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J. Differential Equations 267 (2019) 1731–1747

Journal of Differential Equations

www.elsevier.com/locate/jde

The 2D Boussinesq equations with vertical dissipation and linear stability of shear flows

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Received 18 May 2017; revised 1 February 2019
Available online 5 March 2019

Abstract

This paper studies the linear stability of a steady-state solution with the velocity being a shear flow to the 2D Boussinesq equations with only vertical dissipation. The Boussinesq equations model many fluid phenomena when the Boussinesq approximation applies such as the Rayleigh-Benard convection, atmospheric fronts and oceanic circulation. The vertically dissipative 2D Boussinesq equations model geophysical fluids in certain physical regimes. Whether or not the vertical dissipation can damp perturbations near the equilibrium with the velocity being a shear and the temperature being zero is an important but difficult problem. Assuming the spatial domain is periodic in the horizontal direction and half-line in the vertical direction with no flux boundary condition, we show that any perturbation satisfying the linearized equation around this equilibrium is infinitely smooth in the $x-$variable and decays exponentially in time and in the horizontal Fourier mode, even though the linearized system involves only vertical dissipation.

MSC: 35Q35; 76F10

Keywords: 2D Boussinesq equations; Linear stability; Shear flow; Vertical dissipation

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https://doi.org/10.1016/j.jde.2019.02.020
0022-0396/

1. Introduction

The Boussinesq equations play an important role in the study of many phenomena in fluids and geophysical fluids including atmospheric fronts and oceanic circulation (see, e.g., [10,16,18]). In addition, the Boussinesq equations are the foundation for understanding the Rayleigh-Benard convection, the most frequently studied convection phenomenon (see, e.g., [7,9]). The standard two-dimensional Boussinesq equations can be written as

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\theta + (\mathbf{u}\cdot\nabla)\theta = \eta\Delta\theta, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ denotes the 2D velocity field, p the pressure, θ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, ν the viscosity, η the thermal diffusivity, and $\mathbf{e}_2$ is the unit vector in the vertical direction. The first equation in (1.1) is the Navier-Stokes equation with buoyancy forcing in the vertical direction. The second equation reflects the mass conservation while the third equation is a balance of the temperature convection and diffusion.

In certain physical regime and under suitable scaling, the kinematic dissipation and thermal diffusion may become partial (given by part of Laplacian), as in the notable example of Prandtl boundary layer equation, in which the horizontal velocity equation involves only the vertical dissipation. This paper focuses on the vertically dissipated Boussinesq equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\partial_{yy}\mathbf{u} + \theta\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\theta + (\mathbf{u}\cdot\nabla)\theta = \eta\partial_{yy}\theta. \end{cases} \tag{1.2}$$

When the spatial domain is the whole plane $\mathbb{R}^2$, the paper of Cao and Wu [6], together with two previous papers of Adhikari, Cao and Wu [1] and [2], proved that any H^2 initial data (u_0, θ_0) leads to a unique global solution. Li and Titi [12] were able to weaken the initial regularity assumption to H^1.

This paper attempts to understand the stability of perturbations near the decaying shear profile $(\mathbf{u}_{sh}, \theta_{sh})$ of (1.2), where

$$\mathbf{u}_{sh} = (\bar{U}(y,t), 0), \quad \theta_{sh} = 0, \quad \bar{U}(y,t) = e^{\nu t\partial_{yy}}U(y)$$

with $U(y)$ being a smooth function. Clearly $\mathbf{u}_{sh}$ represents the standard decaying shear flow (the heat evolution of the shear) and $\bar{U}(y,t) = U(y)$ when $U(y)$ is linear. The perturbations

$$\widetilde{u}(x,t) = u(x,t) - \bar{U}(y,t),\ \widetilde{v}(x,t) = v(x,t),\ \widetilde{p}(x,t) = p(x,t),\ \widetilde{\theta}(x,t) = \theta(x,t)$$

then satisfy

$$\begin{cases} \widetilde{u}_t + \bar{U}\partial_x \widetilde{u} + \partial_y \bar{U}\,\widetilde{v} + \widetilde{u}\widetilde{u}_x + \widetilde{v}\widetilde{u}_y = -\widetilde{p}_x + \nu\,\widetilde{u}_{yy}, \\ \widetilde{v}_t + \bar{U}\partial_x \widetilde{v} + \widetilde{u}\widetilde{v}_x + \widetilde{v}\widetilde{v}_y = -\widetilde{p}_y + \nu\,\widetilde{v}_{yy} + \widetilde{\theta}, \\ \widetilde{u}_x + \widetilde{v}_y = 0, \\ \widetilde{\theta}_t + \bar{U}\partial_x \widetilde{\theta} + \widetilde{u}\widetilde{\theta}_x + \widetilde{v}\widetilde{\theta}_y = \eta\,\widetilde{\theta}_{yy}. \end{cases} \tag{1.3}$$

The spatial domain Ω is either the 2D periodic box $\mathbb{T}^2$ or the half infinite pipe $\Omega = \mathbb{T} \times \mathbb{R}^+$ with the no flux boundary condition

$$\partial_y \widetilde{u} = 0, \quad \partial_y \widetilde{v} = 0 \quad \text{and} \quad \partial_y \widetilde{\theta} = 0 \quad \text{on } y = 0. \tag{1.4}$$

Since (1.3) involves only vertical dissipation, the rationale for imposing the no flux boundary condition (or the Neumann boundary condition) is to eliminate the influence of the environment on the evolution of the perturbation. As we recall, the no flux boundary condition (or the Neumann boundary condition) for the half line 1D heat equation represents no heat exchange between the environment and the half-line rod. The corresponding vorticity

$$\widetilde{\omega} = \partial_x \widetilde{v} - \partial_y \widetilde{u}$$

satisfies

$$\partial_t \widetilde{\omega} + \bar{U}\partial_x \widetilde{\omega} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\omega} = \nu\,\partial_{yy}\widetilde{\omega} + \partial_x \widetilde{\theta} + \widetilde{v}\partial_{yy}\bar{U}.$$

