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Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation

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The stability and large-time behavior problem on some partially dissipated systems is not well-understood. The vorticity gradient of the 2D incompressible Euler equation can grow double exponentially in time while the same quantity to the 2D Navier-Stokes equation decays algebraically in time. However, the stability and large-time behavior of the vorticity gradients of the 2D Navier-Stokes equation with only vertical or horizontal dissipation appears to be unknown. This paper presents a global stability result on perturbations near a background magnetic field to the 2D incompressible magnetohydrodynamic (MHD) equations with vertical dissipation and horizontal magnetic diffusion. This stability result provides a significant example for the stabilizing effects of the magnetic field on electrically conducting fluids. In addition, we obtain an explicit decay rate for the solution of this nonlinear system.

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1. Introduction

This paper establishes the stability and large-time behavior of perturbations near a background magnetic field of the 2D MHD equations with vertical dissipation and horizontal magnetic diffusion. The motivation for this study is twofold. First, there are extensive experimental and numerical investigations on the influence of an external magnetic field on the behavior of electrically conducting fluids (see, e.g., [2–5,13,14]). Mathematically rigorous stability results would help gain insight into these rich numerical and experimental observations. Second, existing tools and techniques designed for fully dissipative partial differential equations (PDEs) may not be applicable to models with only partial dissipation. This study aims at developing new approaches and methods that are effective for partially dissipated systems. PDEs with only partial dissipation arise naturally in the modeling of many phenomena and there have been substantial recent developments on some of the most fundamental problems concerning these PDEs. This paper focuses on a very important partial dissipation case of the 2D incompressible MHD equations.

The MHD equations govern the motion of electrically conducting fluids in the presence of a magnetic field such as plasmas, liquid metals and electrolytes (see, e.g., [8,19]). They are the centerpiece of the magneto-hydrodynamics initiated by Hannes Alfvén [4]. The MHD equations are a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electro-magnetism. Besides their wide physical applicability, the MHD equations are also of great interest in mathematics. As a coupled system, the MHD equations contain much richer structures than the Navier-Stokes equations. They are not merely a combination of two parallel Navier-Stokes type equations but an interactive and integrated system. Their distinctive features make analytic studies a great challenge but offer new opportunities.

Foundational work on the well-posedness of the fully dissipative MHD equations has long been laid by Duvaut and Lions [12] and Sermange and Temam [26]. There are substantial recent developments on the global well-posedness problem as well as on the stability problem for the MHD equations with only partial or fractional dissipation. Some of the results can be found in a recent review paper [30]. This paper aims at the stability problem on the following 2D MHD equations with vertical dissipation and horizontal magnetic diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u = \nu\,\partial_{22} u - \nabla P + B\cdot\nabla B, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t B + u\cdot\nabla B = \eta\,\partial_{11} B + B\cdot\nabla u, \quad x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = \nabla\cdot B = 0, \quad x\in\mathbb{R}^2,\ t>0, \end{cases} \tag{1.1}$$

where u represents the velocity field of the fluid, B the magnetic field and P the pressure, and $\nu > 0$ denotes the kinematic viscosity and $\eta > 0$ the magnetic diffusivity. For notational convenience, we have written ∂_1 and ∂_2 for the partial derivatives ∂_{x_1} and ∂_{x_2}, respectively. The global existence and regularity problem on (1.1) was successfully solved by Cao and Wu [7]. Their result states that any initial data $(u_0, B_0) \in H^2(\mathbb{R}^2)$ yields a unique global solution (u, B) of (1.1) satisfying, for any $T > 0$,

$$(u, B) \in L^\infty(0, T; H^2(\mathbb{R}^2)), \quad \partial_2 u,\, \partial_1 B \in L^2(0, T; H^2(\mathbb{R}^2)).$$

In spite of this result, many interesting problems on (1.1) remain open. Among them are the stability problem on perturbations near the trivial steady state $(u, B) \equiv (0, 0)$ and their precise large-time behavior. The upper bound for $\|(u(t), B(t))\|_{H^2}$ in [7] is obtained via Gronwall's inequality and depends exponentially on t. Here for simplicity, we have written $\|(u(t), B(t))\|_{H^2}^2$ for $\|u(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2$. A different approach is needed in order to understand the stability and large-time behavior problem on (1.1). It is worth mentioning a very interesting special case of (1.1). When $B \equiv 0$, (1.1) reduces to the 2D Navier-Stokes equations with only vertical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = \nu\, \partial_{22} u - \nabla P, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

The H^2-stability problem on perturbations near the trivial solution $u \equiv 0$ of (1.2) appears to be open. In particular, the precise large-time behavior of $\nabla\omega(t)$ is unknown, where $\omega = \nabla \times u$ denotes the corresponding vorticity. When $\nu = 0$, (1.2) becomes the 2D Euler equation and $\nabla\omega(t)$ could grow double exponentially, according to several beautiful work (see [10,18,37]). In contrast, when $\partial_{22} u$ is replaced by the full dissipation Δu, then $\nabla\omega(t)$ actually decays algebraically (see, e.g., [25]).

The focus of this paper is the stability of perturbations near a background magnetic field. Clearly, $(u^{(0)}, B^{(0)})$ with

$$u^{(0)} \equiv 0, \quad B^{(0)} \equiv e_1 := (1, 0)$$

is a steady solution of (1.1). The perturbation (u, b) with

$$b := B - B^{(0)}$$

solves the MHD system

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nu\, \partial_{22} u - \nabla P + (b \cdot \nabla) b + \partial_1 b, & x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b + (u \cdot \nabla) b = \eta\, \partial_{11} b + (b \cdot \nabla) u + \partial_1 u, & x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, & x \in \mathbb{R}^2,\ t > 0. \end{cases} \tag{1.3}$$

(1.3) supplemented with the initial data

$$u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x)$$

will be the focus in the rest of our paper. (1.3) differs from (1.1) only by two extra terms $\partial_1 b$ and $\partial_1 u$. These two terms seemingly would make no difference on the stability problem, but they do give us more maneuver. In comparison with (1.2), the magnetic field in (1.3) actually stabilizes the fluid. This can be seen from the linearization of (1.3),

$$\begin{cases} \partial_t u = \nu\, \partial_{22} u + \partial_1 b, \\ \partial_t b = \eta\, \partial_{11} b + \partial_1 u. \end{cases} \tag{1.4}$$

Differentiating (1.4) in time and making suitable substitutions, we can convert (1.4) into

$$\begin{cases} \partial_{tt} u - (\nu\partial_{22} + \eta\partial_{11})\partial_t u + \nu\eta\partial_{1122} u - \partial_{11} u = 0, \\ \partial_{tt} b - (\nu\partial_{22} + \eta\partial_{11})\partial_t b + \nu\eta\partial_{1122} b - \partial_{11} b = 0. \end{cases} \tag{1.5}$$

(1.5) provides much more regularization and damping effect than the linearized part of (1.2), namely $\partial_t u = \nu\, \partial_{22} u$ can give. It is the stabilization and regularization effect of the magnetic field that allows us to establish the global stability for (1.3). More precisely, we obtain the following theorem.

Theorem 1.1. *Suppose that* $(u_0, b_0) \in H^2(\mathbb{R}^2)$ *satisfies* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$*. Then there exists* $\delta = \delta(\nu, \eta) > 0$ *such that, if*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \delta, \tag{1.6}$$

then (1.3) has a unique global solution $(u, b) \in C\big([0, \infty); H^2(\mathbb{R}^2)\big)$ *satisfying*

$$\|(u(t), b(t))\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u(s)\|_{H^2}^2\, ds + 2\eta \int_0^t \|\partial_1 b(s)\|_{H^2}^2 ds \leq C\delta^2$$

for any $t > 0$ *and some uniform constant* C*.*

We explain the challenges we encounter in the proof of Theorem 1.1. The framework of the proof is the bootstrap argument. The centerpiece is the following global energy inequality, for any $t > 0$,

$$E(t) \leq E(0) + C_1\, E(0)^{\frac{3}{2}} + C_2\, E(t)^{\frac{3}{2}} + C_3\, E(t)^2, \tag{1.7}$$

where C_1, C_2 and C_3 are positive constants, and

$$\begin{aligned} E(t) = {} & \sup_{0 \leq \tau \leq t} \Big(\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2 \Big) \\ & + 2\nu \int_0^t \|\partial_2 u(s)\|_{H^2}^2\, ds + 2\eta \int_0^t \|\partial_1 b(s)\|_{H^2}^2\, ds. \end{aligned} \tag{1.8}$$

Once (1.7) is established, an application of the bootstrap argument would imply the desired global stability. The details are given in Section 2. The main efforts are devoted to proving (1.7). Due to the presence of the anisotropic dissipation and magnetic diffusion, we make use of anisotropic estimates for triple products and for the L^∞-norms involving 2D functions (see Lemma 2.1 and Lemma 2.2 in Section 2). The global H^1-bound is relatively easy to obtain compared to the estimates of the second-order derivatives, which is difficult and extremely tedious. To illustrate the difficulty, we consider the 2D Navier-Stokes equation with only vertical dissipation again, namely

$$\partial_t u + u\cdot\nabla u = \nu\partial_{22}u - \nabla P, \quad \nabla\cdot u = 0. \tag{1.9}$$

As aforementioned, in contrast to the 2D Navier-Stokes equations with full dissipation, the problem of whether or not $u_0 \in H^2$ (even small) leads to a global solution u with a uniform-in-time H^2-bound remains open. The difficulty is how to obtain a uniform bound for the L^2-norm of the second-order derivatives of u, or equivalently, the first-order derivative of the corresponding vorticity $w = \nabla\times u$, which obeys

$$\partial_t w + u\cdot\nabla w = \nu\partial_{22}w.$$

If we estimate $\nabla\omega$ in L^2 by the energy method, namely

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \nu\|\partial_2\nabla\omega\|_{L^2}^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx,$$

the trouble is then how to bound the first two terms in

$$\begin{aligned}&\int_{\mathbb{R}^2} \nabla w\cdot\nabla u\cdot\nabla w\,dx \qquad (1.10)\\ &:= \int_{\mathbb{R}^2} \partial_1 u_1\,(\partial_1 w)^2\,dx + \int_{\mathbb{R}^2} \partial_1 u_2 \partial_1 w\,\partial_2 w\,dx + \int_{\mathbb{R}^2} \partial_2 u\cdot\nabla w\partial_2 w\,dx.\end{aligned}$$

Due to the lack of the horizontal dissipation, we do not know how to obtain a suitable bound for the first two terms. As a consequence, the H^2-stability problem for (1.9) is open. This is also why the global stability problem for (1.1) remains open. (1.3) differs from (1.1) only in two terms $\partial_1 b$ and $\partial_1 u$, but they give us extra maneuver. We explain why we can bound the H^2-norm of solutions to (1.3) suitably. To control the homogeneous H^2-norm, we still have to bound the same term as in (1.10). Bounding the first two terms in (1.10) is still highly non-trivial, but we can replace $\partial_1 u_1$ and $\partial_1 u_2$ via the second equation in (1.3), namely

$$\partial_1 u = \partial_t b + (u\cdot\nabla)b - \eta\,\partial_{11}b - (b\cdot\nabla)u.$$

With this substitution, the first term in (1.10) is then given by

$$\int_{\mathbb{R}^2} \partial_1 u_1 \, (\partial_1 w)^2 \, dx = \int (\partial_t b_1 + (u\cdot\nabla) b_1 - \eta \partial_1^2 b_1 - (b\cdot\nabla) u_1)(\partial_1 w)^2 dx \tag{1.11}$$

This makes the estimating process more complicated, but this substitution does allow us to obtain a suitable bound for this seemingly impossible term. The actual process is very tedious. For example, in order to handle the first term in (1.11), we further integrate by parts and use the equation of w. That is,

$$\int \partial_t b_1 (\partial_1 w)^2 dx = \frac{d}{dt}\int b_1 \ (\partial_1 w)^2 dx - 2\int b_1 \partial_1 w \ \partial_t \partial_1 w dx$$

and

$$\int b_1 \partial_1 w \partial_t \partial_1 w \, dx = \int b_1 \partial_1 w \Big[-\partial_1 (u\cdot \nabla w) + \nu \partial_2^2 \partial_1 w + \partial_1 (b\cdot\nabla j) + \partial_1^2 j \Big] dx.$$

These substitutions allow us to rewrite the seemingly impossible term $\int \partial_t b_1 (\partial_1 w)^2 dx$ into eight terms. The details of how these terms are further controlled are complex and left to Section 3.

The global stability result stated in Theorem 1.1 is among one of the very few stability results on the ideal or partially dissipated MHD equations that are currently available. The stability problem on the ideal MHD equations near a background magnetic field was successfully solved by several celebrated papers [5,6,15,29]. The stability problem for the MHD equations with no magnetic diffusion was first studied in [20], which inspired many further investigations. The stability has now been successfully established by several authors via different approaches (see, e.g., [1,9,16,17,20–23,27,31,32,34–36]). Very recently, Wu and Zhu were able to solve the stability problem for the 3D MHD equations with horizontal dissipation and vertical magnetic diffusion [33].

The second goal of this paper is to obtain the precise large-time behavior of the solution established in Theorem 1.1. It is generally understood that, for a partially dissipated system, we need to assume that the initial data is in Sobolev spaces of suitable negative indices in order to obtain the precise decay rates of the solutions. In addition, for a system with degenerate dissipation such as (1.3), we need to make higher regularity assumption on the initial data in order to obtain the decay rates on the H^2-norm of the solution. Here we assume $(u_0, b_0) \in H^4(\mathbb{R}^2)$, which appears to be necessary. We recall the definition of the fractional Laplacian, for any real number β and $k = 1, 2$,

$$\widehat{\Lambda_k^{\beta} f}(\xi) = |\xi_k|^{\beta} \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2).$$

We use $\|f\|_{L^p_{x_k}}$ with $k = 1, 2$ to denote the L^p-norm with respect to x_k only, and

$$\|f\|_{L^p_{x_k} L^q_{x_m}} := \| \|f\|_{L^q_{x_m}} \|_{L^p_{x_k}}.$$

The decay estimate can then be stated as follows.

