A GLOBAL REGULARITY RESULT FOR THE 2D BOUSSINESQ EQUATIONS WITH CRITICAL DISSIPATION

By

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Abstract. This paper examines the global regularity problem on the two-dimensional incompressible Boussinesq equations with fractional dissipation, given by $\Lambda^\alpha u$ in the velocity equation and by $\Lambda^\beta\theta$ in the temperature equation, where $\Lambda = \sqrt{-\Delta}$ denotes the Zygmund operator. We establish the global existence and smoothness of classical solutions when (α, β) is in the critical range: $\alpha > (\sqrt{1777} - 23)/24 = 0.798103\ldots$, $\beta > 0$ and $\alpha + \beta = 1$. This result improves previous work which obtained the global regularity for $\alpha > (23 - \sqrt{145})/12 \approx 0.9132$, $\beta > 0$, and $\alpha + \beta = 1$.

1 Introduction

This paper aims at the global regularity issue on the two-dimensional (2D) Boussinesq equations with fractional dissipation

$$
(1.1)\qquad \begin{cases} \partial_t u + u\cdot\nabla u + \nu\,\Lambda^\alpha u = -\nabla p + \theta\mathbf{e}_2 & x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t\theta + u\cdot\nabla\theta + \kappa\,\Lambda^\beta\theta = 0, & x\in\mathbb{R}^2,\ t>0,\\ u(x,0) = u_0(x),\ \theta(x,0) = \theta_0(x), & x\in\mathbb{R}^2, \end{cases}
$$

where $u = u(x, t)$ denotes the 2D velocity, $p = p(x, t)$ the pressure, $\theta = \theta(x, t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu > 0$, $\kappa > 0$, $0 < \alpha \le 2$ and $0 < \beta \le 2$ are real parameters. Here $\Lambda = \sqrt{-\Delta}$ represents the Zygmund operator with Λ^α being defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha\,\widehat{f}(\xi),$$

[1]Partially supported by NSF grant DMS 1313107.

[2]Partially supported by NSF grant DMS 1209153 and the AT&T Foundation at Oklahoma State University.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 137 (2019)
DOI 10.1007/s11854-018-0073-4

where the Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

When $\alpha = \beta = 2$, (1.1) reduces to the standard 2D Boussinesq equations with Laplacian dissipation. The standard Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation and also play an important role in the study of Raleigh-Bernard convection (see, e.g., [12, 20, 36, 42, 47, 48]).

Although (1.1) with fractional dissipation appears to be a purely mathematical generalization, it may be physically relevant. Firstly, closely related equations such as the surface quasi-geostrophic equation model important geophysical phenomena (see, e.g., [13, 22, 42]). Secondly, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. Flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian; see [6, 20]. Thirdly, it may be possible to derive the Boussinesq equations with fractional dissipation from the Boltzmann-type equations using suitable rescalings. A recent paper [23] derives the fractional Stokes and Stokes-Fourier systems as the incompressible limit of the Boltzmann equation.

Mathematically, the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Navier-Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [37], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows. It is hoped that the study on the 2D Boussinesq equations will shed light on the mysterious global existence and smoothness problem on the 3D Navier-Stokes and Euler equations. The generalization to include the fractional dissipation facilitates this purpose by allowing the simultaneous study of a whole parameter family of equations.

One main pursuit in the study of (1.1) has been to obtain the global regularity of its solutions for the smallest α and β. Intuitively, the smaller α and β are, the harder the global regularity problem is. When there is no dissipation, namely, $\nu = \kappa = 0$ in (1.1), the global regularity problem remains open. The standard idea of proving the global a priori bounds in Sobolev spaces fails. Potential finite time singularities have been explored from different perspectives including boundary effects and 1D models ([9, 10, 35, 44]).

At the other extreme, when $\nu > 0$, $\kappa > 0$, $\alpha = \beta = 2$, the global regularity can be easily obtained, in a similar fashion as for the 2D Navier-Stokes equations ([19, 37]). It is natural to examine (1.1) with intermediate dissipation, which has attracted considerable attention in the last few years ([1, 2, 3, 7, 8, 14, 16, 17, 18, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 38, 41, 50, 49, 51, 52, 54, 55]). In [8] and [27], it is shown that one full Laplacian dissipation in (1.1) is sufficient for the global regularity. More precisely, (1.1) with $\alpha = 2$ and $\kappa = 0$ or with $\beta = 2$ and $\nu = 0$ always admits global classical solutions.

More recent work further reduces the values of α and β, and existing research appears to indicate that one-derivative dissipation is critical. Here, one-derivative dissipation refers to the case when $\alpha + \beta = 1$ in (1.1). For the convenience of description, $\alpha+\beta = 1$ is referred to as the critical case, $\alpha+\beta > 1$ as the subcritical case, and $\alpha + \beta < 1$ as the supercritical case. To position our work in a suitable context, we describe some recent results for these three cases.

We start with the subcritical case. Even this case is not easy. The global regularity has so far been established only for three subcritical cases ([14, 38, 53]). The global existence and regularity problem for the critical case is more difficult. Two particular critical cases, $\alpha = 1$, $\kappa = 0$ and $\beta = 1$, $\nu = 0$, were studied by [25] and [26], which introduced a combined quantity of the vorticity and the Riesz transform of the temperature and were able to establish the global regularity for both cases. For the more general critical case when the one derivative dissipation is split between the velocity equation and the temperature equation, the situation becomes more complex. The general critical case was recently dealt with by Jiu, Miao, Wu, and Zhang and a global regularity result was obtained [29]. By reducing the global regularity issue on the critical Boussinesq system to a parallel problem for an active scalar equation with critical dissipation or, more precisely, the critical surface quasi-geostrophic (SQG) equation and taking advantage of the recent advances on the SQG equation, Jiu, Miao, Wu, and Zhang obtained the global regularity in the critical regime: $\alpha + \beta = 1$ and $\alpha > \alpha_0$, where $\alpha_0 = \frac{23-\sqrt{145}}{12} \approx 0.9132$. Attempts have also been made to go beyond the critical case and the global regularity has been established when the dissipation is logarithmically more singular than the critical case ([24, 31]). The global well-posedness problem for the supercritical case $\alpha + \beta < 1$ is completely open. The only result currently available is the eventual regularity of weak solutions of (1.1) with $\alpha + \beta < 1$ and $\alpha > \alpha_0$ [30].

This paper establishes the global existence and regularity of classical solutions of (1.1) when α and β are in the critical range: $\alpha + \beta = 1$ and $1 > \alpha > \frac{\sqrt{1777}-23}{24} = 0.798103\ldots$. This result improves the work of Jiu, Miao, Wu, and Zhang [29] by allowing α to vary in a bigger interval, by keeping the relation $\alpha + \beta = 1$. The precise statement of our result is given in the following theorem.

Theorem 1.1. Consider (1.1) with $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$ for $\sigma > 2$. If the parameters in (1.1) satisfies $\nu > 0$, $\kappa > 0$, $0.798103\ldots = \frac{\sqrt{1777}-23}{24} < \alpha < 1$, $\alpha + \beta = 1$, then (1.1) has a unique global solution (u, θ) satisfying, for any $T > 0$, $(u, \theta) \in C([0, T]; H^\sigma(\mathbb{R}^2))$.