In the special case when $U(y)$ is linear, namely $U(y) = ay + b$, then

$$\bar{U}(y,t) = U(y) = ay + b$$

and the vorticity perturbation $\widetilde{\omega}$, together with $\widetilde{\theta}$, satisfies

$$\begin{cases} \partial_t \widetilde{\omega} + U(y)\partial_x \widetilde{\omega} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\omega} = \nu\,\partial_{yy}\widetilde{\omega} + \partial_x \widetilde{\theta}, \\ \partial_t \widetilde{\theta} + U(y)\partial_x \widetilde{\theta} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\theta} = \eta\,\partial_{yy}\widetilde{\theta}. \end{cases}$$

The full nonlinear stability problem appears to be out of reach at this moment and this paper focuses on the linear stability problem. The spatial domain is $\Omega = \mathbb{T}^2$ or $\Omega = \mathbb{T} \times \mathbb{R}^+$. For the sake of conciseness, all results are presented for $\Omega = \mathbb{T} \times \mathbb{R}^+$, but they are also valid when $\Omega = \mathbb{T}^2$. The periodic boundary conditions on both directions, namely the case $\Omega = \mathbb{T}^2$ may need some justification since the shear flow $(y, 0)$ is simply not periodic. One justification is that any disturbance or perturbations can be discomposed into frequencies and understanding the stability of the periodic frequencies plays a crucial role in the investigation of the stability of more general perturbations. For notational convenience, the tilde will be dropped for the rest of the paper.

We set $U(y) = y$ and focus on the initial and boundary-value problem for the linearized equations

$$\begin{cases} \partial_t \omega + y\partial_x \omega = \nu\,\partial_{yy}\omega + \partial_x \theta, \\ \partial_t \theta + y\partial_x \theta = \eta\,\partial_{yy}\theta, \\ \partial_y \omega|_{y=0} = 0, \quad \partial_y \theta|_{y=0} = 0, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.5}$$

We show that any solution (ω, θ) of (1.5) is stable, smooth in both variables even though (1.5) involves only vertical dissipation and any non-zero horizontal Fourier modes decay exponentially in time and in the modes. More precisely, we obtain the linear stability and regularity results in the following two theorems.

Theorem 1.1. *Let the spatial domain $\Omega = \mathbb{T} \times \mathbb{R}^+$. Assume ω_0 and θ_0 satisfy*

$$\omega_0 \in L^2(\Omega), \quad \partial_y \omega_0 \in L^2(\Omega), \quad \theta_0 \in L^2(\Omega), \quad \partial_y \theta_0 \in L^2(\Omega).$$

Let (ω, θ) be the corresponding solution of (1.5). Then, for any integer $k \neq 0$ and any $t > 0$,

$$\|\widehat{\theta}(k,\cdot,t)\|_{L_y^2} \le C\left(\|\widehat{\theta}_0(k,\cdot)\|_{L_y^2} + \eta^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_0(k,\cdot)\|_{L_y^2}\right) e^{-\frac{2}{3}\eta^{\frac{1}{3}}k^{\frac{2}{3}}t},$$

$$\|\partial_y\widehat{\theta}(k,\cdot,t)\|_{L_y^2} \le C\left(\eta^{-2/3}k^{2/3}\|\widehat{\theta}_0(k,\cdot)\|_{L_y^2} + \|\partial_y\widehat{\theta}_0(k,\cdot)\|_{L_y^2}\right) e^{-\frac{2}{3}\eta^{\frac{1}{3}}k^{\frac{2}{3}}t}$$

and

$$\|\widehat{\omega}(k,\cdot,t)\|_{L_y^2} \le C\,\Phi_{Combo,\omega_{i,0}}(0)\, e^{-\frac{2}{3}\nu^{\frac{1}{3}}k^{\frac{2}{3}}t},$$

$$\|\partial_y\widehat{\omega}(k,\cdot,t)\|_{L_y^2} \le C\,\Phi_{Combo,\omega_{0,0}}(0)\,\nu^{-\frac{2}{3}}k^{\frac{2}{3}}\, e^{-\frac{2}{3}\nu^{\frac{1}{3}}k^{\frac{2}{3}}t},$$

where C is a constant independent of k and t. Here $\Phi_{Combo,\omega_{0,0}}(t)$ with any $t \ge 0$ is defined as

$$\Phi_{Combo,\omega_{0,0}}(t) = \Phi_{\omega_{0,0}}(t) + (36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3} + 18\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{\theta_{1,0}}(t) + (27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3} + 13.5\,\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{\theta_{2,0}}(t),$$

where $\Phi_{\omega_{0,0}}$, $\Phi_{\theta_{1,0}}$ and $\Phi_{\theta_{2,0}}$ are given by

$$\Phi_{\omega_{0,0}} = k^2\gamma\|\widehat{\omega}\|_{L^2}^2 + \alpha_\nu\|\partial_y\widehat{\omega}\|_{L^2}^2 + k\beta_\nu Re\langle \mathbf{i}\widehat{\omega}, \partial_y\widehat{\omega}\rangle,$$
$$\Phi_{\theta_{1,0}} = k^2\gamma\|\widehat{\partial_x\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y\widehat{\partial_x\theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\widehat{\partial_x\theta}, \partial_y\widehat{\partial_x\theta}\rangle,$$
$$\Phi_{\theta_{2,0}} = k^2\gamma\|\widehat{\partial_x^2\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y\widehat{\partial_x^2\theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\widehat{\partial_x^2\theta}, \partial_y\widehat{\partial_x^2\theta}\rangle$$

with $\mathbf{i}$ being the unit imaginary number, and α_ν, β_ν, α_η, β_η and γ given by

$$\alpha_\nu = \frac{1}{3}\nu^{2/3}k^{-2/3},\ \beta_\nu = \nu^{1/3}k^{-4/3},\ \alpha_\eta = \frac{1}{3}\eta^{2/3}k^{-2/3},\ \beta_\eta = \eta^{1/3}k^{-4/3},\ \gamma = k^{-2}.$$

In the case when $k = 0$, $\widehat{\omega}(0, y, t)$ and $\partial_y\widehat{\omega}(0, y, t)$ solve the 1D heat equation

$$\partial_t f = \nu\partial_{yy} f,$$

while $\widehat{\theta}(0, y, t)$ and $\partial_y\widehat{\theta}(0, y, t)$ solve the 1D heat equation

$$\partial_t f = \eta\partial_{yy} f.$$

In the statement of Theorem 1.1, $\widehat{\omega}(k, y, t)$ and $\widehat{\theta}(k, y, t)$ are the Fourier transforms of ω and θ with respect to x, respectively, and $\|f\|_{L^2_y}$ denotes the L^2-norm with respective to y. Theorem 1.1 indicates that, even though initially ω_0, θ_0, $\partial_y\theta_0$ and $\partial_y\omega_0$ are only in L^2 and (1.5) does not have any regularization in the horizontal direction, the corresponding solution (ω, θ) becomes infinitely smooth in the horizontal direction (actually in a Gevrey class). The second theorem establishes the decay and regularity for higher-order derivatives of solutions to (1.5).