Theorem 1.2. *Let $(u_0, b_0) \in H^4(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that, for some sufficiently small constant $\delta > 0$,*

$$\|u_0\|_{H^4} \le \delta, \qquad \|b_0\|_{H^4} \le \delta, \tag{1.12}$$

and, for $\sigma \ge 5/2$, $k = 1, 2$ and $m = 0, 1, 2$,

$$\begin{aligned} &\|\partial_k^m \Lambda_1^{-\sigma} u_0\|_{L^2_{x_2} L^1_{x_1}} \le \delta, \qquad \|\partial_k^m \Lambda_2^{-\sigma} u_0\|_{L^2_{x_1} L^1_{x_2}} \le \delta, \\ &\|\partial_k^m \Lambda_1^{-\sigma} b_0\|_{L^2_{x_2} L^1_{x_1}} \le \delta, \qquad \|\partial_k^m \Lambda_2^{-\sigma} b_0\|_{L^2_{x_1} L^1_{x_2}} \le \delta. \end{aligned} \tag{1.13}$$

Let (u, b) be the corresponding global solution of (1.3) obtained in Theorem 1.1. Then (u, b) obeys the following decay estimate

$$\|u(t)\|_{H^2} + \|b(t)\|_{H^2} \le C\, \delta\, (1+t)^{-\frac{1}{4} - \frac{\sigma}{2}},$$

where C is a pure constant independent of δ and t.

Due to the degeneracy in the viscous dissipation and also in the magnetic diffusion, classical approaches such as Schonbek's Fourier splitting method [24,25] no longer apply. Furthermore, since the velocity equation in (1.3) contains the linear term $\partial_1 b$ while the equation of b contains $\partial_1 u$, it appears to be fruitless to rewrite each of the equations in (1.3) in an integral form via the one-dimensional heat operator. The efficient approach is to separate the linear terms in (1.3) from the nonlinear ones, solve the linearied system and then represent the full system into an integral form via the Duhamel's principle. The desired decay estimate are based on this integral representation and obtained through the bootstrapping argument. We reveal the key steps here but leave the details to the proof of Theorem 1.2.

Taking the Fourier transform of (1.3), we find

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{M_1} \\ \widehat{M_2} \end{pmatrix}, \tag{1.14}$$

where we have projected the velocity equation onto the divergence-free vector fields

$$A = \begin{pmatrix} -\nu \xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta \xi_1^2 \end{pmatrix}, \qquad M_1 = \mathbb{P}(b \cdot \nabla b - u \cdot \nabla u), \qquad M_2 = b \cdot \nabla u - u \cdot \nabla b.$$

Here $\mathbb{P}$ is the Leray projection onto the divergence-free vector fields. By Duhamel's principle, (1.14) can be represented in the integral form

$$\begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{M_1}(\tau) \\ \widehat{M_2}(\tau) \end{pmatrix} d\tau.$$

The spectra of A, given by

$$\lambda_1 = \frac{-(\nu\xi_2^2 + \eta\xi_1^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\nu\xi_2^2 + \eta\xi_1^2) + \sqrt{\Gamma}}{2}$$

with

$$\Gamma := (\nu\xi_2^2 + \eta\xi_1^2)^2 - 4\nu\eta\xi_1^2\,\xi_2^2 - 4\xi_1^2 = (\nu\xi_2^2 - \eta\xi_1^2)^2 - 4\xi_1^2$$

play a crucial role in the large-time behavior. Clearly, they are both anisotropic and strongly frequency dependent. By computing the corresponding eigenvectors, we can make the representation more explicit, namely

$$\widehat{u}(\xi,t) = \widehat{Q}_1(t)\widehat{u}_0 + \widehat{Q}_2(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_1(t-\tau)\widehat{M_1}(\tau) + \widehat{Q}_2(t-\tau)\widehat{M_2}(\tau)\big)\,d\tau,$$

$$\widehat{b}(\xi,t) = \widehat{Q}_2(t)\widehat{u}_0 + \widehat{Q}_3(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_2(t-\tau)\widehat{M_1}(\tau) + \widehat{Q}_3(t-\tau)\widehat{M_2}(\tau)\big)\,d\tau.$$

The kernel functions $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ are all explicit in terms of λ_1 and λ_2,

$$\widehat{Q}_1(t) = -\nu\xi_2^2 G_1 + G_2, \quad \widehat{Q}_2(t) = i\xi_1 G_1, \quad \widehat{Q}_3(t) = \nu\xi_2^2 G_1 + G_3$$

and

$$G_1 = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad G_2 = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1,$$
$$G_3 = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} + \lambda_2 G_1 = e^{\lambda_2 t} + \lambda_1 G_1.$$

Due to the degeneracy in the viscous dissipation and in the magnetic diffusion, $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ are very anisotropic and their behavior depends strongly on the Fourier frequency. By dividing the Fourier space into sub-domains, we can make the behavior of $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ in each sub-domain definite and transparent. The detailed division into subdomains and the precise behavior of these kernel functions are provided in Proposition 4.1.

The proof of Theorem 1.2 starts with the proof that, when (1.12) holds, the solution (u, b) in H^4 remains uniformly small, namely

$$\|(u(t), b(t))\|_{H^4} \le C\,\delta$$

for a constant C independent of δ. The proof of the desired decay estimate is obtained via the bootstrapping argument applied to the integral representation of u and b. This argument starts with the ansatz that

$$\|u(t)\|_{H^2} \le C_0\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \qquad \|b(t)\|_{H^2} \le C_0\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \tag{1.15}$$

where C_0 is a suitably selected pure constant and will be specified in the proof of Theorem 1.2. We then show by using the ansatz in (1.15) and the integral representation of u and b that

$$\|u(t)\|_{H^2} \le \frac{C_0}{2}\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \qquad \|b(t)\|_{H^2} \le \frac{C_0}{2}\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}. \tag{1.16}$$

Then the bootstrapping argument would imply that (1.16) indeed holds and the desired decay estimate is then achieved. The proof for (1.16) is technical and involves many optimal estimates in order to make up for the loss of decay due to the anisotropicity and the inhomogeneity of the kernel functions. The details are provided in Section 4.

The rest of this paper is divided into three sections. Section 2 applies the bootstrap argument to prove Theorem 1.1 and prepares several anisotropic inequalities to be used subsequently. Section 3 proves the major estimate in (1.7). Section 4 presents the proof of Theorem 1.2.

2. Proof of Theorem 1.1 and anisotropic estimates

This section applies the bootstrap argument to prove Theorem 1.1. In addition, we provide several anisotropic inequalities to be used in the proof of (1.7) in the subsequent section.

Proof of Theorem 1.1. Roughly speaking, the bootstrap argument starts with an ansatz that $E(t)$ is bounded, say

$$E(t) \le M$$

and shows that $E(t)$ actually admits a smaller bound, say

$$E(t) \le \frac{1}{2}M$$

when the initial condition is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [28, p.21]). To apply the bootstrap argument to (1.7), we assume that

$$E(t) \le M := \min\left\{\frac{1}{16C_2^2}, \frac{1}{4C_3}\right\}. \tag{2.1}$$

When (2.1) holds, we have

$$C_2 E^{\frac{1}{2}}(t) \le \frac{1}{4} \quad \text{and} \quad C_3 E(t) \le \frac{1}{4}.$$

It then follows from (1.7) that

$$E(t) \leq E(0) + C_1 E(0)^{\frac{3}{2}} + \frac{1}{2} E(t) \quad \text{or} \quad E(t) \leq 2E(0) + 2C_1 E(0)^{\frac{3}{2}}. \tag{2.2}$$

If we choose $\delta > 0$ sufficiently small such that

$$\delta^2 + C_1 \delta^3 \leq \frac{M}{4},$$

then (1.6) and (2.2) imply that

$$E(t) \leq \frac{1}{2} M.$$

The bootstrap argument then leads to the desired global bound

$$E(t) \leq \frac{1}{2} M := \frac{1}{2} \min \left\{ \frac{1}{16C_2^2}, \frac{1}{4C_3} \right\}.$$

This completes the proof of Theorem 1.1. □

The rest of this section provides several anisotropic Sobolev type inequalities. The MHD system examined in this paper has anisotropic dissipation. Anisotropic inequalities appears to be necessary to deal with such partially dissipated systems. The following anisotropic inequality introduced in [7] has proven to be extremely useful.

Lemma 2.1. *Assume $f, g, h, \partial_1 g, \partial_2 h \in L^2(\mathbb{R}^2)$. Then, for a constant $C > 0$,*

$$\iint fgh\, dx_1 dx_2 \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|\partial_1 g\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|\partial_2 h\|_2^{\frac{1}{2}}.$$

We prove two additional anisotropic Sobolev inequalities to be used in the subsequent section.

Lemma 2.2. *Suppose that $v, \partial_1 v, \partial_2 v$ are all in $H^1(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\|v\|_\infty \leq C \|v\|_{H^1}^{\frac{1}{2}} \|\partial_1 v\|_{H^1}^{\frac{1}{2}}, \quad \|v\|_{L^\infty} \leq C \|v\|_{H^1}^{\frac{1}{2}} \|\partial_2 v\|_{H^1}^{\frac{1}{2}}.$$

Proof. To prove the first inequality, we start with the elementary bound

$$\|v\|_{L^\infty_{x_1}} \leq \sqrt{2} \|v\|_{L^2_{x_1}}^{\frac{1}{2}} \|\partial_{x_1} v\|_{L^2_{x_1}}^{\frac{1}{2}}.$$

By Minkowski's inequity, Hölder's inequality and Sobolev's inequity $\|v\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{H^1(\mathbb{R})}$,

$$
\begin{aligned}
\|v\|_\infty &= \Big\| \|v\|_{L^\infty_{x_1}} \Big\|_{L^\infty_{x_2}} \le \sqrt{2} \Big\| \|v\|^{\frac12}_{L^2_{x_1}} \|\partial_1 v\|^{\frac12}_{L^2_{x_1}} \Big\|_{L^\infty_{x_2}} \\
&\le C \Big\| \|v\|_{L^\infty_{x_2}} \Big\|^{\frac12}_{L^2_{x_1}} \Big\| \|\partial_1 v\|_{L^\infty_{x_2}} \Big\|^{\frac12}_{L^2_{x_1}} \\
&\le C \Big\| \|v\|_{H^1_{x_2}} \Big\|^{\frac12}_{L^2_{x_1}} \Big\| \|\partial_1 v\|_{H^1_{x_2}} \Big\|^{\frac12}_{L^2_{x_1}} \\
&\le C \|v\|^{\frac12}_{H^1} \|\partial_1 v\|^{\frac12}_{H^1}.
\end{aligned}
$$

This completes the proof of the first inequality. The proof for the second one is similar. This completes the proof of Lemma 2.2. □

3. Proof of (1.7)

This section proves the major estimate in (1.7), namely

$$E(t) \le E(0) + C_1\, E(0)^{\frac32} + C_2\, E(t)^{\frac32} + C_3 E^2(t),$$

where $E(t)$ is defined in (1.8). The core of the proof is to bound the H^2-norm of (u, b) suitably. This process involves many terms and is very lengthy. For the sake of clarity, we divide the whole proof into two parts. The first part estimates the H^1-norm of (u, b) while the second part works with the second-order derivatives of (u, b). The H^1-estimates are relatively simple, but the estimates of the second-order derivatives are extremely complex. They involve taking advantage of the special structure of (1.3), making quite a few substitutions and repeatedly applying anisotropic inequalities provided in the previous section.

In what follows we will use the notation $\|\nabla^2 u\|_2$ defined by

$$\|\nabla^2 u\|_2^2 := \sum_{i,j,k=1}^{2} \|\partial_i \partial_j u_k\|_2^2.$$

In addition, the following simple facts will also be used repeatedly. For $\nabla \cdot u = \nabla \cdot b = 0$, $w = \nabla \times u$ and $j = \nabla \times b$,

$$
\begin{aligned}
&\|\nabla u\|_2 = \|w\|_2, \quad \|\partial_2 \nabla u\|_2 = \|\partial_2 w\|_2, \\
&\|\nabla^2 u\|_2 = \|\Delta u\|_2 = \|\nabla w\|_2, \quad \|\partial_2 \nabla^2 u\|_2 = \|\partial_2 \Delta u\|_2 = \|\partial_2 \nabla w\|_2, \\
&\|\nabla b\|_2 = \|j\|_2, \ \ \|\partial_1 \nabla b\|_2 = \|\partial_1 j\|_2, \\
&\|\nabla^2 b\|_2 = \|\Delta b\|_2 = \|\nabla j\|_2, \quad \|\partial_1 \nabla^2 b\|_2 = \|\partial_1 \Delta b\|_2 = \|\partial_1 \nabla j\|_2.
\end{aligned}
$$

These identities can be easily proven by integration by parts. The rest of this section is divided into two subsections with the first devoted to the H^1 bound while the second to the second-order derivatives.

