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Existence and uniqueness results for the 2-D dissipative quasi-geostrophic equation

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Abstract

This paper concerns itself with Besov space solutions of the 2-D quasi-geostrophic (QG) equation with dissipation induced by a fractional Laplacian $(-\Delta)^\alpha$. The goal is threefold: first, to extend a previous result on solutions in the inhomogeneous Besov space $B^r_{2,q}$ [J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, SIAM J. Math. Anal. 36 (2004–2005) 1014–1030] to cover the case when $r = 2 - 2\alpha$; second, to establish the global existence of solutions in the homogeneous Besov space $\mathring{B}^r_{p,q}$ with general indices p and q; and third, to determine the uniqueness of solutions in any one of the four spaces: $B^s_{2,q}$, $\mathring{B}^r_{p,q}$, $L^q((0,T); B^{s+\frac{2\alpha}{q}}_{2,q})$ and $L^q((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$, where $s \geq 2 - 2\alpha$ and $r = 1 - 2\alpha + \frac{2}{p}$.
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1. Introduction

The 2-D dissipative quasi-geostrophic (QG) equation concerned here assumes the form

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \tag{1.1}$$

where $\kappa > 0$ and $\alpha \geq 0$ are parameters, $\theta = \theta(x, t)$ is a scalar function of $x \in \mathbb{R}^2$ and $t \geq 0$, and u is a 2-D velocity field determined by θ through the relations

$$u = (u_1, u_2) = (-\partial_{x_2} \psi, \partial_{x_1} \psi) \quad \text{and} \quad (-\Delta)^{\frac{1}{2}} \psi = \theta. \tag{1.2}$$

The fractional Laplacian operator $(-\Delta)^\beta$ for a real number β is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\beta f}(\xi) = (2\pi |\xi|)^{2\beta} \widehat{f}(\xi)$$

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where the Fourier transform $\widehat{f}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}x.$$

For notational convenience, we write Λ for $(-\Delta)^{\frac{1}{2}}$ and combine the relations in (1.2) into

$$u = \nabla^{\perp}\Lambda^{-1}\theta,$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. Physically, (1.1) models the temperature evolution on the 2-D boundary of a 3-D quasi-geostrophic flow and is sometimes referred to as the surface QG equation [8,13].

Fundamental mathematical issues concerning the 2-D dissipative QG equation (1.1) include the global existence of classical solutions and the uniqueness of solutions in weaker senses. In the subcritical case $\alpha > \frac{1}{2}$, these issues have been more or less resolved [9,14]. When $\alpha \le \frac{1}{2}$, the issue on the global existence of classical solutions becomes extremely difficult. For the critical case $\alpha = \frac{1}{2}$, this issue was first dealt with by Constantin et al. [7] and later studied in [3,6,10,11,17] and other works. A recent work of Kiselev et al. [12] appears to have resolved this issue (in the periodic case) by removing the L^{∞}-smallness condition of [7]. Another recent progress on the critical dissipative QG equation was given in the work by Caffarelli and Vasseur [1]. We also mention other interesting investigations on related issues (see cf. [2,15,16]). The supercritical case $\alpha < \frac{1}{2}$ remains a big challenge. This paper is mainly devoted to understanding the behavior of solutions of (1.1) with $\alpha < \frac{1}{2}$. Although our attention is mainly focused on the case when $\alpha < \frac{1}{2}$, the results presented here also hold for $\alpha \ge \frac{1}{2}$. We attempt to accomplish three major goals that we now describe.

In [17], we established the global existence of solutions of (1.1) in the inhomogeneous Besov space $B^r_{2,q}$ with $1 \le q \le \infty$ and $r > 2 - 2\alpha$ when the corresponding initial data θ_0 satisfies

$$\|\theta_0\|_{B^r_{2,q}} \le C\kappa$$

for some suitable constant C. Our first goal is to extend this result to cover the case when $r = 2-2\alpha$. For this purpose, we derive a new a priori bound on solutions of (1.1) in $B^{2-2\alpha}_{2,q}$. When combined with a procedure detailed in [17], this new bound yields the global existence of solutions in $B^{2-2\alpha}_{2,q}$. As a special consequence of this result, the 2-D critical QG equation ((1.1) with $\alpha = \frac{1}{2}$) possesses a global H^1-solution if the initial datum is comparable to κ.

Our second goal is to explore solutions of (1.1) in the homogeneous Besov space $\mathring{B}^r_{p,q}$ with general indices $2 \le p < \infty$ and $1 \le q \le \infty$. This study was partially motivated by the lower bound

$$\int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^{\alpha} f\,\mathrm{d}x \ge C 2^{2\alpha j}\|f\|^p_{L^p} \tag{1.3}$$

valid for any function f that decays sufficiently fast at infinity and satisfies

$$\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\},$$

where $0 < K_1 \le K_2$ are constants and j is an integer. This inequality, recently established in [5,18], provides a lower bound for the integral generated by the dissipative term when we estimate solutions of (1.1) in $\mathring{B}^r_{p,q}$. Combining this lower bound with suitable upper bounds for the nonlinear term, we are able to derive a priori estimates for solutions of (1.1) in $\mathring{B}^r_{p,q}$. Applying the method of successive approximation, we then establish the existence and uniqueness of solutions emanating from initial data θ_0 satisfying

$$\|\theta_0\|_{\mathring{B}^r_{p,q}} \le C\kappa,$$

where $2 \le p < \infty$, $1 \le q \le \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$. Setting $q = p$, we obtain as a special consequence the global solutions in the homogeneous Sobolev space $\mathring{W}^{p,r}$, where $2 \le p < \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$.

The third goal is to determine the uniqueness of solutions of (1.1) in the spaces

$$B^s_{2,q}, \qquad \mathring{B}^r_{p,q}, \qquad L^q\left((0,T); B^{s+\frac{2\alpha}{q}}_{2,q}\right) \quad \text{and} \quad L^q\left((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right) \tag{1.4}$$

where $s \geq 2 - 2\alpha$ and $r = 1 - 2\alpha + \frac{2}{p}$. For two solutions in any one of these spaces, we establish suitable bounds for their difference which yield as a special consequence the uniqueness. We conclude that two solutions θ and $\widetilde{\theta}$ emanating from the same initial datum must coincide if they satisfy

$$\|\theta\|_{B^r_{2,q}} \leq C\kappa \quad \text{and} \quad \|\widetilde{\theta}\|_{B^r_{2,q}} \leq C\kappa \tag{1.5}$$

for some constant C. A parallel result holds for solutions in $\mathring{B}^r_{p,q}$. In addition, we prove that any two solutions in the space $L^q((0,T); B^{r+\frac{2\alpha}{q}}_{2,q})$ must be identical if they satisfy the same initial condition. The same conclusion can be drawn for solutions in $L^q((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$. A special consequence is the uniqueness of H^1 (or $\mathring{H}^1$) solutions of the critical QG equation if their norms are comparable to κ. A more significant corollary is the uniqueness of solutions of the critical QG equation in $L^2((0,T); H^{\frac{3}{2}})$ (or $L^2((0,T); \mathring{H}^{\frac{3}{2}})$). Since H^1 (or $\mathring{H}^1$) solutions of the critical QG equation are in general also in $L^2((0,T); H^{\frac{3}{2}})$ (or $L^2((0,T); \mathring{H}^{\frac{3}{2}})$), this corollary indicates the uniqueness of H^1 (or $\mathring{H}^1$) solutions that do not necessarily satisfy (1.5).

The rest of this paper is divided into four sections and an Appendix A. Section 2 is further divided into three subsections. Section 2.1 recalls the definitions of Besov spaces, Section 2.2 presents the definitions of two spaces involving time and the relations between them, and Section 2.3 provides the lower bound (1.3). Section 3 focuses on solutions in $B^r_{2,q}$ while Section 4 is devoted to solutions in $\mathring{B}^r_{p,q}$. Section 5 deals with the uniqueness of solutions in the spaces in (1.4). The Appendix A proves a Bernstein inequality involving fractional Laplacians and derives a commutator estimate that is used in Sections 3 through 5.

We finally remark that after the completion of this manuscript, we learned that a result related to Theorem 4.2 in Section 4 was obtained by Chen et al. [5].

2. Function spaces and a lower bound

This section makes necessary preparations for the subsequent sections. It is divided into three subsections. Section 2.1 provides the definition of Besov spaces. Section 2.2 presents two spaces involving time and their relations. The last subsection recalls a lower bound for an integral involving fractional Laplacians.

2.1. Besov spaces

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x) x^\gamma \,\mathrm{d}x = 0, |\gamma| = 0, 1, 2, \ldots\right\}$$

and $\mathcal{S}'_0$ denotes its dual. $\mathcal{S}'_0$ can be identified as

$$\mathcal{S}'_0 = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^{j+1}\}. \tag{2.1}$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp}\widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\Delta_j f = \Phi_j * f, \quad j \in \mathbb{Z}. \tag{2.2}$$

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$.

To define the inhomogeneous Besov space, we set

$$\Delta_j' f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots. \end{cases} \tag{2.4}$$

The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j' f\|_{L^p}\|_{l^q} < \infty.$$

For notational convenience, we will write Δ_j for Δ_j'. There will be no confusion if we keep in mind that Δ_j's associated with the homogeneous Besov spaces is defined in (2.2) while those associated with the inhomogeneous Besov spaces are defined in (2.4).

We will need the following characteristic properties of the Δ_j's defined in (2.2) or in (2.4)

$$\begin{aligned}&\Delta_{j_1}\Delta_{j_2}=0 \quad \text{if } |j_1-j_2|\geq 2,\\ &S_k\equiv\sum_{j>-\infty}^{k-1}\Delta_j\to I \quad \text{as } k\to\infty,\\ &\Delta_j(S_kf\,\Delta_kf)=0 \quad \text{if } |j-k|\geq 3.\end{aligned} \tag{2.5}$$

In addition, the following embedding relations of Besov spaces will be useful.

Proposition 2.1. *Let $s\in\mathbb{R}$, $1\leq p\leq\infty$ and $1\leq q\leq\infty$.*

(1) *If $s>0$, then $B^s_{p,q}\subset\mathring{B}^s_{p,q}$ and*

$$\|f\|_{B^s_{p,q}}=\|f\|_{L^p}+\|f\|_{\mathring{B}^s_{p,q}}.$$

(2) *If $s_1\leq s_2$, then $B^{s_2}_{p,q}\subset B^{s_1}_{p,q}$. This inclusion relation is false for the homogeneous Besov spaces.*
(3) *If $1\leq q_1\leq q_2\leq\infty$, then $B^s_{p,q_1}\subset B^s_{p,q_2}$ and $\mathring{B}^s_{p,q_1}\subset\mathring{B}^s_{p,q_2}$.*
(4) *If $1\leq p_1\leq p_2\leq\infty$ and $s_1=s_2+d(\frac{1}{p_1}-\frac{1}{p_2})$, then*

$$B^{s_1}_{p_1,q}(\mathbb{R}^d)\subset B^{s_2}_{p_1,q}(\mathbb{R}^d),\qquad \mathring{B}^{s_1}_{p_1,q}(\mathbb{R}^d)\subset\mathring{B}^{s_2}_{p_1,q}(\mathbb{R}^d).$$

Finally, we remark that many frequently used function spaces are special cases of Besov spaces. The Sobolev spaces $\mathring{H}^s$ and H^s defined by

$$\mathring{H}^s=\{f\in\mathcal{S}':|\xi|^s|\widehat{f}(\xi)|\in L^2\},\qquad H^s=\{f\in\mathcal{S}':(1+|\xi|^2)^{\frac{s}{2}}|\widehat{f}(\xi)|\in L^2\}$$

can be identified as

$$\mathring{H}^s=\mathring{B}^s_{2,2},\qquad H^s=B^s_{2,2}.$$

For $0<s<1$, $B^s_{\infty,\infty}$ and $\mathring{B}^s_{\infty,\infty}$ are the same as the usual Hölder spaces C^s and $\mathring{C}^s$, respectively, where $\mathring{C}^s$ is a subspace of continuous functions with a seminorm. $B^1_{\infty,\infty}$ is bigger than the space of Lipschitz functions and can be identified with the Zygmund class Zyg characterized by the inequality

$$|f(x-h)-2f(x)+f(x+h)|\leq c|h| \quad \text{for some constant } c \text{ and all } x.$$

2.2. Two types of spaces involving time and their relations

In this subsection, we analyze the relationship between two types of function spaces that map time to Besov spaces. For $1\leq\rho\leq\infty$, $-\infty\leq a<b\leq\infty$ and a real Banach space X, the space $L^\rho(a,b;X)$ consists of measurable functions $f:(a,b)\to X$ with

$$\|f\|_{L^\rho((a,b);X)}=\|\|f\|_X\|_{L^\rho(a,b)}\equiv\left(\int_a^b\|f(\cdot,t)\|_X^\rho\,\mathrm{d}t\right)^{\frac{1}{\rho}}<\infty.$$

We are mainly interested in the cases when $X=B^s_{p,q}$ or $\mathring{B}^s_{p,q}$.

