot)\|_{L^2(\Sigma)}\Big(\|u(t,\cdot)\|_{L^2(\Sigma)}\|\omega(t,\cdot)\|_{L^2(\Sigma)} + \|u(t,\cdot)\|^2_{L^2(\Sigma)}\Big) \\ &\quad + \|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \|\nabla\theta(t,\cdot)\|^2_{L^2(\Sigma)} + S^2\|\theta(t,\cdot)\|^2_{L^2(\Sigma)} \end{aligned}$$

$$
\begin{aligned}
&= 2Rc\|\omega(t,\cdot)\|^2_{L^2(\Sigma)}\|u(t,\cdot)\|_{L^2(\Sigma)} + 2Rc\|\omega(t,\cdot)\|_{L^2(\Sigma)}\|u(t,\cdot)\|^2_{L^2(\Sigma)} \\
&\qquad + \|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \|\nabla\theta(t,\cdot)\|^2_{L^2(\Sigma)} + S^2\|\theta(t,\cdot)\|^2_{L^2(\Sigma)} \\
&\le Rc\|\omega(t,\cdot)\|^2_{L^2(\Sigma)}\left(\|u(t,\cdot)\|^2_{L^2(\Sigma)} + 1\right) \\
&\qquad + Rc\left(\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + 1\right)\|u(t,\cdot)\|^2_{L^2(\Sigma)} \\
&\qquad + \|\omega(t,\cdot)\|^2_{L^2(\Sigma)} + \|\nabla\theta(t,\cdot)\|^2_{L^2(\Sigma)} + S^2\|\theta(t,\cdot)\|^2_{L^2(\Sigma)} \\
&= \|\omega(t,\cdot)\|^2_{L^2(\Sigma)}\left(2RcK_0(t) + Rc + 1\right) + K_0(t) \\
&\qquad + \|\nabla\theta(t,\cdot)\|^2_{L^2(\Sigma)} + S^2\|\theta(t,\cdot)\|^2_{L^2(\Sigma)},
\end{aligned}
\tag{67}
$$

where we also applied (63) for the estimate of $\|u(t,\cdot)\|^2_{L^2(\Sigma)}$. In addition, by (64) with $p = 2$, we can get that

$$
\|\theta(t,\cdot)\|^2_{L^2(\Sigma)} + 2\kappa\int_0^t\int_\Sigma |\nabla\theta(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau = \|\theta^\circ\|^2_{L^2(\Sigma)}.
\tag{68}
$$

Then by applying Grönwall inequality to (67) and using (68), we have

$$
\begin{aligned}
&\|\omega(t,\cdot)\|^2_{L^2(\Sigma)} \\
&\quad \le \exp\left\{2Rc\int_0^t K_0(\tau)\mathrm{d}\tau + Rct + t\right\}\times \\
&\quad\quad \times\left(\int_0^t K_0(\tau)\mathrm{d}\tau + \left(\frac{1}{2\kappa} + S^2 t\right)\|\theta^\circ\|^2_{L^2(\Sigma)}\right) =: K_{12}(t).
\end{aligned}
\tag{69}
$$

3. We deduce from Eq. (65) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa\int_\Sigma |\nabla\nabla\theta(t,x)|^2|\nabla\theta(t,x)|^{p-2}\,\mathrm{d}x \\
&\le p^2\int_\Sigma |u(t,x)||\nabla\theta(t,x)|^{p-1}|\nabla\nabla\theta(t,x)|\,\mathrm{d}x + p\kappa R_1\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\
&\le \frac{p\kappa}{2}\int_\Sigma |\nabla\theta(t,x)|^{p-2}|\nabla\nabla\theta(t,x)|^2\,\mathrm{d}x \\
&\quad + \frac{p^3}{2\kappa}\int_\Sigma |u(t,x)|^2|\nabla\theta(t,x)|^p\,\mathrm{d}x + p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)},
\end{aligned}
\tag{70}
$$

via integration by parts, Hölder and Cauchy inequalities. Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \le \frac{p^3}{2\kappa}\|u(t,\cdot)\|^2_{L^\infty(\Sigma)}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} + p\kappa R\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}.
\tag{71}
$$

The $\|u(t,\cdot)\|_{L^\infty(\Sigma)}$ term can be controlled by Brezis–Wainger inequality (Lemma 2.3), Calderón–Zygmund estimate (Lemma 2.4), Eqs. (63) and (69) as

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^\infty(\Sigma)} &\le c\left(1 + \|\omega(t,\cdot)\|_{L^2(\Sigma)} + \|u(t,\cdot)\|_{L^2(\Sigma)}\right) \\
&\qquad \times\sqrt{1 + \log_+(\|\omega(t,\cdot)\|_{L^p(\Sigma)} + \|u(t,\cdot)\|_{L^p(\Sigma)})} + c\|u(t,\cdot)\|_{L^2(\Sigma)} \\
&\le c\left(1 + \sqrt{K_{12}(t)} + \sqrt{K_0(t)}\right) \\
&\qquad \times\sqrt{1 + \log_+(\|\omega(t,\cdot)\|_{L^p(\Sigma)} + \|u(t,\cdot)\|_{L^p(\Sigma)})} + c\sqrt{K_0(t)}.
\end{aligned}
\tag{72}
$$

By substituting (72) into (71), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)} \le \frac{p^2 c}{2\kappa}\Big\{\big(1 + K_{12}(t) + K_0(t)\big)\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}\times
$$

$$
\begin{aligned}
&\times \left[1+\log_+(\|\omega(t,\cdot)\|_{L^p(\Sigma)}+\|u(t,\cdot)\|_{L^p(\Sigma)})\right] \Big\} \\
&+\kappa R\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}. \qquad (73)
\end{aligned}
$$

Moreover, from (66) we deduce that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} \leq\ & pR\|u(t,\cdot)\|^2_{L^{2p}(\Sigma)}\|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)} \\
&+p\|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)}\left(\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+S\|\theta(t,\cdot)\|_{L^p(\Sigma)}\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|_{L^p(\Sigma)} &\leq R\|u(t,\cdot)\|^2_{L^\infty(\Sigma)}(\mathrm{Vol}\,\Sigma)^{\frac{1}{p}}+\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+S\|\theta(t,\cdot)\|_{L^p(\Sigma)} \\
&\leq Rc(\mathrm{Vol}\,\Sigma)^{\frac{1}{p}}\Big\{\big(1+K_{12}(t)+K_0(t)\big) \\
&\qquad\times\left[1+\log_+(\|\omega(t,\cdot)\|_{L^p(\Sigma)}+\|u(t,\cdot)\|_{L^p(\Sigma)})\right]\Big\} \\
&\quad+\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+S\|\theta(t,\cdot)\|_{L^p(\Sigma)}. \qquad (74)
\end{aligned}
$$

4. By combining (73) and (74), and using (64), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+\|\omega(t,\cdot)\|_{L^p(\Sigma)}\right) \\
&\leq \max\left\{\frac{p^2c}{2\kappa}, Rc(\mathrm{Vol}\,\Sigma)^{\frac{1}{p}}\right\}\left(\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+1\right)\times \\
&\quad\times\Big\{\big(1+K_{12}(t)+K_0(t)\big)\left[1+\log_+(\|\omega(t,\cdot)\|_{L^p(\Sigma)}+\|u(t,\cdot)\|_{L^p(\Sigma)})\right]\Big\} \\
&\quad+(\kappa R+1)\|\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}+S\|\theta^\circ\|_{L^p(\Sigma)}. \qquad (75)
\end{aligned}
$$

Due to the Sobolev embedding $H^1(\Sigma)\hookrightarrow L^p(\Sigma)$, and the Calderón-Zygmund estimate (Lemma 2.3), we know that $\|u(t,\cdot)\|_{L^p(\Sigma)}\lesssim\|\omega(t,\cdot)\|_{L^2(\Sigma)}+\|u(t,\cdot)\|_{L^2(\Sigma)}$ for any $2<p<\infty$. According to (63) and (69), we deduce that $\|u(t,\cdot)\|_{L^p(\Sigma)}\leq K_{13}(t)$. Hence, for any $0<T<\infty$ and any $t\in[0,T]$, it holds that $\|u(t,\cdot)\|_{L^p(\Sigma)}\leq K_{13}(T)$.