3.1. The H^1-estimate

We estimate $\|(u,b)\|_{H^1}$. First we have the L^2-estimate

$$\|(u,b)(t)\|_2^2 + 2\nu \int_0^t \|\partial_2 u(s)\|_2^2 ds + 2\eta \int_0^t \|\partial_1 b(s)\|_2^2 ds = \|(u_0,b_0)\|_2^2, \tag{3.1}$$

which follows from the inner product of (1.3) with (u,b) and integration by parts. To estimate $\|\nabla u\|_2$ and $\|\nabla b\|_2$, we will resort to the equations of the vorticity $w = \nabla \times u$ and the current density $j = \nabla \times b$,

$$\begin{cases} \partial_t w + (u\cdot\nabla)w = \nu\partial_{22}w + (b\cdot\nabla)j + \partial_1 j, \\ \partial_t j + (u\cdot\nabla)j = \eta\partial_{11}j + (b\cdot\nabla)w + \partial_1 w + Q, \end{cases} \tag{3.2}$$

where

$$Q = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1).$$

Taking the L^2 inner product of (3.2) with (w,j) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(w,j)\|_2^2 + \nu\|\partial_2 w\|_2^2 + \eta\|\partial_1 j\|_2^2 = \int Qj\,dx \\ &= 2\int \left[\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)\ j - \partial_1 u_1 \partial_1 b_2 j\right] dx - 2\int \partial_1 u_1 \partial_2 b_1 j\,dx, \end{aligned} \tag{3.3}$$

where we have used the facts

$$\int (b\cdot\nabla)j\ w\,dx + \int (b\cdot\nabla)w\ j\,dx = 0, \quad \int \partial_1 j\ w\,dx + \int \partial_1 w\ j\,dx = 0.$$

By Lemma 2.1, the first integral in (3.3) can be bounded by

$$\begin{aligned} &\int 2\left[\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)\ j - \partial_1 u_1 \partial_1 b_2 j\right] dx \\ &\le 2\int |\partial_1 b_1||\nabla u||j|\,dx + 2\int |\partial_1 u_1||\partial_1 b_2||j|\,dx \\ &\le C\|\partial_1 b_1\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\partial_2 \nabla u\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_1 j\|_2^{\frac{1}{2}} \\ &\quad + C\|\partial_2 u_2\|_2 \|\partial_1 b_2\|_2^{\frac{1}{2}} \|\partial_2\partial_1 b_2\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_1 j\|_2^{\frac{1}{2}} \\ &\le C(\|\nabla u\|_2 + \|j\|_2)(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^1}^2) \\ &\le C(\|\nabla u\|_2 + \|j\|_2)V(t), \end{aligned} \tag{3.4}$$

where, for notational convenience, we have written

$$V(t) := 2\nu\|\partial_2 u(t)\|_{H^2}^2 + 2\eta\|\partial_1 b(t)\|_{H^2}^2.$$

In order to bound the second integral term, we first rewrite it by inserting $j = \partial_1 b_2 - \partial_2 b_1$, and then integrate by parts in the second term and apply Lemma 2.1 to obtain

$$\begin{aligned}
&-2\int \partial_1 u_1 \partial_2 b_1 j dx = -2\int \partial_1 u_1 \partial_2 b_1 (\partial_1 b_2 - \partial_2 b_1) dx \\
&= -2\int \partial_1 u_1 \partial_2 b_1 \partial_1 b_2 dx - 4\int u_1 \partial_2 b_1 \partial_2 \partial_1 b_1 dx \\
&\le C\|\partial_2 u_2\|_2 \|\partial_2 b_1\|_2^{\frac12} \|\partial_2\partial_1 b_1\|_2^{\frac12} \|\partial_1 b_2\|_2^{\frac12} \|\partial_1\partial_2 b_2\|_2^{\frac12} \\
&\quad + C\|\partial_1\partial_2 b_1\|_2 \|u_1\|_2^{\frac12} \|\partial_2 u_1\|_2^{\frac12} \|\partial_2 b_1\|_2^{\frac12} \|\partial_1\partial_2 b_1\|_2^{\frac12} \\
&\le C(\|u\|_2 + \|\nabla b\|_2)(\|\partial_2 u\|_2^2 + \|\partial_1\partial_2 b\|_2^2) \\
&\le C(\|u\|_2 + \|\nabla b\|_2) V(t).
\end{aligned} \tag{3.5}$$

Substituting (3.4), (3.5) in (3.3) and using $\|w\|_2 = \|\nabla u\|_2$ and $\|j\|_2 = \|\nabla b\|_2$, we have

$$\begin{aligned}
&\frac12 \frac{d}{dt}\|(\nabla u, \nabla b)\|_2^2 + \nu\|\partial_2 \nabla u\|_2^2 + \eta\|\partial_1 \nabla b\|_2^2 \\
&\le C(\|u\|_{H^1} + \|\nabla b\|_2) V(t).
\end{aligned} \tag{3.6}$$

Integrating (3.6) over $[0, t]$ and combining with (3.1) yield

$$\begin{aligned}
&\|(u,b)\|_{H^1}^2 + 2\nu\int_0^t \|\partial_2 u(s)\|_{H^1}^2 ds + 2\eta \int_0^t \|\partial_1 b(s)\|_{H^1}^2 ds \\
&\le C\|(u_0, b_0)\|_{H^1}^2 + C\int_0^t \big(\|u(s)\|_{H^1} + \|\nabla b(s)\|_2\big) V(s) ds.
\end{aligned} \tag{3.7}$$

3.2. Estimates for the second-order derivatives of u and b

Taking the inner product of (3.2) with $(-\Delta w, -\Delta j)$ and integrating by parts, we find

$$\begin{aligned}
&\frac12 \frac{d}{dt}\|(\nabla w, \nabla j)\|_2^2 + \nu\|\partial_2 \nabla w\|_2^2 + \eta\|\partial_1 \nabla j\|_2^2 \\
&= \int (u\cdot\nabla) w\, \Delta w dx - \int (b\cdot\nabla) j\, \Delta w dx + \int (u\cdot\nabla) j\, \Delta j dx - \int (b\cdot\nabla) w\, \Delta j dx \\
&\quad - 2\int \partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) \Delta j dx + 2\int \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1) \Delta j dx \\
&:= I_1 + I_2 + \cdots + I_6.
\end{aligned} \tag{3.8}$$

Here we have used the fact that

$$\int \partial_1 j \Delta w dx + \int \partial_1 w \Delta j dx = 0.$$

To estimate I_1, we decompose it into three terms,

$$\begin{aligned} I_1 = -\int (\nabla u \cdot \nabla w)\nabla w dx &= -\int \partial_1 u_1 (\partial_1 w)^2 dx \\ &- \int \partial_1 u_2 \partial_2 w \partial_1 w dx - \int (\partial_2 u \cdot \nabla w)\partial_2 w dx \\ &:= I_{11} + I_{12} + I_{13}. \end{aligned} \tag{3.9}$$

By Hölder's inequality and Sobolev's embedding,

$$\begin{aligned} I_{13} &\le \|\partial_2 u\|_{L^4} \|\nabla w\|_2 \|\partial_2 w\|_{L^4} \le C\|\partial_2 u\|_{H^1} \|\nabla w\|_2 \|\partial_2 w\|_{H^1} \\ &\le C\|\nabla w\|_2 \|\partial_2 u\|_{H^2}^2 \le C\|\nabla^2 u\|_2 V(t). \end{aligned} \tag{3.10}$$

As aforementioned in the introduction, I_{11} and I_{12} are the two most difficult terms due to the lack of full dissipation. It does not appear possible to bound them directly. The trick is to replace $\partial_1 u$ by the other terms in the equation of the magnetic field,

$$\partial_1 u = \partial_t b + (u \cdot \nabla)b - \eta \partial_1^2 b - (b \cdot \nabla)u.$$

Then I_{11} becomes

$$\begin{aligned} I_{11} &= -\int (\partial_t b_1 + (u \cdot \nabla)b_1 - \eta \partial_1^2 b_1 - (b \cdot \nabla)u_1)(\partial_1 w)^2 dx \\ &= -\frac{d}{dt}\int b_1 \ (\partial_1 w)^2 dx + 2\int b_1 \partial_1 w \ \partial_t \partial_1 w dx - \int (u \cdot \nabla) b_1 (\partial_1 w)^2 dx \\ &\quad + \eta \int \partial_1^2 b_1 (\partial_1 w)^2 dx + \int (b \cdot \nabla) u_1 (\partial_1 w)^2 dx \end{aligned} \tag{3.11}$$

By the vorticity equation in (3.2), the second term in (3.11) can be written as

$$\int b_1 \partial_1 w \partial_t \partial_1 w \, dx = \int b_1 \partial_1 w \Big[-\partial_1 (u \cdot \nabla w) + \nu \partial_2^2 \partial_1 w + \partial_1 (b \cdot \nabla j) + \partial_1^2 j \Big] dx.$$

Therefore,

$$\begin{aligned} & I_{11} + \frac{d}{dt}\int b_1 \ (\partial_1 w)^2 dx \\ & = -2\int b_1 \partial_1 w \ \partial_1 (u \cdot \nabla w) dx + 2\nu \int b_1 \partial_1 w \ \partial_2^2 \partial_1 w dx \end{aligned}$$

$$
\begin{aligned}
&+2\int b_1\partial_1 w\ \partial_1(b\cdot\nabla j)dx+2\int b_1\partial_1 w\ \partial_1^2 j dx\\
&-\int (u\cdot\nabla)b_1(\partial_1 w)^2dx+\eta\int \partial_1^2 b_1(\partial_1 w)^2dx+\int (b\cdot\nabla)u_1(\partial_1 w)^2dx\\
:=\ &K_1+K_2+\cdots+K_7. \qquad (3.12)
\end{aligned}
$$

Now we bound K_1 through K_7 one by one. First,

$$K_1=-2\int b_1\partial_1 w\ (\partial_1 u\cdot\nabla)wdx-2\int b_1\partial_1 w\ (u\cdot\nabla)\partial_1 wdx.$$

By integration by parts,

$$\int b_1\partial_1 w\ (u\cdot\nabla)\partial_1 wdx=-\int b_1\partial_1 w\ (u\cdot\nabla)\partial_1 wdx-\int (u\cdot\nabla b_1)(\partial_1 w)^2dx$$

or

$$\int b_1\partial_1 w\ (u\cdot\nabla)\partial_1 wdx=-\frac{1}{2}\int (u\cdot\nabla b_1)(\partial_1 w)^2dx.$$

Thus, by integration by parts,

$$
\begin{aligned}
K_1&=-2\int b_1\partial_1 w\ (\partial_1 u\cdot\nabla)wdx+\int (u\cdot\nabla b_1)(\partial_1 w)^2dx\\
&=-2\int b_1\partial_1 w\ \partial_1 u_1\partial_1 wdx-2\int b_1\partial_1 w\ \partial_1 u_2\partial_2 wdx\\
&\quad+\int u_1\partial_1 b_1(\partial_1 w)^2dx+\int u_2\partial_2 b_1(\partial_1 w)^2dx\\
&=-2\int b_1\partial_1 w\ \partial_1 u_1\partial_1 wdx-2\int b_1\partial_1 w\ \partial_1 u_2\partial_2 wdx\quad+\int u_1\partial_1 b_1(\partial_1 w)^2dx\\
&\quad-\int b_1\left[\partial_2 u_2(\partial_1 w)^2+2u_2\partial_1 w\partial_1\partial_2 w\right]dx\\
&=\int b_1\partial_2 u_2(\partial_1 w)^2dx-2\int b_1\partial_1 w\ \partial_1 u_2\partial_2 wdx\\
&\quad+\int u_1\partial_1 b_1(\partial_1 w)^2dx-2\int b_1 u_2\partial_1 w\partial_1\partial_2 wdx.
\end{aligned}
$$

By Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
K_1\le C\|b_1\|_{L^\infty}\Big(&\|\partial_1 w\|_2\|\partial_2 u_2\|_2^{\frac12}\|\partial_1\partial_2 u_2\|_2^{\frac12}\|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12}\\
&+\|\partial_1 w\|_2\|\partial_1 u_2\|_2^{\frac12}\|\partial_1\partial_2 u_2\|_2^{\frac12}\|\partial_2 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12}\\
&+\|u_2\|_2^{\frac12}\|\partial_1 u_2\|_2^{\frac12}\|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac32}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|u_1\|_{L^\infty}\|\partial_1 w\|_2\|\partial_1 b_1\|_2^{\frac{1}{2}}\|\partial_1^2 b_1\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\le C\|b_1\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_1\|_{H^1}^{\frac{1}{2}}\Big(\|\partial_1 w\|_2\|\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ \|\partial_1 w\|_2\|\partial_1 u_2\|_2^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_2 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ \|u_2\|_2^{\frac{1}{2}}\|\partial_1 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{3}{2}}\Big) \\
&+ C\|u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_1 w\|_2\|\partial_1 b_1\|_2^{\frac{1}{2}}\|\partial_1^2 b_1\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\le C\big(\|b\|_{H^1}^2 + \|u\|_{H^2}^2\big)\big(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2\big) \\
&\le C\big(\|b\|_{H^1}^2 + \|u\|_{H^2}^2\big)V(t).
\end{aligned} \tag{3.13}
$$

By integration by parts and applying Lemma 2.1,

$$
\begin{aligned}
K_2 &= -2\nu\int(\partial_2 b_1\partial_1 w + b_1\ \partial_1\partial_2 w)\partial_1\partial_2 w dx \\
&\le C\|\partial_1\partial_2 w\|_2\|\partial_2 b_1\|_2^{\frac{1}{2}}\|\partial_1\partial_2 b_1\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} + \|b_1\|_{L^\infty}\|\partial_1\partial_2 w\|_2^2 \\
&\le C(\|b\|_{H^2} + \|\nabla w\|_2)(\|\partial_1\partial_2 w\|_2^2 + \|\partial_1\partial_2 b_1\|_2^2) \\
&\le C(\|b\|_{H^2} + \|\nabla w\|_2)V(t).
\end{aligned} \tag{3.14}
$$

To bound K_5, we first write it as

$$
\begin{aligned}
K_5 &= -\int u_1\partial_1 b_1(\partial_1 w)^2 dx - \int u_2\partial_2 b_1(\partial_1 w)^2 dx \\
&= -\int u_1\partial_1 b_1(\partial_1 w)^2 dx + \int \big(2u_2\partial_1 w\partial_1\partial_2 w + \partial_2 u_2(\partial_1 w)^2\big) b_1 dx.
\end{aligned}
$$

By Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
K_5 &\le C\|u\|_{L^\infty}\|\partial_1 w\|_2\|\partial_1 b_1\|_2^{\frac{1}{2}}\|\partial_1^2 b_1\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ C\Big(\|\partial_1\partial_2 w\|_2\|u_2\|_2^{\frac{1}{2}}\|\partial_1 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ \|\partial_1 w\|_2\|\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}}\Big)\|b_1\|_{L^\infty} \\
&\le C\|u\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_1\|_2^{\frac{1}{2}}\|\partial_1^2 b_1\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{3}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ C\Big(\|\partial_1\partial_2 w\|_2\|u_2\|_2^{\frac{1}{2}}\|\partial_1 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&+ \|\partial_1 w\|_2\|\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_2^{\frac{1}{2}}\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}}\Big)\|b_1\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_1\|_{H^1}^{\frac{1}{2}} \\
&\le C(\|u\|_{H^2}^2 + \|b\|_{H^1}^2)(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 b_1\|_{H^1}^2) \\
&\le C(\|u\|_{H^2}^2 + \|b\|_{H^1}^2)V(t).
\end{aligned} \tag{3.15}
$$