In [4], Chemin introduced the space $\widetilde{L^\rho}((a,b);B^s_{p,q})$, which consists of functions f for which the norm

$$\|f\|_{\widetilde{L^\rho}((a,b);B^s_{p,q})}\equiv\|2^{js}\|\Delta_jf\|_{L^\rho((a,b);L^p_x)}\|_{l^q}$$

in finite. The space $\widetilde{L^\rho}((a,b);\mathring{B}^s_{p,q})$ is similarly defined. For notational convenience, we sometimes write $\widetilde{L^\rho_t}(B^s_{p,q})$ for $\widetilde{L^\rho}((0,t);B^s_{p,q})$, $L^\rho_t(B^s_{p,q})$ for $L^\rho((0,t);B^s_{p,q})$, etc.

We investigate how $L^\rho((a,b);\mathring{B}^s_{p,q})$ is related to $\widetilde{L^\rho}((a,b);\mathring{B}^s_{p,q})$ and how $L^\rho((a,b);B^s_{p,q})$ is related to $\widetilde{L^\rho}((a,b);B^s_{p,q})$. We start with some elementary facts.

Lemma 2.2. *Let $\{f_j\}$ be a sequence of measurable functions on an interval (a,b). Assume $f_j \geq 0$ on (a,b) for each j. Then*

$$\sum_j \int_a^b f_j(\tau)\,d\tau = \int_a^b \sum_j f_j(\tau)d\tau, \tag{2.6}$$

$$\sup_j \int_a^b f_j(\tau)\,d\tau \leq \int_a^b \sup_j f_j(\tau)\,d\tau. \tag{2.7}$$

Proof. (2.6) can be obtained by applying the Monotone Convergence theorem to the sequence $\{g_k\}$, where

$$g_k = \sum_{j\leq k} f_j.$$

(2.7) also follows from the Monotone Convergence theorem. In fact,

$$\sup_j \int_a^b f_j(\tau)\,d\tau = \lim_{k\to\infty} \max_{j\leq k} \int_a^b f_j d\tau \leq \lim_{k\to\infty} \int_a^b \max_{j\leq k} f_j d\tau = \int_a^b \sup_j f_j(\tau)\,d\tau. \quad \square$$

The following proposition is a consequence of (2.6).

Proposition 2.3. *Let $s \in \mathbb{R}$ and $\rho, p, q \in [1,\infty]$. If $\rho = q$, then*

$$L^\rho((a,b); \mathring{B}^s_{p,q}) = \widetilde{L^\rho}((a,b); \mathring{B}^s_{p,q}), \qquad L^\rho((a,b); B^s_{p,q}) = \widetilde{L^\rho}((a,b); B^s_{p,q}).$$

Proof. In the case when $1 \leq \rho = q < \infty$,

$$\|f\|_{L^\rho((a,b);\mathring{B}^s_{p,q})} = \|f\|_{\widetilde{L^\rho}((a,b);\mathring{B}^s_{p,q})} = \left(\int_a^b \sum_j 2^{jsq}\|\Delta_j f\|^q_{L^p} d\tau\right)^{\frac{1}{q}}$$

according to (2.6). In the case when $\rho = q = \infty$,

$$\|f\|_{L^\infty((a,b);\mathring{B}^s_{p,\infty})} = \|f\|_{\widetilde{L^\infty}((a,b);\mathring{B}^s_{p,\infty})} = \sup_j \sup_{t\in(a,b)} 2^{js}\|\Delta_j f\|_{L^p}. \quad \square$$

The inclusion relation in the following proposition follows from (2.7).

Proposition 2.4. *For any $s \in \mathbb{R}$ and $p \in [1,\infty]$,*

$$L^1((a,b); \mathring{B}^s_{p,\infty}) \subset \widetilde{L^1}((a,b); \mathring{B}^s_{p,\infty}), \qquad L^1((a,b); B^s_{p,\infty}) \subset \widetilde{L^1}((a,b); B^s_{p,\infty}).$$

Proof. By (2.7),

$$\begin{aligned}\|f\|_{\widetilde{L^1}((a,b);\mathring{B}^s_{p,\infty})} &= \sup_j 2^{js} \int_a^b \|\Delta_j f\|_{L^p} d\tau \\ &\leq \int_a^b \sup_j 2^{js}\|\Delta_j f\|_{L^p} d\tau = \|f\|_{L^1((a,b);\mathring{B}^s_{p,\infty})}.\end{aligned}$$

The proof for $L^1((a,b); B^s_{p,\infty}) \subset \widetilde{L^1}((a,b); B^s_{p,\infty})$ is the same. $\square$

2.3. A lower bound for an integral of fractional Laplacians

When we estimate solutions to partial differential equations with fractional Laplacian dissipation in L^p-related spaces, we often encounter an integral of the form

$$D_p(f) \equiv \int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^\alpha f \, \mathrm{d}x.$$

For $\alpha = 1$, lower bounds for this integral can be derived through integration by parts. For a general fraction $\alpha > 0$, $(-\Delta)^\alpha$ is a nonlocal operator and lower bounds no longer follow from integration by parts. The following lower bound was recently established in [5,18].

Proposition 2.5. *Assume either $\alpha \ge 0$ and $p = 2$ or $0 \le \alpha \le 1$ and $2 < p < \infty$. Assume f decays sufficiently fast at infinity and $\widehat{f}$ satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and real numbers $0 < K_1 \le K_2$. Then we have the lower bound

$$D_p(f) \ge C 2^{2\alpha j} \|f\|_{L^p}^p$$

for some constant C depending on d, α, K_1 and K_2.

3. Solutions in $B^r_{2,q}$ for $r \ge 2 - 2\alpha$

This section is concerned with solutions of the initial-value problem (IVP) for the 2-D dissipative QG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ u = \nabla^\perp \Lambda^{-1}\theta, \quad x \in \mathbb{R}^2,\ t > 0, \\ \theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^2 \end{cases} \tag{3.1}$$

in the inhomogeneous Besov space $B^r_{2,q}$. The major results are presented in Theorem 3.2, which asserts the global existence and uniqueness of solutions in $B^r_{2,q}$. The proof of this theorem relies on several a priori estimates, which are stated in Theorem 3.1.

Theorem 3.1. *Let $\alpha > 0$. Let θ solve the IVP (3.1). Then θ satisfies the following a priori estimates.*

(1) *In the case when $r = 2 - 2\alpha$ and $q = \infty$, we have*

$$\|\theta(t)\|_{B^{2-2\alpha}_{2,\infty}} + C\kappa \|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})} \le \|\theta_0\|_{B^{2-2\alpha}_{2,\infty}} + C\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})}. \tag{3.2}$$

(2) *In the case when $r > 2 - 2\alpha$ and $q = \infty$, we have for any $s \in \mathbb{R}$,*

$$\|\theta(t)\|_{B^s_{2,\infty}} + C\kappa \|\theta\|_{\widetilde{L}^1_t(B^{s+2\alpha}_{2,\infty})} \le \|\theta_0\|_{B^s_{2,\infty}} + C\|\theta\|_{L^\infty_t(B^s_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^{r+2\alpha}_{2,\infty})}. \tag{3.3}$$

In particular, we have by setting $s = r$,

$$\|\theta(t)\|_{B^r_{2,\infty}} + C\kappa \|\theta\|_{\widetilde{L}^1_t(B^{r+2\alpha}_{2,\infty})} \le \|\theta_0\|_{B^r_{2,\infty}} + C\|\theta\|_{L^\infty_t(B^r_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^{r+2\alpha}_{2,\infty})}.$$

(3) *In the case when $r \ge 2 - 2\alpha$ and $1 \le q < \infty$, we have*

$$\|\theta(t)\|^q_{B^r_{2,q}} + Cq\kappa \|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})} \le \|\theta_0\|^q_{B^r_{2,q}} + Cq\|\theta\|_{L^\infty_t(B^r_{2,q})} \|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}. \tag{3.4}$$

Remark. Because of Proposition 2.3, the norm $\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})$ in (3.4) can be replaced by $L^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})$.

Proof of Theorem 3.1. For $j \in \mathbb{Z}$, we apply Δ_j to the first equation in (3.1) to obtain

$$\partial_t \Delta_j\theta + u \cdot \nabla\Delta_j\theta + \kappa(-\Delta)^\alpha \Delta_j\theta = [u \cdot \nabla, \Delta_j]\theta, \tag{3.5}$$

where the brackets [ ] represent the commutator operator, namely

$$[u \cdot \nabla, \Delta_j]\theta \equiv u \cdot \nabla(\Delta_j \theta) - \Delta_j(u \cdot \nabla\theta).$$

Multiplying both sides by $2\Delta_j\theta$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^2}^2 + 2\kappa\int_{\mathbb{R}^2}\Delta_j\theta(-\Delta)^\alpha\Delta_j\theta\,\mathrm{d}x = 2\int_{\mathbb{R}^2}\Delta_j\theta[u\cdot\nabla,\Delta_j]\theta\,\mathrm{d}x. \tag{3.6}$$

By Bernstein's inequality (Theorem A.1),

$$\int_{\mathbb{R}^2}\Delta_j\theta(-\Delta)^\alpha\Delta_j\theta\,\mathrm{d}x = \int_{\mathbb{R}^2}|\Lambda^\alpha\Delta_j\theta|^2\,\mathrm{d}x \geq 2^{2\alpha j}\,\|\Delta_j\theta\|_{L^2}^2. \tag{3.7}$$

By Hölder's inequality,

$$2\int_{\mathbb{R}^2}\Delta_j\theta[u\cdot\nabla,\Delta_j]\theta\,\mathrm{d}x \leq C\,\|\Delta_j\theta\|_{L^2}\|[u\cdot\nabla,\Delta_j]\theta\|_{L^2}. \tag{3.8}$$

Inserting (3.7) and (3.8) in (3.6), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^2} + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|_{L^2} \leq C\|[u\cdot\nabla,\Delta_j]\theta\|_{L^2}.$$