Let us define for any $t\in[0,T]$,

$$
\mathbf{Z}(t):=\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}+\|\omega(t,\cdot)\|^p_{L^p(\Sigma)}+K_{13}(T).
$$

Then we derive from (75) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Z}(t)\leq C\left[1+\log_+(\mathbf{Z}(t))\right]\mathbf{Z}(t),
$$

where $C=C(p,c,\kappa,R,\mathrm{Vol}\,\Sigma,S,\theta^\circ,T)$, from which we may deduce

$$
\mathbf{Z}(t)\leq K_{14}(t)\leq K_{14}(T). \qquad (76)
$$

5. Next, we derive the L^p-estimates for $\nabla\nabla\theta$, the Hessian of θ. We adopt Einstein's summation convention for $i,j,k,l,\ldots\in\{1,2\}$.

Taking two covariant derivatives to Eq. (2), one obtains

$$
\partial_t\nabla_i\nabla_j\theta+\nabla_i\nabla_j(u^k\nabla_k\theta)-\kappa\nabla_i\nabla_j\Delta\theta=0. \qquad (77)
$$

We then utilise the Ricci identities (Lemma 2.1) repeatedly to deduce

$$
\begin{aligned}
&\nabla_i\nabla_j\Delta\theta-\Delta\nabla_i\nabla_j\theta \\
&\quad=\nabla_l(\mathrm{Riem}^m_{lij}\nabla_m\theta)+\nabla_i(\mathrm{Riem}^m_{jll}\nabla_m\theta)+\mathrm{Riem}^m_{ill}\nabla_m\nabla_j\theta. \qquad (78)
\end{aligned}
$$

Thus, multiplying $p|\nabla\nabla\theta|^{p-2}\nabla_i\nabla_j\theta$ to Eq.(77) and taking $\sum_{1\leq i,j\leq 2}$, we get

$$\begin{aligned}\partial_t\Big(|\nabla\nabla\theta|^p\Big) &= p\kappa|\nabla\nabla\theta|^{p-2}\Big(\Delta\nabla\nabla\theta : \nabla\nabla\theta\Big)\\ &+ p\kappa|\nabla\nabla\theta|^{p-2}\Big\{\nabla\nabla\theta : \Big[\nabla\mathrm{Riem}\star\nabla\theta + \mathrm{Riem}\star\nabla\nabla\theta\Big]\Big\}\\ &- p|\nabla\nabla\theta|^{p-2}\Big(\nabla\nabla\theta : \nabla\nabla[u\cdot\nabla\theta]\Big) =: J_1+J_2+J_3. \end{aligned}\tag{79}$$

In what follows we estimate J_1, J_2 and J_3 one by one.

First, due to the Leibniz' rule, there holds

$$J_1 = p\kappa\,\mathrm{div}\Big\{|\nabla\nabla\theta|^{p-2}\nabla\nabla\nabla\theta : \nabla\nabla\theta\Big\} - p(p-1)\kappa|\nabla\nabla\nabla\theta|^2|\nabla\nabla\theta|^{p-2}.$$

Second, it is clear that

$$|J_2| \leq p\kappa R\Big\{|\nabla\nabla\theta|^p + |\nabla\nabla\theta|^{p-1}|\nabla\theta|\Big\},$$

where R depends only on $\|Riem\|_{W^{1,\infty}(\Sigma)}$ as before.

Third, we note that $J_3 = J_{31}+J_{32}+J_{33}$, where

$$J_{31} = p\nabla_i\Big\{|\nabla\nabla\theta|^{p-2}(\nabla_i\nabla_j\theta)\nabla_j[u^k\nabla_k\theta]\Big\}$$

is of the divergence form,

$$J_{32} = -p(p-2)|\nabla\nabla\theta|^{p-4}(\nabla_l\nabla_m\theta)(\nabla_i\nabla_l\nabla_m\theta)(\nabla_i\nabla_j\theta)\nabla_j[u^k\nabla_k\theta],$$

and

$$J_{33} = -p|\nabla\nabla\theta|^{p-2}(\nabla_i\nabla_i\nabla_j\theta)\nabla_j[u^k\nabla_k\theta].$$

Clearly we have

$$|J_{32}+J_{33}| \leq p(p-1)|\nabla\nabla\theta|^{p-2}|\nabla\nabla\nabla\theta||\nabla[u\cdot\nabla\theta]|;$$

thus, we may deduce the following by integrating Eq. (79) over space-time:

$$\begin{aligned}&\int_\Sigma|\nabla\nabla\theta(t,x)|^p\,\mathrm{d}x + p(p-1)\kappa\int_0^t\int_\Sigma|\nabla\nabla\nabla\theta(\tau,x)|^2|\nabla\nabla\theta(\tau,x)|^{p-2}\,\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq p\kappa R\int_0^t\int_\Sigma\Big\{|\nabla\nabla\theta(\tau,x)|^p + |\nabla\nabla\theta(\tau,x)|^{p-1}|\nabla\theta(\tau,x)|\Big\}\,\mathrm{d}x\mathrm{d}\tau\\ &\quad\quad+ p(p-1)\int_0^t\int_\Sigma\Big\{|u(\tau,x)||\nabla\nabla\theta(\tau,x)|^{p-1}|\nabla\nabla\nabla\theta(\tau,x)|\Big\}\,\mathrm{d}x\mathrm{d}\tau\\ &\quad\quad+ p(p-1)\int_0^t\int_\Sigma\Big\{|\nabla u(\tau,x)||\nabla\theta(\tau,x)||\nabla\nabla\theta(\tau,x)|^{p-2}|\nabla\nabla\nabla\theta(\tau,x)|\Big\}\,\mathrm{d}x\mathrm{d}\tau\\ &\quad=: J_4+J_5+J_6. \end{aligned}\tag{80}$$

6. We continue the estimate $\|\nabla\nabla\theta(t,\cdot)\|_{L^p(\Sigma)}$ by bounding J_4, J_5 and J_6 in order. Indeed, by Hölder and Young inequalities, one can bound

$$J_4 \leq \Big(p\kappa R + (p-1)\kappa R\Big)\int_0^t\int_\Sigma|\nabla\nabla\theta(\tau,x)|^p\,\mathrm{d}x\mathrm{d}\tau + \kappa R\int_0^t\int_\Sigma|\nabla\theta(\tau,x)|^p\,\mathrm{d}x\mathrm{d}\tau,$$

$$\begin{aligned}J_5 &\leq \frac{p(p-1)\kappa}{2}\int_0^t\int_\Sigma|\nabla\nabla\nabla\theta(\tau,x)|^2|\nabla\nabla\theta(\tau,x)|^{p-2}\,\mathrm{d}x\mathrm{d}\tau\\ &\quad+\frac{p(p-1)}{2\kappa}\int_0^t\int_\Sigma|u(\tau,x)|^2|\nabla\nabla\theta(\tau,x)|^p\,\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned} J_6 \leq & \frac{p(p-1)\kappa}{4} \int_0^t \int_\Sigma |\nabla\nabla\nabla\theta(\tau,x)|^2 |\nabla\nabla\theta(\tau,x)|^{p-2} \,\mathrm{d}x\mathrm{d}\tau \\ & + \frac{p(p-1)}{\kappa} \int_0^t \int_\Sigma |\nabla u(\tau,x)|^2 |\nabla\theta(\tau,x)|^2 |\nabla\nabla\theta(\tau,x)|^{p-2} \,\mathrm{d}x\mathrm{d}\tau =: J_{61} + J_{62}. \end{aligned}$$