We outline the main idea in the proof of this theorem and explain how to improve [29]. A large portion of the efforts are devoted to obtaining global a priori bounds for (u, θ). Because $\nabla \cdot u = 0$, the L^2-level global bounds for (u, θ) follow from easy energy estimates,

$$\begin{aligned}
&\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{for } q \in [1, \infty], \\
&\|\theta(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\Lambda^{\frac{\beta}{2}} \theta(\tau)\|_{L^2}^2 \, d\tau = \|\theta_0\|_{L^2}^2, \\
&\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^{\frac{\alpha}{2}} u(\tau)\|_{L^2}^2 \, d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.
\end{aligned} \tag{1.2}$$

Naturally, the next target is the global H^1-bound for u or, equivalently, global L^2-bound for the vorticity $\omega = \nabla \times u$, which satisfies

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_1 \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega \quad \text{or} \quad u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \tag{1.3}$$

Because of the presence of the "vortex stretching" term $\partial_1 \theta$, direct energy estimates do not yield the desired global bound for $0 < \alpha < 1$. For notational convenience, throughout the rest of this paper, we set $\nu = \kappa = 1$ in (1.1). The strategy is to hide $\partial_1 \theta$ by considering the combined quantity $G = \omega - \mathcal{R}_\alpha \theta$ with $\mathcal{R}_\alpha = \Lambda^{-\alpha} \partial_1$. It is easy to check that G satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla] \theta + \Lambda^{\beta - \alpha} \partial_1 \theta. \tag{1.4}$$

Here we have used the standard commutator notation

$$[\mathcal{R}_\alpha, u \cdot \nabla] \theta = \mathcal{R}_\alpha (u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.$$

Although (1.4) appears to be more complicated than the vorticity equation, the commutator term $[\mathcal{R}_\alpha, u \cdot \nabla] \theta$ is less singular than $\partial_1 \theta$ in the vorticity equation. In fact, we are able to show the global bound for $\|G\|_{L^2}$ whenever $\alpha > 3/4$ and $\alpha + \beta = 1$. The major contribution of this paper is on the global L^6-bound for G. Previously, for $\alpha > 4/5$ and $\alpha + \beta = 1$, [29] obtained a global bound for $\|G\|_{L^q}$ for q in the range

$$2 \le q < q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}. \tag{1.5}$$

Obviously, $q_0 \in (2, 4)$ when $\alpha \in (4/5, 1)$. We are able to enlarge the range of q significantly. More precisely, we prove the following proposition.

Proposition 1.2. Let $0 < \beta < 1$ and $\alpha + \beta = 1$, with

$$1 > \alpha > \frac{\sqrt{1777} - 23}{24} = 0.798103\ldots.$$

Let (u_0, θ_0) be as specified in Theorem 1.1, and let (u, θ) be the corresponding smooth solution of (1.1). Assume G satisfies (1.4). Then, for any $T > 0$ and $t \leq T$,

$$\|G(t)\|_{L^q} \leq C \quad \text{for } 2 \leq q \leq 6,$$

where C is a constant depending on T and the initial data (u_0, θ_0).

The proof of Proposition 1.2 involves the decomposition of the velocity field

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta \equiv u_G + u_\theta, \tag{1.6}$$

commutator estimates, and various functional inequalities.

Proposition 1.2 is crucial in further showing that G is actually globally regular in the sense that

$$\|G(t)\|_{B^s_{q,\infty}(\mathbb{R}^2)} \leq C \quad \text{for} \quad 0 \leq s \leq 3\alpha - 2 \quad \text{and} \quad 2 \leq q \leq 6, \tag{1.7}$$

and for any $T > 0$ and $t \leq T$, where C is a constant depending on T and the initial data (u_0, θ_0). Here $B^s_{q,\infty}$ denotes an inhomogeneous Besov space; see Section 2 for more details on Besov spaces. Once Proposition 1.2 is established, the proof of (1.7) is similar to that of [29, Proposition 7.1]. A special consequence of (1.7) is that $G \in B^0_{\infty,1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, which in turn implies that u_G, defined in (1.6), is Lipschitz, $\|\nabla u_G(t)\|_{L^\infty} \leq C$. Then (1.6) with the equation of θ can be treated as a generalized critical SQG equation, which, following a similar approach as in [29], leads to the global regularity of (u, θ).

The rest of this paper is divided into two main sections. Section 2 recalls the Littlewood-Paley decomposition, the definition of Besov spaces and some other related relevant facts. It also presents several commutator estimates and a global L^2-bound for G, which serve as a preparation for the proof of Proposition 1.2. Section 3 contains the proof of Proposition 1.2 and Theorem 1.1.

2 Preliminaries

This section includes several parts. It recalls the Littlewood-Paley theory, introduces the Besov spaces, provides Bernstein inequalities and Kato-Ponce estimates, and proves several commutator estimates and a global L^2-bound for G. We start with the definitions of some of the functional spaces and related facts that are used in the subsequent sections. Materials on Besov space and related facts presented here can be found in several books and many papers; see, e.g., [4, 5, 39, 43, 46].

2.1 Fourier transform and the Littlewood-Paley theory. We start with notation. The symbol $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions; $\mathcal{S}_0$ denotes the subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma\, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\},$$

and $\mathcal{S}_0'$ denotes its dual. The space $\mathcal{S}_0'$ can be identified with the space $\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ denotes the space of multinomials. On the Schwartz class, we can define the Fourier transform and its inverse via

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx, \quad f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi,$$

respectively.

To introduce the Littlewood-Paley decomposition, we write, for each $j \in \mathbb{Z}$,

$$A_j = \left\{\xi \in \mathbb{R}^d : 2^{j-1} \le |\xi| < 2^{j+1}\right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}}$ in $\mathcal{S}$ such that $\operatorname{supp} \widehat{\Phi}_j \subset A_j$, $\widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi)$ or $\Phi_j(x) = 2^{jd}\Phi_0(2^j x)$, and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & , \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & , \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have $\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi)$ for $\xi \in \mathbb{R}^d \setminus \{0\}$. Moreover, if $\psi \in \mathcal{S}_0$, then $\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi)$ for all $\xi \in \mathbb{R}^d$; i.e., for $\psi \in \mathcal{S}_0$, $\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$, and hence $\sum_{j=-\infty}^{\infty} \Phi_j * f = f$, $f \in \mathcal{S}_0'$ in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \quad j \in \mathbb{Z}. \tag{2.1}$$

2.2 Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the **homogeneous Besov space** $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that $\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi)$, $\xi \in \mathbb{R}^d$. Then, for all $\psi \in \mathcal{S}$, $\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$, and hence $\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$ in $\mathcal{S}'$ for all $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots. \end{cases} \tag{2.2}$$

Definition 2.2. The **inhomogeneous Besov space** $B^s_{p,q}$, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \leq p, q \leq \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt\right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt\right)^{1/q},$$

respectively. When $q = \infty$, the expressions are interpreted as suprema instead of integrals.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 2.3. For all $s \in \mathbb{R}$, $\mathring{H}^s \sim \mathring{B}^s_{2,2}$, $H^s \sim B^s_{2,2}$. For all $s \in \mathbb{R}$ and $1 < q < \infty$, $\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}$. In particular,

$$\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}.$$

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. There should be no confusion if one bears in mind that the Δ_j's associated with the homogeneous Besov spaces are defined in (2.1), while those associated with the inhomogeneous Besov spaces are defined in (2.2).