Theorem 1.2. *Let the spatial domain $\Omega = \mathbb{T} \times \mathbb{R}^+$. Let $i \geq 0$ and $j > 0$ be integers and assume ω_0 and θ_0 satisfy, for $0 \leq m \leq j$ and $0 \leq l \leq i + 3$,*

$$\partial_y^m \partial_x^l \omega_0 \in L^2(\Omega), \quad \partial_y^m \partial_x^l \theta_0 \in L^2(\Omega).$$

Consider the initial and boundary-value problem

$$\begin{cases} \partial_t\omega + y\partial_x\omega = \nu\, \partial_{yy}\omega + \partial_x\theta, \\ \partial_t\theta + y\partial_x\theta = \eta\, \partial_{yy}\theta, \\ \partial_y^m\omega|_{y=0} = 0, \quad \partial_y^m\theta|_{y=0} = 0, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases}$$

Then, for $0 \leq m \leq j + 1$ and $0 \leq l \leq i + 3$, any integer $k \neq 0$ and any $t > 0$,

$$\|\partial_y^m \partial_x^l \widehat{\theta}(k, \cdot, t)\|_{L^2_y} \leq P\left(t, k^{-\frac{2}{3}}, k^{\frac{2}{3}}\right) e^{-\frac{2}{3}\nu^{1/3}k^{2/3}t}, \tag{1.6}$$

$$\|\partial_y^m \partial_x^l \widehat{\omega}(k, \cdot, t)\|_{L^2_y} \leq Q\left(t, k^{-\frac{2}{3}}, k^{\frac{2}{3}}\right) e^{-\frac{2}{3}\eta^{1/3}k^{2/3}t}, \tag{1.7}$$

where P and Q are polynomials of t, $k^{-\frac{2}{3}}$ and $k^{\frac{2}{3}}$ depending on ν, η and the initial norms. In the case when $k = 0$, $\partial_y^m \partial_x^l \theta(0, y, t)$ and $\partial_y^m \partial_x^l \omega(0, y, t)$ solve the 1D heat equations.

This smooth effect shown in the theorems above comes from hypoellipticity [11]. Hypoellipticity and enhanced dissipation have been explored for the Navier-Stokes and other equations, and significant results have been obtained (see, e.g., [3–5,8,13–15,17,20]). As explained below, the situation with the 2D Boussinesq equations is more complex due to the presence of the buoyancy forcing term in the velocity equation. The regularity and decay estimates in Theorems 1.1 and 1.2 do not follow from direct energy estimates. The proof uses the "hypocoercivity" approach developed by C. Villani [19]. This approach is applicable to linear operators of the form $L = A^*A + B$ in a Hilbert space X, where B is skew-symmetric. This approach ensures that, under suitable assumptions, the spectral properties of L are comparable to those of $\widetilde{L} = A^*A + C^*C$, where $C = [A, B]$ is the commutator.

The proofs of Theorem 1.1 and 1.2 are divided into three main steps. The first step is to prove the regularity and decay estimates for $\widehat{\theta}$. To do so, we project the equation of θ onto the $k-$th Fourier mode of the x-variable,

$$\partial_t\widehat{\theta} + y\partial_x\widehat{\theta} = \eta\partial_{yy}\widehat{\theta}. \tag{1.8}$$

(1.8) can be written as

$$\partial_t\widehat{\theta} + L\widehat{\theta} = 0,$$

where

$$L = AA^* + B\,, \quad A = \sqrt{\eta}\partial_y\,, \quad B = y\partial_x$$

with AA^* being the symmetric part and B being the antisymmetric part. We define $C = [A, B] = \sqrt{\eta}\partial_x$ and seek a functional proportional to $\|A\theta\|_{L^2}$, $\|C\theta\|_{L^2}$ and $Re\langle A\theta, C\theta\rangle$:

$$\Phi(t) = k^2\gamma\|\widehat{\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y\widehat{\theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle. \tag{1.9}$$

Here, $\langle f, g\rangle = \int f\bar{g}dy$. α, β and γ are parameters such that

$$\alpha_\eta = \frac{1}{3}\eta^{2/3}k^{-2/3}\,, \quad \beta_\eta = \eta^{1/3}k^{-4/3}\,, \quad \gamma = k^{-2}\,. \tag{1.10}$$

The notation $k^2\gamma$, albeit being 1, is more suitable for the later derivations. For any $k \neq 0$, we are able to show that

$$\frac{d}{dt}\Phi(t) \le -\frac{2}{3}\eta^{2/3}k^{2/3}\Phi(t),$$

which implies the desired exponential decay for $\widehat{\theta}$ and $\partial_y\widehat{\theta}$. Since $\partial_x^i\theta$ for any integer $i > 0$ satisfies the same equation as θ, $\partial_x^i\theta$ and $\partial_y\partial_x^i\theta$ obey the similar estimates as those for $\widehat{\theta}$ and $\partial_y\widehat{\theta}$.