Applying Proposition A.2 to the right-hand side yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^2} + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|_{L^2} \leq C\left(2^{2j}\|\Delta_j\theta\|_{L^2}^2 + \|\Delta_j\theta\|_{L^2}\sum_{k\leq j-1}2^{2k}\|\Delta_k\theta\|_{L^2} + 2^{2j}\sum_{k\geq j-1}\|\Delta_k\theta\|_{L^2}^2\right). \tag{3.9}$$

We now prove (1). Multiplying (3.9) by $2^{(2-2\alpha)j}$, integrating over $[0, t]$ and taking supremum over $j \geq -1$, we obtain

$$\|\theta(t)\|_{B^{2-2\alpha}_{2,\infty}} + C\kappa\|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})} \leq \|\theta_0\|_{B^{2-2\alpha}_{2,\infty}} + I_1 + I_2 + I_3, \tag{3.10}$$

where I_1 and I_2 are given by

$$I_1 = \sup_j\int_0^t 2^{(2-2\alpha)j}\,2^{2j}\|\Delta_j\theta\|_{L^2}^2\,\mathrm{d}\tau,$$

$$I_2 = \sup_j\int_0^t 2^{(2-2\alpha)j}\|\Delta_j\theta\|_{L^2}\sum_{k\leq j-1}2^{2k}\|\Delta_k\theta\|_{L^2}\,\mathrm{d}\tau,$$

$$I_3 = \sup_j\int_0^t 2^{(2-2\alpha)j}2^{2j}\sum_{k\geq j-1}\|\Delta_k\theta\|_{L^2}^2\,\mathrm{d}\tau.$$

Clearly,

$$\begin{aligned} I_1 &\leq \sup_j\sup_{0\leq\tau\leq t}\{2^{(2-2\alpha)j}\|\Delta_j\theta\|_{L^2}\}\int_0^t 2^{2j}\|\Delta_j\theta\|_{L^2}\,\mathrm{d}\tau \\ &\leq \sup_{0\leq\tau\leq t}\sup_j 2^{(2-2\alpha)j}\|\Delta_j\theta\|_{L^2}\sup_j 2^{2j}\int_0^t\|\Delta_j\theta\|_{L^2}\,\mathrm{d}\tau \\ &= \|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})}\|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})}. \end{aligned} \tag{3.11}$$

$$\begin{aligned} I_2 &\leq \sup_j\int_0^t 2^{2j}\|\Delta_j\theta\|_{L^2}\sum_{k\leq j-1}2^{2\alpha(k-j)}\,2^{(2-2\alpha)k}\|\Delta_k\theta\|_{L^2}\,\mathrm{d}\tau \\ &\leq \sup_j\sup_{0\leq\tau\leq t}\sum_{k\leq j-1}2^{2\alpha(k-j)}2^{(2-2\alpha)k}\|\Delta_k\theta\|_{L^2}\sup_j\int_0^t 2^{2j}\|\Delta_j\theta\|_{L^2}\,\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&\leq C \sup_{0\leq\tau\leq t} \sup_j 2^{(2-2\alpha)j} \|\Delta_j\theta\|_{L^2} \|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})} \\
&= C\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})}
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
I_3 &\leq \sup_j \int_0^t \sum_{k\geq j-1} 2^{(2-2\alpha)k} \|\Delta_k\theta\|_{L^2} 2^{2k} \|\Delta_k\theta\|_{L^2} 2^{(4-2\alpha)(j-k)}\, \mathrm{d}\tau \\
&\leq \sup_{0\leq\tau\leq t} \sup_k 2^{(2-2\alpha)k} \|\Delta_k\theta\|_{L^2} \sup_j \sum_{k\geq j-1} \int_0^t 2^{2k} \|\Delta_k\theta\|_{L^2}\, \mathrm{d}\tau\, 2^{(4-2\alpha)(j-k)}.
\end{aligned}
$$

By Young's inequality for the convolution of sequences,

$$
\begin{aligned}
I_3 &\leq C\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})} \sup_j \int_0^t 2^{2j} \|\Delta_j\theta\|_{L^2}\, \mathrm{d}\tau \sum_{j\leq 0} 2^{(4-2\alpha)j} \\
&= C\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^2_{2,\infty})}.
\end{aligned} \tag{3.13}
$$

Combining (3.10)–(3.13) yields the estimate (3.2).

To prove (2), we multiply both sides of (3.9) by 2^{sj}, integrate with respect to t and take the supremum over all $j \geq -1$ to obtain

$$
\|\theta(t)\|_{B^s_{2,\infty}} + C\kappa \|\theta\|_{\widetilde{L}^1_t(B^{s+2\alpha}_{2,\infty})} \leq \|\theta_0\|_{B^s_{2,\infty}} + I_4 + I_5 + I_6, \tag{3.14}
$$

where

$$
\begin{aligned}
I_4 &= \sup_j \int_0^t 2^{sj} 2^{2j} \|\Delta_j\theta\|^2_{L^2}\, \mathrm{d}\tau, \\
I_5 &= \sup_j \int_0^t 2^{sj} \|\Delta_j\theta\|_{L^2} \sum_{k\leq j-1} 2^{2k} \|\Delta_k\theta\|_{L^2}\, \mathrm{d}\tau, \\
I_6 &= \sup_j \int_0^t 2^{sj} 2^{2j} \sum_{k\geq j-1} \|\Delta_k\theta\|^2_{L^2}\, \mathrm{d}\tau.
\end{aligned}
$$

We bound I_4 and I_5 as follows

$$
\begin{aligned}
I_4 &\leq \sup_j \sup_{0\leq\tau\leq t} \{2^{sj} \|\Delta_j\theta\|_{L^2}\} \int_0^t 2^{2j} \|\Delta_j\theta\|_{L^2}\, \mathrm{d}\tau \\
&\leq \sup_{0\leq\tau\leq t} \sup_j 2^{sj} \|\Delta_j\theta\|_{L^2} \sup_j 2^{(r+2\alpha)j} \int_0^t \|\Delta_j\theta\|_{L^2}\, \mathrm{d}\tau \\
&= \|\theta\|_{L^\infty_t(B^s_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^{r+2\alpha}_{2,\infty})} \quad \text{for any } r \geq 2-2\alpha.
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
I_5 &\leq \sup_j \int_0^t 2^{sj} \|\Delta_j\theta\|_{L^2} \sum_{k\leq j-1} 2^{2k} \|\Delta_k\theta\|_{L^2}\, \mathrm{d}\tau \\
&\leq \sup_j \sup_{0\leq\tau\leq t} 2^{sj} \|\Delta_j\theta\|_{L^2} \sup_j \sum_{k=-1}^{j-1} 2^{2k} \int_0^t \|\Delta_k\theta\|_{L^2}\, \mathrm{d}\tau \\
&\leq \sup_{0\leq\tau\leq t} \sup_j 2^{sj} \|\Delta_j\theta\|_{L^2} \sup_j \max_{-1\leq k\leq j-1} 2^{(r+2\alpha)k} \int_0^t \|\Delta_k\theta\|_{L^2}\, \mathrm{d}\tau \sum_{k=-1}^{j-1} 2^{(2-2\alpha-r)k} \\
&= C\|\theta\|_{L^\infty_t(B^s_{2,\infty})} \|\theta\|_{\widetilde{L}^1_t(B^{r+2\alpha}_{2,\infty})} \quad \text{for any } r > 2-2\alpha.
\end{aligned} \tag{3.16}
$$

I_6 can be similarly bounded. Putting these bounds together gives the estimate (3.3).

To prove (3), we multiply both sides of (3.9) by $q2^{rqj}\|\Delta_j\theta\|_{L^2}^{q-1}$, integrate over $[0,t]$ and sum over j to obtain

$$\|\theta(t)\|^q_{B^r_{2,q}} + Cq\kappa\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})} \le \|\theta_0\|^q_{B^r_{2,q}} + I_7 + I_8 + I_9, \tag{3.17}$$

where

$$I_7 = q\sum_j \int_0^t 2^{rqj+2j}\|\Delta_j\theta\|^{q+1}_{L^2}\,\mathrm{d}\tau,$$

$$I_8 = q\sum_j \int_0^t 2^{rqj}\|\Delta_j\theta\|^q_{L^2}\sum_{k\le j-1} 2^{2k}\|\Delta_k\theta\|_{L^2}\,\mathrm{d}\tau,$$

$$I_9 = q\sum_j \int_0^t 2^{rqj}\,2^{2j}\sum_{k\ge j-1}\|\Delta_k\theta\|^2_{L^2}\,\mathrm{d}\tau.$$

We bound I_7 and I_8 as follows

$$\begin{aligned}
I_7 &= q\sum_j \int_0^t 2^{rj}\|\Delta_j\theta\|_{L^2} 2^{(r+\frac{2-r}{q})qj}\|\Delta_j\theta\|^q_{L^2}\,\mathrm{d}\tau\\
&\le q\sum_j \sup_{0\le\tau\le t} 2^{rj}\|\Delta_j\theta\|_{L^2}\int_0^t 2^{(r+\frac{2-r}{q})qj}\|\Delta_j\theta\|^q_{L^2}\,\mathrm{d}\tau\\
&\le q\sup_j \sup_{0\le\tau\le t} 2^{rj}\|\Delta_j\theta\|_{L^2}\sum_j 2^{(r+\frac{2-r}{q})qj}\int_0^t \|\Delta_j\theta\|^q_{L^2}\,\mathrm{d}\tau\\
&\le Cq\sup_{0\le\tau\le t}\sup_j 2^{rj}\|\Delta_j\theta\|_{L^2}\sum_j 2^{(r+\frac{2\alpha}{q})qj}\int_0^t \|\Delta_j\theta\|^q_{L^2}\,\mathrm{d}\tau\\
&\le Cq\sup_{0\le\tau\le t}\left(\sum_j 2^{rjq}\|\Delta_j\theta\|^q_{L^2}\right)^{1/q}\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}\\
&= Cq\|\theta\|_{L^\infty_t(B^r_{2,q})}\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}.\\
I_8 &= q\sum_j \int_0^t 2^{2\alpha j+rqj}\|\Delta_j\theta\|^q_{L^2}\sum_{k\le j-1} 2^{2\alpha(k-j)}2^{(2-2\alpha)k}\|\Delta_k\theta\|_{L^2}\,\mathrm{d}\tau\\
&\le q\sup_j \sup_{0\le\tau\le t}\sum_{k\le j-1} 2^{2\alpha(k-j)}\,2^{(2-2\alpha)k}\|\Delta_k\theta\|_{L^2}\sum_j\int_0^t 2^{(r+\frac{2\alpha}{q})qj}\|\Delta_j\theta\|^q_{L^2}\,\mathrm{d}\tau\\
&\le Cq\sup_{0\le\tau\le t}\sup_j 2^{(2-2\alpha)j}\|\Delta_j\theta\|_{L^2}\,\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}\\
&\le Cq\sup_{0\le\tau\le t}\sup_j 2^{rj}\|\Delta_j\theta\|_{L^2}\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}\\
&= Cq\|\theta\|_{L^\infty_t(B^r_{2,q})}\|\theta\|^q_{\widetilde{L}^q_t(B^{r+\frac{2\alpha}{q}}_{2,q})}.
\end{aligned}$$

I_9 is similarly bounded. After inserting these estimates in (3.17), we obtain (3.4). □

As in the proof of Theorem 3.1 of [17], we can establish the following theorem concerning solutions of the IVP (3.1) in the inhomogeneous Besov space $B^r_{2,q}$. The proof combines the a priori estimates of Theorem 3.1 and the method of successive approximation. We omit the details of the proof.