The partial sum S_j is also a useful notation. For an integer j, the partial sum S_j is defined by $S_j \equiv \sum_{k=-1}^{j-1} \Delta_k$, where Δ_k is given by (2.2).

For all $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

2.3 Bernstein inequalities and Kato-Ponce estimates. Bernstein's inequalities are useful tools for dealing with Fourier localized functions, and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition 2.4. Let $\alpha \geq 0$, $1 \leq p \leq q \leq \infty$, and $f : \mathbb{R}^d \to \mathbb{C}$.

1) If $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K 2^j\}$, for some integer j and constant $K > 0$, then
$$\|\Lambda^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$
where C_1 is a constant depending on K, α, p and q only.

2) If $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$ for some integer j and constants $0 < K_1 \leq K_2$, then
$$C_1\, 2^{\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|\Lambda^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$
where C_2 is a constant depending on K_1, K_2, α, p and q only.

We use the following Kato-Ponce type estimate (also known as the fractional Leibniz rule) extensively:

$$\|\Lambda^s (fg)\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{q_1}} \|g\|_{L^{r_1}} + \|\Lambda^s g\|_{L^{q_2}} \|f\|_{L^{r_2}}), \tag{2.3}$$

whenever $s > 0$, $1 < p, q_1, q_2 < \infty$, $1 < r_1, r_2 \leq \infty$, $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$. This inequality can be found in many references; see [32] and [21] for recent results and survey of the literature on the topic. As a corollary, we can deduce the following estimate, at least for all integers m:

$$\|\Lambda^s (f^m)\|_{L^p} \leq C \|\Lambda^s f\|_{L^q} \|f\|_{L^{r(m-1)}}^{m-1}, \tag{2.4}$$

whenever $1 < p, q, r < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

2.4 Commutator estimates. As we have seen already, the equation (1.4) involves commutators. Thus, we need to develop corresponding estimates so that we can bound the commutators suitably.

Lemma 2.5. Let $1 > \alpha > 1/2$ and $1 < p < \infty$, $1 < p_1, p_2 \leq \infty : \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For any integer k and u_G and u_θ as defined in (1.6),

$$\|\Delta_k([\mathcal{R}_\alpha, u_G \cdot \nabla]\theta)\|_{L^p} \leq C 2^{(1-\alpha)k} \|G\|_{L^{p_1}} \|\theta\|_{L^{p_2}}; \tag{2.5}$$

$$\|\Delta_k([\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi)\|_{L^p} \leq C 2^{(2-2\alpha)k} \|\theta\|_{L^{p_1}} \|\psi\|_{L^{p_2}}. \tag{2.6}$$

More generally, for $0 \le s \le 1-\alpha$,

$$\|\Delta_k([\mathcal{R}_\alpha, u_G \cdot \nabla]\theta)\|_{L^p} \le C_s 2^{(1-\alpha-s)k}\|G\|_{L^{p_1}}\|\Lambda^s\theta\|_{L^{p_2}}; \tag{2.7}$$

$$\|\Delta_k([\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi)\|_{L^p} \le C_s 2^{(2-2\alpha-s)k}\|\Lambda^s\theta\|_{L^{p_1}}\|\psi\|_{L^{p_2}}. \tag{2.8}$$

Proof. Most of the proof is devoted to establishing (2.5) and (2.6). At the end, we indicate what minor modifications are needed to establish (2.7) and (2.8).

To simplify notation, we write $u_j := \Delta_j u$, $u_{<k} := S_k u$, and $u_{[k-A,k+B]} := \sum_{j=k-A}^{k+B} u_j$. If $A, B < 10$, we denote $u_{[k-A,k+B]}$ by $u_{\sim k}$, etc. We use the following paraproduct decomposition for the product of two functions:

$$\Delta_k(fg) = \Delta_k(f_{<k-10}g_{\sim k}) + \Delta_k(f_{\sim k}g_{<k+10}) + \Delta_k\Big(\sum_{l=k+10}^{\infty} f_l g_{\sim l}\Big).$$

We refer to the first term as low-high interaction, the second term is high-low interaction, and the third term is high-high interaction. Only the low-high interaction term is not straightforward and requires the commutator structure. Note that, according to the definition (1.6), $u_G \sim \Lambda^{-1}G$ and $u_\theta \sim \Lambda^{-\alpha}\theta$. More precisely, for all $1 \le p \le \infty$,

$$\|\Delta_l(u_G)\|_{L^p} \sim 2^{-l}\|\Delta_l G\|_{L^p} \le C2^{-l}\|G\|_{L^p};$$
$$\|\Delta_l(u_\theta)\|_{L^p} \sim 2^{-\alpha l}\|\Delta_l \theta\|_{L^p} \le C2^{-\alpha l}\|\theta\|_{L^p}.$$

High-low interactions. For (2.5), we have, by Hölder's inequality,

$$\begin{aligned}\|\Delta_k([\mathcal{R}_\alpha, (u_G)_{\sim k}\cdot\nabla]\theta_{<k+10})\|_{L^p} &\le C\|\mathcal{R}_\alpha \Delta_k[(u_G)_{\sim k}\cdot\nabla\theta_{<k+10}]\|_{L^p} \\ &\quad + \|[(u_G)_{\sim k}\cdot\nabla]\mathcal{R}_\alpha\theta_{<k+10}\|_{L^p}) \\ &\le C2^{k(1-\alpha)}\|(u_G)_{\sim k}\|_{L^{p_1}}\|\nabla\theta_{<k+10}\|_{L^{p_2}} \\ &\quad + C\|(u_G)_{\sim k}\|_{L^{p_1}}\|\Lambda^{2-\alpha}\theta_{<k+10}\|_{L^{p_2}}.\end{aligned}$$

But $\|(u_G)_{\sim k}\|_{L^{p_1}} \le C2^{-k}\|G\|_{L^{p_1}}$, and

$$\|\nabla\theta_{<k+10}\|_{L^{p_2}} \le C2^k\|\theta\|_{L^{p_2}}, \quad \|\Lambda^{2-\alpha}\theta_{<k+10}\|_{L^{p_2}} \le C2^{(2-\alpha)k}\|\theta\|_{L^{p_2}}. \tag{2.9}$$

Putting everything together yields the desired inequality

$$\|\Delta_k([\mathcal{R}_\alpha, (u_G)_{\sim k}\cdot\nabla]\theta_{<k+10})\|_{L^p} \le C2^{k(1-\alpha)}\|G\|_{L^{p_1}}\|\theta\|_{L^{p_2}}.$$