The second main step is to prove the regularity and decay estimates for ω. The process is more complex due to the presence of the extra term $\partial_x\theta$ in the vorticity equation,

$$\partial_t\omega + y\partial_x\omega = \nu\partial_{yy}\omega + \partial_x\theta.$$

Since the term $\partial_x\theta$ can not be incorporated into the operator form, we treat it as an external forcing term and the functional construction is more sophisticated. Naturally we attempt a similar functional as the one in (1.9),

$$\Phi_{\omega_{0,0}}(t) = k^2\gamma\|\widehat{\omega}\|_{L^2}^2 + \alpha_\nu\|\partial_y\widehat{\omega}\|_{L^2}^2 + k\beta_\nu Re\langle \mathbf{i}\widehat{\omega}, \partial_y\widehat{\omega}\rangle,$$

where we have written $\Phi_{\omega_{i,j}}$ for the functional associated with $\partial_x^i\partial_y^j\omega$ and $\Phi_{\omega_{0,0}}$ is just the functional for ω. If we differentiate $\Phi_{\omega_{0,0}}(t)$, due to $\partial_x\theta$ term, $\partial_t\Phi_{\omega_{0,0}}(t)$ involves the derivatives of θ, which makes a closed-form estimate impossible. It appears that we need to construct a combined functional of ω and θ. It is a very tedious process to figure out the exact combination. By carefully examining the terms in $\partial_t\Phi_{\omega_{0,0}}(t)$, we find a combined functional that would serve our purpose,

$$\begin{aligned}\Phi_{Combo,\omega_{0,0}}(t) = \Phi_{\omega_{0,0}}(t) &+ (36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3} + 18\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{\theta_{1,0}}(t)\\ &+(27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3} + 13.5\,\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{\theta_{2,0}}(t),\end{aligned} \tag{1.11}$$

where $\Phi_{\theta_{1,0}}$ and $\Phi_{\theta_{2,0}}$ denote the functionals associated with $\partial_x\theta$ and $\partial_x^2\theta$, respectively, namely (1.9) with $\partial_x\theta$ and $\partial_x^2\theta$ replacing θ in the formula of Φ. Differentiating (1.11) in time and evaluating the terms meticulously, we obtain

$$\frac{d}{dt}\Phi_{Combo,\omega_{0,0}}(t) \leqslant -\frac{2}{3}\nu^{1/3}k^{2/3}\Phi_{Combo,\omega_{0,0}}(t),$$

which yields the desired decay estimates for ω. A similar approach works for $\partial_x^i\omega$ and $\partial_y\partial_x^i\omega$ for any integer $i > 0$.

The third step proves the global bounds for $\partial_y^j\partial_x^i\theta$ and $\partial_y^j\partial_x^i\omega$. $\partial_y^j\partial_x^i\theta$ satisfies

$$\partial_t(\partial_y^j\partial_x^i\theta) + j\partial_x^{i+1}(\partial_y^{j-1})\theta + y\partial_x^{i+1}(\partial_y^j\theta) = \eta\partial_{yy}(\partial_y^j\partial_x^i\theta),$$

which suggests that we make an induction on j. By constructing suitable combined functionals of θ and carefully evaluating each term, we are able to prove the decay bounds for $\partial_y^j\partial_x^i\theta$. The proof for the exponential decay of $\partial_y^j\partial_x^i\omega$ involves very tedious evaluations of many terms. More details are given in the section that follows. The rest of this paper proves Theorems 1.1 and 1.2.

2. Proof of Theorems 1.1 and 1.2

This section proves Theorems 1.1 and 1.2. For the sake of clarity, we divide the whole section into three subsections. The first subsection focuses on the decay estimates for $\|\partial_x^i\theta\|_{L_y^2}$ and for $\|\partial_y\partial_x^i\theta\|_{L_y^2}$ for any nonnegative integer i. The second subsection establishes the regularity and decay bounds for $\|\partial_x^i\omega\|_{L_y^2}$ and for $\|\partial_y\partial_x^i\omega\|_{L_y^2}$ while the last subsection deals with the exponential decay for derivatives $\partial_y^j\partial_x^i\theta$ and $\partial_y^j\partial_x^i\omega$ with $j \geq 1$.

2.1. *Exponential decay for $\partial_x^i\theta$ and $\partial_y\partial_x^i\theta$*

This subsection is devoted to proving the decay estimates for $\partial_x^i\theta$ and $\partial_y\partial_x^i\theta$ for any nonnegative integer i. More precisely, we prove the following proposition.

Proposition 2.1. *Let the spatial domain $\Omega = \mathbb{T}\times\mathbb{R}^+$. Let $i \geq 0$ be an integer and assume $\theta_0 \in L^2(\Omega)$ and $\partial_x^i\theta_0 \in L^2(\Omega)$. Consider the initial and boundary-value problem*

$$\begin{cases} \partial_t\theta + y\partial_x\theta = \eta\,\partial_{yy}\theta, \\ \partial_y\theta|_{y=0} = 0, \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{2.1}$$

Then, for any integer $k \neq 0$,

$$\|\partial_x^i\widehat{\theta}(k,\cdot,t)\|_{L_y^2} \leq C\left(\|\partial_x^i\widehat{\theta_0}(k,\cdot)\|_{L_y^2} + \eta^{2/3}k^{-2/3}\|\partial_y\partial_x^i\widehat{\theta_0}(k,\cdot)\|_{L_y^2}\right)e^{-\frac{2}{3}\eta^{\frac{1}{3}}k^{\frac{2}{3}}t}, \tag{2.2}$$

$$\|\partial_y\partial_x^i\widehat{\theta}(k,\cdot,t)\|_{L_y^2} \leq C\left(\eta^{-2/3}k^{2/3}\|\partial_x^i\widehat{\theta_0}(k,\cdot)\|_{L_y^2} + \|\partial_y\partial_x^i\widehat{\theta_0}(k,\cdot)\|_{L_y^2}\right)e^{-\frac{2}{3}\eta^{\frac{1}{3}}k^{\frac{2}{3}}t}, \tag{2.3}$$

where C is a constant independent of k and t. In the case when $k=0$, $\partial_x^i\widehat{\theta}(0,y,t)$ and $\partial_y\partial_x^i\widehat{\theta}(0,y,t)$ solve the 1D heat equation,

$$\partial_t\partial_x^i\widehat{\theta}(0,y,t)=\eta\partial_{yy}\partial_x^i\widehat{\theta}(0,y,t),\quad \partial_t\partial_y\partial_x^i\widehat{\theta}(0,y,t)=\eta\partial_{yy}\partial_y\partial_x^i\widehat{\theta}(0,y,t).$$