For J_{62}, one applies the Hölder inequality, Gagliardo–Nirenberg interpolation inequality, and Calderón-Zygmund estimate (Lemma 2.3) to deduce, for any $t \in [0,T]$,

$$\begin{aligned} J_{62} \leq & \frac{p(p-1)}{\kappa} \int_0^t \|\nabla\theta(\tau,\cdot)\|^2_{L^\infty(\Sigma)} \Big(\int_\Sigma |\nabla u(\tau,x)|^p \,\mathrm{d}x \Big)^{2/p} \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{p}} \mathrm{d}\tau \\ \leq & c\kappa^{-1} \int_0^t \|\nabla u(\tau,\cdot)\|^2_{L^p(\Sigma)} \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p^2-p+2}{p^2}} \Big(\int_\Sigma |\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{p^2}} \mathrm{d}\tau \\ & + c\kappa^{-1} \int_0^t \|\nabla u(\tau,\cdot)\|^2_{L^p(\Sigma)} \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{p}} \|\theta(\tau,\cdot)\|^2_{L^2(\Sigma)} \,\mathrm{d}\tau \\ \leq & c\kappa^{-1} \|\theta^\circ\|^{\frac{p-2}{p}}_{L^p(\Sigma)} \int_0^t \Big(\|\omega(\tau,\cdot)\|^2_{L^p(\Sigma)} + \|u(\tau,\cdot)\|^2_{L^p(\Sigma)} \Big) \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p^2-p+2}{p^2}} \mathrm{d}\tau \\ & + c\kappa^{-1} \|\theta^\circ\|^2_{L^2(\Sigma)} \int_0^t \Big(\|\omega(\tau,\cdot)\|^2_{L^p(\Sigma)} + \|u(\tau,\cdot)\|^2_{L^p(\Sigma)} \Big) \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{p}} \mathrm{d}\tau \\ \leq & c\kappa^{-1} \int_0^t \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p^2-p+2}{p^2}} + \Big(\int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \Big)^{\frac{p-2}{p}} \mathrm{d}\tau, \end{aligned}$$

where we applied (76), and the constant $c = c(p, \Sigma, \theta^\circ, T)$. For $p > 2$, by applying the Young inequality, we have

$$J_{62} \leq c\kappa^{-1} \int_0^t \int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \,\mathrm{d}\tau + c\kappa^{-1} T.$$

Substituting the above estimates into Eq. (80) and using Eq. (64), one has

$$\begin{aligned} & \int_\Sigma |\nabla\nabla\theta(t,x)|^p \,\mathrm{d}x + \frac{p(p-1)\kappa}{4} \int_0^t \int_\Sigma |\nabla\nabla\nabla\theta(\tau,x)|^2 |\nabla\nabla\theta(\tau,x)|^{p-2} \,\mathrm{d}x\mathrm{d}\tau \\ & \leq 2pR\kappa \int_0^t \int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x\mathrm{d}\tau + \kappa R \int_0^t \int_\Sigma |\nabla\theta(\tau,x)|^p \,\mathrm{d}x\mathrm{d}\tau \\ & \quad + \frac{p(p-1)}{2\kappa} \int_0^t \|u(\tau,\cdot)\|^2_{L^\infty(\Sigma)} \int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x\mathrm{d}\tau \\ & \quad + c\kappa^{-1} \int_0^t \int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x \,\mathrm{d}\tau + c\kappa^{-1} T. \end{aligned}$$

Applying the Sobolev embedding $W^{1,p}(\Sigma) \hookrightarrow L^\infty(\Sigma)$ plus (76) to $\|u(\tau,\cdot)\|_{L^\infty(\Sigma)}$, we obtain the differential inequality

$$\int_\Sigma |\nabla\nabla\theta(t,x)|^p \,\mathrm{d}x \leq c \int_0^t \int_\Sigma |\nabla\nabla\theta(\tau,x)|^p \,\mathrm{d}x\mathrm{d}\tau + c\kappa^{-1} T, \tag{81}$$

for each $t \in [0,T]$. Therefore, we conclude from Grönwall inequality that

$$\int_\Sigma |\nabla\nabla\theta(t,x)|^p \,\mathrm{d}x \leq K_{15}(T) \qquad \text{for all } t \in [0,T]. \tag{82}$$

Here K_{15} depends on $\Sigma, T, \kappa, p, u^\circ$ and θ°; note that $K_{15}(T) \to \infty$ as $\kappa \to 0^+$.

7. Again, in view of the breakdown criterion (Theorem 3.4), the Sobolev–Morrey embedding $W^{2,p}(\Sigma) \hookrightarrow W^{1,\infty}(\Sigma)$ for $p > 2$ and the uniqueness Theorems 3.2 and 3.3, the proof is now complete. □

6. **Vanishing viscosity and diffusivity limits.** In this section we study two singular limits of the Boussinesq equations on surfaces.

Throughout, we let (u, θ, P) be the strong solution to the non-degenerate Boussinesq system (1)–(3) on $[0, T] \times \Sigma$, (u_N, θ_N, P_N) be the strong solution to the same system with degenerate viscosity ($\nu = 0$), and (u_K, θ_K, P_K) with degenerate thermal diffusivity ($\kappa = 0$). That is, on $[0, T] \times \Sigma$ there hold

$$\begin{cases} \partial_t u_N + u_N \cdot \nabla u_N + \nabla P_N = \theta_N \mathbf{e}, \\ \partial_t \theta_N + u \cdot \nabla \theta_N - \kappa \Delta \theta_N = 0, \\ \operatorname{div} u_N = 0, \end{cases} \tag{83}$$

and

$$\begin{cases} \partial_t u_K + u_K \cdot \nabla u_K - \nu \Delta u_K + \nabla P_K = \theta_K \mathbf{e}, \\ \partial_t \theta_K + u_K \cdot \nabla \theta_K = 0, \\ \operatorname{div} u_K = 0. \end{cases} \tag{84}$$

We impose the same initial conditions:

$$(u, \theta)|_{t=0} = (u_N, \theta_N)|_{t=0} = (u_K, \theta_K)|_{t=0} = (u^\circ, \theta^\circ), \tag{85}$$

where $\operatorname{div} u^\circ = 0$ for the sake of compatibility. Let us emphasise that we require $\kappa > 0$ in (83) and $\nu > 0$ in (84).

We first establish the vanishing viscosity limit:

Theorem 6.1. *Let Σ be a closed surface with Lipschitz curvature, and let $T > 0$ be arbitrary. Let (u, θ) be the strong solution to the non-degenerate Boussinesq equations (1)–(3), and let (u_N, θ_N) be the strong solution to Eq. (83) with zero viscosity, with the same initial data $(u^\circ, \theta^\circ) \in H^3(\Sigma; T\Sigma) \times H^3(\Sigma)$ satisfying $\operatorname{div} u^\circ = 0$. Assume $u \in C^0(0, T; H^3(\Sigma; T\Sigma) \cap \mathcal{H}) \cap L^2(0, T; H^4(\Sigma; T\Sigma))$, $u_N \in C^0(0, T; H^3(\Sigma; T\Sigma) \cap \mathcal{H})$ and $\theta, \theta_N \in C^0(0, T; H^3(\Sigma; T\Sigma))$.*

Then $u \to u_N, \theta \to \theta_N$ in $C^0(0, T; H^j(\Sigma))$ as $\nu \to 0^+$ for each $j < 3$.