Similarly, for (2.6), we have, by Hölder's inequality,

$$\begin{aligned}\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{\sim k}\cdot\nabla]\psi_{<k+10})\|_{L^p} &\le C\|\mathcal{R}_\alpha[(u_\theta)_{\sim k}\cdot\nabla\psi_{<k+10}]\|_{L^p} \\ &\quad + \|[(u_\theta)_{\sim k}\cdot\nabla]\mathcal{R}_\alpha\psi_{<k+10}\|_{L^p} \\ &\le C2^{k(1-\alpha)}\|(u_\theta)_{\sim k}\|_{L^{p_1}}\|\nabla\psi_{<k+10}\|_{L^{p_2}} \\ &\quad + C\|(u_\theta)_{\sim k}\|_{L^{p_1}}\|\Lambda^{2-\alpha}\psi_{<k+10}\|_{L^{p_2}}.\end{aligned}$$

Again, $\|(u_\theta)_{\sim k}\|_{L^{p_1}} \le C 2^{-k\alpha}\|\theta\|_{L^{p_1}}$, which, in conjunction with (2.9), yields

$$\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{\sim k}\cdot\nabla]\psi_{<k+10})\|_{L^p} \le C 2^{2(1-\alpha)k}\|\theta\|_{L^{p_1}}\|\psi\|_{L^{p_2}}.$$

High-high interactions. For (2.5), we have that $u\cdot\nabla\theta = \nabla\cdot[u\theta]$ (since $\nabla\cdot u = 0$). Thus, by Hölder's inequality,

$$\begin{aligned}\Big\|\sum_{l=k+10}^{\infty}\Delta_k([\mathcal{R}_\alpha, (u_G)_l\cdot\nabla]\theta_{\sim l})\Big\|_{L^p} &\le C\sum_{l=k+10}^{\infty}\|\mathcal{R}_\alpha\nabla\Delta_k[(u_G)_l\cdot\theta_{\sim l}]\|_{L^p}\\ &\quad + C\sum_{l=k+10}^{\infty}\|\nabla\Delta_k[(u_G)_l\cdot]\mathcal{R}_\alpha\theta_{\sim l}\|_{L^p}\\ &\le C2^{k(2-\alpha)}\sum_{l=k+10}^{\infty}\|(u_G)_l\|_{L^{p_1}}\|\theta_{\sim l}\|_{L^{p_2}}\\ &\quad + C2^k\sum_{l=k+10}^{\infty}\|(u_G)_l\|_{L^{p_1}}\|\Lambda^{1-\alpha}\theta_{\sim l}\|_{L^{p_2}}\\ &\le C\|G\|_{L^{p_1}}\|\theta\|_{L^{p_2}}\sum_{l=k+10}^{\infty}(2^{k(2-\alpha)}2^{-l}+2^k2^{-\alpha l})\\ &\le C2^{k(1-\alpha)}\|G\|_{L^{p_1}}\|\theta\|_{L^{p_2}}.\end{aligned}$$

For (2.6), we proceed in the same way, but note that near the end, we need $\alpha > 1/2$. We have

$$\begin{aligned}\Big\|\sum_{l=k+10}^{\infty}\Delta_k([\mathcal{R}_\alpha, (u_\theta)_l\cdot\nabla]\psi_{\sim l})\Big\|_{L^p} &\le C\sum_{l=k+10}^{\infty}\|\mathcal{R}_\alpha\nabla\Delta_k[(u_\theta)_l\cdot\psi_{\sim l}]\|_{L^p}\\ &\quad + C\sum_{l=k+10}^{\infty}\|\nabla\Delta_k[(u_\theta)_l\cdot]\mathcal{R}_\alpha\psi_{\sim l}\|_{L^p}\\ &\le C2^{k(2-\alpha)}\sum_{l=k+10}^{\infty}\|(u_\theta)_l\|_{L^{p_1}}\|\psi_{\sim l}\|_{L^{p_2}}\\ &\quad + C2^k\sum_{l=k+10}^{\infty}\|(u_\theta)_l\|_{L^{p_1}}\|\Lambda^{1-\alpha}\psi_{\sim l}\|_{L^{p_2}}\\ &\le C\|G\|_{L^{p_1}}\|\theta\|_{L^{p_2}}\sum_{l=k+10}^{\infty}(2^{k(2-\alpha)}2^{-\alpha l}+2^k2^{-l(2\alpha-1)})\\ &\le C2^{2k(1-\alpha)}\|G\|_{L^{p_1}}\|\theta\|_{L^{p_2}}.\end{aligned}$$

Low-high interactions. Now we need to estimate

$$\|\Delta_k([\mathcal{R}_\alpha, (u_G)_{<k-10}\cdot\nabla]\theta_{\sim k})\|_{L^p}.$$

As before, we use the divergence free condition to reduce to

$$\begin{aligned}\| \Delta_k([\mathcal{R}_\alpha, (u_G)_{<k-10} \cdot \nabla]\theta_{\sim k})\|_{L^p} &= \|\nabla \Delta_k([\mathcal{R}_\alpha, (u_G)_{<k-10}\cdot]\theta_{\sim k})\|_{L^p}\\ &\leq C2^k \| \Delta_k([\mathcal{R}_\alpha, (u_G)_{<k-10}\cdot]\theta_{\sim k})\|_{L^p},\end{aligned}$$

so that now, we need to check

$$\| \Delta_k([\mathcal{R}_\alpha, (u_G)_{<k-10}\cdot]\theta_{\sim k})\|_{L^p} \leq C2^{-k\alpha} \|G\|_{L^{p_1}} \|\theta\|_{L^{p_2}}. \tag{2.10}$$

In addition, $\mathcal{R}_\alpha = \partial_1 \Lambda^{-\alpha}$, so we use the (standard) product rule to conclude

$$\begin{aligned}[\mathcal{R}_\alpha, (u_G)_{<k-10}\cdot]\theta_{\sim k} &= \mathcal{R}_\alpha[(u_G)_{<k-10} \cdot \theta_{\sim k}] - (u_G)_{<k-10} \cdot \mathcal{R}_\alpha \theta_{\sim k}\\ &= \Lambda^{-\alpha}[\partial_1 (u_G)_{<k-10} \cdot \theta_{\sim k}] + [\Lambda^{-\alpha}, (u_G)_{<k-10}\cdot]\partial_1 \theta_{\sim k}.\end{aligned}$$

Clearly, the first term satisfies the required estimates, since

$$\begin{aligned}\| \Delta_k \Lambda^{-\alpha}[\partial_1 (u_G)_{<k-10} \cdot \theta_{\sim k}]\|_{L^p} &\leq 2^{-k\alpha} \|\partial_1 (u_G)_{<k-10}\|_{L^{p_1}} \|\theta_{\sim k}\|_{L^{p_2}}\\ &\leq C2^{-k\alpha} \|G\|_{L^{p_1}} \|\theta\|_{L^{p_2}},\end{aligned}$$

which is (2.10). It then remains to show

$$\| \Delta_k[\Lambda^{-\alpha}, f_{<k-10}]g_{\sim k}\|_{L^p} \leq C2^{-k(1+\alpha)} \|\nabla f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \tag{2.11}$$

Indeed, applying (2.11) to $f = u_G$, $g = \partial_1 \theta$ yields the desired result.