Theorem 3.2. *Let $\kappa > 0$ and $\alpha > 0$. Let $1 \le q \le \infty$ and $r \ge 2 - 2\alpha$. Consider the IVP (3.1) with $\theta_0 \in B^r_{2,q}(\mathbb{R}^2)$. There exists a constant C_0 depending on α, r and q only such that if*

$$\|\theta_0\|_{B^r_{2,q}} \le C_0\kappa,$$

then the IVP (3.1) has a unique solution $\theta \in B^r_{2,q}(\mathbb{R}^2)$ satisfying

$$\|\theta(\cdot, t)\|_{B^r_{2,q}} \le C_1\kappa$$

for any $t > 0$ and some constant C_1 depending on α, r and q. In addition, $\theta \in \widetilde{L^1}((0, \infty); B^{r+2\alpha}_{2,\infty})$ in the case when $q = \infty$ and $\theta \in L^q((0, \infty); B^{r+\frac{2\alpha}{q}}_{2,q})$ in the case when $q < \infty$.

Setting $q = 2$ and $\alpha = \frac{1}{2}$ in Theorem 3.2, we obtain the following corollary on H^1-solutions of the 2-D critical QG equation.

Corollary 3.3. *Consider the IVP (3.1) with $\alpha = \frac{1}{2}$ and $\theta_0 \in H^1(\mathbb{R}^2)$. There exists a constant C_2 such that if*

$$\|\theta_0\|_{H^1} \le C_2\kappa,$$

then the IVP (3.1) has a unique solution θ satisfying

$$\theta \in C([0, \infty); H^1) \cap L^2((0, \infty); H^{\frac{3}{2}})$$

and, for some constant C_3,

$$\|\theta(\cdot, t)\|_{H^1} \le C_3\kappa \quad \text{for any } t > 0.$$

4. Solutions in $\mathring{B}^r_{p,q}$

In this section we study solutions of the IVP (3.1) in the homogeneous Besov space $\mathring{B}^{1-2\alpha+\frac{2}{p}}_{p,q}$ for $p \in [2, \infty)$ and $q \in [1, \infty]$. We start with the following a priori estimates.

Theorem 4.1. *Assume either $\alpha > 0$ and $p = 2$ or $0 < \alpha \le 1$ and $2 < p < \infty$. Let $1 \le q \le \infty$, and $-\infty < r < \infty$. Consider the IVP (3.1) with $\theta_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^2)$. Then the corresponding solution θ of (3.1) obeys the following a priori estimates.*

(1) *In the case when $q < \infty$, we have*

$$\|\theta(t)\|^q_{\mathring{B}^r_{p,q}} + C_1\kappa\|\theta\|^q_{L^q_t(\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})} \le \|\theta_0\|^q_{\mathring{B}^r_{p,q}} + C_2\|\theta\|_{L^\infty_t(\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,q})}\|\theta\|^q_{L^q_t(\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})}.$$

(2) *In the case when $q = \infty$, we have*

$$\|\theta(t)\|_{\mathring{B}^r_{p,\infty}} + C_1\kappa\|\theta\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})} \le \|\theta_0\|_{\mathring{B}^r_{p,\infty}} + C_2\|\theta\|_{L^\infty_t(\mathring{B}^{1-2\alpha+\frac{d}{p}}_{p,\infty})}\|\theta\|_{\widetilde{L}^1_t(\mathring{B}^{r+2\alpha}_{p,\infty})}.$$

Proof. We first consider the case when $1 \le q < \infty$. Multiplying (3.5) by $p|\Delta_j\theta|^{p-2}\Delta_j\theta$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|^p_{L^p} + \kappa p J_1 = pJ_2, \tag{4.1}$$

where J_1 and J_2 are given by

$$J_1 = \int_{\mathbb{R}^2} |\Delta_j\theta|^{p-2}\Delta_j\theta(-\Delta)^\alpha\Delta_j\theta\,\mathrm{d}x,$$

$$J_2 = \int_{\mathbb{R}^2} |\Delta_j\theta|^{p-2}\Delta_j\theta[u\cdot\nabla, \Delta_j]\theta\,\mathrm{d}x.$$

Applying the lower bound in Proposition 2.5, we have

$$J_1 \geq C\, 2^{2\alpha j}\, \|\Delta_j \theta\|_{L^p}^p. \tag{4.2}$$

By Hölder's inequality,

$$J_2 \leq C\, \|\Delta_j \theta\|_{L^p}^{p-1} \|[u\cdot\nabla, \Delta_j]\theta\|_{L^p}. \tag{4.3}$$

Inserting (4.2) and (4.3) in (4.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^p} + C\kappa 2^{2\alpha j}\, \|\Delta_j\theta\|_{L^p} \leq C\|[u\cdot\nabla, \Delta_j]\theta\|_{L^p}.$$

Applying Proposition A.2 to the right-hand side yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\theta\|_{L^p} + C\kappa 2^{2\alpha j}\, \|\Delta_j\theta\|_{L^p} \leq C\left(2^{(1+\frac{2}{p})j}\|\Delta_j\theta\|_{L^p}^2 + \|\Delta_j\theta\|_{L^p} \sum_{-\infty<k\leq j-1} 2^{(1+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p}\right). \tag{4.4}$$

For the sake of brevity, we have intentionally omitted the interaction term of high–high frequencies. As we have seen in the previous section, this term can be similarly dealt with. Multiplying this inequality by $q2^{rqj}\, \|\Delta_j\theta\|_{L^p}^{q-1}$, integrating over $[0, t]$ and summing over $j \in \mathbb{Z}$, we obtain

$$\|\theta(t)\|^q_{\mathring{B}^r_{p,q}} + C\kappa q\|\theta\|^q_{L^q_t(\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})} \leq \|\theta_0\|^q_{\mathring{B}^r_{p,q}} + J_3 + J_4,$$

where J_3 and J_4 are given by

$$J_3 = C\, q\sum_j \int_0^t 2^{rqj} 2^{(1+\frac{2}{p})j}\|\Delta_j\theta\|_{L^p}^{q+1}\, \mathrm{d}\tau,$$

$$J_4 = C\, q\sum_j \int_0^t 2^{rqj}\|\Delta_j\theta\|_{L^p}^{q} \sum_{-\infty<k\leq j-1} 2^{(1+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p}\, \mathrm{d}\tau.$$

To bound J_3, we first rewrite it as

$$J_3 = Cq\sum_j \int_0^t 2^{(1-2\alpha+\frac{2}{p})j}\|\Delta_j\theta\|_{L^p} 2^{rqj+2\alpha j}\|\Delta_j\theta\|_{L^p}^{q}\, \mathrm{d}\tau.$$

According to Lemma 2.2,

$$\begin{aligned} J_3 &\leq Cq\int_0^t \sum_j \|\theta(\tau)\|_{\mathring{B}^{1-2\alpha+\frac{2}{p}}_{p,q}} 2^{(r+\frac{2\alpha}{q})qj}\|\Delta_j\theta\|_{L^p}^q\, \mathrm{d}\tau \\ &\leq Cq\sup_{\tau\in[0,t]}\|\theta(\tau)\|_{\mathring{B}^{1-2\alpha+\frac{2}{p}}_{p,q}} \int_0^t\sum_j 2^{(r+\frac{2\alpha}{q})qj}\|\Delta_j\theta\|_{L^p}^q\, \mathrm{d}\tau \\ &= Cq\|\theta\|_{L^\infty_t(\mathring{B}^{1-2\alpha+\frac{2}{p}}_{p,q})}\|\theta\|^q_{L^q_t(\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})}. \end{aligned}$$

To bound J_4, we start by writing it as

$$J_4 = Cq\sum_j\int_0^t 2^{(r+\frac{2\alpha}{q})qj}\|\Delta_j\theta\|_{L^p}^q \sum_{k\leq j-1} 2^{2\alpha(k-j)}\, 2^{(1-2\alpha+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p}\, \mathrm{d}\tau.$$

Since

$$\begin{aligned} \sum_{k\leq j-1} 2^{2\alpha(k-j)}\, 2^{(1-2\alpha+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p} &\leq C\left(\sum_{k\leq j-1}[2^{(1-2\alpha+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p}]^q\right)^{\frac{1}{q}} \\ &\leq C\, \|\theta\|_{\mathring{B}^{1-2\alpha+\frac{2}{p}}_{p,q}}, \end{aligned}$$

we have

$$J_4 \le C\, q \|\theta\|_{L_t^\infty(\mathring{B}_{p,q}^{1-2\alpha+\frac{2}{p}})} \|\theta\|^q_{L_t^q(\mathring{B}_{p,q}^{r+\frac{2\alpha}{q}})}.$$

This completes the proof for (1).

In the case when $q = \infty$, we multiply (4.4) by 2^{rj}, integrate over $[0, t]$ and take the supremum over $j \in \mathbb{Z}$. This results in the inequality

$$\|\theta(t)\|_{\mathring{B}^r_{p,\infty}} + C\kappa \|\theta\|_{\widetilde{L}_t^1(\mathring{B}^{r+2\alpha}_{p,\infty})} \le \|\theta_0\|_{\mathring{B}^r_{p,\infty}} + C J_5 + C J_6,$$

where J_5 and J_6 are given by

$$J_5 = \sup_j \int_0^t 2^{rj} 2^{\left(1+\frac{2}{p}\right)j} \|\Delta_j\theta\|^2_{L^p}\, \mathrm{d}\tau,$$

$$J_6 = \sup_j \int_0^t 2^{rj} \|\Delta_j\theta\|_{L^p} \sum_{k\le j-1} 2^{(1+\frac{2}{p})k} \|\Delta_k\theta\|_{L^p}\, \mathrm{d}\tau.$$

J_5 is bounded by

$$J_5 \le \sup_j \sup_{\tau\in[0,t]} 2^{(1-2\alpha+\frac{2}{p})j} \|\Delta_j\theta(\tau)\|_{L^p} \|\theta\|_{\widetilde{L}_t^1(\mathring{B}^{r+2\alpha}_{p,\infty})} = \|\theta\|_{L_t^\infty(\mathring{B}_{p,\infty}^{1-2\alpha+\frac{2}{p}})} \|\theta\|_{\widetilde{L}_t^1(\mathring{B}^{r+2\alpha}_{p,\infty})}.$$

J_6 can be bounded as follows

$$\begin{aligned} J_6 &= \sup_j \int_0^t 2^{(r+2\alpha)j}\, \|\Delta_j\theta\|_{L^p} \sum_{k\le j-1} 2^{2\alpha(k-j)}\, 2^{(1-2\alpha+\frac{2}{p})k} \|\Delta_k\theta\|_{L^p}\, \mathrm{d}\tau \\ &\le \sup_j \int_0^t 2^{(r+2\alpha)j}\, \|\Delta_j\theta\|_{L^p} \sup_k 2^{(1-2\alpha+\frac{2}{p})k} \|\Delta_k\theta\|_{L^p} \sum_{k\le j-1} 2^{2\alpha(k-j)}\, \mathrm{d}\tau \\ &= C\, \|\theta\|_{L_t^\infty(\mathring{B}_{p,\infty}^{1-2\alpha+\frac{2}{p}})} \|\theta\|_{\widetilde{L}_t^1(\mathring{B}^{r+2\alpha}_{p,\infty})}. \end{aligned}$$

This completes the proof for the case $q = \infty$ and thus the proof of Theorem 4.1. □

Combining these a priori estimates with the method of successive approximation allows us to prove the following existence and uniqueness result.