Similarly, K_6 can be bounded by

$$\begin{aligned}
K_6 &= -\eta \int \partial_1\partial_2 b_2 (\partial_1 w)^2 dx = 2\eta \int \partial_1 b_2 \partial_1 w \partial_1\partial_2 w dx \\
&\leq C\|\partial_1\partial_2 w\|_2 \|\partial_1 b_2\|_2^{\frac{1}{2}} \|\partial_1^2 b_2\|_2^{\frac{1}{2}} \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\leq C(\|\nabla b\|_2 + \|\nabla w\|_2)(\|\partial_1\partial_2 w\|_2^2 + \|\partial_1^2 b_1\|_2^2) \\
&\leq C(\|\nabla b\|_2 + \|\nabla^2 u\|_2)V(t). \qquad (3.16)
\end{aligned}$$

Again, by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
K_3 &= 2\int b_1 \partial_1 w \ (\partial_1 b \cdot \nabla) j dx + 2\int b_1 \partial_1 w \ (b\cdot\nabla)\partial_1 j dx \\
&\leq C\|b_1\|_{L^\infty} \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \Big(\|\partial_1 b\|_2^{\frac{1}{2}} \|\partial_1^2 b\|_2^{\frac{1}{2}} \|\nabla j\|_2 + \|b\|_2^{\frac{1}{2}} \|\partial_1 b\|_2^{\frac{1}{2}} \|\nabla \partial_1 j\|_2 \Big) \\
&\leq C\|b_1\|_{H^1}^{\frac{1}{2}} \|\partial_1 b_1\|_{H^1}^{\frac{1}{2}} \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\quad \times \Big(\|\partial_1 b\|_2^{\frac{1}{2}} \|\partial_1^2 b\|_2^{\frac{1}{2}} \|\nabla j\|_2 + \|b\|_2^{\frac{1}{2}} \|\partial_1 b\|_2^{\frac{1}{2}} \|\nabla\partial_1 j\|_2 \Big) \\
&\leq C(\|b\|_{H^2}^2 + \|\nabla w\|_2^2)(\|\partial_1 b\|_{H^1}^2 + \|\partial_1\partial_2 w\|_2^2 + \|\partial_1 \nabla j\|_2^2) \\
&\leq C(\|b\|_{H^2}^2 + \|\nabla^2 u\|_2^2)V(t). \qquad (3.17)
\end{aligned}$$

We combine the estimates of K_4 and K_7 in (3.12). Due to $\|\nabla u_1\|_2 = \|\partial_2 u\|_2$,

$$\begin{aligned}
K_4 + K_7 &\leq C\|\partial_1^2 j\|_2 \|b_1\|_2^{\frac{1}{2}} \|\partial_1 b_1\|_2^{\frac{1}{2}} \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\quad + C\|b\|_{L^\infty} \|\nabla u_1\|_2^{\frac{1}{2}} \|\nabla\partial_1 u_1\|_2^{\frac{1}{2}} \|\partial_1 w\|_2 \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\leq C\|\partial_1^2 j\|_2 \|b_1\|_2^{\frac{1}{2}} \|\partial_1 b_1\|_2^{\frac{1}{2}} \|\partial_1 w\|_2^{\frac{1}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\quad + C\|b\|_{H^1}^{\frac{1}{2}} \|\partial_1 b\|_{H^1}^{\frac{1}{2}} \|\nabla u_1\|_2^{\frac{1}{2}} \|\nabla\partial_1 u_1\|_2^{\frac{1}{2}} \|\partial_1 w\|_2^{\frac{3}{2}} \|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\leq C\big(\|b\|_2 + \|\nabla w\|_2 + \|b\|_{H^1}^2 + \|\nabla w\|_2^2\big)\big(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\big) \\
&\leq C\big(\|b\|_2 + \|\nabla w\|_2 + \|b\|_{H^1}^2 + \|\nabla w\|_2^2\big)V(t). \qquad (3.18)
\end{aligned}$$

Combining all the estimates (3.13) through (3.18) yields

$$I_{11} + \frac{d}{dt}\int b_1 \ (\partial_1 w)^2 dx \leq C\big(\|u\|_{H^2} + \|b\|_{H^2} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2\big)V(t). \qquad (3.19)$$

We proceed to bound I_{12}. As in the estimates of I_{11}, we utilize the equation of b to rewrite it as

$$I_{12} = -\int (\partial_t b_2 + u\cdot\nabla b_2 - \eta\partial_1^2 b_2 - b\cdot\nabla u_2)\partial_1 w \partial_2 w dx$$

$$
\begin{aligned}
&= -\frac{d}{dt}\int b_2\ \partial_1 w\partial_2 w dx + \int b_2\ \partial_1 w\partial_2 w_t dx \\
&\quad + \int b_2\ \partial_1 w_t\partial_2 w dx - \int u\cdot\nabla b_2\partial_1 w\partial_2 w dx \\
&\quad + \eta\int \partial_1^2 b_2\partial_1 w\partial_2 w dx + \int b\cdot\nabla u_2\partial_1 w\partial_2 w dx.
\end{aligned}
$$

That is,

$$
\begin{aligned}
I_{12} + \frac{d}{dt}\int b_2\ \partial_1 w\partial_2 w dx &= \int b_2\ \partial_t\partial_1 w\partial_2 w dx \\
&\quad + \int b_2\ \partial_1 w\partial_t\partial_2 w dx - \int u\cdot\nabla b_2\partial_1 w\partial_2 w dx \\
&\quad + \eta\int \partial_1^2 b_2\partial_1 w\partial_2 w dx + \int b\cdot\nabla u_2\partial_1 w\partial_2 w dx \\
&:= H_1 + H_2 + \cdots + H_5.
\end{aligned}\tag{3.20}
$$

The estimates for H_3, H_4 and H_5 are not too difficult. By Hölder's inequality, Sobolev's inequality and the fact $\|\nabla b_2\|_{H^1} = \|\partial_1 b\|_{H^1}$,

$$
\begin{aligned}
H_3 + H_4 &\le \|u\|_{L^\infty}\|\nabla b_2\|_4\|\partial_1 w\|_2\|\partial_2 w\|_4 + \eta\|\partial_1^2 b_2\|_4\|\partial_1 w\|_2\|\partial_2 w\|_4 \\
&\le C\|u\|_{H^2}\|\nabla b_2\|_{H^1}\|\partial_1 w\|_2\|\partial_2 w\|_{H^1} + C\|\partial_1^2 b_2\|_{H^1}\|\partial_1 w\|_2\|\partial_2 w\|_{H^1} \\
&\le C(\|\nabla w\|_2 + \|u\|_{H^2}^2)\big(\|\partial_2 w\|_{H^1}^2 + \|\partial_1 b\|_{H^2}^2\big) \\
&\le C(\|\nabla^2 u\|_2 + \|u\|_{H^2}^2)\|_2 V(t).
\end{aligned}\tag{3.21}
$$

By Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
H_5 &= \int b_1\partial_1 u_2\partial_1 w\partial_2 w dx + \int b_2\partial_2 u_2\partial_1 w\partial_2 w dx \\
&\le C\|b\|_{L^\infty}\Big(\|\partial_1 u_2\|_2\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}}\|\partial_2 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2^{\frac{1}{2}} \\
&\qquad + \|\partial_2 u_2\|_4\|\partial_1 w\|_2\|\partial_2 w\|_4\Big) \\
&\le C\|b\|_{H^1}^{\frac{1}{2}}\|\partial_1 b\|_{H^1}^{\frac{1}{2}}\Big(\|\partial_1 u_2\|_2\|\partial_1 w\|_2^{\frac{1}{2}}\|\partial_1\partial_2 w\|_2\|\partial_2 w\|_2^{\frac{1}{2}} \\
&\qquad + \|\partial_2 u_2\|_{H^1}\|\partial_1 w\|_2\|\partial_2 w\|_{H^1}\Big) \\
&\le C(\|w\|_{H^1}^2 + \|b\|_{H^2}^2)\big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2\big) \\
&\le C(\|u\|_{H^2}^2 + \|b\|_{H^2}^2)V(t).
\end{aligned}\tag{3.22}
$$

Next we deal with H_1 and H_2. Invoking the equation of w in (3.2), H_1 can be written as

$$\begin{aligned}
H_1 &= \int b_2\partial_2 w\Big[-\partial_1(u\cdot\nabla w)+\nu\partial_2^2\partial_1 w+\partial_1(b\cdot\nabla j)+\partial_1^2 j\Big]dx\\
&= -\int b_2\partial_2 w\partial_1(u\cdot\nabla w)dx+\nu\int b_2\partial_2 w\partial_2^2\partial_1 w dx\\
&\quad+\int b_2\partial_2 w\partial_1(b\cdot\nabla j)dx+\int b_2\partial_2 w\partial_1^2 j dx\\
&:= H_{11}+H_{12}+H_{13}+H_{14}.
\end{aligned}$$

By integration by parts, Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
H_{11} &= \int(\partial_1 b_2\partial_2 w+b_2\partial_2\partial_1 w)u\cdot\nabla w dx\\
&\le \|u\|_{L^\infty}\|\partial_1 b_2\|_4\|\partial_2 w\|_4\|\nabla w\|_2\\
&\quad+\|u\|_{L^\infty}\|b_2\|_{L^\infty}\|\nabla w\|_2\|\partial_1\partial_2 w\|_2\\
&\le \|u\|_{H^2}\|\partial_1 b_2\|_{H^1}\|\partial_2 w\|_{H^1}\|\nabla w\|_2\\
&\quad+\|u\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^1}^{\frac12}\|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12}\|\nabla w\|_2\|\partial_1\partial_2 w\|_2\\
&\le C(\|u\|_{H^2}^2+\|b\|_{H^1}^2)\Big(\|\partial_1 b\|_{H^1}^2+\|\partial_2 u\|_{H^2}^2\Big)\\
&\le C(\|u\|_{H^2}^2+\|b\|_{H^1}^2)V(t).
\end{aligned}\tag{3.23}$$

Similarly,

$$\begin{aligned}
H_{13} &= -\int(\partial_1 b_2\partial_2 w+b_2\partial_2\partial_1 w)b\cdot\nabla j dx\\
&\le \|b\|_{L^\infty}\|\partial_1 b_2\|_4\|\partial_2 w\|_4\|\nabla j\|_2+\|b\|_{L^\infty}^2\|\partial_1\partial_2 w\|_2\|\nabla j\|_2\\
&\le \|b\|_{H^2}\|\partial_1 b_2\|_{H^1}\|\partial_2 w\|_{H^1}\|\nabla j\|_2+\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|\partial_1\partial_2 w\|_2\|\nabla j\|_2\\
&\le C\|b\|_{H^2}^2\Big(\|\partial_1 b\|_{H^1}^2+\|\partial_2 w\|_{H^1}^2\Big)\le C\|b\|_{H^2}^2 V(t).
\end{aligned}\tag{3.24}$$

Again, by integration by parts, Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
H_{12} &= -\nu\int(\partial_1 b_2\partial_2 w+b_2\partial_2\partial_1 w)\partial_2^2 w dx\\
&\le \nu\|\partial_1 b_2\|_4\|\partial_2 w\|_4\|\partial_2^2 w\|_2+\nu\|b_2\|_{L^\infty}\|\partial_2\partial_1 w\|_2\|\partial_2^2 w\|\\
&\le C\|\partial_1 b_2\|_{H^1}\|\partial_2 w\|_{H^1}\|\partial_2^2 w\|_2+C\|b_2\|_{H^2}\|\partial_2\nabla w\|_2^2\\
&\le C\|b\|_{H^2}\|\partial_2 w\|_{H^1}^2\le C\|b\|_{H^2}V(t).
\end{aligned}\tag{3.25}$$

By Lemma 2.1,

$$\begin{aligned}
H_{14} &\le C\|\partial_1^2 j\|_2\|b_2\|_2^{\frac12}\|\partial_1 b_2\|_2^{\frac12}\|\partial_2 w\|_2^{\frac12}\|\partial_2^2 w\|_2^{\frac12}\\
&\le C(\|b\|_2+\|\nabla w\|_2)(\|\partial_1 b\|_{H^2}^2+\|\partial_2^2 w\|_2^2)
\end{aligned}$$

$$\le C(\|b\|_2 + \|\nabla w\|_2)V(t). \tag{3.26}$$

In summary, we have from (3.23) through (3.26)

$$H_1 \le C(\|b\|_{H^2} + \|\nabla^2 u\|_2 + \|u\|_{H^2}^2 + \|b\|_{H^2}^2)V(t). \tag{3.27}$$

The estimates of H_2 bear some similarities to H_1, but there are differences. Invoking the equation of w in (3.2), we have

$$\begin{aligned}
H_2 &= \int b_2 \partial_1 w \Big[-\partial_2(u\cdot\nabla w) + \nu\partial_2^3 w + \partial_2(b\cdot\nabla j) + \partial_1\partial_2 j\Big] dx \\
&= -\int b_2\partial_1 w \partial_2(u\cdot\nabla w)dx + \nu\int b_2\partial_1 w\partial_2^3 w dx \\
&\quad + \int b_2\partial_1 w\partial_2(b\cdot\nabla j)dx + \int b_2\partial_1 w\partial_1\partial_2 j dx \\
&:= H_{21} + H_{22} + H_{23} + H_{24}.
\end{aligned}$$

By Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
H_{21} &= -\int b_2\partial_1 w(\partial_2 u\cdot\nabla)w dx - \int b_2\partial_1 w(u\cdot\nabla)\partial_2 w dx \\
&\le C\|b_2\|_{L^\infty}\Big(\|\nabla w\|_2 \|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12}\|\partial_2 u\|_2^{\frac12}\|\partial_1\partial_2 u\|_2^{\frac12} \\
&\quad + \|u\|_{L^\infty}\|\partial_1 w\|_2\|\nabla\partial_2 w\|_2\Big) \\
&\le C\|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12}\Big(\|\nabla w\|_2^{\frac32}\|\partial_1\partial_2 w\|_2^{\frac12}\|\partial_2 u\|_2^{\frac12}\|\partial_1\partial_2 u\|_2^{\frac12} \\
&\quad + \|u\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^1}^{\frac12}\|\partial_1 w\|_2\|\nabla\partial_2 w\|_2\Big) \\
&\le C(\|b\|_{H^1}^2 + \|u\|_{H^2}^2)\big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 b\|_{H^1}^2\big) \\
&\le C(\|b\|_{H^1}^2 + \|u\|_{H^2}^2)V(t).
\end{aligned} \tag{3.28}$$