To establish (2.11), write

$$\Delta_k[\Lambda^{-\alpha}, f_{<k-10}]g_{\sim k} = \Delta_k \Lambda^{-\alpha}[f_{<k-10} g_{\sim k}] - \Delta_k[f_{<k-10} \Lambda^{-\alpha} g_{\sim k}]$$

Denote the multiplier of $g_{\sim k}$ by $\tilde{\Delta}_k$. Note that, by the support properties of the corresponding multipliers, $\tilde{\Delta}_k \Delta_k = \Delta_k$. Thus,

$$\begin{aligned}\Delta_k[\Lambda^{-\alpha}, f_{<k-10}]g_{\sim k} &= \Delta_k \tilde{\Delta}_k \Lambda^{-\alpha}[f_{<k-10} g_{\sim k}] - \Delta_k[f_{<k-10} \Lambda^{-\alpha} g_{\sim k}]\\ &= \Delta_k([\tilde{\Delta}_k \Lambda^{-\alpha}, f_{<k-10}]g_{\sim k}).\end{aligned}$$

Thuerefore, it suffices to estimate $\|[\tilde{\Delta}_k \Lambda^{-\alpha}, f_{<k-10}]g_{\sim k}\|_{L^p}$. We have $\tilde{\Delta}_k \Lambda^{-\alpha} = 2^{-k\alpha} P_k$, where $\widehat{P_k f}(\xi) = \tilde{\chi}(2^{-k}\xi)\hat{f}(\xi)$ and $\tilde{\chi}$ is a C^∞ function supported in $\{\xi : |\xi| \in (1/2, 2)\}$. Thus, we need to show

$$\|[P_k, f]g\|_{L^p} \leq C2^{-k} \|\nabla f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \tag{2.12}$$

But this is a standard result in harmonic analysis. For the sake of completeness, we provide a short proof. We begin with

$$\begin{aligned}[P_k, f]g(x) &= 2^{kd} \int_{\mathbb{R}^d} \hat{\chi}(2^k(x-y))(f(y) - f(x))g(y)dy\\ &= 2^{kd} \int_{\mathbb{R}^d} \hat{\chi}(2^k(x-y))g(y)\Big(\int_0^1 \langle y - x, \nabla f(x - \rho(x-y))\rangle d\rho\Big)dy.\end{aligned}$$

It follows that

$$|[P_k, f]g(x)| \le \int_0^1 \int_{\mathbb{R}^d} |\nabla f(x-\rho z)||g(x-z)|2^{kd}|z||\hat{\chi}(2^k z))|dzd\rho.$$

By Hölder's inequlality,

$$\|[P_k, f]g\|_{L^p} \le C\|\nabla f\|_{L^{p_1}}\|g\|_{L^{p_2}} \int_{\mathbb{R}^d} 2^{kd}|z||\hat{\chi}(2^k z))|dz = C2^{-k}\|\nabla f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

This finishes the proof of (2.12) and hence of (2.5).

For the low-high interaction term of (2.6), we reduce similarly. More precisely, by the divergence free condition,

$$\begin{aligned}\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10}\cdot\nabla]\psi_{\sim k})\|_{L^p} &= \|\nabla\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10}\cdot]\psi_{\sim k})\|_{L^p}\\ &\le C2^k\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10}\cdot]\psi_{\sim k})\|_{L^p},\end{aligned}$$

Also,

$$\begin{aligned}[\mathcal{R}_\alpha, (u_\theta)_{<k-10}\cdot]\psi_{\sim k}) &= \mathcal{R}_\alpha[(u_\theta)_{<k-10}\cdot\psi_{\sim k}] - (u_\theta)_{<k-10}\cdot\mathcal{R}_\alpha\psi_{\sim k}\\ &= \Lambda^{-\alpha}[\partial_1(u_\theta)_{<k-10}\cdot\psi_{\sim k}] + [\Lambda^{-\alpha}, (u_\theta)_{<k-10}\cdot]\partial_1\psi_{\sim k}.\end{aligned}$$

For the first term, since $\|\partial_1(u_\theta)_{<k-10}\|_{L^{p_1}} \le C2^{k(1-\alpha)}\|\theta\|_{L^{p_1}}$, we have

$$\begin{aligned}\|\Delta_k\Lambda^{-\alpha}[\partial_1(u_\theta)_{<k-10}\cdot\psi_{\sim k}]\|_{L^p} &\le 2^{-k\alpha}\|\partial_1(u_\theta)_{<k-10}\|_{L^{p_1}}\|\psi_{\sim k}\|_{L^{p_2}}\\ &\le C2^{k(1-2\alpha)}\|\theta\|_{L^{p_1}}\|\psi\|_{L^{p_2}}.\end{aligned}$$

For the second term, we can reduce, in a similar way, making it necessary to prove only an estimate in the form

$$\|[\tilde{\Delta}_k\Lambda^{-\alpha}, (u_\theta)_{<k-10}]\psi_{\sim k}\|_{L^p} \le 2^{-2\alpha k}\|\theta\|_{L^{p_1}}\|\psi\|_{L^{p_2}}.$$

Recalling $\tilde{\Delta}_k\Lambda^{-\alpha} = 2^{-k\alpha}P_k$, by (2.12) and $\|\nabla(u_\theta)_{<k-10}\|_{L^{p_1}} \le C2^{k(1-\alpha)}\|\theta\|_{L^{p_1}}$, we have

$$\begin{aligned}\|[\tilde{\Delta}_k\Lambda^{-\alpha}, (u_\theta)_{<k-10}]\psi_{\sim k}\|_{L^p} &= 2^{-k\alpha}\|P_k, (u_\theta)_{<k-10}]\psi_{\sim k}\|_{L^p}\\ &\le C2^{-k(1+\alpha)}\|\nabla(u_\theta)_{<k-10}\|_{L^{p_1}}\|\psi\|_{L^{p_2}}\\ &\le C2^{-2k\alpha}\|\theta\|_{L^{p_1}}\|\psi\|_{L^{p_2}}.\end{aligned}$$

Regarding the proofs of (2.7) and (2.8), one just needs to go back to the arguments presented above and trace the derivatives. More precisely, for (2.7), things are clear in the high-high and the low high interaction cases, since we use (2.5) and the inequality $2^{k(1-\alpha)}\|\theta_{\ge k-10}\|_{L^{p_2}} \le C2^{k(1-\alpha-s)}\|\Lambda^s\theta\|_{L^{p_2}}$. In the high-low interaction case, note that we have used $\|\nabla\theta_{<k+10}\|_{L^{p_2}} \le 2^k\|\theta\|_{L^{p_2}}$. If we instead use $\|\nabla\theta_{<k+10}\|_{L^{p_2}} \le 2^{k(1-s)}\|\Lambda^s\theta\|_{L^{p_2}}$, we obtain (2.7), instead of (2.5). The arguments for (2.8) are of similar nature, so we omit them. □

Corollary 2.6. Let $1 > \alpha > 1/2$ and $1 < p_2 < \infty$, $1 < p_1, p_3 \leq \infty$, so that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. For every $s_1 : 0 \leq s_1 < 1-\alpha$ and $s_2 : s_2 > 1-\alpha-s_1$, there exists a $C = C(p_1, p_2, p_3, s_1, s_2)$ such that

$$\Big| \int_{d} F[\mathcal{R}_\alpha, u_G \cdot \nabla]\theta dx \Big| \leq C \|\Lambda^{s_1}\theta\|_{L^{p_1}} \|F\|_{W^{s_2,p_2}} \|G\|_{L^{p_3}}. \tag{2.13}$$