Theorem 4.2. *Assume either $\alpha > 0$ and $p = 2$ or $0 < \alpha \le 1$ and $2 < p < \infty$. Let $1 \le q \le \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$. Consider the IVP (3.1) with $\theta_0 \in \mathring{B}^r_{p,q}$. Then there exists a constant C_0 such that if*

$$\|\theta_0\|_{\mathring{B}^r_{p,q}} \le C_0\kappa,$$

then the IVP (3.1) has a unique solution θ satisfying

$$\|\theta\|^q_{L^\infty([0,\infty);\mathring{B}^r_{p,q})} + C_1\kappa \|\theta\|^q_{L^q((0,\infty),\mathring{B}_{p,q}^{r+\frac{2\alpha}{q}})} \le C_2\, \kappa^q$$

in the case when $1 \le q < \infty$, and

$$\|\theta\|_{L^\infty([0,\infty);\mathring{B}^r_{p,\infty})} + C_1\kappa \|\theta\|_{\widetilde{L}^1((0,\infty),\mathring{B}^{r+2\alpha}_{p,\infty})} \le C_2\, \kappa$$

in the case when $q = \infty$, where C_1 and C_2 are constants depending on α, p and q only.

Remark. Although Theorem 4.2 does not cover the case when $1 \le p < 2$, the global existence of solutions for $\theta_0 \in \mathring{B}^r_{p,q}$ with $1 \le p < 2$ can be established by combining this theorem with the Besov embedding stated in Proposition 2.1. In fact, for any $1 \le p_1 < 2$ and $r_1 = 1 - 2\alpha + \frac{2}{p_1}$, we can choose $p_2 \ge 2$ and $r_2 = 1 - 2\alpha + \frac{2}{p_2}$ such that $r_1 - \frac{2}{p_1} = r_2 - \frac{2}{p_2}$. By the Besov embedding,

$$\theta_0 \in \mathring{B}^{r_1}_{p_1,q}(\mathbb{R}^2) \subset \mathring{B}^{r_2}_{p_2,q}(\mathbb{R}^2).$$

Theorem 4.2 then concludes that θ_0 leads to a global solution.

Proof of Theorem 4.2. We sketch the proof of this theorem very briefly. It consists of two major steps. The first step is to consider a successive approximation sequence $\{\theta^{(n)}\}_{n=1}^{\infty}$ satisfying

$$\begin{cases} \theta^{(1)} = S_2\,\theta_0, \\ u^{(n)} = \nabla^{\perp}\Lambda^{-1}\,\theta^{(n)}, \\ \partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} + \kappa(-\Delta)^{\alpha}\theta^{(n+1)} = 0, \\ \theta^{(n+1)}(x,0) = \theta_0^{(n+1)}(x) = S_{n+2}\theta_0(x) \end{cases}$$

and show that $\{\theta^{(n)}\}_{n=1}^{\infty}$ is bounded uniformly in $L^{\infty}([0,\infty);\mathring{B}^r_{p,q})$. More precisely, we show that if $\|\theta_0\|_{\mathring{B}^r_{p,q}} \le C_0\kappa$, then

$$\|\theta^{(n)}\|^q_{L^{\infty}([0,\infty);\mathring{B}^r_{p,q})} + C_1\kappa\|\theta^{(n)}\|^q_{L^q((0,\infty),\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})} \le C_2\,\kappa^q$$

in the case when $1 \le q < \infty$, and

$$\|\theta^{(n)}\|_{L^{\infty}([0,\infty);\mathring{B}^r_{p,\infty})} + C_1\kappa\|\theta^{(n)}\|_{\widetilde{L^1}((0,\infty),\mathring{B}^{r+2\alpha}_{p,\infty})} \le C_2\,\kappa$$

in the case when $q = \infty$, where C_1 and C_2 are constants independent of κ and n.

The second step proves that $\{\theta^{(n)}\}$ is a Cauchy sequence in $L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,q})$. That is, we show the sequence $\{\eta^{(n)}\}$ with $\eta^{(n)} = \theta^{(n)} - \theta^{(n-1)}$ satisfies

$$\|\eta^{(n)}\|_{L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,q})} \le C\,\|\theta_0\|_{\mathring{B}^r_{p,q}}\,2^{-n}.$$

For this purpose, we consider the equations that $\{\eta^{(n)}\}$ satisfies

$$\begin{cases} \eta^{(1)} = S_2\theta_0 - \theta_0, \\ w^{(n)} = \nabla^{\perp}\Lambda^{-1}\,\eta^{(n)}, \\ \partial_t\eta^{(n+1)} + u^{(n)}\cdot\nabla\eta^{(n+1)} + \kappa(-\Delta)^{\alpha}\eta^{(n+1)} = w^{(n)}\cdot\nabla\theta^{(n)}, \\ \eta^{(n+1)}(x,0) = \eta_0^{(n+1)}(x) = \Delta_{n+1}\theta_0 \end{cases}$$

and prove that

$$\|\eta^{(n)}\|^q_{L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,q})} + C_1\kappa\|\eta^{(n)}\|^q_{L^q((0,\infty),\mathring{B}^{r-1+\frac{2\alpha}{q}}_{p,q})} \le C_2\,\kappa^q\,2^{-qn}$$

in the case when $1 \le q < \infty$, and

$$\|\eta^{(n)}\|_{L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,\infty})} + C_1\kappa\|\eta^{(n)}\|_{\widetilde{L^1}((0,\infty),\mathring{B}^{r-1+2\alpha}_{p,\infty})} \le C_2\,\kappa 2^{-n}$$

in the case when $q = \infty$. Therefore, there exists

$$\theta \in L^{\infty}([0,\infty);\mathring{B}^r_{p,q}) \cap L^q\left((0,\infty),\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right)$$

for $1 \le q < \infty$ and

$$\theta \in L^{\infty}([0,\infty);\mathring{B}^r_{p,\infty}) \cap \widetilde{L^1}\left((0,\infty),\mathring{B}^{r+2\alpha}_{p,\infty}\right)$$

for $q = \infty$ such that

$$\theta^{(n)} \to \theta \quad \text{in } L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,q}) \cap L^q\left((0,\infty),\mathring{B}^{r-1+\frac{2\alpha}{q}}_{p,q}\right) \text{ for } 1 \le q < \infty,$$

$$\theta^{(n)} \to \theta \quad \text{in } L^{\infty}([0,\infty);\mathring{B}^{r-1}_{p,\infty}) \cap \widetilde{L^1}\left((0,\infty),\mathring{B}^{r-1+2\alpha}_{p,\infty}\right) \text{ for } q = \infty.$$

One can then easily verify that θ satisfies the 2-D QG equation

$$\partial_t\theta + u\cdot\nabla\theta + \kappa(-\Delta)^{\alpha}\theta = 0$$

in $\mathring{B}^{r-1}_{p,q}$, where $u = \nabla^{\perp}\Lambda^{-1}\,\theta$. We omit further details. □

5. Uniqueness

This section addresses the issue of the uniqueness of solutions to the IVP (3.1) in Besov spaces. The major results are presented in four theorems.

Theorem 5.1. *Let* $\alpha > 0$. *Let* $r \geq 2 - 2\alpha$ *and* $1 \leq q \leq \infty$. *Let* $T > 0$ *and let*

$$\theta \in L^\infty([0,T); B^r_{2,q}(\mathbb{R}^2)) \quad \text{and} \quad \widetilde{\theta} \in L^\infty([0,T); B^r_{2,q}(\mathbb{R}^2))$$

be the solutions of the IVP (3.1) corresponding to the initial data

$$\theta_0 \in B^r_{2,q}(\mathbb{R}^2) \quad \text{and} \quad \widetilde{\theta}_0 \in B^r_{2,q}(\mathbb{R}^2),$$

respectively. Let $s < 1 - \frac{2\alpha}{q}$. *If* $\eta_0 = \widetilde{\theta}_0 - \theta_0$ *is in* $B^s_{2,q}$, *then the difference*

$$\eta = \widetilde{\theta} - \theta$$

satisfies for $1 \leq q < \infty$

$$\|\eta\|^q_{L^\infty_t(B^s_{2,q})} + C\kappa \|\eta\|^q_{L^q_t(B^{s+\frac{2\alpha}{q}}_{2,q})} \leq \|\eta_0\|^q_{B^s_{2,q}} + C\left(\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,q})} + \|\widetilde{\theta}\|_{L^\infty_t(B^{2-2\alpha}_{2,q})}\right)\|\eta\|^q_{L^q_t(B^{s+\frac{2\alpha}{q}}_{2,q})} \tag{5.1}$$

and for $q = \infty$

$$\|\eta\|_{L^\infty_t(B^s_{2,\infty})} + C\kappa\|\eta\|_{\widetilde{L}^1_t(B^{s+2\alpha}_{2,\infty})} \leq \|\eta_0\|_{B^s_{2,\infty}} + C\left(\|\theta\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})} + \|\widetilde{\theta}\|_{L^\infty_t(B^{2-2\alpha}_{2,\infty})}\right)\|\eta\|_{\widetilde{L}^1_t(B^{s+2\alpha}_{2,\infty})} \tag{5.2}$$

for any $t \leq T$. *In particular, if* $\theta_0 = \widetilde{\theta}_0$ *and*

$$\|\theta\|_{L^\infty_T(B^{2-2\alpha}_{2,q})} \leq C\kappa, \qquad \|\widetilde{\theta}\|_{L^\infty_T(B^{2-2\alpha}_{2,q})} \leq C\kappa$$

for some suitable constant C, *then* $\theta = \widetilde{\theta}$.

In the special case when $\alpha = \frac{1}{2}$ and $r = 1$, this theorem reduces to a corollary on the uniqueness of H^1 solutions of the 2-D critical QG equation.

Corollary 5.2. *Let* $\alpha = \frac{1}{2}$ *and* $T > 0$. *Assume that*

$$\theta \in L^\infty([0,T); H^1(\mathbb{R}^2)) \quad \text{and} \quad \widetilde{\theta} \in L^\infty([0,T); H^1(\mathbb{R}^2))$$

are solutions of the IVP (3.1) corresponding to the initial data $\theta_0 \in H^1(\mathbb{R}^2)$ *and* $\widetilde{\theta}_0 \in H^1(\mathbb{R}^2)$, *respectively. Then there exists a constant* C *such that if*

$$\|\theta\|_{L^\infty_T(H^1)} \leq C\kappa \quad \text{and} \quad \|\widetilde{\theta}\|_{L^\infty_T(H^1)} \leq C\kappa,$$

then $\theta_0 = \widetilde{\theta}_0$ *implies* $\theta = \widetilde{\theta}$.