Proof. Define

$$v := u - u_N, \qquad \zeta := \theta - \theta_N. \tag{86}$$

Taking the difference between the non-degenerate and zero-diffusivity Boussinesq equations, we obtain the following system on $[0, T] \times \Sigma$:

$$\partial_t v + u \cdot \nabla v + v \cdot \nabla u_N - \nu \Delta v - \nu \Delta u_N + \nabla (p - p_N) = \zeta \mathbf{e}, \tag{87}$$

$$\partial_t \zeta + u \cdot \nabla \zeta + v \cdot \nabla \theta_N - \kappa \Delta \zeta = 0, \tag{88}$$

$$\operatorname{div} v = 0. \tag{89}$$

Multiplying ζ to Eq. (88), one gets

$$\frac{1}{2} \partial_t (\zeta^2) + \zeta v \cdot \nabla \theta_N + \kappa |\nabla \zeta|^2 + \operatorname{div} \Big(\frac{1}{2} u \zeta^2 - \kappa \zeta \nabla \zeta \Big) = 0.$$

Thus, by the Stokes' theorem and integration by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\Sigma |\zeta(t, x)|^2 \, \mathrm{d}x \Big) + 2\kappa \int_\Sigma |\nabla \zeta(t, x)|^2 \, \mathrm{d}x = \\ - 2 \int_\Sigma \zeta(t, x) v(t, x) \cdot \nabla \theta_N(t, x) \, \mathrm{d}x. \tag{90}$$

Similarly, the standard L^2 estimate for v gives us

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Sigma |v(t,x)|^2\,\mathrm{d}x\Big)+2\nu\int_\Sigma|\nabla v(t,x)|^2\,\mathrm{d}x\\&=2\int_\Sigma\Big\{\big\langle v(t,x),\mathbf{e}(x)\big\rangle\zeta(t,x)-\nu\nabla v(t,x):\nabla u_N(t,x)\\&\qquad-\nabla u_N(t,x):\Big(v(t,x)\otimes v(t,x)\Big)\Big\}\,\mathrm{d}x.\end{aligned}\tag{91}$$

Adding Eqs. (90) and (91) together, we find that the L^2-energy

$$\mathbf{E}(t):=\int_\Sigma\Big\{|v(t,x)|^2+|\zeta(t,x)|^2\Big\}\,\mathrm{d}x\tag{92}$$

verifies the differential inequality

$$\begin{aligned}&\mathbf{E}'(t)+2\kappa\int_\Sigma|\nabla\zeta(t,x)|^2\,\mathrm{d}x+2\nu\int_\Sigma|\nabla v(t,x)|^2\,\mathrm{d}x\\&\quad\le\big(1+2\|\nabla\theta_N(t,\cdot)\|_{L^\infty(\Sigma)}\big)\|v(t,\cdot)\|_{L^2(\Sigma)}\|\zeta(t,\cdot)\|_{L^2(\Sigma)}\\&\qquad+\nu\|\nabla u_N(t,\cdot)\|_{L^2(\Sigma)}\|\nabla v(t,\cdot)\|_{L^2(\Sigma)}+\|\nabla u_N(t,\cdot)\|_{L^\infty(\Sigma)}\|v(t,\cdot)\|^2_{L^2(\Sigma)}.\end{aligned}$$

Applying the usual Cauchy's inequality to the penultimate term, we deduce

$$\mathbf{E}'(t)\le\Big(1+2\|\nabla\theta_N(t,\cdot)\|_{L^\infty(\Sigma)}+\|\nabla u_N(t,\cdot)\|_{L^\infty(\Sigma)}\Big)\mathbf{E}(t)+\frac{\nu}{4}\|\nabla u_N(t,\cdot)\|^2_{L^2(\Sigma)}.\tag{93}$$

Now let us invoke Theorem 5.1 to establish the existence of strong solutions (u_N,θ_N) to the limiting system. Indeed, we have $u_N\in C^0(0,T;H^3(\Sigma;T\Sigma))\cap L^2(0,T;H^4(\Sigma;T\Sigma))$ and $\theta_N\in C^0(0,T;H^3(\Sigma))$, where, in particular, the indicated norms are independent of ν. Let Λ denote a generic constant that depends only on these norms. Then

$$\mathbf{E}(t)\le\Lambda\nu,$$

by the Grönwall inequality and $\mathbf{E}(0)=0$. Sending $\nu\to0^+$, we obtain the convergence of the L^2 energy.

For higher energies, by interpolation one has

$$\|v(t,\cdot)\|_{H^j(\Sigma)}\le K_{16}\|v(t,\cdot)\|^{1-j/3}_{L^2(\Sigma)}\|v(t,\cdot)\|^{j/3}_{H^k(\Sigma)}+K_{16}\|v(t,\cdot)\|_{L^2(\Sigma)},$$

where $K_{16}=K(\Sigma,j)$. Let us bound the H^3 norm of v by $\|u(t,\cdot)\|_{H^3(\Sigma)}$ and $\|u_N(t,\cdot)\|_{H^k(\Sigma)}$, following the arguments in §5, this bound is *independent of* ν. Therefore, for every $t\in[0,T]$,

$$\|v(t,\cdot)\|_{H^j(\Sigma)}\le K_{17}\nu^{\frac{3-j}{6}}+K_{16}(\Lambda\nu)^{\frac12}\longrightarrow0\qquad\text{as }\nu\to0^+,\tag{94}$$

with the constant $K_{17}=K(j,\Sigma,\Lambda,\kappa)$. The same convergence result holds for $\|\zeta(t,\cdot)\|_{H^j(\Sigma)}$. Hence the assertion is proved. □

Next, we prove the vanishing thermal diffusivity limit:

Theorem 6.2. *Let Σ be a closed surface with Lipschitz curvature, and let $T>0$ be arbitrary. Let (u,θ) be the strong solution to the non-degenerate Boussinesq equations (1)–(3) and let (u_K,θ_K) be the strong solution to Eq. (84) with zero thermal diffusivity, with the same initial data $(u^\circ,\theta^\circ)\in H^3(\Sigma;T\Sigma)\times H^3(\Sigma)$ satisfying* $\operatorname{div}u^\circ=0$. *Assume that $u,u_K\in C^0(0,T;H^3(\Sigma;T\Sigma)\cap\mathcal{H})\cap L^2(0,T;H^4(\Sigma;T\Sigma))$ and $\theta,\theta_K\in C^0(0,T;H^3(\Sigma;T\Sigma))$.*

Then $u\to u_K,\theta\to\theta_K$ in $C^0(0,T;H^j(\Sigma))$ as $\kappa\to0^+$ for each $j<3$.

Proof. The arguments are analogous to the proof of Theorem 6.1 in the large, hence some details are safely omitted.