By integration by parts, Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
H_{23} &= -\int \partial_2 b_2\partial_1 w\ b\cdot\nabla j dx - \int b_2\partial_1\partial_2 w\ (b\cdot\nabla)j dx \\
&\le C\|b\|_{L^\infty}\Big(\|\nabla j\|_2\|\partial_2 b_2\|_2^{\frac12}\|\partial_1\partial_2 b_2\|_2^{\frac12}\|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12} \\
&\quad + \|\partial_1\partial_2 w\|_2\|b_2\|_2^{\frac12}\|\partial_2 b_2\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\partial_1\nabla j\|_2^{\frac12}\Big) \\
&\le C\|b\|_{H^1}^{\frac12}\|\partial_1 b\|_{H^1}^{\frac12}\Big(\|\nabla j\|_2\|\partial_2 b_2\|_2^{\frac12}\|\partial_1\partial_2 b_2\|_2^{\frac12}\|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12} \\
&\quad + \|\partial_1\partial_2 w\|_2\|b_2\|_2^{\frac12}\|\partial_2 b_2\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\partial_1\nabla j\|_2^{\frac12}\Big)
\end{aligned}$$

$$\le C(\|b\|_{H^2}^2+\|\nabla w\|_2^2)\big(\|\partial_1 b\|_{H^2}^2+\|\partial_1\partial_2 w\|_2^2\big)$$
$$\le C(\|b\|_{H^2}^2+\|\nabla^2 u\|_2^2)V(t). \tag{3.29}$$

To bound H_{22} and H_{24}, we make use of the bounds (3.25) and (3.26) for H_{12} and H_{14} to get

$$\begin{aligned}H_{22}+H_{24}&=-\nu\int(\partial_2 b_2\partial_1 w+b_2\partial_2\partial_1 w)\partial_2^2 w dx+\int b_2\ \partial_1 w\partial_1\partial_2 j dx\\&\le C(\|\partial_2 b_2\|_{L^\infty}\|\partial_1 w\|_2+\|b_2\|_{L^\infty}\|\partial_2\partial_1 w\|_2)\|\partial_2^2 w\|_2\\&+\|\partial_1\partial_2 j\|_2\|b_2\|_2^{\frac12}\|\partial_1 b_2\|_2^{\frac12}\|\partial_1 w\|_2^{\frac12}\|\partial_1\partial_2 w\|_2^{\frac12}\\&\le C(\|b\|_{H^2}+\|\nabla w\|_2)V(t).\end{aligned} \tag{3.30}$$

(3.28), (3.29) and (3.30) yield

$$H_2\le C(\|b\|_{H^2}+\|\nabla^2 u\|_2+\|u\|_{H^2}^2+\|b\|_{H^2}^2)V(t). \tag{3.31}$$

Combining (3.21), (3.22), (3.27) and (3.31), we deduce

$$I_{12}+\frac{d}{dt}\int b_2\ \partial_1 w\partial_2 w dx\le C(\|b\|_{H^2}+\|\nabla^2 u\|_2+\|u\|_{H^2}^2+\|b\|_{H^2}^2)V(t). \tag{3.32}$$

Substituting (3.10), (3.19) and (3.32) into (3.9), we find

$$\begin{aligned}&I_1+\frac{d}{dt}\int b_1\ (\partial_1 w)^2 dx+\frac{d}{dt}\int b_2\ \partial_1 w\partial_2 w dx\\&\le C(\|u\|_{H^2}+\|b\|_{H^2}+\|u\|_{H^2}^2+\|b\|_{H^2}^2)V(t).\end{aligned} \tag{3.33}$$

This completes the estimate for I_1. Now we turn back to handle the other terms in (3.8). We first bound I_2 and I_4 together. By integration by parts and $\nabla\cdot b=0$, we write I_2+I_4 as

$$\begin{aligned}I_2+I_4&=\int(\nabla b\cdot\nabla)j\ \nabla w dx+\int(\nabla b\cdot\nabla)w\ \nabla j dx\\&=\int\partial_1 b_1\partial_1 j\partial_1 w dx+\int\partial_1 b_2\partial_2 j\partial_1 w dx+\int\partial_2 b_1\partial_1 j\partial_2 w dx\\&\quad+2\int\partial_2 b_2\partial_2 j\partial_2 w dx+\int\partial_1 b\cdot\nabla w\partial_1 j dx+\int\partial_2 b_1\partial_1 w\partial_2 j dx\\&=\int\partial_1 b_1\partial_1 j\partial_1 w dx-\int\big(\partial_1^2 b_2\partial_2 j+\partial_1 b_2\partial_1\partial_2 j\big)w dx+\int\partial_2 b_1\partial_1 j\partial_2 w dx\\&\quad+2\int\partial_2 b_2\partial_2 j\partial_2 w dx+\int\partial_1 b\cdot\nabla w\partial_1 j dx\\&\quad-\int\big(\partial_1\partial_2 b_1\partial_2 j+\partial_2 b_1\partial_1\partial_2 j\big)w dx.\end{aligned}$$

Thus,

$$
\begin{aligned}
I_2+I_4 \leq\ & 2\int|\partial_1 b||\partial_1 j||\nabla w|dx+2\int|\partial_2 w|\big(|\nabla b||\partial_1 j|+|\partial_1 b_1||\nabla j|\big)dx\\
&+2\int|w|\big(|\partial_1\nabla b||\nabla j|+|\nabla b||\partial_1\partial_2 j|\big)dx.
\end{aligned}
$$

By Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
&2\int|\partial_1 b||\partial_1 j||\nabla w|dx+2\int|\partial_2 w|\big(|\nabla b||\partial_1 j|+|\partial_1 b_1||\nabla j|\big)dx\\
&\leq 2\|\partial_1 b\|_4\|\partial_1 j\|_4\|\nabla w\|_2+2\|\partial_2 w\|_4\big(\|\partial_1 j\|_4\|\nabla b\|_2+\|\partial_1 b\|_4\|\nabla j\|_2\big)\\
&\leq C\|\partial_1 b\|_{H^1}\|\partial_1 j\|_{H^1}\|\nabla w\|_2+C\|\partial_2 w\|_{H^1}\big(\|\partial_1 j\|_{H^1}\|\nabla b\|_2+\|\partial_1 b\|_{H^1}\|\nabla j\|_2\big)\\
&\leq C(\|\nabla w\|_2+\|\nabla b\|_{H^1})(\|\partial_1 b\|_{H^2}^2+\|\partial_2 w\|_{H^1}^2)\\
&\leq C(\|\nabla^2 u\|_2+\|\nabla b\|_{H^1})V(t).
\end{aligned}
$$

For the last term, we apply Lemma 2.1 to obtain

$$
\begin{aligned}
&2\int|w|\big(|\partial_1\nabla b||\nabla j|+|\nabla b||\partial_1\partial_2 j|\big)dx\\
&\leq C\|w\|_2^{\frac12}\|\partial_2 w\|_2^{\frac12}\Big(\|\nabla j\|_2^{\frac12}\|\partial_1\nabla j\|_2^{\frac12}\|\partial_1\nabla b\|_2+\|\nabla b\|_2^{\frac12}\|\partial_1\nabla b\|_2^{\frac12}\|\partial_1\partial_2 j\|_2\Big)\\
&\leq C(\|w\|_2+\|j\|_{H^1})(\|\partial_2 w\|_2^2+\|\partial_1 b\|_{H^2}^2)\\
&\leq C(\|\nabla u\|_2+\|\nabla b\|_{H^1})V(t).
\end{aligned}
$$

Consequently,

$$
I_2+I_4\leq C\big(\|\nabla u\|_{H^1}+\|\nabla b\|_{H^1}\big)V(t). \tag{3.34}
$$

By integration by parts, Hölder's inequality, Sobolev's inequality and Lemma 2.1,

$$
\begin{aligned}
I_3 &= -\int(\nabla u\cdot\nabla)j\ \nabla j dx\\
&= -\int\partial_1 u_1(\partial_1 j)^2dx-\int\partial_1 u_2\partial_2 j\partial_1 j dx\\
&\quad -\int\partial_2 u_1\partial_1 j\partial_2 j dx-\int\partial_2 u_2(\partial_2 j)^2dx\\
&= \int\partial_2 u_2(\partial_1 j)^2dx+\int u_2(\partial_1\partial_2 j\partial_1 j+\partial_2 j\partial_1^2 j)dx\\
&\quad -\int\partial_2 u_1\partial_1 j\partial_2 j dx-2\int u_1\partial_2 j\partial_1\partial_2 j dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\int|\partial_2 u||\partial_1 j||\nabla j|dx+4\int|u||\partial_1\nabla j||\nabla j|dx\\
&\leq 2\|\partial_2 u\|_4\|\partial_1 j\|_4\|\nabla j\|_2+4\|\partial_1\nabla j\|_2\|u\|_2^{\frac{1}{2}}\|\partial_2 u\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\partial_1\nabla j\|_2^{\frac{1}{2}}\\
&\leq C\|\partial_2 u\|_{H^1}\|\partial_1 j\|_{H^1}\|\nabla j\|_2+C\|u\|_2^{\frac{1}{2}}\|\partial_2 u\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\partial_1\nabla j\|_2^{\frac{3}{2}}\\
&\leq C(\|u\|_2+\|\nabla j\|_2)\Big(\|\partial_2 u\|_{H^1}^2+\|\partial_1 j\|_{H^1}^2\Big)\\
&\leq C(\|u\|_2+\|\nabla^2 b\|_2)V(t). \qquad (3.35)
\end{aligned}
$$

For I_5, we first split it into three terms and then estimated similarly as I_3 to obtain

$$
\begin{aligned}
I_5&=-2\int\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)\partial_1^2 j dx-2\int\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)\partial_2^2 j dx\\
&=-2\int\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)\partial_1^2 j dx\\
&\quad+2\int\Big[\partial_1\partial_2 b_1(\partial_1 u_2+\partial_1 u_1)+\partial_1 b_1(\partial_2\partial_1 u_2+\partial_2^2 u_1)\Big]\partial_2 j dx\\
&\leq 2\int|\partial_1 b_1||\nabla u||\partial_1^2 j|dx+2\int|\partial_1\partial_2 b||\nabla u||\nabla j|dx+2\int|\partial_1 b_1||\partial_2\nabla u||\nabla j|dx\\
&\leq 2\|\partial_1 b\|_4\|\nabla u\|_4\|\partial_1^2 j\|_2+C\|\partial_1\partial_2 b\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\partial_2\nabla u\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\partial_1\nabla j\|_2^{\frac{1}{2}}\\
&\quad+2\|\partial_1 b\|_4\|\partial_2\nabla u\|_4\|\nabla j\|_2\\
&\leq 2\|\partial_1 b\|_{H^1}\|\nabla u\|_{H^1}\|\partial_1^2 j\|_2+C\|\partial_1\partial_2 b\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\partial_2\nabla u\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\partial_1\nabla j\|_2^{\frac{1}{2}}\\
&\quad+2\|\partial_1 b\|_{H^1}\|\partial_2\nabla u\|_{H^1}\|\nabla j\|_2\\
&\leq C(\|\nabla u\|_{H^1}+\|\nabla j\|_2)(\|\partial_2\nabla u\|_{H^1}^2+\|\partial_1 b\|_{H^2}^2)\\
&\leq C(\|\nabla u\|_{H^1}+\|\nabla^2 b\|_2)V(t). \qquad (3.36)
\end{aligned}
$$

It remains to estimate I_6. The handling for I_6 is subtle. We rewrite it as

$$
\begin{aligned}
I_6&=2\int\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)\partial_1^2 j dx+2\int\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)\partial_2^2 j dx\\
&=2\int\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)\partial_1^2 j dx-2\int\partial_2 j\partial_2\partial_1 u_1\partial_1 b_2 dx\\
&\quad-2\int\partial_2 j\partial_2\partial_1 u_1\partial_2 b_1 dx-2\int\partial_2 j\partial_1 u_1\partial_2\partial_1 b_2 dx-2\int\partial_2 j\partial_1 u_1\partial_2^2 b_1 dx\\
&=2\int\partial_1 u_1\big(\partial_1 b_2+\partial_2 b_1\big)\partial_1^2 j dx-2\int\partial_2 j\partial_2\partial_1 u_1\partial_1 b_2 dx\\
&\quad+2\int\partial_2 u_1\big(\partial_1\partial_2 j\partial_2 b_1+\partial_2 j\partial_1\partial_2 b_1\big)dx\\
&\quad-2\int\partial_2 j\partial_1 u_1\partial_2\partial_1 b_2 dx+2\int u_1\big(\partial_1\partial_2 j\partial_2^2 b_1+\partial_2 j\partial_2^2\partial_1 b_1\big)dx
\end{aligned}
$$

$$:= I_{61} + I_{62} + I_{63} + I_{64} + I_{65}.$$

The first four terms $I_{61} + I_{62} + I_{63} + I_{64}$ are bounded by

$$\begin{aligned}
& I_{61} + I_{62} + I_{63} + I_{64} \\
& \le C \int |\partial_2 u||\nabla b||\partial_1 \nabla j| dx + 2 \int |\nabla j||\partial_2 \nabla u||\partial_1 b| dx + C \int |\partial_2 u||\partial_1 \nabla b||\nabla j| dx \\
& \le C\|\partial_2 u\|_4 \|\nabla b\|_4 \|\partial_1 \nabla j\|_2 + C\|\nabla j\|_2 \|\partial_2 \nabla u\|_4 \|\partial_1 b\|_4 + C\|\partial_2 u\|_4 \|\partial_1 \nabla b\|_4 \|\nabla j\|_2 \\
& \le C\|\partial_2 u\|_{H^1} \|\nabla b\|_{H^1} \|\partial_1 \nabla j\|_2 + C\|\nabla j\|_2 \|\partial_2 \nabla u\|_{H^1} \|\partial_1 b\|_{H^1} \\
& \quad + C\|\partial_2 u\|_{H^1} \|\partial_1 \nabla b\|_{H^1} \|\nabla j\|_2 \\
& \le C\|\nabla b\|_{H^1} (\|\partial_2 u\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2) \le C\|\nabla b\|_{H^1} V(t).
\end{aligned}$$