Proof of Theorem 5.1. Let u and $\widetilde{u}$ be the corresponding velocities, namely

$$u = \nabla^\perp \Lambda^{-1}\theta \quad \text{and} \quad \widetilde{u} = \nabla^\perp \Lambda^{-1}\widetilde{\theta}.$$

The difference $\eta = \widetilde{\theta} - \theta$ satisfies the equation

$$\partial_t \eta + \widetilde{u}\cdot\nabla\eta + w\cdot\nabla\theta + \kappa(-\Delta)^\alpha\eta = 0,$$

where $w = \widetilde{u} - u$. Applying Δ_j to this equation, we have

$$\partial_t\Delta_j\eta + \widetilde{u}\cdot\nabla\Delta_j\eta + \kappa(-\Delta)^\alpha\Delta_j\eta = [\widetilde{u}\cdot\nabla, \Delta_j]\eta - \Delta_j(w\cdot\nabla\theta).$$

Multiplying by $\Delta_j\eta$, integrating over $\mathbb{R}^2$, bounding the dissipative term from below and applying the Hölder inequality to the terms on the right, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\eta\|_{L^2} + C\kappa 2^{2\alpha j}\|\Delta_j\eta\|_{L^2} \leq C\|[\widetilde{u}\cdot\nabla, \Delta_j]\eta\|_{L^2} + C\|\Delta_j(w\cdot\nabla\theta)\|_{L^2}. \tag{5.3}$$

For $1 \le q < \infty$, we multiply this inequality by $2^{sjq}\|\Delta_j\eta\|_{L^2}^{q-1}$, integrate over $[0,t]$ and sum over j to obtain

$$\|\eta(t)\|_{B^s_{2,q}}^q + C\kappa\|\eta\|^q_{L^q_t(B^{s+\frac{2\alpha}{q}}_{2,q})} \le K_1 + K_2,$$

where K_1 and K_2 are given by

$$K_1 = C\sum_j \int_0^t 2^{sjq}\|\Delta_j\eta\|_{L^2}^{q-1}\,\|[\widetilde{u}\cdot\nabla, \Delta_j]\eta\|_{L^2}\,\mathrm{d}\tau,$$

$$K_2 = C\sum_j \int_0^t 2^{sjq}\|\Delta_j\eta\|_{L^2}^{q-1}\,\|\Delta_j(w\cdot\nabla\theta)\|_{L^2}\,\mathrm{d}\tau.$$

To estimate K_1, we first change the order of summation and time integration and then apply Proposition A.2 to obtain

$$K_1 \le \int_0^t (K_{11} + K_{12} + K_{13})\,\mathrm{d}t$$

with K_{11}, K_{12} and K_{13} being given by

$$K_{11} = C\sum_j 2^{sjq}\|\Delta_j\eta\|_{L^2}^q \sum_{m\le j-1} 2^{2m}\|\Delta_m\widetilde{u}\|_{L^2},$$

$$K_{12} = C\sum_j 2^{sjq}\|\Delta_j\eta\|_{L^2}^{q-1}\,\|\Delta_j\widetilde{u}\|_{L^2} \sum_{m\le j-1} 2^{2m}\|\Delta_m\eta\|_{L^2},$$

$$K_{13} = C\sum_j 2^{sjq}\|\Delta_j\eta\|_{L^2}^{q-1}\,\|\Delta_j\widetilde{u}\|_{L^2}\,2^{2j}\|\Delta_j\eta\|_{L^2}.$$

For K_{11}, we have

$$\begin{aligned} K_{11} &= C\sum_j 2^{(s+\frac{2\alpha}{q})jq}\|\Delta_j\eta\|_{L^2}^q \sum_{m\le j-1} 2^{2\alpha(m-j)}\,2^{(2-2\alpha)m}\|\Delta_m\widetilde{u}\|_{L^2} \\ &\le C\sum_j 2^{(s+\frac{2\alpha}{q})jq}\|\Delta_j\eta\|_{L^2}^q \left(\sum_{k\le j-1} 2^{2\alpha(m-j)q/(q-1)}\right)^{1-\frac{1}{q}} \|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}} \\ &= C\,\|\eta\|^q_{B^{s+\frac{2\alpha}{q}}_{2,q}}\,\|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}}. \end{aligned} \tag{5.4}$$

For $s < 2 - \frac{2\alpha}{q}$, K_{12} can be bounded as follows

$$\begin{aligned} K_{12} &= C\sum_j 2^{(s+\frac{2\alpha}{q})j(q-1)}\|\Delta_j\eta\|_{L^2}^{q-1}\,2^{(2-2\alpha)j}\|\Delta_j\widetilde{u}\|_{L^2} \sum_{m\le j-1} 2^{(2-s-\frac{2\alpha}{q})(m-j)}\,2^{(s+\frac{2\alpha}{q})m}\|\Delta_m\eta\|_{L^2} \\ &\le C\,\|\eta\|^{q-1}_{B^{s+\frac{2\alpha}{q}}_{2,q}}\,\|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}} \left(\sum_{k\le j-1} 2^{(2-s-\frac{2\alpha}{q})(m-j)q/(q-1)}\right)^{1-\frac{1}{q}} \left(\sum_{k\le j-1} 2^{(s+\frac{2\alpha}{q})mq}\|\Delta_m\eta\|_{L^2}^q\right)^{\frac{1}{q}} \\ &\le C\,\|\eta\|^q_{B^{s+\frac{2\alpha}{q}}_{2,q}}\,\|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}}. \end{aligned}$$

$$K_{13} = \sum_j 2^{\left(s+\frac{2\alpha}{q}\right)jq}\|\Delta_j\eta\|_{L^2}^q\,2^{(2-2\alpha)j}\|\Delta_j\widetilde{u}\|_{L^2} \le C\,\|\eta\|^q_{B^{s+\frac{2\alpha}{q}}_{2,q}}\,\|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}}.$$

To bound K_2, we obtain after applying Proposition A.2

$$K_2 \le C\int_0^t (K_{21} + K_{22} + K_{23})\,\mathrm{d}\tau,$$

where K_{21}, K_{22} and K_{23} are given by

$$K_{21} = \sum_j 2^{sjq} \|\Delta_j \eta\|_{L^2}^{q-1} \|S_{j-1} w\|_{L^\infty} \|\nabla \Delta_j \theta\|_{L^2},$$

$$K_{22} = \sum_j 2^{sjq} \|\Delta_j \eta\|_{L^2}^{q-1} \|\Delta_j w\|_{L^2} \|\nabla S_{j-1} \theta\|_{L^\infty},$$

$$K_{23} = \sum_j 2^{sjq} \|\Delta_j \eta\|_{L^2}^{q-1} \|\Delta_j w\|_{L^2} \|\nabla \Delta_j \theta\|_{L^\infty}.$$

For $s < 1 - \frac{2\alpha}{q}$, K_{21} is bounded by

$$\begin{aligned} K_{21} &\le \sum_j 2^{sjq} \|\Delta_j \eta\|_{L^2}^{q-1} 2^j \|\Delta_j \theta\|_{L^2} \sum_{m \le j-1} 2^m \|\Delta_m w\|_{L^2} \\ &= \sum_j 2^{(s+\frac{2\alpha}{q})j(q-1)} \|\Delta_j \eta\|_{L^2}^{q-1} 2^{(2-2\alpha)j} \|\Delta_j \theta\|_{L^2} \sum_{m \le j-1} 2^{(1-s-\frac{2\alpha}{q})(m-j)} 2^{(s+\frac{2\alpha}{q})m} \|\Delta_m w\|_{L^2} \\ &\le C \|\eta\|^{q-1}_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|\theta\|_{B^{2-2\alpha}_{2,q}} \|w\|_{B^{s+\frac{2\alpha}{q}}_{2,q}}. \end{aligned}$$

For K_{22} and K_{23}, we have

$$\begin{aligned} K_{22} &\le \sum_j 2^{(s+\frac{2\alpha}{q})j(q-1)} \|\Delta_j \eta\|_{L^2}^{q-1} 2^{(s+\frac{2\alpha}{q})j} \|\Delta_j w\|_{L^2} \sum_{m \le j-1} 2^{2\alpha(m-j)} 2^{(2-2\alpha)m} \|\Delta_m \theta\|_{L^2} \\ &\le C \|\eta\|^{q-1}_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|w\|_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|\theta\|_{B^{2-2\alpha}_{2,q}}, \\ K_{23} &\le \sum_j 2^{(s+\frac{2\alpha}{q})j(q-1)} \|\Delta_j \eta\|_{L^2}^{q-1} 2^{(s+\frac{2\alpha}{q})j} \|\Delta_j w\|_{L^2} 2^{(2-2\alpha)j} \|\Delta_j \theta\|_{L^2} \\ &\le C \|\eta\|^{q-1}_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|w\|_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|\theta\|_{B^{2-2\alpha}_{2,q}}. \end{aligned}$$

Combining the estimates for K_1 and K_2, we obtain

$$\begin{aligned} \|\eta(t)\|^q_{B^s_{2,q}} + C\kappa \|\eta\|^q_{L^q_t(B^{s+\frac{2\alpha}{q}}_{2,q})} &\le \|\eta_0\|^q_{B^s_{2,q}} + C\int_0^t \|\eta\|^q_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}} \, \mathrm{d}\tau \\ &\quad + C\int_0^t \|\eta\|^{q-1}_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|w\|_{B^{s+\frac{2\alpha}{q}}_{2,q}} \|\theta\|_{B^{2-2\alpha}_{2,q}} \, \mathrm{d}\tau. \end{aligned} \tag{5.5}$$

Since $\widetilde{u} = \nabla^\perp \Lambda^{-1} \widetilde{\theta}$ and $w = \nabla^\perp \Lambda^{-1} \eta$ are Riesz transforms of $\widetilde{\theta}$ and η, respectively,

$$\|\widetilde{u}\|_{B^{2-2\alpha}_{2,q}} \le \|\widetilde{\theta}\|_{B^{2-2\alpha}_{2,q}} \quad \text{and} \quad \|w\|_{B^{s+\frac{2\alpha}{q}}_{2,q}} \le \|\eta\|_{B^{s+\frac{2\alpha}{q}}_{2,q}}$$

according to the boundedness of Riesz transforms on L^2. Inserting these estimates in (5.5), we establish (5.1).

In the case when $q = \infty$, we integrate (5.3) with respect to t, multiply by 2^{sj} and take the supremum over j to obtain

$$\|\eta(t)\|_{B^s_{2,\infty}} + C\kappa \|\eta\|_{\widetilde{L}^1_t(B^{s+2\alpha}_{2,\infty})} \le K_3 + K_4,$$

where K_3 and K_4 are given by

$$K_3 = \sup_j 2^{sj} \int_0^t \|[\widetilde{u} \cdot \nabla, \Delta_j]\eta\|_{L^2} \, \mathrm{d}\tau, \qquad K_4 = C \sup_j 2^{sj} \int_0^t \|\Delta_j (w \cdot \nabla \theta)\|_{L^2} \, \mathrm{d}\tau.$$

As in the estimates for K_1 and K_2, we split each of K_3 and K_4 into three terms. For the sake of brevity, we shall only provide the details for K_{31}, a term in K_3.

$$\begin{aligned}
K_{31} &= C \sup_j 2^{sj} \int_0^t \|\Delta_j \eta\|_{L^2} \sum_{m\le j-1} 2^{2m} \|\Delta_m \widetilde{u}\|_{L^2}\, \mathrm{d}\tau \\
&= C \sup_j 2^{(s+2\alpha)j} \int_0^t \|\Delta_j \eta\|_{L^2} \sum_{m\le j-1} 2^{2\alpha(m-j)}\, 2^{(2-2\alpha)m} \|\Delta_m \widetilde{u}\|_{L^2}\, \mathrm{d}\tau \\
&\le C \sup_{0\le\tau\le t} \sup_m 2^{(2-2\alpha)m} \|\Delta_m \widetilde{u}\|_{L^2} \sup_j 2^{(s+2\alpha)j} \int_0^t \|\Delta_j \eta\|_{L^2}\, \mathrm{d}\tau \\
&= C\, \|\widetilde{u}\|_{L_t^\infty(B^{2-2\alpha}_{2,\infty})} \|\eta\|_{\widetilde{L}_t^1(B^{s+2\alpha}_{2,\infty})}.
\end{aligned}$$

(5.2) is then established after combining the estimates for K_3 and K_4. □

Theorem 5.3. *Let $\alpha > 0$. Let $r \ge 2 - 2\alpha$ and $1 \le q < \infty$. Let $T > 0$ and let*

$$\theta \in L^q\left((0,T); B^{r+\frac{2\alpha}{q}}_{2,q}(\mathbb{R}^2)\right) \quad \textit{and} \quad \widetilde{\theta} \in L^q\left((0,T); B^{r+\frac{2\alpha}{q}}_{2,q}(\mathbb{R}^2)\right)$$

be the solutions of the IVP (3.1) corresponding to the initial data θ_0 and $\widetilde{\theta}_0$, respectively. Let $s < 2$. If $\eta_0 = \widetilde{\theta}_0 - \theta_0$ is in $B^s_{2,q}$, then the difference

$$\eta = \widetilde{\theta} - \theta$$

satisfies, for any $t \le T$,

$$\|\eta(t)\|^q_{B^s_{2,q}} \le \|\eta_0\|^q_{B^s_{2,q}} + C \int_0^t \left(\|\theta(\tau)\|^q_{B^{2-2\alpha+\frac{2\alpha}{q}}_{2,q}} + \|\widetilde{\theta}(\tau)\|^q_{B^{2-2\alpha+\frac{2\alpha}{q}}_{2,q}} \right) \|\eta(\tau)\|^q_{B^s_{2,q}}\, \mathrm{d}\tau. \tag{5.6}$$

In particular, if $\theta_0 = \widetilde{\theta}_0$, then

$$\theta = \widetilde{\theta}. \tag{5.7}$$

For the sake of brevity, we did not include the case when $q = \infty$ in this theorem. We now state as a corollary a special consequence of this theorem.