First, define

$$w := u - u_K, \qquad \chi := \theta - \theta_K. \tag{95}$$

These variables satisfy

$$\partial_t w + u \cdot \nabla w + w \cdot \nabla u_K - \nu \Delta w + \nabla(p - p_K) = \chi \mathbf{e}, \tag{96}$$

$$\partial_t \chi + w \cdot \nabla \theta_K + u \cdot \nabla \chi - \kappa \Delta \chi - \kappa \Delta \theta_K = 0, \tag{97}$$

$$\operatorname{div} w = 0 \qquad \text{on } [0,T] \times \Sigma, \tag{98}$$

which can be seen by subtracting Eq. (84) from Eqs. (1)–(3).

Standard L^2 energy estimates lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Sigma |w(t,x)|^2 \,\mathrm{d}x\Big) + 2\nu \int_\Sigma |\nabla w(t,x)|^2 \,\mathrm{d}x \\
&\qquad \leq 2\int_\Sigma \Big\{ |w(t,x)|^2 |\nabla u_K(t,x)| + |\chi(t,x)||w(t,x)| \Big\} \,\mathrm{d}x,
\end{aligned}$$

together with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Sigma |\chi(t,x)|^2 \,\mathrm{d}x\Big) + 2\kappa \int_\Sigma |\nabla \chi(t,x)|^2 \,\mathrm{d}x \\
&\qquad \leq 2\int_\Sigma \Big\{ |\chi(t,x)||w(t,x)||\nabla\theta_K(t,x)| + \kappa |\nabla\theta_K(t,x)||\nabla\chi(t,x)| \Big\} \,\mathrm{d}x,
\end{aligned}$$

where we have applied integration by parts and the Stokes' theorem. Denoting the total L^2 energy by

$$\mathbf{F}(t) := \int_\Sigma \Big\{ |w(t,x)|^2 + |\chi(t,x)|^2 \Big\} \,\mathrm{d}x, \tag{99}$$

one obtains

$$\begin{aligned}
&\mathbf{F}'(t) + 2\nu \int_\Sigma |\nabla w(t,x)|^2 \,\mathrm{d}x + 2\kappa \int_\Sigma |\nabla \chi(t,x)|^2 \,\mathrm{d}x \\
&\qquad \leq \Big(1 + 2\|\nabla u_K(t,\cdot)\|_{L^\infty(\Sigma)} + \|\nabla\theta_K(t,\cdot)\|_{L^\infty(\Sigma)}\Big)\mathbf{F}(t) \\
&\qquad\quad + 2\kappa \int_\Sigma |\nabla\theta_K(t,x)||\nabla\chi(t,x)| \,\mathrm{d}x.
\end{aligned}$$

Hence, we infer from Cauchy's inequality that

$$\begin{aligned}
\mathbf{F}'(t) &\leq \Big(1 + 2\|\nabla u_K(t,\cdot)\|_{L^\infty(\Sigma)} + \|\nabla\theta_K(t,\cdot)\|_{L^\infty(\Sigma)}\Big)\mathbf{F}(t) \\
&\quad + \frac{\kappa}{4}\|\nabla\theta_K(t,\cdot)\|^2_{L^2(\Sigma)}.
\end{aligned} \tag{100}$$

Similarly as in the proof of Theorem 6.1 above, we may now invoke Theorem 5.1 to deduce the existence of strong solution $u_K \in C^0(0,T;H^3(\Sigma;T\Sigma)) \cap L^2(0,T;H^4(\Sigma;T\Sigma))$ and $\theta_K \in C^0(0,T;H^3(\Sigma))$. Let Λ' denote an upper bound for the indicated norms, modulo a uniform constant. Then, Grönwall's inequality implies

$$\mathbf{F}(t) \leq \Lambda'\kappa.$$

By an interpolation argument, for each $t \in [0,T]$ and $j < 3$ we may now infer

$$\|w(t,\cdot)\|_{H^j(\Sigma)} + \|\chi(t,\cdot)\|_{H^j(\Sigma)} \leq K_{18}\kappa^{\frac{3-j}{6}} + K_{18}(\Lambda'\kappa)^{\frac{1}{2}} \longrightarrow 0 \quad \text{as } \kappa \to 0^+. \tag{101}$$

Here $K_{18} = K(j,\Sigma,\Lambda',\nu)$. Hence the assertion follows. □

7. **Conclusion.** We have studied the Cauchy problem for the Boussinesq equations on a closed surface. By utilising energy methods, we established a group of results concerning the global well-posedness and breakdown criteria of large data classical solutions to the Boussinesq equations with non-degenerate and partially degenerate dissipation. The results appear to be among the first ones concerning the Boussinesq equations on manifolds. The proofs adopt classical approaches for the 2-dimensional Boussinesq equations in Euclidean space, yet are considerably more involved due to the geometric complications which appear in the energy estimates for the higher order derivatives of the solutions.

In passing, we remark that the theorems of this paper appear to remain valid if we take the Bochner Laplacian in place of the Hodge Laplacian in the equations, namely

$$\Delta' := -\nabla^* \nabla$$

in Eq. (1), where ∇^* is the adjoint of the Levi-Civita connection. This is because

$$\Delta u - \Delta' u = \mathrm{Riem} \star u$$

due to the Bochner–Weitzenböck formulae ([43]), hence the only extra terms are of lower order in the energy estimates.

We would also like to remark that the long-time behaviours of the global-in-time solutions constructed in this paper have not been studied. Technically, the proofs for the case of Riemannian manifolds are significantly different from and more difficult than the case of Euclidean space (*cf.* [26, 54, 62]), again due to the geometric complications appearing in higher order energy estimates. In addition, the Boussinesq equations on Riemannian manifolds with fractional dissipation (*i.e.*, fractional Laplace–Beltrami operators $\Lambda_g^\alpha u, \Lambda_g^\alpha \theta$) are also of considerable interests. We leave the investigation for the future.

Appendix. In the Appendix we prove Theorem 3.4. The strategy for the proof is similar to that for the breakdown criterion of the Boussinesq equations on $\mathbb{R}^2$. We adapt the arguments from Chae–Nam [18]; also see Chae–Kim–Nam [19]. These works were motivated in turn by the classical paper [9] due to Beale, Kato and Majda. We need more delicate estimates to account for the non-trivial geometry of Σ. The heart of the matter is the commutator identity (110).

Indeed, we shall establish a more general result:

Theorem 7.1. *Let Σ be a closed surface with Lipschitz curvature, and let $T > 0$ be arbitrary. Let $(u^\circ, \theta^\circ) \in H^m(\Sigma; T\Sigma) \times H^m(\Sigma)$ be initial data satisfying $\operatorname{div} u^\circ = 0$ for some $m > 2$. Assume that (u, θ) is a strong solution to the Boussinesq equations (1)–(4) on $[0, T] \times \Sigma$:*

$$u \in C^0\Big(0, T; H^m(\Sigma, T\Sigma) \cap \mathcal{H}\Big), \qquad \theta \in C^0\Big(0, T; H^m(\Sigma)\Big).$$

We allow either $\kappa \geq 0$ or $\nu \geq 0$ to degenerate. In addition, assume that the $W^{m-2,\infty}$-norms of the curvature and the vector field $\mathbf{e}$ are finite (cf. Assumptions 7.2, 7.3 below).

Then, if

$$\int_0^T \|\nabla \theta(t, \cdot)\|_{L^\infty(\Sigma)} \,\mathrm{d}t < \infty, \tag{102}$$

then the strong solution can be continued to $[0, T + \epsilon[$ for some $\epsilon > 0$.

Theorem 3.4 corresponds to the special case of $m = 3$ in Theorem 7.1.

Proof. We divide the proof into eleven steps. The generic constants A_i depend on the geometry of Σ and the lifespan T, and they remain finite when $\nu, \kappa \to 0^+$.