Similarly, for every $s_1 : 0 \leq s_1 < 1-\alpha$ and $s_2 : s_2 > 2-2\alpha-s_1$,

$$\Big| \int_{d} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi dx \Big| \leq C \|\Lambda^{s_1}\theta\|_{L^{p_1}} \|F\|_{W^{s_2,p_2}} \|\psi\|_{L^{p_3}}. \tag{2.14}$$

Proof. Recall that $\sum_j \mathring{\Delta}_j = Id$. Take $\tilde{\Delta}_j$ with similar properties, so that $\tilde{\Delta}_j \mathring{\Delta}_j = \mathring{\Delta}_j$. For (2.13), we have by (2.7),

$$\begin{aligned}
\Big| \int_{d} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi dx \Big| &\leq \sum_j \int |\tilde{\Delta}_j F| |\mathring{\Delta}_j [\mathcal{R}_\alpha, u_G \cdot \nabla]\theta]| dx \\
&\leq C \sum_j 2^{j(1-\alpha-s_1)} \|\tilde{\Delta}_j F\|_{L^{p_2}} \|G\|_{L^{p_3}} \|\Lambda^{s_1}\theta\|_{L^{p_1}} \\
&\leq C \|G\|_{L^{p_3}} \|\Lambda^{s_1}\theta\|_{L^{p_1}} \max(\|F_{<10}\|_{L^{p_2}}, \sup_{j\geq 0} 2^{js_2} \|\tilde{\Delta}_j F\|_{L^{p_2}}),
\end{aligned}$$

which of course implies (2.13). The proof of (2.14) is similar. □

2.5 L^2 bound for G. We present the global L^2 bound for G which improves the corresponding L^2-bound of [29, Theorem 5.1] by relaxing the condition from $\alpha > 4/5$ to $\alpha > 3/4$.

Lemma 2.7. Let $\alpha > 3/4$ and (u, θ) be the solution of (1.1) in some interval $[0, T]$. Then G defined in (1.4) satisfies

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}} G(\tau)\|_{L^2}^2 d\tau \leq C(T, u_0, \theta_0)$$

for every $0 \leq t \leq T$.

Proof. The proof is similar to that of [29, Theorem 5.1], but we make use of the global bound (see (1.2))

$$\int_0^T \|\Lambda^{\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2 d\tau < C(T, \theta_0).$$

Taking the inner product of (1.4) with G, we obtain, after integrating by parts,

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 = J_1 + J_2, \tag{2.15}$$

where

$$J_1 = \int G\Lambda^{1-2\alpha}\partial_1\theta\,dx, \quad J_2 = \int G[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,dx.$$

Applying Hölder's inequality and noting that the Riesz transform $\Lambda^{-1}\partial_1$ is bounded in L^q for all $1 < q < \infty$, we obtain, since $3/4 < \alpha < 1$,

$$|J_1| \le \|\Lambda^{2-\frac{5}{2}\alpha}\theta\|_{L^2}\,\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2} \le \frac{1}{4}\,\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C\,\|\theta\|^2_{H^{\frac{\beta}{2}}}.$$

To bound J_2, we write $u = u_G + u_\theta$ as in (1.6). Again, since $3/4 < \alpha < 1$, we can choose $1-\alpha < s < \alpha/2$ and then apply Corollary 2.6 to obtain

$$\begin{aligned}\Big|\int G[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,dx\Big| &\le C\,\|\theta\|_{L^\infty}\|G\|_{L^2}\|G\|_{H^s} \le C\|\theta_0\|_{L^\infty}\|G\|_{L^2}\|G\|_{H^{\frac{\alpha}{2}}}\\ &\le \frac{1}{4}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C\,\|G\|_{L^2}^2.\end{aligned}$$

Applying Corollary 2.6 with $s_1 = \frac{(1-\alpha)}{2}$ and $\frac{3}{2}(1-\alpha) < s_2 < \frac{\alpha}{2}$, we have

$$\begin{aligned}\Big|\int G\,[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta\,dx\Big| &\le C\,\|\Lambda^{s_1}\theta\|_{L^2}\,\|\theta\|_{L^\infty}\|G\|_{H^{s_2}} \le C\|\theta_0\|_{L^\infty}\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^2}\|G\|_{H^{\frac{\alpha}{2}}}\\ &\le \frac{1}{4}\,\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^2}^2.\end{aligned}$$

Therefore,

$$|J_2| \le \frac{1}{2}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C\,\|G\|_{L^2}^2 + C\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^2}^2.$$

Inserting the bounds for J_1 and J_2 into (2.15) and applying Gronwall's inequality yield the desired bound. □

3 On the L^6 bound for G

This section provides a proof of Proposition 1.2, which provides a global L^6-bound for G. Once Proposition 1.2 is established, Theorem 1.1 can be proved similarly to its counterpart in [29]. Nevertheless we give a brief outline of its proof here.

Let us prepare the proof of Proposition 1.2 with the following observation. By the estimates for the evolution of θ in (1.2), we have control of $\|\theta(t)\|_{L^\infty}$ and $\|\Lambda^{(1-\alpha)/2}\theta\|_{L^2_tL^2_x}$. By the Gagliardo-Nirenberg inequality [40], for all $\gamma \in (0, 1/2)$,

$$\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_t^{\frac{1}{\gamma}}L_x^{\frac{1}{\gamma}}} \le \|\Lambda^{(1-\alpha)/2}\theta\|^{2\gamma}_{L^2_tL^2_x}\|\theta_0\|^{1-2\gamma}_{L^\infty_{tx}}. \tag{3.1}$$

Proof of Proposition 1.2. Consider $\gamma \in (0, 1/2)$ to be fixed momentarily, and let α_{cr} be the solution of $(2-\gamma)(1-\alpha) = \frac{\alpha}{2}$, i.e., $\alpha_{cr} = (4-2\gamma)/(5-2\gamma)$. Note that for each $\alpha > \alpha_{cr}$, $(2-\gamma)(1-\alpha) < \frac{\alpha}{2}$. We henceforth assume $\alpha > \alpha_{cr}$.

In view of Lemma 2.7, it suffices to consider the case $q = 6$. Multiplying (1.4) by $G|G|^4 = G^5$ and integrating in x, we obtain

$$\frac{1}{q}\partial_t \|G(t)\|_{L^6}^6 + \int G^5 \Lambda^\alpha G\,dx = \int G^5[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,dx + \int G^5\Lambda^{1-2\alpha}\partial_1\theta\,dx. \tag{3.2}$$

By the maximum principle of [15] and Sobolev embedding, we have

$$\int G^5\Lambda^\alpha G\,dx \ge C\int |\Lambda^{\frac{\alpha}{2}}G^3|^2 dx \ge C\|G\|^6_{L^{\frac{12}{2-\alpha}}}.$$

Next, we deal with the second term on the right-hand side of (3.2). Note that $\partial_1\Lambda^{-1}$ is the Riesz transform in the first variable, which is bounded on all L^p spaces, $1 < p < \infty$. By Hölder's inequality and the Kato-Ponce estimate (2.4),

$$\begin{aligned}\left|\int G^5\Lambda^{1-2\alpha}\partial_1\theta dx\right| &\le \|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|\partial_1\Lambda^{-1}\Lambda^{(2-\gamma)(1-\alpha)}(G^5)\|_{L^{\frac{1}{1-\gamma}}}\\ &\le C\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|\Lambda^{(2-\gamma)(1-\alpha)}G\|_{L^2}\|G\|^4_{L^{\frac{8}{1-2\gamma}}}\\ &\le C\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|G\|_{H^{\alpha/2}}\|G\|^4_{L^{\frac{8}{1-2\gamma}}},\end{aligned}$$

where in the last line, we have used our assumption that $\alpha > \alpha_{cr}$.