Corollary 5.4. *Let $\alpha = \frac{1}{2}$ and $T > 0$. Let θ and $\widetilde{\theta}$ satisfying*

$$\theta \in L^2((0,T); H^{\frac{3}{2}}(\mathbb{R}^2)) \quad \textit{and} \quad \widetilde{\theta} \in L^2((0,T); H^{\frac{3}{2}})$$

be two solutions of the IVP for the 2-D critical QG equation (3.1) corresponding to the initial data θ_0 and $\widetilde{\theta}_0$, respectively. If $\theta_0 = \widetilde{\theta}_0$, then $\theta = \widetilde{\theta}$.

Proof of Theorem 5.3. We estimate the difference $\eta = \widetilde{\theta} - \theta$ in $B^s_{2,q}$. We start with the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta_j \eta\|_{L^2} + C\kappa 2^{2\alpha j} \|\Delta_j \eta\|_{L^2} \le C \|[\widetilde{u}\cdot\nabla, \Delta_j]\eta\|_{L^2} + C\|\Delta_j(w\cdot\nabla\theta)\|_{L^2}.$$

Integrating with respect to t, we obtain

$$\|\Delta_j \eta\|_{L^2} \le E_j(t)\|\Delta_j \eta_0\|_{L^2} + C\int_0^t E_j(t-\tau)\|[\widetilde{u}\cdot\nabla, \Delta_j]\eta\|_{L^2}\, \mathrm{d}\tau + C\int_0^t E_j(t-\tau)\|\Delta_j(w\cdot\nabla\theta)\|_{L^2}\, \mathrm{d}\tau,$$

where $E_j(t) = \exp(-C\,\kappa 2^{2\alpha j} t)$. Multiplying both sides by 2^{sj}, raising them to the qth power and summing over j, we obtain

$$\|\eta(t)\|^q_{B^s_{2,q}} \le \|\eta_0\|^q_{B^s_{2,q}} + L_1 + L_2, \tag{5.8}$$

where

$$L_1 = C \sum_j 2^{sjq} \left(\int_0^t E_j(t-\tau) \| [\widetilde{u} \cdot \nabla, \Delta_j] \eta \|_{L^2} \, \mathrm{d}\tau \right)^q,$$

$$L_2 = C \sum_j 2^{sjq} \left(\int_0^t E_j(t-\tau) \| \Delta_j (w \cdot \nabla \theta) \|_{L^2} \, \mathrm{d}\tau \right)^q.$$

Applying Proposition A.2, we have

$$L_1 \le L_{11} + L_{12} + L_{13},$$

where

$$L_{11} = C \sum_j 2^{sjq} \left(\int_0^t E_j(t-\tau) \| \Delta_j \eta \|_{L^2} \sum_{m \le j-1} 2^{2m} \| \Delta_m \widetilde{u} \|_{L^2} \, \mathrm{d}\tau \right)^q,$$

$$L_{12} = C \sum_j 2^{sjq} \left(\int_0^t E_j(t-\tau) \| \Delta_j \widetilde{u} \|_{L^2} \sum_{m \le j-1} 2^{2m} \| \Delta_m \eta \|_{L^2} \, \mathrm{d}\tau \right)^q,$$

$$L_{13} = C \sum_j 2^{sjq} \left(\int_0^t E_j(t-\tau) \| \Delta_j \widetilde{u} \|_{L^2} 2^{2j} \| \Delta_j \eta \|_{L^2} \, \mathrm{d}\tau \right)^q.$$

We now provide the estimate for L_{11}. By Hölder's inequality,

$$\begin{aligned}
L_{11} &\le C \sum_j 2^{sjq} \left(\int_0^t E_j^{\frac{q}{q-1}}(t-\tau) \mathrm{d}s \right)^{q-1} \int_0^t \left(\| \Delta_j \eta \|_{L^2} \sum_{m \le j-1} 2^{2m} \| \Delta_m \widetilde{u} \|_{L^2} \right)^q \mathrm{d}\tau \\
&= C \sum_j 2^{sjq - 2\alpha j(q-1)} \int_0^t \left(\| \Delta_j \eta \|_{L^2} \sum_{m \le j-1} 2^{2m} \| \Delta_m \widetilde{u} \|_{L^2} \right)^q \mathrm{d}\tau \\
&= C \int_0^t \sum_j 2^{sjq} \, \| \Delta_j \eta \|_{L^2}^q \left(\sum_{m \le j-1} 2^{(2\alpha - \frac{2\alpha}{q})(m-j)} 2^{(2-2\alpha+\frac{2\alpha}{q})m} \| \Delta_m \widetilde{u} \|_{L^2} \right)^q \mathrm{d}\tau \\
&\le C \int_0^t \| \eta(\tau) \|_{B^s_{2,q}}^q \, \| \widetilde{u}(\tau) \|^q_{B^{(2-2\alpha+\frac{2\alpha}{q})}_{2,q}} \, \mathrm{d}\tau.
\end{aligned}$$

For L_{12} and L_{13}, we have

$$\begin{aligned}
L_{12} &\le C \sum_j 2^{sjq} \, 2^{-2\alpha j(q-1)} \int_0^t \left(\| \Delta_j \widetilde{u} \|_{L^2} \sum_{m \le j-1} 2^{2m} \| \Delta_m \eta \|_{L^2} \right)^q \mathrm{d}\tau \\
&= C \int_0^t \sum_j 2^{(2-2\alpha+\frac{2\alpha}{q})jq} \| \Delta_j \widetilde{u} \|_{L^2}^q \left(\sum_{m \le j-1} 2^{(2-s)(m-j)} 2^{sm} \| \Delta_m \eta \|_{L^2} \right)^q \mathrm{d}\tau \\
&\le C \int_0^t \| \widetilde{u}(\tau) \|^q_{B^{(2-2\alpha+\frac{2\alpha}{q})}_{2,q}} \| \eta(\tau) \|^q_{B^s_{2,q}} \, \mathrm{d}\tau. \\
L_{13} &\le C \sum_j 2^{sjq} \, 2^{-2\alpha j(q-1)} \int_0^t \left(\| \Delta_j \widetilde{u} \|_{L^2} 2^{2j} \| \Delta_j \eta \|_{L^2} \right)^q \mathrm{d}\tau \\
&= C \int_0^t \sum_j 2^{(2-2\alpha+\frac{2\alpha}{q})jq} \| \Delta_j \widetilde{u} \|_{L^2}^q \, 2^{sjq} \| \Delta_j \eta \|_{L^2}^q \, \mathrm{d}\tau \\
&\le C \int_0^t \| \widetilde{u}(\tau) \|^q_{B^{(2-2\alpha+\frac{2\alpha}{q})}_{2,q}} \| \eta(\tau) \|^q_{B^s_{2,q}} \, \mathrm{d}\tau.
\end{aligned}$$

L_2 can be similarly estimated and is bounded by

$$L_2 \le C \int_0^t \|\theta(\tau)\|^q_{B_{2,q}^{(2-2\alpha+\frac{2\alpha}{q})}} \|w(\tau)\|^q_{B^s_{2,q}} \,\mathrm{d}\tau.$$

Combining (5.8) with the estimates for L_1 and L_2 and using the fact that

$$\|\widetilde{u}(\tau)\|^q_{B_{2,q}^{(2-2\alpha+\frac{2\alpha}{q})}} \le \|\widetilde{\theta}(\tau)\|^q_{B_{2,q}^{(2-2\alpha+\frac{2\alpha}{q})}} \quad \text{and} \quad \|w(\tau)\|^q_{B^s_{2,q}} \le \|\eta(\tau)\|^q_{B^s_{2,q}},$$

we establish (5.6). (5.7) is obtained by applying Gronwall's inequality to (5.6). □

The following two theorems assert the uniqueness of solutions of the 2-D QG equation in homogeneous Besov spaces. We omit their proofs since they are similar to those of Theorems 5.1 and 5.3.

Theorem 5.5. *Assume either $\alpha > 0$ and $p = 2$ or $0 < \alpha \le 1$ and $2 < p < \infty$. Let $1 \le q \le \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$. Let $T > 0$. Let*

$$\theta \in L^\infty([0,T); \mathring{B}^r_{p,q}(\mathbb{R}^2)) \quad \text{and} \quad \widetilde{\theta} \in L^\infty([0,T); \mathring{B}^r_{p,q}(\mathbb{R}^2))$$

be the solutions of the IVP (3.1) corresponding to the initial data

$$\theta_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^2) \quad \text{and} \quad \widetilde{\theta}_0 \in \mathring{B}^r_{p,q}(\mathbb{R}^2),$$

respectively. Let $s < 1 - \frac{2\alpha}{q}$. If $\eta_0 = \widetilde{\theta}_0 - \theta_0$ is in $\mathring{B}^s_{p,q}(\mathbb{R}^2)$, then the difference

$$\eta = \widetilde{\theta} - \theta$$

satisfies for $1 \le q < \infty$

$$\|\eta\|^q_{L_t^\infty(\mathring{B}^s_{\mathring{B},q})} + C\kappa \|\eta\|^q_{L_t^q(\mathring{B}^{s+\frac{2\alpha}{q}}_{p,q})} \le \|\eta_0\|^q_{\mathring{B}^s_{p,q}} + C\left(\|\theta\|_{L_t^\infty(\mathring{B}^r_{p,q})} + \|\widetilde{\theta}\|_{L_t^\infty(\mathring{B}^r_{p,q})}\right) \|\eta\|^q_{L_t^q(\mathring{B}^{s+\frac{2\alpha}{q}}_{p,q})} \tag{5.9}$$

and for $q = \infty$

$$\|\eta\|_{L_t^\infty(\mathring{B}^s_{p,\infty})} + C\kappa \|\eta\|_{\widetilde{L}_t^1(\mathring{B}^{s+2\alpha}_{p,\infty})} \le \|\eta_0\|_{\mathring{B}^s_{p,\infty}} + C\left(\|\theta\|_{L_t^\infty(\mathring{B}^r_{p,\infty})} + \|\widetilde{\theta}\|_{L_t^\infty(\mathring{B}^r_{p,\infty})}\right) \|\eta\|_{\widetilde{L}_t^1(\mathring{B}^{s+2\alpha}_{p,\infty})} \tag{5.10}$$

for any $t \le T$. In particular, if $\theta_0 = \widetilde{\theta}_0$ and

$$\|\theta\|_{L_T^\infty(\mathring{B}^r_{2,q})} \le C\kappa, \qquad \|\widetilde{\theta}\|_{L_T^\infty(\mathring{B}^r_{2,q})} \le C\kappa$$

for some suitable constant C, then

$$\theta = \widetilde{\theta}.$$

Theorem 5.6. *Assume either $\alpha > 0$ and $p = 2$ or $0 < \alpha \le 1$ and $2 < p < \infty$. Let $1 \le q < \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$. Let $T > 0$. Let*

$$\theta \in L^q\left((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}(\mathbb{R}^2)\right) \quad \text{and} \quad \widetilde{\theta} \in L^q\left((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}(\mathbb{R}^2)\right)$$

be the solutions of the IVP (3.1) corresponding to the initial data θ_0 and $\widetilde{\theta}_0$, respectively. Let $s < 2$. If $\eta_0 = \widetilde{\theta}_0 - \theta_0$ is in $\mathring{B}^s_{p,q}(\mathbb{R}^2)$, then the difference

$$\eta = \widetilde{\theta} - \theta$$

satisfies, for any $t \le T$,

$$\|\eta(t)\|^q_{\mathring{B}^s_{p,q}} \le \|\eta_0\|^q_{\mathring{B}^s_{p,q}} + C\int_0^t \left(\|\theta(\tau)\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}} + \|\widetilde{\theta}(\tau)\|^q_{\mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}}\right) \|\eta(\tau)\|^q_{\mathring{B}^s_{p,q}} \,\mathrm{d}\tau.$$