Proof. We start with the case $i=0$. For notational convenience, we simply write $\|f\|_{L^2}$ for $\|f\|_{L^2_y}$ to denote the L^2-norm in the y-variable. As aforementioned in the introduction, we define $\Phi(t)$ as in (1.9), namely

$$\Phi(t)=k^2\gamma\|\widehat{\theta}\|_{L^2}^2+\alpha\|\partial_y\widehat{\theta}\|_{L^2}^2+k\beta Re\langle \mathbf{i}\widehat{\theta},\partial_y\widehat{\theta}\rangle$$

with the parameters α, β and γ given by

$$\alpha=\frac{1}{3}\eta^{2/3}k^{-2/3},\quad \beta=\eta^{1/3}k^{-4/3},\quad \gamma=k^{-2}.$$

The notation $k^2\gamma$, albeit being 1, is more suitable for the following derivation. With these choices of parameters, we have

$$\begin{aligned}
k\beta Re\langle \mathbf{i}\widehat{\theta},\partial_y\widehat{\theta}\rangle &= \frac{\sqrt{3}\eta^{1/3}k^{-4/3}}{\eta^{1/3}k^{-1/3}}k\left\|\frac{1}{\sqrt{3}}\eta^{1/3}k^{-1/3}\partial_y\widehat{\theta}\right\|_{L^2}\|\widehat{\theta}\|_{L^2}\\
&\leqslant \frac{\sqrt{3}}{2}\left\|\frac{1}{\sqrt{3}}\eta^{1/3}k^{-1/3}\partial_y\widehat{\theta}\right\|_{L^2}^2+\frac{\sqrt{3}}{2}\|\widehat{\theta}\|_{L^2}^2\\
&= \frac{\sqrt{3}}{2}\alpha\|\partial_y\widehat{\theta}\|_{L^2}^2+\frac{\sqrt{3}}{2}k^2\gamma\|\widehat{\theta}\|_{L^2}^2,
\end{aligned}\tag{2.4}$$

which implies the equivalence between $\Phi(t)$ and $\|\widehat{\theta}\|_{L^2}^2+\alpha\|\partial_y\widehat{\theta}\|_{L^2}^2$,

$$\Phi(t)\geq\left(1-\frac{\sqrt{3}}{2}\right)\left(k^2\gamma\|\widehat{\theta}\|_{L^2}^2+\alpha\|\partial_y\widehat{\theta}\|_{L^2}^2\right)$$

and

$$\Phi(t)\leq\left(1+\frac{\sqrt{3}}{2}\right)\left(k^2\gamma\|\widehat{\theta}\|_{L^2}^2+\alpha\|\partial_y\widehat{\theta}\|_{L^2}^2\right).$$

Therefore, to prove (2.2) and (2.3) with $i=0$, it suffices to prove the decay for $\Phi(t)$.

We estimate the time derivative of Φ:

$$\begin{aligned}
\frac{d}{dt}\Phi(t)=&\,k^2\gamma\left(\langle\partial_t\widehat{\theta},\widehat{\theta}\rangle+\langle\widehat{\theta},\partial_t\widehat{\theta}\rangle\right)\\
&+\alpha\left(\langle\partial_t\partial_y\widehat{\theta},\partial_y\widehat{\theta}\rangle+\langle\partial_y\widehat{\theta},\partial_t\partial_y\widehat{\theta}\rangle\right)\\
&+k\beta\left(Re\langle\mathbf{i}\,\partial_t\widehat{\theta},\partial_y\widehat{\theta}\rangle+Re\langle\mathbf{i}\widehat{\theta},\partial_t\partial_y\widehat{\theta}\rangle\right).
\end{aligned}\tag{2.5}$$

Integrating by parts and making use of the boundary condition $\partial_y\theta=0$ on $y=0$, we have

$$
\begin{aligned}
\langle \partial_t \widehat{\theta}, \widehat{\theta}\rangle + \langle \widehat{\theta}, \partial_t \widehat{\theta}\rangle &= \langle \eta \partial_{yy}\widehat{\theta} - y\partial_x \widehat{\theta}, \widehat{\theta}\rangle + \langle \widehat{\theta}, \eta\partial_{yy}\widehat{\theta} - y\partial_x\widehat{\theta}\rangle \\
&= -2\eta \|\partial_y \widehat{\theta}\|_{L^2}^2 + \int -iky\widehat{\theta}\bar{\widehat{\theta}} + \widehat{\theta} iky \bar{\widehat{\theta}}\, dy \\
&= -2\eta \|\partial_y \widehat{\theta}\|_{L^2}^2 .
\end{aligned}
$$

Due to the equation on $\partial_y \widehat{\theta}$:

$$
\partial_t \partial_y \widehat{\theta} + ik\widehat{\theta} + y\partial_{xy}\widehat{\theta} = \eta \partial_{yyy}\widehat{\theta},
$$

we have

$$
\begin{aligned}
&\langle \partial_t \partial_y \widehat{\theta}, \partial_y \widehat{\theta}\rangle + \langle \partial_y \widehat{\theta}, \partial_t \partial_y \widehat{\theta}\rangle \\
&\quad = \langle -\partial_x \widehat{\theta} - y\partial_{xy}\widehat{\theta} + \eta \partial_{yyy}\widehat{\theta}, \partial_y\widehat{\theta}\rangle + \langle \partial_y \widehat{\theta}, -\partial_x\widehat{\theta} - y\partial_{xy}\widehat{\theta} + \eta\partial_{yyy}\widehat{\theta}\rangle \\
&\quad = -2\eta \|\partial_{yy}\widehat{\theta}\|_{L^2}^2 + \int -ik\widehat{\theta}\partial_y \bar{\widehat{\theta}} + \partial_y \widehat{\theta} ik \bar{\widehat{\theta}}\, dy \\
&\qquad + \int -iky\partial_y \widehat{\theta}\partial_y \bar{\widehat{\theta}} + \partial_y\widehat{\theta} iky \partial_y \bar{\widehat{\theta}}\, dy \\
&\quad = -2\eta \|\partial_{yy}\widehat{\theta}\|_{L^2}^2 - 2kRe\langle \mathbf{i}\widehat{\theta}, \partial_y \widehat{\theta}\rangle .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&Re\langle \mathbf{i}\,\partial_t \widehat{\theta}, \partial_y \widehat{\theta}\rangle + Re\langle \mathbf{i}\widehat{\theta}, \partial_t\partial_y\widehat{\theta}\rangle \\
&\quad = Re\langle \mathbf{i}\,\eta \partial_{yy}\widehat{\theta} - iy\partial_x\widehat{\theta}, \partial_y\widehat{\theta}\rangle + Re\langle i\widehat{\theta}, -\partial_x\widehat{\theta} - y\partial_{xy}\widehat{\theta} + \eta\partial_{yyy}\widehat{\theta}\rangle \\
&\quad = Re\int i\eta\partial_{yy}\widehat{\theta}\partial_y\bar{\widehat{\theta}}\, dy + Re\int -i\partial_y\widehat{\theta}\eta\partial_{yy}\bar{\widehat{\theta}}\, dy \\
&\qquad + Re\int ky\widehat{\theta}\partial_y\bar{\widehat{\theta}}\, dy + Re\int i\widehat{\theta} ik\bar{\widehat{\theta}}\, dy + Re\int i\widehat{\theta} yik\partial_y \bar{\widehat{\theta}}\, dy \\
&\quad = 2Re\langle \mathbf{i}\,\eta\partial_{yy}\widehat{\theta}, \partial_y\widehat{\theta}\rangle - k\|\widehat{\theta}\|_{L^2}^2 .
\end{aligned}
$$