For the last term I_{65}, we have, by Lemma 2.1,

$$\begin{aligned}
I_{65} & \le C\|u_1\|_2^{\frac{1}{2}} \|\partial_2 u_1\|_2^{\frac{1}{2}} \left[\|\partial_1 \partial_2 j\|_2 \|\partial_2^2 b_1\|_2^{\frac{1}{2}} \|\partial_1 \partial_2^2 b_1\|_2^{\frac{1}{2}} + \|\partial_2 j\|_2^{\frac{1}{2}} \|\partial_2 \partial_1 j\|_2^{\frac{1}{2}} \|\partial_1 \partial_2^2 b_1\|_2 \right] \\
& \le C(\|u\|_2 + \|\nabla^2 b\|_2)(\|\partial_2 u\|_2^2 + \|\partial_1 \partial_2 j\|_2^2) \le C(\|u\|_2 + \|\nabla^2 b\|_2) V(t).
\end{aligned}$$

Therefore,

$$I_6 \le C(\|u\|_2 + \|\nabla b\|_{H^1}) V(t). \tag{3.37}$$

Inserting (3.33) through (3.37) in (3.8), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|(\nabla w, \nabla j)\|_2^2 + \frac{d}{dt} \int b_1 \ (\partial_1 w)^2 dx + \frac{d}{dt} \int b_2 \ \partial_1 w \partial_2 w dx \\
& \quad + \nu \|\partial_2 \nabla w\|_2^2 + \eta \|\partial_1 \nabla j\|_2^2 \le C(\|u\|_{H^2} + \|b\|_{H^2} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2) V(t).
\end{aligned} \tag{3.38}$$

Integrating (3.38) over $[0, t]$ leads to

$$\begin{aligned}
& \|(\nabla w, \nabla j)\|_2^2 + 2 \int b_1 \ (\partial_1 w)^2 dx + 2 \int b_2 \ \partial_1 w \partial_2 w dx \\
& \qquad + 2\nu \int_0^t \|\partial_2 \nabla w(s)\|_2^2 ds + 2\eta \int_0^t \|\partial_1 \nabla j(s)\|_2^2 ds \\
& \le C \int_0^t \left(\|u(s)\|_{H^2} + \|b(s)\|_{H^2} + \|u(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^2 \right) V(s) ds \\
& \qquad + \|(\nabla w_0, \nabla j_0)\|_2^2 + 2 \int b_{01} \ (\partial_1 w_0)^2 dx + 2 \int b_{02} \ \partial_1 w_0 \partial_2 w_0 dx.
\end{aligned} \tag{3.39}$$

Combining (3.39) with (3.7) gives

$$\begin{aligned}
&\|u,b\|_{H^2}^2 + 2\int b_1\ (\partial_1 w)^2 dx + 2\int b_2\ \partial_1 w\partial_2 w dx \\
&\quad + 2\nu\int_0^t \|\partial_2 u(s)\|_{H^2}^2 ds + 2\eta\int_0^t \|\partial_1 b(s)\|_{H^2}^2 ds \\
&\le C\int_0^t \Big(\|u(s)\|_{H^2} + \|b(s)\|_{H^2} + \|u(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^2\Big)V(s)ds \\
&\quad + \|(u_0,b_0)\|_{H^2}^2 + 2\int b_{01}\ (\partial_1 w_0)^2 dx + 2\int b_{02}\ \partial_1 w_0\partial_2 w_0 dx.
\end{aligned}$$

By Sobolev's inequality,

$$\begin{aligned}
&\|u,b\|_{H^2}^2 + 2\nu\int_0^t \|\partial_2 u(s)\|_{H^2}^2 ds + 2\eta\int_0^t \|\partial_1 b(s)\|_{H^2}^2 ds \\
&\le C\int_0^t \Big(\|u(s)\|_{H^2} + \|b(s)\|_{H^2} + \|u(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^2\Big)V(s)ds \\
&\quad + 4\|b\|_{L^\infty}\|\nabla w\|_2^2 + \|(u_0,b_0)\|_{H^2}^2 + 4\|b_0\|_{L^\infty}\|\nabla w_0\|_2^2 \\
&\le C\int_0^t \Big(\|u(s)\|_{H^2} + \|b(s)\|_{H^2} + \|u(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^2\Big)V(s)ds \\
&\quad + C\,\|b\|_{H^2}\|u\|_{H^2}^2 + \|(u_0,b_0)\|_{H^2}^2 + C\,\|b_0\|_{H^2}\|\nabla w_0\|_2^2 \\
&\le \|(u_0,b_0)\|_{H^2}^2 + C\,\|b_0\|_{H^2}\|u_0\|_{H^2}^2 + C\,\|b\|_{H^2}\|u\|_{H^2}^2 \\
&\quad + C\int_0^t \Big(\|u(s)\|_{H^2} + \|b(s)\|_{H^2} + \|u(s)\|_{H^2}^2 + \|b(s)\|_{H^2}^2\Big)V(s)ds, \qquad (3.40)
\end{aligned}$$

where C's are constants. (3.40) is the desired inequality in (1.7). This completes the proof of the key energy inequality in (1.7).

4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. As explained in the introduction, due to the degeneracy of the viscous dissipation and the magnetic diffusion in (1.3), classical approaches designed for fully dissipated systems no longer work here. It also appears fruitless if we separately write the velocity equation and the magnetic field equation into integral forms via the one-dimensional heat operators due to the presence of the linear terms $\partial_1 b$ and $\partial_1 u$.

The strategy here is to separate the linear and the nonlinear parts in (1.3), solve the linearized system and represent the nonlinear systems in an integral form via the Duhamel's principle. Taking the Fourier transform of (1.3), we find

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{M_1} \\ \widehat{M_2} \end{pmatrix},$$

where A comes from the linear operators, and M_1 and M_2 are nonlinear terms,

$$A = \begin{pmatrix} -\nu\xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta\xi_1^2 \end{pmatrix}, \qquad M_1 = \mathbb{P}(b \cdot \nabla b - u \cdot \nabla u), \qquad M_2 = b \cdot \nabla u - u \cdot \nabla b.$$

The spectra of A, given by

$$\lambda_1 = \frac{-(\nu\xi_2^2 + \eta\xi_1^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\nu\xi_2^2 + \eta\xi_1^2) + \sqrt{\Gamma}}{2} \tag{4.1}$$

with

$$\Gamma := (\nu\xi_2^2 + \eta\xi_1^2)^2 - 4\nu\eta\xi_1^2\,\xi_2^2 - 4\xi_1^2 = (\nu\xi_2^2 - \eta\xi_1^2)^2 - 4\xi_1^2 \tag{4.2}$$

play a crucial role in the large-time behavior. Clearly, they are both anisotropic and strongly frequency dependent. By computing the corresponding eigenvectors and diagonalizing A, we obtain

$$\widehat{u}(\xi,t) = \widehat{Q}_1(t)\widehat{u}_0 + \widehat{Q}_2(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_1(t-\tau)\widehat{M_1}(\tau) + \widehat{Q}_2(t-\tau)\widehat{M_2}(\tau)\big)\, d\tau, \tag{4.3}$$

$$\widehat{b}(\xi,t) = \widehat{Q}_2(t)\widehat{u}_0 + \widehat{Q}_3(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_2(t-\tau)\widehat{M_1}(\tau) + \widehat{Q}_3(t-\tau)\widehat{M_2}(\tau)\big)\, d\tau, \tag{4.4}$$

where the kernel functions $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ depend crucially on λ_1 and λ_2,

$$\widehat{Q}_1(t) = -\nu\xi_2^2 G_1 + G_2, \quad \widehat{Q}_2(t) = i\xi_1 G_1, \quad \widehat{Q}_3(t) = \nu\xi_2^2 G_1 + G_3 \tag{4.5}$$

with

$$\begin{aligned} G_1 &= \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad G_2 = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1, \\ G_3 &= \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} + \lambda_2 G_1 = e^{\lambda_2 t} + \lambda_1 G_1. \end{aligned} \tag{4.6}$$

We explain the framework of the proof of Theorem 1.2. We first show that, when (1.12) holds, the solution (u, b) in H^4 remains uniformly small, namely

$$\|(u(t),b(t))\|_{H^4} \le C\,\delta$$

for a constant C independent of δ. The proof of the desired decay estimate is obtained via the bootstrapping argument applied to the integral representation. This argument starts with the ansatz that

$$\|u(t)\|_{H^2} \le C_0\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \qquad \|b(t)\|_{H^2} \le C_0\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \tag{4.7}$$

where C_0 is a suitably selected pure constant and will be specified in the proof of Theorem 1.2. We then show by using the ansatz in (1.15) and the integral representation of u and b that

$$\|u(t)\|_{H^2} \le \frac{C_0}{2}\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}, \qquad \|b(t)\|_{H^2} \le \frac{C_0}{2}\,\delta\,(1+t)^{-\frac{1}{4}-\frac{\delta}{2}}. \tag{4.8}$$

The proof of (4.8) is not trivial due to the degeneracy, anisotropicity and inhomogeneity of the kernel functions. In particular, $\widehat{Q}_1(\xi,t)$, $\widehat{Q}_2(\xi,t)$ and $\widehat{Q}_3(\xi,t)$ behave differently for different frequencies ξ. As a preparation for the estimates, we need to divide the frequency space into subdomains and classify the behavior of the kernel functions in each subdomain.

Proposition 4.1. *Let Γ be defined as in (4.2). We split $\mathbb{R}^2$ into two domains*

$$\Omega_1 = \left\{\xi\in\mathbb{R}^2,\ \sqrt{\Gamma} \le \frac{\nu\xi_2^2+\eta\xi_1^2}{2}, \quad \text{i.e.,} \quad (\nu\xi_2^2+\eta\xi_1^2)^2 \le \frac{16}{3}\xi_1^2(\nu\eta\xi_2^2+1)\right\},$$
$$\Omega_2 = \left\{\xi\in\mathbb{R}^2,\ \sqrt{\Gamma} > \frac{\nu\xi_2^2+\eta\xi_1^2}{2}, \quad \text{i.e.,} \quad (\nu\xi_2^2+\eta\xi_1^2)^2 > \frac{16}{3}\xi_1^2(\nu\eta\xi_2^2+1)\right\}.$$

Let λ_1 and λ_2 be given by (4.1), G_1, G_2 and G_3 by (4.6) and Q_1, Q_2 and Q_3 by (4.5). Then the behavior of λ_1 and λ_2, G_1, G_2 and G_3, and $\widehat{Q}_1$, $\widehat{Q}_2$ and $\widehat{Q}_3$ can be classified as follows.

(a) *For any $\xi\in\Omega_1$, we have*

$$\begin{aligned}
&\lambda_1 \le -\frac{\nu\xi_2^2+\eta\xi_1^2}{2}, \qquad \lambda_2 \le -\frac{\nu\xi_2^2+\eta\xi_1^2}{4},\\
&|G_1| \le Cte^{-a(\nu\xi_2^2+\eta\xi_1^2)t}, \quad a\in(1/4,1/2),\\
&G_2 = e^{\lambda_1 t} - \lambda_1 G_1, \qquad G_3 = e^{\lambda_2 t}+\lambda_1 G_1,\\
&|\widehat{Q}_1| \le C\,e^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t} \qquad |\widehat{Q}_3| \le C\,e^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t}, \quad c_0>0,\\
&|\widehat{Q}_2| = |\xi_1 G_1| \le C\,e^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t}.
\end{aligned}$$

(b) *For any* $\xi \in \Omega_2$,

$$\lambda_1 \le -\frac{\nu\xi_2^2+\eta\xi_1^2}{2}, \qquad \lambda_2 \le -\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2},$$
$$|G_1| \le 2\,(\nu\xi_2^2+\eta\xi_1^2)^{-2}(e^{\lambda_1 t}+e^{\lambda_2 t}),$$
$$|\widehat{Q}_1|, |\widehat{Q}_2|, |\widehat{Q}_3| \le C\,(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}).$$

We further split Ω_2 *into three subdomains.* Ω_{21} *denotes the subdomain when* $|\sqrt{\eta}\xi_1|$ *is comparable to* $|\sqrt{\nu}\xi_2|$, Ω_{21} *denotes the subdomain when* $|\sqrt{\nu}\xi_2|$ *is far bigger than* $|\sqrt{\eta}\xi_1|$ *while* Ω_{23} *is the remaining part. That is,*

$$\Omega_{21} = \{\xi \in \Omega_2, \quad |\sqrt{\eta}\xi_1| \sim |\sqrt{\nu}\xi_2|\},$$
$$\Omega_{22} = \{\xi \in \Omega_2, \quad |\sqrt{\nu}\xi_2| >> |\sqrt{\eta}\xi_1|\},$$
$$\Omega_{23} = \{\xi \in \Omega_2, \quad |\sqrt{\nu}\xi_2| << |\sqrt{\eta}\xi_1|\}.$$

Then, for some $c_0 > 0$,

$$|\widehat{Q}_1|, |\widehat{Q}_2|, |\widehat{Q}_3| \le C\,e^{-c_0|\xi|^2 t}, \quad \text{if} \quad \xi \in \Omega_{21},$$
$$|\widehat{Q}_1|, |\widehat{Q}_2|, |\widehat{Q}_3| \le C\,e^{-c_0\xi_1^2 t}, \quad \text{if} \quad \xi \in \Omega_{22},$$
$$|\widehat{Q}_1|, |\widehat{Q}_2|, |\widehat{Q}_3| \le C\,e^{-c_0\xi_2^2 t}, \quad \text{if} \quad \xi \in \Omega_{23}.$$

Proof of Proposition 4.1. We start with the case when $\xi \in \Omega_1$. (4.1) obviously implies

$$\lambda_1 \le -\frac{\nu\xi_2^2+\eta\xi_1^2}{2}, \qquad \lambda_2 \le -\frac{\nu\xi_2^2+\eta\xi_1^2}{4}.$$