In particular, if $\theta_0 = \widetilde{\theta}_0$, *then*

$$\theta = \widetilde{\theta}.$$

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Appendix A

This appendix proves a Bernstein type inequality for fractional derivatives and a commutator estimate that has been used in the previous sections.

Theorem A.1. *Let* $\alpha \geq 0$. *Let* $1 \leq p \leq q \leq \infty$.

(1) *If* f *satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K 2^j\},$$

for some integer j *and a constant* $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If* f *satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\} \tag{A.1}$$

for some integer j *and constants* $0 < K_1 \leq K_2$, *then*

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 *and* C_2 *are constants depending on* α, p *and* q *only.*

Proof. We prove (2) and the proof of (1) is similar. To prove (2), it suffices to show

$$\|f\|_{L^q} \leq C 2^{jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p} \tag{A.2}$$

and

$$C 2^{2\alpha j} \|f\|_{L^p} \leq \|(-\Delta)^\alpha f\|_{L^p} \leq C 2^{2\alpha j} \|f\|_{L^p}. \tag{A.3}$$

Because of (A.1), there exists a Φ_j such that

$$\widehat{f} = \widehat{\Phi}_j\, \widehat{f}, \tag{A.4}$$

where Φ_j is as defined in Section 2. That is, $f = \Phi_j * f$. By Young's inequality

$$\|f\|_{L^q} \leq \|\Phi_j\|_{L^{p_1}} \|f\|_{L^p},$$

where $\frac{1}{p_1} = 1 + \frac{1}{q} - \frac{1}{p}$. Noticing that $\Phi_j(x) = 2^{jd}\, \Phi_0(2^j x)$, we have

$$\|\Phi_j\|_{L^{p_1}} = 2^{jd(\frac{1}{p} - \frac{1}{q})} \|\Phi_0\|_{L^{p_1}}$$

and this proves (A.2).

To prove (A.3), we choose Φ_j such that

$$\widehat{(-\Delta)^\alpha} f(\xi) = (2\pi|\xi|)^{2\alpha}\, \widehat{f}(\xi) = \widehat{\Phi}_j(\xi)(2\pi|\xi|)^{2\alpha}\, \widehat{f}(\xi). \tag{A.5}$$

That is,

$$(-\Delta)^\alpha f = K_j * f, \tag{A.6}$$

where

$$K_j(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\,\widehat{\Phi}_j(\xi)(2\pi|\xi|)^{2\alpha}\,d\xi.$$

Since $\widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi)$, we have

$$K_j(x) = 2^{2\alpha j}2^{dj}(2\pi)^{2\alpha}\int_{\mathbb{R}^d} e^{2\pi i 2^j x\cdot\xi}\,\widehat{\Phi}_0(\xi)|\xi|^{2\alpha}\,d\xi.$$

By repeated integration by parts, we obtain

$$|K_j(x)| \le C\,2^{2\alpha j}2^{dj}|2^j x|^{-s}$$

for any $s > 0$. Therefore,

$$\|K_j\|_{L^1} \le C\,2^{2\alpha j}. \tag{A.7}$$

Applying Young's inequality to (A.6) and using (A.7), we prove the right half of (A.3). To prove the left half of (A.3), we have from (A.5) that

$$\widehat{f}(\xi) = (\widehat{\Phi}_j(\xi)(2\pi|\xi|)^{2\alpha})^{-1}\widehat{(-\Delta)^{\alpha} f}(\xi), \quad \xi \in A_j$$

where A_j is defined in (2.1). This, in turn, implies

$$f = L_j * (-\Delta)^{\alpha} f \quad \text{with } L_j(x) = \int_{A_j} e^{2\pi i x\cdot\xi}(\widehat{\Phi}_j(\xi)(2\pi|\xi|)^{2\alpha})^{-1}\,d\xi.$$

The rest is then similar to the proof for the right half of (A.3). This completes the proof of Theorem A.1. □

We now state and prove the commutator estimate.

Proposition A.2. *Let j be an integer. Let $1 \le p < \infty$ and $1 \le r \le \infty$. Let u be a divergence free vector field. Then*

$$\begin{aligned}\|[u\cdot\nabla, \Delta_j]\theta\|_{L^p} \le C\Bigg(& 2^{(1+\frac{d}{r})j}\|\Delta_j u\|_{L^p}\,\|\Delta_j\theta\|_{L^r} + \|\Delta_j\theta\|_{L^p}\sum_{k\le j-1} 2^{(1+\frac{d}{r})k}\|\Delta_k u\|_{L^r}\\ &+ \|\Delta_j u\|_{L^p}\sum_{k\le j-1} 2^{(1+\frac{d}{r})k}\|\Delta_k\theta\|_{L^r} + 2^{(1+\frac{d}{r})j}\sum_{k\ge j-1}\|\Delta_k u\|_{L^r}\|\Delta_k\theta\|_{L^p}\Bigg),\end{aligned} \tag{A.8}$$

where the brackets [ ] *represent the commutator operator, namely*

$$[u\cdot\nabla, \Delta_j]\theta \equiv u\cdot\nabla(\Delta_j\theta) - \Delta_j(u\cdot\nabla\theta).$$

In particular, if $d = 2$, $2 \le p < \infty$ and $u = \nabla^{\perp}\Lambda^{-1}\theta$, then

$$\begin{aligned}\|[u\cdot\nabla, \Delta_j]\theta\|_{L^p} \le C\Bigg(& 2^{(1+\frac{2}{p})j}\|\Delta_j\theta\|_{L^p}^2 + \|\Delta_j\theta\|_{L^p}\sum_{k\le j-1} 2^{(1+\frac{2}{p})k}\|\Delta_k\theta\|_{L^p}\\ &+ 2^{(1+\frac{2}{p})j}\sum_{k\ge j-1}\|\Delta_k\theta\|_{L^p}^2\Bigg).\end{aligned} \tag{A.9}$$

Proof. Splitting $[u\cdot\nabla, \Delta_j]\theta$ into paraproducts, we have

$$[u\cdot\nabla, \Delta_j]\theta = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \sum_k S_{k-1}u\cdot\nabla\Delta_j\Delta_k\theta - \Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta),$$

$$I_2 = \sum_k \Delta_k u\cdot\nabla\Delta_j S_{k-1}\theta,$$

$$I_3 = \sum_k \Delta_j(\Delta_k u \cdot \nabla S_{k-1}\theta),$$

$$I_4 = \sum_k \sum_{|k-l|\le 1} \Delta_k u \cdot \nabla \Delta_j \Delta_l \theta,$$

$$I_5 = \sum_k \sum_{|k-l|\le 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l \theta).$$

We bound the L^p-norms of these terms. According to (2.5), the summation in I_1 is only over k satisfying $|k-j| \le 2$. Using the definition of Δ_j, we can write

$$I_1 = \sum_{|k-j|\le 2} \int_{\mathbb{R}^d} \Phi_j(x-y)(S_{k-1}u(x) - S_{k-1}u(y)) \cdot \nabla \Delta_k \theta(y)\, \mathrm{d}y.$$

We integrate by parts and use the fact that $\nabla \cdot u = 0$ to obtain

$$I_1 = -\sum_{|k-j|\le 2} \int_{\mathbb{R}^d} \nabla \Phi_j(x-y) \cdot (S_{k-1}u(x) - S_{k-1}u(y)) \Delta_k \theta(y)\, \mathrm{d}y.$$

By Young's inequality,

$$\begin{aligned}\|I_1\|_{L^p} &\le C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k \theta\|_{L^p} \int_{\mathbb{R}^d} |x||\nabla \Phi_j(x)|\, \mathrm{d}x \\ &= C \sum_{|k-j|\le 2} \|\Delta_k \theta\|_{L^p} \|\nabla S_{k-1}u\|_{L^\infty}.\end{aligned}$$

Similarly, the summation in I_2, I_3 and I_4 are also only over k satisfying $|k-j| \le 2$. The estimates for these terms are simple and their L^p-norms are both bounded by

$$\|I_2\|_{L^p}, \|I_3\|_{L^p} \le C \sum_{|k-j|\le 2} \|\Delta_k u\|_{L^p} \|\nabla S_{k-1}\theta\|_{L^\infty},$$

$$\|I_4\|_{L^p} \le C \sum_{|k-j|\le 2, |k-l|\le 1} \|\Delta_k u\|_{L^p} \|\nabla \Delta_l \theta\|_{L^\infty}.$$

The estimates for I_5 are slightly different. The summation is over all $k \ge j-1$, namely

$$I_5 = \sum_{k \ge j-1, |k-l|\le 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l \theta).$$

Since u is divergence free, we obtain after applying Bernstein's inequality

$$\|I_5\|_{L^p} \le C 2^{(1+\frac{d}{r})j} \sum_{k \ge j-1, |k-l|\le 1} \|\Delta_j(\Delta_k u \Delta_l \theta)\|_{L^q},$$

where q satisfies $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$. By Hölder's inequality

$$|I_5| \le C\, 2^{(1+\frac{d}{r})j} \sum_{k \ge j-1, |k-l|\le 1} \|\Delta_k u\|_{L^r} \|\Delta_l \theta\|_{L^p}.$$

Since the summations in the bounds for I_1 through I_4 are only over a finite number of k's, it suffices to consider the typical term with $k = j$ in our further estimates. It follows from Bernstein's inequalities that

$$\|\nabla \Delta_j \theta\|_{L^\infty} \le C\, 2^{(1+\frac{d}{r})j} \|\Delta_j \theta\|_{L^r},$$

$$\|\nabla S_{j-1}\theta\|_{L^\infty} \le \sum_{k \le j-1} \|\nabla \Delta_k \theta\|_{L^\infty} \le C \sum_{k \le j-1} 2^{(1+\frac{d}{r})k} \|\Delta_k \theta\|_{L^r}.$$

(A.9) is a consequence of (A.8) since $\|\Delta_j u\|_{L^p} \le C\, \|\Delta_j \theta\|_{L^p}$. $\square$

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