Then, (2.5) becomes

$$
\begin{aligned}
\frac{d}{dt}\Phi(t) &= -2\eta\, k^2\gamma\, \|\partial_y\widehat{\theta}\|_{L^2}^2 - 2\alpha\eta \|\partial_{yy}\widehat{\theta}\|_{L^2}^2 - 2\alpha k Re\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle \\
&\quad + 2k\beta\eta Re\langle \mathbf{i}\,\partial_{yy}\widehat{\theta}, \partial_y\widehat{\theta}\rangle - k^2\beta\|\widehat{\theta}\|_{L^2}^2 \\
&\leqslant -2\eta\|\partial_y\widehat{\theta}\|_{L^2}^2 - \frac{2}{3}\eta^{5/3}k^{-2/3}\|\partial_{yy}\widehat{\theta}\|_{L^2}^2 - \frac{2}{3}\eta^{2/3}k^{-2/3}kRe\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle \\
&\quad + 2\eta^{4/3}k^{-1/3}Re\langle \partial_{yy}\widehat{\theta}, \partial_y\widehat{\theta}\rangle - \eta^{1/3}k^{2/3}\|\widehat{\theta}\|_{L^2}^2 \\
&\leqslant -2\eta\|\partial_y\widehat{\theta}\|_{L^2}^2 - \frac{2}{3}\eta^{5/3}k^{-2/3}\|\partial_{yy}\widehat{\theta}\|_{L^2}^2 - \frac{2}{3}\eta^{2/3}k^{-2/3}kRe\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle \\
&\quad + \frac{2}{3}\eta^{5/3}k^{-2/3}\|\partial_{yy}\widehat{\theta}\|_{L^2}^2 + \frac{3}{2}\eta\|\partial_y\widehat{\theta}\|_{L^2}^2 - \eta^{1/3}k^{2/3}\|\widehat{\theta}\|_{L^2}^2
\end{aligned}
$$

$$= -\eta^{1/3}k^{2/3}\|\widehat{\theta}\|_{L^2}^2 - \frac{3}{2}\cdot\frac{1}{3}\eta\|\partial_y\widehat{\theta}\|_{L^2}^2 - \frac{2}{3}\eta^{2/3}k^{-2/3}k\,Re\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle$$
$$= -\frac{2}{3}\eta^{1/3}k^{2/3}\left(\frac{3}{2}\|\widehat{\theta}\|_{L^2}^2 + \frac{9}{4}\cdot\frac{1}{3}\eta^{\frac{2}{3}}k^{-\frac{2}{3}}\|\partial_y\widehat{\theta}\|_{L^2}^2 + \eta^{\frac{1}{3}}k^{-\frac{4}{3}}k\,Re\langle \mathbf{i}\widehat{\theta}, \partial_y\widehat{\theta}\rangle\right).$$

Recalling the definitions of α, β and γ in (1.10), we are led to

$$\frac{d}{dt}\Phi(t) \leqslant -\frac{2}{3}\eta^{1/3}k^{2/3}\Phi(t),$$

which implies the desired exponential decay $\Phi(t) \leqslant \Phi(0)e^{-\frac{2}{3}\eta^{1/3}k^{2/3}t}$. By the equivalence of the norms (2.4), we have that both $\|\widehat{\theta}\|_{L^2}$ and $\|\partial_y\widehat{\theta}\|_{L^2}$ decay exponentially,

$$\left(1 - \frac{\sqrt{3}}{2}\right)\left(k^2\gamma\|\widehat{\theta}(\cdot,t)\|_{L^2} + \alpha\|\partial_y\widehat{\theta}(\cdot,t)\|_{L^2}\right) \leqslant \Phi(0)e^{-\frac{1}{3}\eta^{1/3}k^{2/3}t}.$$

This completes the proof for the case when $i = 0$. For a general positive integer i, $\partial_x^i\theta$ satisfies the same equation as that of θ,

$$\partial_t\partial_x^i\theta + L(\partial_x^i\theta) = 0, \qquad \partial_x^i\theta(x,y,0) = \partial_x^i\theta_0(x,y).$$

As a consequence, $\|\partial_x^i\theta\|_{L^2}$ and $\|\partial_y\partial_x^i\theta\|_{L^2}$ obey the same decay rate in k and in t. When $k = 0$, $\partial_x^i\widehat{\theta}(0,y,t)$ and $\partial_y\partial_x^i\widehat{\theta}(0,y,t)$ satisfy the 1D heat equation,

$$\partial_t\partial_x^i\widehat{\theta}(0,y,t) = \eta\partial_{yy}\partial_x^i\widehat{\theta}(0,y,t), \quad \partial_t\partial_y\partial_x^i\widehat{\theta}(0,y,t) = \eta\partial_{yy}\partial_y\partial_x^i\widehat{\theta}(0,y,t),$$

which yields the desired representations for $k = 0$. This completes the proof of Proposition 2.1. □

2.2. *Exponential decay for ω*

This subsection proves the decay estimates for $\widehat{\omega}$ and $\partial_y\widehat{\omega}$, as stated in Theorem 1.1. The precise statement is provided in the following proposition.