The upper bound for G_1 follows from the definition of G_1 in (4.6) and the mean-value property. The estimates for $|\widehat{Q}_1|$ and $|\widehat{Q}_3|$ are not difficult,

$$|\widehat{Q}_1| = |G_2 - \nu\xi_2^2 G_1| = |e^{\lambda_1 t} - \lambda_1 G_1 - \nu\xi_2^2 G_1| \le Ce^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t},$$
$$|\widehat{Q}_3| = |e^{\lambda_1 t} + \lambda_2 G_1 + \nu\xi_2^2 G_1| \le e^{-\frac{1}{2}(\nu\xi_2^2+\eta\xi_1^2)t} + \nu|\xi|^2 te^{-a(\nu\xi_2^2+\eta\xi_1^2)t}$$
$$\le Ce^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t},$$

where we have used the simple fact that $x\,e^{-x} \le C$ for any $x \ge 0$. The estimate for $\widehat{Q}_2(\xi,t)$ with $\xi \in \Omega_1$ is more elaborate. Recall that

$$\Gamma = (\nu\xi_2^2+\eta\xi_1^2)^2 - 4\xi_1^2(1+\nu\eta\xi_2^2).$$

To obtain the desired bound, we divide the consideration into two cases:

$$|\sqrt{\Gamma}| \ge |\xi_1| \quad \text{or} \quad |\sqrt{\Gamma}| < |\xi_1|.$$

In the case when $|\sqrt{\Gamma}| \geq |\xi_1|$, we have

$$|\widehat{Q}_2| = |\xi_1 G_1| = |\xi_1| \frac{|e^{\lambda_2 t} - e^{\lambda_1 t}|}{|\sqrt{\Gamma}|} \leq 2\, e^{-\frac{1}{4}(\nu\xi_2^2+\eta\xi_1^2)t}. \tag{4.9}$$

In the case when $|\sqrt{\Gamma}| < |\xi_1|$ or, by the definition of Γ,

$$\left|(\nu\xi_2^2+\eta\xi_1^2)^2 - 4\xi_1^2(1+\nu\eta\xi_2^2)\right| < \xi_1^2,$$

which is equivalent to either

$$4\xi_1^2(1+\nu\eta\xi_2^2) \leq (\nu\xi_2^2+\eta\xi_1^2)^2 < 4\xi_1^2(1+\nu\eta\xi_2^2) + \xi_1^2 \tag{4.10}$$

or

$$(\nu\xi_2^2+\eta\xi_1^2)^2 \leq 4\xi_1^2(1+\nu\eta\xi_2^2) < (\nu\xi_2^2+\eta\xi_1^2)^2 + \xi_1^2. \tag{4.11}$$

Any one of them, (4.10) or (4.11), implies that

$$(\nu\xi_2^2+\eta\xi_1^2)^2 \geq 3\xi_1^2 + 4\nu\eta\xi_1^2\xi_2^2 \quad \text{or} \quad \nu\xi_2^2+\eta\xi_1^2 \geq |\xi_1|\sqrt{3+4\nu\eta\xi_2^2}.$$

Therefore,

$$\begin{aligned}
|\widehat{Q}_2| &\leq |\xi_1| t\, e^{-\frac{a}{2}(\nu\xi_2^2+\eta\xi_1^2)t}\; e^{-\frac{a}{2}(\nu\xi_2^2+\eta\xi_1^2)t} \\
&\leq |\xi_1| t\, e^{-\frac{a}{2}|\xi_1| t\sqrt{3+4\nu\eta\xi_2^2}}\; e^{-\frac{a}{2}(\nu\xi_2^2+\eta\xi_1^2)t} \\
&\leq C\, e^{-\frac{a}{2}(\nu\xi_2^2+\eta\xi_1^2)t},
\end{aligned} \tag{4.12}$$

where we have used the simple fact that $x\, e^{-x} \leq C$ for any $x \geq 0$. It then follows from (4.9) and (4.12) that, for $c_0 = \min\{1/4, a/2\} > 0$,

$$|\widehat{Q}_2| \leq C\, e^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t}.$$

We now turn to the case when $\xi \in \Omega_2$. Trivially,

$$\lambda_1 \leq -\frac{\nu\xi_2^2+\eta\xi_1^2}{2}.$$

For $\xi \in \Omega_2$, we have $\sqrt{\Gamma} \geq \frac{\nu\xi_2^2+\eta\xi_1^2}{2}$ and

$$\begin{aligned}
\lambda_2 &= \frac{-(\nu\xi_2^2+\eta\xi_1^2)+\sqrt{\Gamma}}{2} = -\frac{2\xi_1^2(\nu\eta\xi_2^2+1)}{\sqrt{\Gamma}+\nu\xi_2^2+\eta\xi_1^2} \leq -\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}, \\
|G_1| &= \left|\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\sqrt{\Gamma}}\right| \leq 2\,(\nu\xi_2^2+\eta\xi_1^2)^{-1}(e^{\lambda_1 t}+e^{\lambda_2 t}).
\end{aligned}$$

As a consequence,

$$|\widehat{Q}_1| = |G_2 - \nu\xi_2^2 G_1| = |e^{\lambda_1 t} - \lambda_1 G_1 - \nu\xi_2^2 G_1| \le C(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}).$$

Clearly,

$$|\widehat{Q}_2| = |i\xi_1 G_1| \le 2|\xi_1|(\nu\xi_2^2 + \eta\xi_1^2)^{-1}(e^{\lambda_2 t} + e^{\lambda_1 t}).$$

Furthermore, since $\sqrt{\Gamma} \ge \frac{\nu\xi_2^2+\eta\xi_1^2}{2}$, we have

$$\frac{3}{4}(\nu\xi_2^2 + \eta\xi_1^2)^2 \ge 4\nu\eta\xi_1^2\xi_2^2 + 4\xi_1^2 \ge 4\xi_1^2$$

and consequently

$$|\widehat{Q}_2| \le C(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}).$$

We finally bound $|\widehat{Q}_3|$. For $\xi \in \Omega_2$, we have

$$\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{(\nu\xi_2^2+\eta\xi_1^2)^2} \le \frac{3}{16}.$$

Therefore,

$$\begin{aligned}
|\widehat{Q}_3| &= |\nu\xi_2^2 G_1 + G_3| = |e^{\lambda_1 t} + \lambda_2 G_1 + \nu\xi_2^2 G_1| \\
&\le C(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}) + C\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{(\nu\xi_2^2+\eta\xi_1^2)^2}(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}) \\
&\le C(e^{-\frac{\nu\xi_2^2+\eta\xi_1^2}{2}t} + e^{-\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2}t}).
\end{aligned}$$

The further division of Ω_2 into Ω_{21}, Ω_{22} and Ω_{23} is to make the upper bound for $|\widehat{Q}_1|, |\widehat{Q}_2|$ and $|\widehat{Q}_3|$ more definite. For $\xi \in \Omega_{21}$, $\eta\xi_1^2 \sim \nu\xi_2^2$ and

$$\frac{\xi_1^2(\nu\eta\xi_2^2+1)}{\nu\xi_2^2+\eta\xi_1^2} \quad \sim \quad |\xi|^2 + 1.$$

The behavior for $\xi \in \Omega_{22}$ and $\xi \in \Omega_{23}$ can be similarly identified. This completes the proof of Proposition 4.1. $\square$

We shall also make use of the following decay estimate for the solution operator associated with a fractional Laplacian (see, e.g., [11]). Recall that the fractional Laplacian operator Λ^ρ with any real number ρ is defined via the Fourier transform,

$$\widehat{\Lambda^\rho f}(\xi) = |\xi|^\rho \,\widehat{f}(\xi).$$

Lemma 4.2. *Assume $\alpha \geq 0$ and $\beta > 0$ are real numbers. Let $1 \leq p \leq q \leq \infty$. Then there is a constant $C > 0$ such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t}\|_{L^q(\mathbb{R}^d)} \leq C\, t^{-\frac{\alpha}{\beta} - \frac{d}{\beta}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The first step is to show that, when $\delta > 0$ is sufficiently small and (u_0, b_0) satisfies (1.12), the solution (u, b) satisfies

$$\|(u(t), b(t))\|_{H^4} \leq C\,\delta \tag{4.13}$$

for a constant C independent of δ. When (1.12) holds, Theorem 1.1 asserts that $(u, b) \in H^2$ and

$$\|(u(t), b(t))\|_{H^2} \leq C\,\delta.$$

To show (4.13), it suffices to bound (u, b) in the homogeneous spaces $\dot{H}^3$ and $\dot{H}^4$. The process of showing the uniform boundedness of $\|(u, b)\|_{\dot{H}^3}$ and $\|(u, b)\|_{\dot{H}^4}$ is similar to the proof of Theorem 1.1. We shall only supply the proof for

$$\|(u(t), b(t))\|_{\dot{H}^3} \leq C\,\delta. \tag{4.14}$$

We define the energy functional

$$\begin{aligned} F(t) = \sup_{0 \leq \tau \leq t} &\Big(\|u(\tau)\|_{\dot{H}^3}^2 + \|b(\tau)\|_{\dot{H}^3}^2\Big) \\ &+ 2\nu \int_0^t \|\partial_2 u(s)\|_{\dot{H}^3}^2\, ds + 2\eta \int_0^t \|\partial_1 b(s)\|_{\dot{H}^3}^2\, ds, \end{aligned}$$

and show that

$$F(t) \leq F(0) + C\,F(0)^{\frac{3}{2}} + C\,F(t)^{\frac{3}{2}} + C\,F(t)^2. \tag{4.15}$$

Due to the norm equivalence

$$\|f\|_{\dot{H}^3}^2 \quad \sim \quad \|\partial_1^3 f\|_{L^2}^2 + \|\partial_2^3 f\|_{L^2}^2,$$

it suffices to consider

$$\|\partial_1^3 (u, b)\|_{L^2}^2 + \|\partial_2^3 (u, b)\|_{L^2}^2.$$

Clearly, we have

$$\frac{d}{dt}(\|\partial_1^3(u,b)\|_{L^2}^2 + \|\partial_2^3(u,b)\|_{L^2}^2) + 2\nu\|\partial_2\partial_1^3 u\|_{L^2}^2 + 2\nu\|\partial_2^4 u\|_{L^2}^2$$
$$+2\eta\|\partial_1^4 b\|_{L^2}^2 + 2\nu\|\partial_1\partial_2^3 b\|_{L^2}^2 = L_1 + \cdots + L_6,$$

where the terms on the right-hand side are given by

$$L_1 = -\int \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\,dx, \quad L_2 = -\int \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\,dx,$$
$$L_3 = -\int \partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\,dx, \quad L_4 = -\int \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\,dx,$$
$$L_5 = \int \partial_1^3(b\cdot\nabla b)\cdot\partial_1^3 u\,dx + \int \partial_1^3(b\cdot\nabla u)\cdot\partial_1^3 b\,dx,$$
$$L_6 = \int \partial_2^3(b\cdot\nabla b)\cdot\partial_2^3 u\,dx + \int \partial_2^3(b\cdot\nabla u)\cdot\partial_2^3 b\,dx.$$

The estimates for the terms are very similar to those in the proof of Theorem 1.1. To avoid unnecessary repetitions, we shall only look at one of the most difficult terms in L_1, namely

$$L_{11} = -\int \partial_1^3 u_2\partial_2 u_2\partial_1^3 u_2\,dx = \int \partial_1 u_1\,(\partial_1^3 u_2)^2\,dx.$$

As in the estimate of I_{11} in (3.11), we replace $\partial_1 u_1$ by the equation

$$\partial_1 u_1 = \partial_t b_1 + u\cdot\nabla b_1 - \eta\partial_1^2 b_1 - b\cdot\nabla u_1$$

and rewrite L_{11} as in (3.12),

$$\begin{aligned}
&L_{11} - \frac{d}{dt}\int b_1\,(\partial_1^3 u_2)^2 dx\\
&= 2\int b_1\partial_1^3 u_2\;\partial_1^3(u\cdot\nabla u_2)dx - 2\nu\int b_1\partial_1^3 u_2\;\partial_2^2\partial_1^3 u_2 dx\\
&\quad -2\int b_1\partial_1^3 u_2\;\partial_1^2(b\cdot\nabla b_2)dx - 2\int b_1\partial_1^3 u_2\;\partial_1^4 b_2 dx\\
&\quad -\int (u\cdot\nabla)b_1(\partial_1^3 u_2)^2 dx - \eta\int \partial_1^2 b_1(\partial_1^3 u_2)^2 dx + \int (b\cdot\nabla)u_1(\partial_1^3 u_2)^2 dx
\end{aligned}$$

All the terms above can be bounded in a very similar way as in the estimates of K_1 through K_7 in the proof of Theorem 1.1. For example,

$$\begin{aligned}
&-2\nu\int b_1\partial_1^3 u_2\;\partial_2^2\partial_1^3 u_2 dx = 2\nu\int b_1\,(\partial_2\partial_1^3 u_2)^2\,dx + 2\nu\int \partial_2 b_1\,\partial_1^3 u_2\,\partial_2\partial_1^3 u_2\,dx\\
&\le C\,\|b_1\|_{L^\infty}\,\|\partial_2\partial_1^3 u_2\|_{L^2}^2 + C\,\|\partial_2 b_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_2 b_1\|_{L^2}^{\frac12}\|\partial_1^3 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 u_2\|_{L^2}^{\frac32}\\
&\le C\,F(t)^{\frac32}.
\end{aligned}$$

We shall not provide more details on the estimates of the terms to avoid unnecessary repetitions. It then follows that (4.15) holds. A bootstrapping argument then establishes the global uniform bound stated in (4.14). Similarly we obtain the global bound for (u, b) in H^4. This proves (4.13).

Next we derive the desired decay estimate. As described at the beginning of this section, we employ the bootstrapping argument. We assume (4.7) and show (4.8) via the integral representation in (4.3) and (4.4). The kernel functions Q_1, Q_2 and Q_3 are inhomogeneous and anisotropic. We make use of the detailed analysis obtained in Proposition 4.1. As Proposition 4.1 indicates, $\widehat{Q}_1(\xi,t), \widehat{Q}_2(\xi,t)$ and $\widehat{Q}_3(\xi,t)$ with $\xi \in \Omega_1$ are bounded by $e^{-c_0(\nu\xi_2^2+\eta\xi_1^2)t}$ for some $c_0 > 0$. Therefore they behave like the heat operator in this part of the frequency domain. For $\xi \in \Omega_{21}$, all of them still behave like the heat operator. For $\xi \in \Omega_{22}$, the upper bound for all of them is $e^{-c_0\xi_1^2 t}$ and for $\xi \in \Omega_{23}$, the upper bound is $e^{-c_0\xi_2^2 t}$. That is, the kernel functions behave like one-dimensional heat operators in either Ω_{22} or Ω_{23}.