1. We consider the Boussinesq equations on $[0,T] \times \Sigma$ for a fixed $T > 0$. It is clear that $\|\nabla\theta\|_{L^1(0,T;L^\infty(\Sigma))} = \infty$ is necessary for the blowup of the strong solution in H^m, $m \geq 2+\delta$ for any $\delta > 0$. So, we assume that $\|\nabla\theta\|_{L^1(0,T;L^\infty(\Sigma))} < \infty$ and prove that the strong solution does not blow up before the fixed time T.

2. Let us first recall the L^2-estimate (31) for u:

$$\|u(t,\cdot)\|^2_{L^2(\Sigma)} \leq K_0(T) = e^T\Big(\|u^\circ\|^2_{L^2(\Sigma)} + \|\theta^\circ\|^2_{L^2(\Sigma)}\Big)$$

for any $t \in]0,T]$.

Next, from the vorticity equation (34), reproduced below:

$$\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega + \mathrm{Riem} \star |u|^2 - \mathrm{rot}\,(\theta\mathbf{e}) = 0,$$

we may estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} \leq pR\int_\Sigma |\omega(t,x)|^{p-1}|u(t,x)|^2\,\mathrm{d}x + pS\int_\Sigma \Big(|\nabla\theta(t,x)| + |\theta(t,x)|\Big)|\omega(t,x)|^{p-1}\,\mathrm{d}x. \tag{103}$$

As before, R, S depend on the Lipschitz norm of the curvature and the vector field $\mathbf{e}$, respectively. The estimate of the first term on the right-hand side is the same as (38):

$$\int_\Sigma |\omega(t,x)|^{p-1}|u(t,x)|^2\,\mathrm{d}x \leq A_1\left(\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1\right), \tag{104}$$

where the constant A_1 depends on Σ, p, T and K_0. On the other hand, the last term in Eq. (103) can be treated by Hölder:

$$\int_\Sigma \Big(|\nabla\theta(t,x)| + |\theta(t,x)|\Big)|\omega(t,x)|^{p-1}\,\mathrm{d}x \leq \|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)}\|\theta(t,\cdot)\|_{W^{1,p}(\Sigma)}.$$

Thus, by Young's inequality

$$\int_\Sigma \Big(|\nabla\theta(t,x)| + |\theta(t,x)|\Big)|\omega(t,x)|^{p-1}\,\mathrm{d}x \leq \frac{p-1}{p}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \frac{1}{p}\|\theta(t,\cdot)\|^p_{W^{1,p}(\Sigma)}. \tag{105}$$

Therefore, putting together Eqs. (104)–(105), we can deduce from Eq. (103) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} \leq A_2\Big(1 + \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \|\theta(t,\cdot)\|^p_{W^{1,p}(\Sigma)}\Big). \tag{106}$$

The constant A_2 depends on Σ, S, p, T, $\|u^\circ\|_{L^2(\Sigma)}$ and $\|\theta^\circ\|_{L^2(\Sigma)}$. Note that Eq. (106) remains valid for $\nu = 0$.

3. Now we derive the differential inequality for the $W^{1,p}$-norm of θ. We have proved (see Eq. (28)) that $\|\theta(t,\cdot)\|_{L^p(\Sigma)}$ is non-increasing in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t,\cdot)\|^p_{L^p(\Sigma)} \leq 0. \tag{107}$$

Moreover, taking the covariant derivative of the temperature equation (2) and utilising Lemma 2.1 again, one gets

$$\partial_t(\nabla\theta) + \nabla\theta\cdot\nabla u + u\cdot\nabla\nabla\theta - \kappa\Delta\nabla\theta + \kappa\mathrm{Riem}\star\nabla\theta = 0.$$

Hence, for any $p \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\ \leq p\|\nabla u(t,\cdot)\|_{L^p(\Sigma)}\|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\|\nabla\theta(t,\cdot)\|^{p-1}_{L^p(\Sigma)} + \kappa Rp\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}.$$

By Eq. (37) we may loosely bound

$$\|\nabla u(t,\cdot)\|_{L^p(\Sigma)} \leq c\|\omega(t,\cdot)\|_{L^p(\Sigma)} + c\|u(t,\cdot)\|_{L^\infty(\Sigma)} \\ \leq c\|\omega(t,\cdot)\|_{L^p(\Sigma)} + cK_0^{\frac{p-2}{2p-2}}\|\omega(t,\cdot)\|^{\frac{p}{2p-2}}_{L^p(\Sigma)} + c$$

for $c = c(\Sigma, p)$. By Young's inequality, we thus deduce for $p \geq 2$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)} \\ \leq A_3\Big\{1 + \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + \|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}\Big\}\|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)} \\ + \kappa Rp\|\nabla\theta(t,\cdot)\|^p_{L^p(\Sigma)}, \tag{108}$$

where $A_3 = A(K_0, p, \Sigma)$. Again, $p \geq 2$ is crucial since we need $\frac{p}{2p-2} \leq 1$.

4. Consider the functional

$$\mathbf{E}(t) := \|\theta(t,\cdot)\|^p_{W^{1,p}(\Sigma)} + \|\omega(t,\cdot)\|^p_{L^p(\Sigma)} + 1, \qquad \text{where } p \geq 2.$$

Adding up Eqs. (106), (107) and (108) together, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) \leq A_4\Big(1 + \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\Big)\mathbf{E}(t).$$

The constant A_4 depends on $\Sigma, \|\mathbf{e}\|_{W^{1,\infty}(\Sigma)}$, p, T, $\|u^\circ\|_{L^2(\Sigma)}$, $\|\theta^\circ\|_{L^2(\Sigma)}$ and κ, ν. Therefore, by Grönwall's inequality, for all $t \in [0, T]$ there holds

$$\mathbf{E}(t) \leq A_5\Big(\|u^\circ\|_{H^m(\Sigma)}, \|\theta^\circ\|_{H^m(\Sigma)}, \kappa, \nu, T, p, \Sigma, \int_0^T \|\nabla\theta(\tau,\cdot)\|_{L^\infty(\Sigma)}\,\mathrm{d}\tau\Big), \tag{109}$$

whenever $m \geq 2 + \delta$. Here, for the initial data we utilised the Sobolev–Morrey embedding $H^m(\Sigma) \hookrightarrow W^{1,p}(\Sigma)$.

In particular, A_4 (hence A_5) does not blow up as $\kappa, \nu \to 0^+$.

5. Next we deduce the energy estimates for higher derivatives of u. For our purpose we only need $\|u(t,\cdot)\|_{H^m(\Sigma)}$ with $m = 3$; nevertheless, let us tackle the case of general m, which is of independent interest. It requires higher regularity assumptions for the curvature than $\|Riem\|_{W^{1,\infty}(\Sigma)} \leq R$. In this step we introduce a crucial geometric identity (Eq. (110) below).

Let $I \in \{1,2\}^{|I|}$ be a multi-index of order $|I|$. By an abuse of notations, we sometimes write $\nabla^{|J|} \equiv \nabla^J$. Denote by ∇^I a generic covariant derivative iterated for $|I|$ times. For instance, $\nabla^I u$ is an $(|I|+1)$-form (or equivalently, a multi-vector field of the same valence).

Applying the Ricci identity (Lemma 2.1) for $|I|$ times, we get

$$[\nabla^I, \nabla]u = \sum_{0\leq j\leq |I|-2} C_j \nabla^{|I|-2-j}u \star \nabla^j\,(\mathrm{Riem}). \tag{110}$$

Here the bracket $[\cdot,\cdot]$ is the commutator of differential operators. This identity also holds if we take a scalar function on Σ in place of the vector field u. Therefore, it is natural to require:

Assumption 7.2. $\|Riem\|_{W^{m-2,\infty}(\Sigma)} \leq R_m < \infty$.