Write $u = u_G + u_\theta$ as in (1.6). Then the first term on the right-hand side of (3.2) splits into two terms. We have, according to (2.14), for every $s > (2-\gamma)(1-\alpha)$,

$$\begin{aligned}\left|\int G^5[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta dx\right| &\le C_s\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L^{\frac{1}{\gamma}}}\|\theta\|_{L^\infty}\|G^5\|_{W^{s,\frac{1}{1-\gamma}}}\\ &\le \|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|\theta_0\|_{L^\infty}\|G\|_{H^s}\|G\|^4_{L^{\frac{8}{1-2\gamma}}}.\end{aligned}$$

We now pick s so close to $(2-\gamma)(1-\alpha)$ that $(2-\gamma)(1-\alpha) < s < \alpha/2$. This is possible, because $\alpha > \alpha_{cr}$. Then

$$\left|\int G^5[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta dx\right| \le C\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|G\|_{H^{\alpha/2}}\|G\|^4_{L^{\frac{8}{1-2\gamma}}}.$$

We now handle the term $\int G^5[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta dx$. By (2.13),

$$\left|\int G^5[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\right| \le \|\Lambda^{(1-\gamma)(1-\alpha)+\rho}(G^5)\|_{L^{p_1}}\|G\|_{L^6}\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}},$$

where $\rho > 0$ is a small parameter and

$$\frac{1}{p_1} = \frac{5}{6} - \gamma. \tag{3.3}$$

By the Kato-Ponce estimate and the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\Lambda^{(1-\gamma)(1-\alpha)+\rho}(G^5)\|_{L^{p_1}} &\le C\,\|\Lambda^{(1-\gamma)(1-\alpha)+\rho}G\|_{L^{p_2}}\,\|G\|^4_{L^{4p_3}}\\ &\le C\|G\|^a_{H^{\frac{\alpha}{2}}}\,\|G\|^{1-a}_{L^6}\,\|G\|^{4b}_{L^6}\,\|G\|^{4(1-b)}_{L^{\frac{12}{2-\alpha}}},\end{aligned}$$

where $a, b \in (0,1)$ and $p_2, p_3 \in (1,\infty)$ satisfy

$$(3.4)\qquad \begin{aligned} (1-\gamma)(1-\alpha)+\rho &= \frac{\alpha}{2}a,\\ \frac{1}{p_2} &= \frac{(1-\gamma)(1-\alpha)+\rho}{2} + a\left(\frac12-\frac{\alpha}{4}\right)+\frac16(1-a) = \frac{1+2a}{6},\\ \frac{1}{p_3} &= \frac{1}{p_1}-\frac{1}{p_2} = \frac{2-a}{3}-\gamma, \end{aligned}$$

$$(3.5)\qquad \frac{1}{4p_3} = \frac16 b + (1-b)\frac{2-\alpha}{12}, \quad \text{or} \quad b = 1-\frac{1}{\alpha}(a+3\gamma).$$

We return later in the proof to check that $a, b \in (0,1)$. Combining (3.1) and (3.4) yields

$$\begin{aligned} \left|\int G^5[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\right| &\le C\|G\|^{4(1-b)}_{L^{\frac{12}{2-\alpha}}}\|G\|^a_{H^{\frac{\alpha}{2}}}\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\|G\|^{2-a+4b}_{L^6}\\ &\le \frac18\|G\|^6_{L^{\frac{12}{2-\alpha}}} + C\left(\|G\|^a_{H^{\frac{\alpha}{2}}}\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\right)^{(\frac{3}{2(1-b)})'}\|G\|^{(2-a+4b)(\frac{3}{2(1-b)})'}_{L^6}, \end{aligned}$$

where $(3/2(1-b))'$ is the conjugate index of $3/2(1-b)$. We collect all the estimates for the right-hand side of (3.2) to obtain

$$\begin{aligned} \partial_t\|G\|^6_{L^6} + C\|G\|^6_{L^{\frac{12}{2-\alpha}}} &\le C\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\alpha/2}}\|G\|^4_{L^{\frac{8}{1-2\gamma}}}\\ &\quad + C\left(\|G\|^a_{H^{\frac{\alpha}{2}}}\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\right)^{(\frac{3}{2(1-b)})'}\|G\|^{(2-a+4b)(\frac{3}{2(1-b)})'}_{L^6}. \end{aligned}$$

By the Gagliardo-Nirbenberg inequality, we have

$$\|G\|_{L^{\frac{8}{1-2\gamma}}} \le \|G\|^{\beta_1}_{L^6}\|G\|^{1-\beta_1}_{L^{\frac{12}{2-\alpha}}},$$

where $\beta_1 = \beta_1(\gamma,\alpha)$ is determined from

$$(3.6)\qquad \frac{1-2\gamma}{8} = \frac{\beta_1}{6} + (1-\beta_1)\frac{2-\alpha}{12} \quad \text{or} \quad \beta_1 = \frac{12}{\alpha}\left(\frac{1-2\gamma}{8}+\frac{\alpha-2}{12}\right).$$

Note that our γ, α need to be such that $\beta_1(\gamma,\alpha) \in (0,1)$. We return later in the proof to check this. We invoke (3.1) and apply Young's inequality to obtain

$$\|\Lambda^{\gamma(1-\alpha)}\theta\|_{L_x^{\frac{1}{\gamma}}}\|G\|_{H^{\alpha/2}}\|G\|^4_{L^{\frac{8}{1-2\gamma}}} \le \frac18\|G\|^6_{L^{\frac{12}{2-\alpha}}} + C(\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\|G\|_{H^{\alpha/2}}\|G\|^{4\beta_1}_{L^6})^{(\frac{3}{2(1-\beta_1)})'}.$$

Thus,

$$\begin{aligned} \partial_t\|G\|^6_{L^6} &\le C\left(\|G\|^a_{H^{\frac{\alpha}{2}}}\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\right)^{(\frac{3}{2(1-b)})'}\|G\|^{(2-a+4b)(\frac{3}{2(1-b)})'}_{L^6}\\ &\quad + C(\|\Lambda^{\frac{1-\alpha}{2}}\theta\|^{2\gamma}_{L^2_x}\|G\|_{H^{\alpha/2}})^{(\frac{3}{2(1-\beta_1)})'}\|G\|^{4\beta_1(\frac{3}{2(1-\beta_1)})'}_{L^6}. \end{aligned}$$