Proposition 2.2. *Let the spatial domain $\Omega = \mathbb{T}\times\mathbb{R}^+$. Let $i \geq 0$ be an integer. Assume ω_0 and θ_0 satisfy*

$$\partial_x^i\omega_0 \in L^2(\Omega), \quad \partial_y\partial_x^i\omega_0 \in L^2(\Omega), \quad \partial_x^{i+1}\theta_0 \in L^2(\Omega), \quad \partial_y\partial_x^{i+1}\theta_0 \in L^2(\Omega),$$
$$\partial_x^{i+2}\theta_0 \in L^2(\Omega), \quad \partial_y\partial_x^{i+2}\theta_0 \in L^2(\Omega).$$

Consider the initial and boundary-value problem

$$\begin{cases} \partial_t\omega + y\partial_x\omega = \nu\,\partial_{yy}\omega + \partial_x\theta, \\ \partial_t\theta + y\partial_x\theta = \eta\,\partial_{yy}\theta, \\ \partial_y\omega|_{y=0} = 0, \quad \partial_y\theta|_{y=0} = 0, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases}$$

Then, for any integer $k \neq 0$ and any $t > 0$,

$$\|\partial_x^i \widehat{\omega}(k,\cdot,t)\|_{L_y^2} \le C\,\Phi_{Combo,\omega_{i,0}}(0)\, e^{-\frac{2}{3}[\min(\nu,\eta)]^{\frac{1}{3}}\, k^{\frac{2}{3}} t},$$

$$\|\partial_y \partial_x^i \widehat{\omega}(k,\cdot,t)\|_{L_y^2} \le C\,\Phi_{Combo,\omega_{i,0}}(0)\, \nu^{-\frac{2}{3}}\, k^{\frac{2}{3}}\, e^{-\frac{2}{3}[\min(\nu,\eta)]^{\frac{1}{3}}\, k^{\frac{2}{3}} t},$$

where C is a constant independent of k and t. Here $\Phi_{Combo,\omega_{i,0}}(t)$ with any $t \ge 0$ is defined as

$$\begin{aligned}\Phi_{Combo,\omega_{i,0}}(t) = \Phi_{\omega_{0,0}}(t) &+ (36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3} + 18\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{\theta_{i+1,0}}(t)\\ &+ (27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3} + 13.5\,\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{\theta_{i+2,0}}(t),\end{aligned}$$

where $\Phi_{\omega_{i,0}}$, $\Phi_{\theta_{i+1,0}}$ and $\Phi_{\theta_{i+2,0}}$ are given by

$$\begin{aligned}\Phi_{\omega_{i,0}} &= k^2\gamma\,\|\widehat{\partial_x^i\omega}\|_{L^2}^2 + \alpha_\nu\|\partial_y\widehat{\partial_x^i\omega}\|_{L^2}^2 + k\beta_\nu Re\langle \mathbf{i}\,\widehat{\partial_x^i\omega}, \partial_y\widehat{\partial_x^i\omega}\rangle,\\ \Phi_{\theta_{i+1,0}} &= k^2\gamma\,\|\widehat{\partial_x^{i+1}\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y\widehat{\partial_x^{i+1}\theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\,\widehat{\partial_x^{i+1}\theta}, \partial_y\widehat{\partial_x^{i+1}\theta}\rangle,\\ \Phi_{\theta_{i+2,0}} &= k^2\gamma\,\|\widehat{\partial_x^{i+2}\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y\widehat{\partial_x^{i+2}\theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\,\widehat{\partial_x^{i+2}\theta}, \partial_y\widehat{\partial_x^{i+2}\theta}\rangle\end{aligned}$$

with α_ν, β_ν, α_η, β_η and γ specified as before. In the case when $k = 0$, $\partial_x^i\widehat{\omega}(0,\cdot,t)$ and $\partial_y\partial_x^i\widehat{\omega}(0,\cdot,t)$ solve the 1D heat equations.

As mentioned in the introduction, the proof of Proposition 2.2 is more complex than that of Proposition 2.1. The equation of ω,

$$\partial_t\omega = -y\partial_x\omega + \nu\partial_{yy}\omega + \partial_x\theta$$

contains an extra term $\partial_x\theta$ and can not be combined into the linear operator L in the previous analysis. Instead $\partial_x\theta$ is treated as a forcing term and we are forced to construct suitable combined functionals of ω and θ.

Proof of Proposition 2.2. We define the combined functional $\Phi_{Combo,\omega_{0,0}}$ as

$$\begin{aligned}\Phi_{Combo,\omega_{0,0}}(t) = \Phi_{\omega_{0,0}}(t) &+ (36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3} + 18\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{\theta_{1,0}}(t)\\ &+ (27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3} + 13.5\,\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{\theta_{2,0}}(t),\end{aligned}$$

where $\Phi_{\omega_{0,0}}$, $\Phi_{\theta_{1,0}}$ and $\Phi_{\theta_{2,0}}$ are as provided in the statement of Proposition 2.2. We differentiate $\Phi_{Combo,\omega_{0,0}}$ and start with $\Phi_{\omega_{0,0}}$.

$$\begin{aligned}\frac{d}{dt}\Phi_{\omega_{0,0}} = k^2\gamma\,&(\langle\partial_t\widehat{\omega},\widehat{\omega}\rangle + \langle\widehat{\omega},\partial_t\widehat{\omega}\rangle)\\ &+\alpha_\nu\left(\langle\partial_t\partial_y\widehat{\omega},\partial_y\widehat{\omega}\rangle + \langle\partial_y\widehat{\omega},\partial_t\partial_y\widehat{\omega}\rangle\right)\\ &+\beta_\nu\left(k Re\langle\mathbf{i}\,\partial_t\widehat{\omega},\partial_y\widehat{\omega}\rangle + k Re\langle\mathbf{i}\,\widehat{\omega},\partial_t\partial_y\widehat{\omega}\rangle\right).\end{aligned}$$