6. Applying ∇^I to Eq. (1), we get

$$\begin{aligned}\partial_t\nabla^I u + u\cdot(\nabla^I u) - \nu\Delta\nabla^I u + \nabla^I\nabla P\\ = \nabla^I(\theta\mathbf{e}) + \nu[\nabla^I,\Delta]u - [\nabla^I, u\cdot\nabla]u.\end{aligned} \tag{111}$$

Taking the inner product with $\nabla^I u$ and integrating by parts, the left-hand side of Eq. (111) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^I u(t,\cdot)\|^2_{L^2(\Sigma)} + \int_\Sigma \big\langle \nabla^I u(t,x), \nabla^I\nabla P(t,x)\big\rangle\,\mathrm{d}x.$$

For the second term we first move the innermost derivative on p to the outside (*i.e.*, obtaining $\nabla\nabla^I P$), then move it to $\operatorname{div}\nabla^I u$ via integration by parts, and finally get $\nabla^I \operatorname{div} u$, which vanishes due to incompressibility. The error for this process is quantified by the identity (110):

$$\begin{aligned}&\left|\int_\Sigma \big\langle \nabla^I u(t,x), \nabla^I\nabla P(t,x)\big\rangle\,\mathrm{d}x\right|\\ &\quad\leq CR_m^2\Big\{\sum_{0\leq|J|\leq|I|-2}\big\|\nabla^{|J|}u(t,\cdot)\big\|_{L^2(\Sigma)}\Big\}\Big\{\sum_{0\leq|K|\leq|I|-2}\big\|\nabla^{|K|}P(t,\cdot)\big\|_{L^2(\Sigma)}\Big\},\end{aligned}$$

where $|I|\leq m$ and C is a combinatorial constant depending only on $|I|$.

To proceed, we need to estimate the H^s-norm of P, where $s\leq m-2$. This is postponed to the next step. We further assume

Assumption 7.3. $\|\mathbf{e}\|_{W^{m-2,\infty}(\Sigma)}\leq S_m<\infty$.

Then, in view of the right-hand side of Eq. (111), we need to bound

$$\begin{aligned}&CS_m\int_\Sigma\big|\nabla^I\theta(t,x)\big|\big|\nabla^I u(t,x)\big|\,\mathrm{d}x + CR_m\nu\int_\Sigma\big|\nabla^I u(t,x)\big|\sum_{0\leq|J|\leq|I|-2}\big|\nabla^J u(t,x)\big|\,\mathrm{d}x\\ &\quad + CR_m\int_\Sigma\sum_{0\leq|K|\leq|I|+1}\big|\nabla^{|K|}u(t,x)\big|\,\mathrm{d}x.\end{aligned}$$

Again, $C=C(|I|)$ is a combinatorial constant, and $|I|\leq m$. The terms involving $(|I|+1)$ derivatives of u come from the nonlinear term $[\nabla^I, u\cdot\nabla]u$.

To summarise, we have the following estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^I u(t,\cdot)\|^2_{L^2(\Sigma)} \leq CR_m^2&\Big\{\sum_{0\leq|J|\leq|I|-2}\big\|\nabla^{|J|}u(t,\cdot)\big\|_{L^2(\Sigma)}\Big\}\times\\ &\times\Big\{\sum_{0\leq|K|\leq|I|-2}\big\|\nabla^{|K|}P(t,\cdot)\big\|_{L^2(\Sigma)}\Big\}\\ &+CS_m\int_\Sigma\big|\nabla^I\theta(t,x)\big|\big|\nabla^I u(t,x)\big|\,\mathrm{d}x\\ &+CR_m\nu\int_\Sigma\big|\nabla^I u(t,x)\big|\sum_{0\leq|J|\leq|I|-2}\big|\nabla^J u(t,x)\big|\,\mathrm{d}x\\ &+CR_m\int_\Sigma\sum_{0\leq|K|\leq|I|+1}\big|\nabla^{|K|}u(t,x)\big|\,\mathrm{d}x.\end{aligned}\tag{112}$$

7. Now we control the L^2-norm of the derivatives of P. This is done by the Poisson equation:

$$\Delta P = -\nabla_i \nabla_j (u^i u^j) + \nabla_i (\theta \mathbf{e}^i).$$

Thus, using the standard elliptic estimates for closed manifolds, we have

$$\begin{aligned}\left\|\nabla^{|K|}P(t,\cdot)\right\|_{L^2(\Sigma)} &\leq \left\|u(t,\cdot)\otimes u(t,\cdot)\right\|_{H^{|K|}(\Sigma)} + S_m\left\|\theta(t,\cdot)\right\|_{H^{|K|-1}(\Sigma)}\\ &\leq C\|u(t,\cdot)\|_{H^{|K|}(\Sigma)}\|u(t,\cdot)\|_{L^\infty(\Sigma)} + S_m\left\|\theta(t,\cdot)\right\|_{H^{|K|-1}(\Sigma)}.\end{aligned}$$

The second line follows from an interpolation result due to Morrey; C depends only on $|K|$. However, the L^∞-norm of u is estimated by Eq. (37) once more:

$$\|u(t,\cdot)\|_{L^\infty(\Sigma)} \leq c\Big\{1 + \|u(t,\cdot)\|_{L^2(\Sigma)}^{\frac{p-2}{2p-2}}\|\omega(t,\cdot)\|_{L^p(\Sigma)}^{\frac{p}{2p-2}}\Big\}.$$

In addition, we have already obtained the uniform bound for $\|\omega(t,\cdot)\|_{L^p(\Sigma)}$ in Eq. (109), as well as $\|u(t,\cdot)\|_{L^2(\Sigma)} \leq \sqrt{K_0}$. So, for some $A_6 = A(T, p, \Sigma, \nu, \kappa, \theta^\circ, u^\circ)$ we have

$$\|u(t,\cdot)\|_{L^\infty(\Sigma)} \leq A_6 \qquad \text{for all } t \in [0,T]. \tag{113}$$

Therefore, for each $|K| \leq |I| - 2$, $|I| \leq m$, we can bound the L^2-norm of $|K|$ derivatives of the pressure by lower order energies of u and θ:

$$\left\|\nabla^{|K|}P(t,\cdot)\right\|_{L^2(\Sigma)} \leq C_{|K|}A_6\|u(t,\cdot)\|_{H^{|K|}(\Sigma)} + S_m\left\|\theta(t,\cdot)\right\|_{H^{|K|-1}(\Sigma)}.$$

The constant A_6 remains finite when $\kappa, \nu \to 0^+$. We can continue the estimate Eq. (112) by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^I u(t,\cdot)\right\|_{L^2(\Sigma)}^2 \leq CR_m^2\Big\{&\sum_{0\leq|J|\leq|I|-2}\left\|\nabla^{|J|}u(t,\cdot)\right\|_{L^2(\Sigma)}\Big\}\times\\ &\times\Big\{\sum_{0\leq|K|\leq|I|-2} C_{|K|}A_6\|u(t,\cdot)\|_{H^{|K|}(\Sigma)} + S_m\left\|\theta(t,\cdot)\right\|_{H^{|K|-1}(\Sigma)}\Big\}\\ &+ CS_m\int_\Sigma \left|\nabla^I\theta(t,x)\right|\left|\nabla^I u(t,x)\right|\,\mathrm{d}x\\ &+ CR_m\nu\int_\Sigma \left|\nabla^I u(t,x)\right|\sum_{0\leq|J|\leq|I|-2}\left|\nabla^J u(t,x)\right|\,\mathrm{d}x\\ &+ CR_m\int_\Sigma\sum_{0\leq|K|\leq|I|+1}\left|\nabla^{|K|}u(t,x)\right|\,\mathrm{d}x.\end{aligned} \tag{114}$$