We estimate $\|u(t)\|_{H^2}$ and $\|b(t)\|_{H^2}$ in each subdomain. We start with $\|u(t)\|_{H^2(\Omega_{22})}$. For $k = 1, 2$ and $m = 0, 1, 2$, according to (4.3),

$$\begin{aligned}\partial_k^m u(t) &= Q_1 * \partial_k^m u_0 + Q_2 * \partial_k^m b_0 \\ &\quad + \int_0^t (Q_1(t-\tau) * \partial_k^m M_1(\tau) + Q_2(t-\tau) * \partial_k^m M_2(\tau))\, d\tau.\end{aligned}$$

We take the $L^2(\Omega_{22})$-norm to obtain

$$\begin{aligned}\|\partial_k^m u(t)\|_{L^2(\Omega_{22})} &\le \|\widehat{Q}_1\, \widehat{\partial_k^m u_0}\|_{L^2(\Omega_{22})} + \|\widehat{Q}_2\, \widehat{\partial_k^m b_0}\|_{L^2(\Omega_{22})} \\ &\quad + \int_0^t \|\widehat{Q}_1(t-\tau)\widehat{\partial_k^m M_1}(\tau)\|_{L^2(\Omega_{22})}\, d\tau \\ &\quad + \int_0^t \|\widehat{Q}_2(t-\tau)\widehat{\partial_k^m M_2}(\tau)\|_{L^2(\Omega_{22})}\, d\tau. \qquad (4.16)\end{aligned}$$

The terms on the right can be bounded as follows. We make use of the assumption that the solution (u, b) remains uniformly bounded in the setting of (1.13). By Proposition 4.1 and for s = t/2,

$$\begin{aligned}\|\widehat{Q}_1\, \widehat{\partial_k^m u_0}\|_{L^2(\Omega_{22})} &\le C\, \|e^{-c_0\xi_1^2 t}\widehat{\partial_k^m u_0}\|_{L^2(\Omega_{22})} \\ &\le C\, \||\xi_1|^\sigma e^{-c_0\xi_1^2 t}\widehat{\partial_k^m \Lambda_1^{-\sigma} u_0}\|_{L^2(\mathbb{R}^2)} \\ &= C\, \Big\| \||\xi_1|^\sigma e^{-c_0\xi_1^2 t}\widehat{\partial_k^m \Lambda_1^{-\sigma} u_0}\|_{L^2_{x_1}} \Big\|_{L^2_{x_2}} \\ &\le C\,(1+t)^{-\frac14-\frac{\sigma}{2}}\, \Big\| \|\partial_k^m \Lambda_1^{-\sigma} u_0\|_{L^1_{x_1}} \Big\|_{L^2_{x_2}}\end{aligned}$$

$$\leq C\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}}, \tag{4.17}$$

where we have used Lemma 4.2 and the condition (1.13) on the initial data. Similarly,

$$\|\widehat{Q}_2\,\widehat{\partial_k^m b_0}\|_{L^2(\Omega_{22})} \leq C\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}}. \tag{4.18}$$

Recalling that $M_1 = \mathbb{P}(-u\cdot\nabla u + b\cdot\nabla b)$ with $\mathbb{P}$ being the Leray projection onto divergence-free vector fields and using the fact that $\mathbb{P}$ is bounded on L^2, we have

$$\begin{aligned}\int_0^t \|\widehat{Q}_1(t-\tau)\widehat{\partial_k^m M_1}(\tau)\|_{L^2(\Omega_{22})}\,d\tau &\leq \int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \\ &\quad + \int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m\widehat{(b\cdot\nabla b)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau.\end{aligned}$$

Applying the bound for $Q_1(\xi, t-\tau)$ with $\xi \in \Omega_{22}$ in Proposition 4.1, we find

$$\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \leq C\int_0^t \|e^{-c_0\xi_1^2(t-\tau)}\partial_k^m\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\mathbb{R}^2)}\,d\tau.$$

By Lemma 4.2,

$$\begin{aligned}&\int_0^t \|e^{-c_0\xi_1^2(t-\tau)}\partial_k^m\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\qquad \leq C\int_0^t (1+t-\tau)^{-\frac{1}{4}}\,\Big\|\|\partial_k^m(u\cdot\nabla u)\|_{L^1_{x_1}}\Big\|_{L^2_{x_2}}\,d\tau.\end{aligned} \tag{4.19}$$

By Minkowski's inequality,

$$\Big\|\|\partial_k^m(u\cdot\nabla u)\|_{L^1_{x_1}}\Big\|_{L^2_{x_2}} \leq \Big\|\|\partial_k^m(u\cdot\nabla u)\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}}.$$

To proceed, we need to distinguish the order of the derivatives. For $m = 0$, by the one-dimensional Sobolev inequality and then the Hölder inequality,

$$\begin{aligned}\Big\|\|\partial_k^m(u\cdot\nabla u)\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} &= \Big\|\|u\cdot\nabla u\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} \leq \Big\|\|u\|_{L^4_{x_2}}\,\|\nabla u\|_{L^4_{x_2}}\Big\|_{L^1_{x_1}} \\ &\leq C\,\Big\|\|u\|_{L^2_{x_2}}^{\frac{3}{4}}\,\|\nabla u\|_{L^2_{x_2}}\,\|\Delta u\|_{L^2}^{\frac{1}{4}}\Big\|_{L^1_{x_1}} \\ &\leq C\,\|u\|_{L^2}^{\frac{3}{4}}\,\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}^{\frac{1}{4}}.\end{aligned}$$

Inserting this bound in (4.19) and invoking the ansatz in (4.7), we find

$$\begin{aligned}
&\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m \widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac14}\,\|u(\tau)\|_{L^2}^{\frac34}\,\|\nabla u(\tau)\|_{L^2}\,\|\Delta u(\tau)\|_{L^2}^{\frac14}\,d\tau \\
&\quad\le C\,C_0^2\,\delta^2\int_0^t (1+t-\tau)^{-\frac14}\,(1+\tau)^{2(-\frac14-\frac{\sigma}{2})}\,d\tau \\
&\quad\le C\,C_0^2\,\delta^2\,(1+t)^{\frac14-\sigma}.
\end{aligned} \tag{4.20}$$

For $m=1$,

$$\begin{aligned}
\Big\|\|\partial_k^m(u\cdot\nabla u)\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} &= \Big\|\|(\partial_k u)\cdot\nabla u + u\cdot\nabla\partial_k u\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} \\
&\le \Big\|\|\nabla u\|_{L^4_{x_2}}^2\Big\|_{L^1_{x_1}} + C\,\Big\|\|u\|_{L^\infty_{x_2}}\|\Delta u\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} \\
&\le \Big\|\|u\|_{H^2_{x_2}}^2\Big\|_{L^1_{x_1}} + C\,\Big\|\|u\|_{L^\infty_{x_2}}\|\Delta u\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} \\
&\le C\,\|u\|_{H^2}^2 + C\,\Big\|\|u\|_{H^2_{x_2}}\|\Delta u\|_{L^2_{x_2}}\Big\|_{L^1_{x_1}} \\
&\le C\,\|u\|_{H^2}^2.
\end{aligned}$$

Again inserting this bound in (4.19) and invoking the ansatz in (4.7), we find

$$\begin{aligned}
&\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m \widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac14}\,\|u(\tau)\|_{H^2}^2\,d\tau \\
&\quad\le C\,C_0^2\,\delta^2\int_0^t (1+t-\tau)^{-\frac14}\,(1+\tau)^{2(-\frac14-\frac{\sigma}{2})}\,d\tau \\
&\quad\le C\,C_0^2\,\delta^2\,(1+t)^{\frac14-\sigma}.
\end{aligned} \tag{4.21}$$

The handling of the case when $m=2$ is slightly different. For $m=2$,

$$\left\| \|\partial_k^m (u\cdot\nabla u)\|_{L^2_{x_2}} \right\|_{L^1_{x_1}} = \left\| \|(\partial_k^2 u)\cdot\nabla u + u\cdot\nabla\partial_k^2 u + 2\partial_k u\cdot\nabla\partial_k u\|_{L^2_{x_2}} \right\|_{L^1_{x_1}}$$
$$\leq \left\| C\,\|\nabla u\|_{L^4_{x_2}}\,\|\Delta u\|_{L^4_{x_2}} \right\|_{L^1_{x_1}} + \left\| \|u\|_{L^\infty_{x_2}} \|\nabla\partial_k^2 u\|_{L^2_{x_2}} \right\|_{L^1_{x_1}}.$$

Since the ansatz in (4.7) does not contain the decay rate for the solution in $\dot{H}^3$, we need to interpolate the $\dot{H}^3$-norm between $\dot{H}^2$ and $\dot{H}^4$. This is exactly where we need the uniform boundedness of $\dot{H}^4$-norm on the solutions. Invoking the following estimates

$$\|\nabla u\|_{L^4_{x_2}} \leq C\,\|u\|_{H^2_{x_2}}, \quad \|\Delta u\|_{L^4_{x_2}} \leq C\,\|\Delta u\|_{L^2_{x_2}}^{\frac78}\,\|u\|_{\dot{H}^4_{x_2}}^{\frac18},$$
$$\|u\|_{L^\infty_{x_2}} \leq C\,\|u\|_{H^2}, \qquad \|\nabla\partial_k^2 u\|_{L^2_{x_2}} \leq \|u\|_{H^2_{x_2}}^{\frac12}\,\|u\|_{\dot{H}^4_{x_2}}^{\frac12}.$$

By Hölder's inequality,

$$\left\| \|\partial_k^m (u\cdot\nabla u)\|_{L^2_{x_2}} \right\|_{L^1_{x_1}} \leq C\,\|u\|_{H^2}^{\frac{15}{8}}\,\|u\|_{\dot{H}^4}^{\frac18} + C\,\|u\|_{H^2}^{\frac32}\,\|u\|_{\dot{H}^4}^{\frac12}.$$

Inserting this bound in (4.19), invoking the ansatz in (4.7) and also making use of the uniform bound for $\|u\|_{\dot{H}^4}$, namely (4.13), we find, for $m=2$,

$$\begin{aligned}
&\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m \widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \\
&\leq C \int_0^t (1+t-\tau)^{-\frac14}\left(\|u(\tau)\|_{H^2}^{\frac{15}{8}}\,\|u(\tau)\|_{\dot{H}^4}^{\frac18} + \|u(\tau)\|_{H^2}^{\frac32}\,\|u(\tau)\|_{\dot{H}^4}^{\frac12}\right) d\tau \\
&\leq C\,C_0^2\,\delta^2 \int_0^t (1+t-\tau)^{-\frac14}\left((1+\tau)^{\frac{15}{8}(-\frac14-\frac{\sigma}{2})} + (1+\tau)^{\frac32(-\frac14-\frac{\sigma}{2})}\right) d\tau \\
&\leq C\,C_0^2\,\delta^2\,(1+t)^{\frac38-\frac34\sigma}.
\end{aligned} \tag{4.22}$$

Here the second to the last integral contains two terms and the second one gives the rate on the last line since the second one is bigger. Combining the bounds in (4.20), (4.21) and (4.22), we obtain, for $k=1,2$ and $m=0,1,2$,

$$\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m \widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \leq C\,C_0^2\,\delta^2\,(1+t)^{\frac38-\frac34\sigma}.$$

Very similarly,

$$\int_0^t \|\widehat{Q}_1(t-\tau)\partial_k^m \widehat{(b\cdot\nabla b)}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \leq C\,C_0^2\,\delta^2\,(1+t)^{\frac38-\frac34\sigma}.$$

Therefore,

$$\int_0^t \|\widehat{Q}_1(t-\tau)\widehat{\partial_k^m M_1}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \le C\,C_0^2\,\delta^2\,(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma}. \tag{4.23}$$

Similarly,

$$\int_0^t \|\widehat{Q}_2(t-\tau)\widehat{\partial_k^m M_2}(\tau)\|_{L^2(\Omega_{22})}\,d\tau \le C\,C_0^2\,\delta^2\,(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma}. \tag{4.24}$$

Combining (4.16), (4.17), (4.18), (4.23) and (4.24), we obtain

$$\|\partial_k^m u(t)\|_{L^2(\Omega_{22})} \le\ C\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}} + C\,C_0^2\,\delta^2\,(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma}. \tag{4.25}$$

The L^2-norm of $\partial_k^m u(t)$ on other domains, Ω_1, Ω_{21} and Ω_{23} can be bounded very similarly. In fact, as stated in Proposition 4.1, the kernel functions Q_1, Q_2 and Q_3 all behave like the heat kernel on Ω_1 and Ω_{21}. The bound for $\|\partial_k^m u(t)\|_{L^2(\Omega_1)}$ and $\|\partial_k^m u(t)\|_{L^2(\Omega_{21})}$ actually has a faster decay rate. On the subdomain Ω_{23}, the kernel functions Q_1, Q_2 and Q_3 all behave like the one-dimensional heat kernel and the L^2-norm of $\partial_k^m u(t)$ on this domain admits pretty much the same bound as in (4.25). Combining the bounds for all these subdomains, we obtain, for $k = 1, 2$ and $m = 0, 1, 2$,

$$\|\partial_k^m u(t)\|_{L^2(\mathbb{R}^2)} \le\ C_1\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}} + C_2\,C_0^2\,\delta^2\,(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma}, \tag{4.26}$$

where C_1 and C_2 are pure constants. $\|\partial_k^m b(t)\|_{L^2(\mathbb{R}^2)}$ can be similarly shown to obey the same bound

$$\|\partial_k^m b(t)\|_{L^2(\mathbb{R}^2)} \le\ C_1\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}} + C_2\,C_0^2\,\delta^2\,(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma}. \tag{4.27}$$

(4.26) and (4.27) are obtained under the ansatz in (4.7). Since $\sigma \ge 5/2$,

$$(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma} \le (1+t)^{-\frac{1}{4}-\frac{\sigma}{2}}.$$

If C_0 is suitably selected and δ is sufficiently small, say

$$C_1 \le \frac{C_0}{64}, \qquad C_2\,C_0\,\delta \le \frac{1}{64},$$

then (4.26) and (4.27) would imply

$$\|(u(t), b(t))\|_{H^2} \le \frac{C_0}{4}\,\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}} + \frac{C_0}{4}\,\delta(1+t)^{\frac{3}{8}-\frac{3}{4}\sigma} \le \frac{C_0}{2}\delta(1+t)^{-\frac{1}{4}-\frac{\sigma}{2}},$$

which is the desired bound in (4.8). The bootstrapping argument then allows us to conclude the desired decay estimate. This completes the proof of Theorem 1.2. $\square$

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