To close the argument, we need to show that the indices satisfy

$$(3.7)\qquad (2-a+4b)\Big(\frac{3}{2(1-b)}\Big)' \le 6, \quad 4\beta_1\Big(\frac{3}{2(1-\beta_1)}\Big)' \le 6,$$

$$(3.8)\qquad (a+2\gamma)\Big(\frac{3}{2(1-b)}\Big)' \le 2, \quad \Big(\frac{3}{2(1-\beta_1)}\Big)' \le \frac{2}{2\gamma+1}.$$

Indeed, by Young's inequality, we have

$$\begin{aligned}\Big(\|G\|_{H^{\frac{\alpha}{2}}}^{a}\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^{2\gamma}\Big)^{(\frac{3}{2(1-b)})'} &\le C(\|G\|_{H^{\alpha/2}}^2+\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^2),\\ (\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^{2\gamma}\|G\|_{H^{\alpha/2}})^{(\frac{3}{2(1-\beta_1)})'} &\le C(\|G\|_{H^{\alpha/2}}^2+\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^2)\end{aligned}$$

and

$$\begin{aligned}\|G\|_{L^6}^{(2-a+4b)(\frac{3}{2(1-b)})'} &\le C(1+\|G\|_{L^6}^6),\\ \|G\|_{L^6}^{4\beta_1(\frac{3}{2(1-\beta_1)})'} &\le C(1+\|G\|_{L^6}^6).\end{aligned}$$

This implies the differential inequality

$$\partial_t\|G\|_{L^6}^6 \le \ C(1+\|G\|_{L^6}^6)\left(\|G\|_{H^{\alpha/2}}^2+\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^2\right).$$

Applying Gronwall's inequality then yields

$$\|G(T)\|_{L^6}^6 \le (1+\|G(0)\|_{L^6}^6)\exp(A(T)),$$

where

$$A(T) = \int_0^T \left(\|G\|_{H^{\alpha/2}}^2+\|\Lambda^{\frac{1-\alpha}{2}}\theta\|_{L_x^2}^2\right)dt < \infty.$$

This finishes the argument.

It remains to analyze the inequalities (3.7) and (3.8). It turns out that (3.7) is always satisfied. The first inequality in (3.7) in the same as

$$2-a+4b \le 6\left(1-\frac{2(1-b)}{3}\right) = 2+4b,$$

which is trivially true for $a > 0$. The second inequality in (3.7) is always true as well, since

$$4\beta_1 \le 6\left(1-\frac{2}{3}(1-\beta_1)\right) = 4\beta_1+2.$$

The first condition in (3.8) simplifies to

$$a+2\gamma \le \frac{2}{3}+\frac{4}{3}b,$$

or, according to (3.4) and (3.5),

$$\left(1+\frac{4}{3\alpha}\right)\frac{2}{\alpha}((1-\gamma)(1-\alpha)+\rho)+\left(2+\frac{4}{\alpha}\right)\gamma \le 2.$$

Since ρ can be taken as small as we wish, we can further reduce this inequality to

$$6(1-\gamma)\alpha^2+(1-7\gamma)\alpha-4(1-\gamma) > 0,$$

which is equivalent to

$$\alpha > \alpha_1(\gamma) \equiv \frac{-(1-7\gamma)+\sqrt{(1-7\gamma)^2+96(1-\gamma)^2}}{12(1-\gamma)}.$$

The second condition in (3.8), namely, $(\frac{3}{2(1-\beta_1)})' \le \frac{2}{2\gamma+1}$ is equivalent to

$$\frac{3}{2(1-\beta_1)} \ge \frac{2}{1-2\gamma}, \quad \text{equivalently,} \beta_1 \ge \frac{1+6\gamma}{4}.$$

Using (3.6) for β_1, we obtain

$$\alpha \ge \frac{12\gamma+2}{3-6\gamma}.$$

This means that α needs to satisfy the inequalities

$$\alpha > \max\left(\alpha_1(\gamma), \quad \frac{12\gamma+2}{3-6\gamma}, \quad \frac{4-2\gamma}{5-2\gamma}\right).$$

The smallest value of this maximum is achieved for $\gamma_0 = \frac{43-\sqrt{1777}}{36}$. Thus the value of α_{cr} is minimized and we get

$$\alpha_{cr} = \frac{4-2\gamma_0}{5-2\gamma_0} = \frac{12\gamma_0+2}{3-6\gamma_0} = \frac{\sqrt{1777}-23}{24} = 0.798103\ldots.$$

Finally, recall that we also need to check that $a, b, \beta_1 \in (0,1)$ for $\gamma = \gamma_0$ and $\alpha \in (\alpha_{cr}, 1)$. Figure 1 below verifies this. □

We now briefly sketch the proof of Theorem 1.1.

Sketch of proof of Theorem 1.1. The global existence and smoothness of solutions is proven in two steps. The first step uses the local well-posedness of (1.1), which can be established through a standard procedure; see, e.g., [37, 50]. The second step extends the local solution of the first step to a global solution via a priori estimates. Proposition 1.2 provides a global L^q-bound for G for each $2 \le q \le 6$. As in [29, Proposition 7.1], we can show that, for all $0 \le s \le 3\alpha - 2$,

$$\sup_{0\le t\le T} \|G(t)\|_{B^s_{6,\infty}} \le C(T, u_0, \theta_0).$$

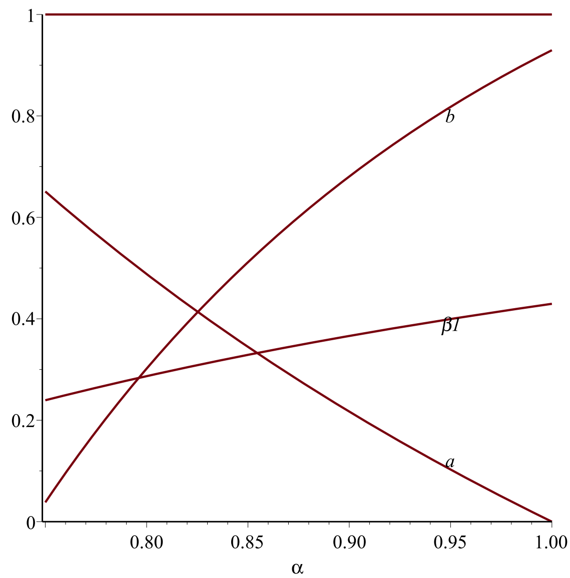

Figure 1. Graphs of a, b, and β_1 for $\gamma = \gamma_0$ and $\alpha \in (\alpha_{cr}, 1)$.

Recall the embedding, $B^{3\alpha-2}_{6,\infty}(\mathbb{R}^2) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^2)$ for $\alpha > 7/9$. In light of it, for $\alpha > (\sqrt{1777} - 23)/24 > 7/9$,

$$\|\nabla u_G\|_{L^\infty} = \|\nabla\nabla^\perp\Delta^{-1}G\|_{L^\infty} \le \|G\|_{B^0_{\infty,1}} \le \|G\|_{B^s_{6,\infty}}.$$

This yields a global Lipschitz bound on u_G. The rest of the proof is the same as that in [29]; we thus omit further details. □

Acknowledgements. Wu thanks Professors Quansen Jiu, Changxing Miao, and Zhifei Zhang for discussions and thanks Drs. Haifeng Shang and Zhuan Ye for comments.

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(Received June 4, 2015)