8. Next we derive the higher order energy estimates for θ.

Taking ∇^I to Eq. (2), we get

$$\partial_t\nabla^I\theta + u\cdot\nabla(\nabla^I\theta) - \kappa\Delta\nabla^I\theta = [u\cdot\nabla, \nabla^I]\theta + \kappa[\Delta, \nabla^I]\theta.$$

Thus, by the identity (110) again, we obtain the following energy estimate:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^I\theta(t,\cdot)\right\|_{L^2(\Sigma)}^2\\ &\quad\leq CR_m\int_\Sigma\sum_{1\leq|J|\leq|I|}\left|\nabla^J u(t,x)\right|\,\mathrm{d}x\\ &\qquad+ C\kappa R_m\int_\Sigma\sum_{0\leq|L|\leq|I|-2}\left|\nabla^L\theta(t,x)\right|\,\mathrm{d}x.\end{aligned} \tag{115}$$

Once more, the constant C depends only on $|I|$.

9. Let us denote by

$$\mathbf{E}_s(t) := 1 + \sum_{0\leq |I| \leq s} \int_\Sigma \Big\{ |\nabla^I u(t,x)|^2 + |\nabla^I \theta(t,x)|^2 \Big\}\, \mathrm{d}x. \tag{116}$$

Eqs. (114)-(115) yield the following: for any $|I| \leq m$ (where $m \geq 2+\delta$ as before), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{|I|} &\leq CR_m^2 A_6 \mathbf{E}_{|I|-2} + CR_m^2 S_m^2 \sqrt{\mathbf{E}_{|I|-2}}\sqrt{\mathbf{E}_{|I|-3}} + CS_m \mathbf{E}_{|I|} \\ &\quad + CR_m \nu \sqrt{\mathbf{E}_{|I|}}\sqrt{\mathbf{E}_{|I|-2}} + CR_m \|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)} \mathbf{E}_{|I|} \\ &\quad + CR_m \sqrt{\operatorname{Vol}\Sigma}\, \mathbf{E}_{|I|} + C\kappa R_m \sqrt{\operatorname{Vol}\Sigma}\, \mathbf{E}_{|I|-2}.\end{aligned}$$

Here we use Cauchy–Schwarz, and C depends only on $|I|$. The $\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)}$ term comes from $\int_\Sigma \sum_{0\leq|K|\leq|I|+1} \left|\nabla^{|K|} u(t,x)\right| \mathrm{d}x$ in the final line of Eq. (114).

Thus, there is a constant

$$A_7 = A\Big(m, \Sigma, T, R_m, S_m, \nu, \kappa, \|u^\circ\|_{H^m(\Sigma)}, \|\theta^\circ\|_{H^m(\Sigma)}\Big)$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{|I|} \leq A_7 \Big(1 + \|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)}\Big) \mathbf{E}_{|I|}(t). \tag{117}$$

This can be seen from a simple induction; notice that $\mathbf{E}_1 \equiv \mathbf{E}$ is already bounded in Step 4 (with $p=2$ therein).

10. It thus remains to bound $\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)}$. This is done by the Brezis–Wainger inequality (Lemma 2.2). From now on we restrict to $p>2$; then

$$\begin{aligned}&\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)} \\ &\quad \leq c\Big(1 + \|\omega(t,\cdot)\|_{L^\infty(\Sigma)}\Big)\Big(1 + \log_+ \|u(t,\cdot)\|_{H^m(\Omega)}\Big) + c\|\omega(t,\cdot)\|_{L^p(\Sigma)}.\end{aligned} \tag{118}$$

for a geometric constant $c>0$. The L^p-norm of ω is again already bounded in Step 4. Moreover, a bound for the L^∞-norm of ω can be directly obtained from Eq. (103) above, reproduced here:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} &\leq pR \int_\Sigma |\omega(t,x)|^{p-1} |u(t,x)|^2\, \mathrm{d}x \\ &\quad + pS \int_\Sigma \Big(|\nabla\theta(t,x)| + |\theta(t,x)|\Big)|\omega(t,x)|^{p-1}\, \mathrm{d}x.\end{aligned}$$

Indeed, noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^p_{L^p(\Sigma)} = p\|\omega(t,\cdot)\|^{p-1}_{L^p(\Sigma)} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|_{L^p(\Sigma)}$$

and invoking the L^∞-bound for u, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|_{L^p(\Sigma)} \leq c\|u(t,\cdot)\|^2_{L^\infty(\Sigma)} + c\|\theta(t,\cdot)\|_{W^{1,\infty}(\Sigma)}, \tag{119}$$

where c depends on R and S. We have already bounded $\|u(t,\cdot)\|_{L^\infty(\Sigma)} \leq A_6$ in Eq. (113). There A_6 depends on some index p', but we can fix p' once and for all.

Sending $p\to\infty$ in Eq. (119), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|_{L^\infty(\Sigma)} \leq A_7\Big(1 + \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\Big).$$

Here we used $\|\theta(t,\cdot)\|_{L^\infty(\Sigma)} \leq \|\theta^\circ\|_{L^\infty(\Sigma)}$ for all t; A_7 is a constant independent of p (and has the same dependence as A_6 apart from p).

Therefore, integrating this differential inequality, we get

$$\|\omega(t,\cdot)\|_{L^\infty(\Sigma)} \leq A_8\Big(\Sigma,\nu,\kappa,T,\int_0^T \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\,\mathrm{d}t, \|u^\circ\|_{H^m(\Sigma)}, \|\theta^\circ\|_{H^m(\Sigma)}\Big) \tag{120}$$

for all $t \in [0,T]$. Here A_8 does not blow up as $\nu,\kappa \to 0$.

As a result, we deduce from Eqs. (118) and (120) that, for arbitrary $p'' > 2$,

$$\|\nabla u(t,\cdot)\|_{L^\infty(\Sigma)} \leq A_9\Big(1+\log_+\|u(t,\cdot)\|_{H^m(\Omega)}\Big), \tag{121}$$

where

$$A_9 = A\Big(p'',\Sigma,\nu,\kappa,T,\int_0^T \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\,\mathrm{d}t, \|u^\circ\|_{H^m(\Sigma)}, \|\theta^\circ\|_{H^m(\Sigma)}\Big).$$

We can also fix p'' once and for all.

11. Finally, by Eqs. (121) and (117), for any $0 \leq |I| \leq m$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_m(t) \leq A_{10}\Big(1+\log_+\mathbf{E}_m(t)\Big)\mathbf{E}_m(t), \tag{122}$$

where A_{10} depends on Σ, ν, κ, R_m, S_m, T, m, $\|u^\circ\|_{H^m(\Sigma)}$, $\|\theta^\circ\|_{H^m(\Sigma)}$, and $\int_0^T \|\nabla\theta(t,\cdot)\|_{L^\infty(\Sigma)}\,\mathrm{d}t$, and it remains finite even if we send κ,ν to 0.

Integrate this differential inequality, one obtains:

$$\mathbf{E}_m(t) \leq \exp\left\{\Big(1+\log_+\mathbf{E}_m(0)\Big)e^{A_{10}t}\right\} < \infty. \tag{123}$$

Thus we can further continue the strong solution in time. The proof is complete. □

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