Clearly,

$$\begin{aligned}
&\langle \partial_t \widehat{\omega}, \widehat{\omega}\rangle + \langle \widehat{\omega}, \partial_t \widehat{\omega}\rangle = -2\nu \|\partial_y \widehat{\omega}\|_{L^2}^2 + 2kRe\langle \mathbf{i}\widehat{\theta}, \widehat{\omega}\rangle\,,\\
&\langle \partial_t \partial_y \widehat{\omega}, \partial_y \widehat{\omega}\rangle + \langle \partial_y \widehat{\omega}, \partial_t \partial_y \widehat{\omega}\rangle = -2\nu \|\partial_{yy} \widehat{\omega}\|_{L^2}^2 - 2kRe\langle \mathbf{i}\widehat{\omega}, \partial_y\widehat{\omega}\rangle + 2Re\langle \mathbf{i}k\partial_y\widehat{\theta}, \partial_y \widehat{\omega}\rangle\,,\\
&kRe\langle \mathbf{i}\,\partial_t \widehat{\omega}, \partial_y \widehat{\omega}\rangle + kRe\langle \mathbf{i}\widehat{\omega}, \partial_t\partial_y \widehat{\omega}\rangle = 2Re\langle \mathbf{i}k\nu\partial_{yy}\widehat{\omega}, \partial_y\widehat{\omega}\rangle - k^2\|\widehat{\omega}\|_{L^2}^2\\
&\qquad\qquad + Re\langle -k^2\widehat{\theta}, \partial_y\widehat{\omega}\rangle + Re\langle \mathbf{i}k\widehat{\omega}, \partial_{xy}\widehat{\theta}\rangle\,.
\end{aligned}$$

Some of the terms can be similarly estimated as in the previous subsection. Those terms related to θ are handled differently.

$$\begin{aligned}
\frac{d}{dt}\Phi_{\omega_{0,0}}(t) \le &-\frac{2}{3}\nu^{1/3}k^{2/3}\left(\frac{3}{2}\|\widehat{\omega}\|_{L^2}^2 + \frac{9}{4}\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\omega}\|_{L^2}^2\right.\\
&\left. +\nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\,\widehat{\omega}, \partial_y\widehat{\omega}\rangle\right)\\
&+\frac{2}{3}\nu^{1/3}k^{2/3}\left(18\,\nu^{-2/3}k^{-4/3}\|\widehat{\theta}_x\|_{L^2}^2 + \frac{1}{8}\|\widehat{\omega}\|_{L^2}^2\right)\\
&+\frac{2}{3}\nu^{1/3}k^{2/3}\left(18\,\nu^{-2/3}k^{-4/3}\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_x\|_{L^2}^2\right.\\
&\left. +\frac{1}{8}\cdot\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\omega}\|_{L^2}^2\right)\\
&+\frac{2}{3}\nu^{1/3}k^{2/3}\left(13.5\,\nu^{-2/3}k^{-10/3}\|\widehat{\theta}_{xx}\|_{L^2}^2 + \frac{1}{8}\cdot\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\omega}\|_{L^2}^2\right)\\
&+\frac{2}{3}\nu^{1/3}k^{2/3}\left(13.5\,\nu^{-2/3}k^{-10/3}\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_{xx}\|_{L^2}^2 + \frac{1}{8}\|\widehat{\omega}\|_{L^2}^2\right).
\end{aligned}$$

We then differentiate $\Phi_{\theta_{1,0}}$ and $\Phi_{\theta_{2,0}}$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\Phi_{Combo,\omega_{0,0}}(t)\\
&\leqslant -\frac{2}{3}\nu^{1/3}k^{2/3}\left(\frac{5}{4}\,\|\widehat{\omega}\|_{L^2}^2 + 2\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\omega}\|_{L^2}^2 + \nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\,\widehat{\omega}, \partial_y\widehat{\omega}\rangle\right)\\
&\quad -36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3}\,\frac{2}{3}\eta^{1/3}k^{2/3}\,\left(\|\widehat{\theta}_x\|_{L^2}^2 + \frac{9}{4}\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_x\|_{L^2}^2\right.\\
&\qquad \left. +\nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\widehat{\theta}_x, \partial_y\widehat{\theta}_x\rangle\right)\\
&\quad -18\,\nu^{1/3}\eta^{-1}k^{-4/3}\,\frac{2}{3}\eta^{1/3}k^{2/3}\,\left(\frac{3}{2}\|\widehat{\theta}_x\|_{L^2}^2 + \frac{5}{4}\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_x\|_{L^2}^2\right.\\
&\qquad \left. +\nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\widehat{\theta}_x, \partial_y\widehat{\theta}_x\rangle\right)\\
&\quad -27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3}\,\frac{2}{3}\eta^{1/3}k^{2/3}\left(\|\widehat{\theta}_{xx}\|_{L^2}^2 + \frac{9}{4}\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_{xx}\|_{L^2}^2\right.
\end{aligned}$$

$$+\nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\widehat{\theta}_{xx}, \partial_y\widehat{\theta}_{xx}\rangle\Big)$$

$$-13.5\,\nu^{1/3}\eta^{-1}k^{-10/3}\,\frac{2}{3}\eta^{1/3}k^{2/3}\,\Big(\frac{3}{2}\,\|\widehat{\theta}_{xx}\|^2_{L^2}+\frac{5}{4}\,\frac{1}{3}\nu^{2/3}k^{-2/3}\|\partial_y\widehat{\theta}_{xx}\|^2_{L^2}$$

$$+\nu^{1/3}k^{-4/3}kRe\langle \mathbf{i}\widehat{\theta}_{xx}, \partial_y\widehat{\theta}_{xx}\rangle\Big)$$

$$\leqslant -\frac{2}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}\Phi_{Combo,\omega_{0,0}}(t).$$

Therefore,

$$\Phi_{Combo,\omega_{0,0}}(t)\leqslant \Phi_{Combo,\omega_{0,0}}(0)e^{-\frac{2}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}t}.$$

Especially,

$$\begin{aligned}\Phi_{\omega_{0,0}}(t)\ &\leqslant e^{-\frac{2}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}t}\\ &\quad\Big(\Phi_{\omega_{0,0}}(t)+(36\,\nu^{-1/3}\eta^{-1/3}k^{-4/3}+18\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{\theta_{1,0}}(t)\\ &\quad+(27\,\nu^{-1/3}\eta^{-1/3}k^{-10/3}+13.5\,\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{\theta_{2,0}}(t)\Big).\end{aligned}$$

Since both $\Phi_{\theta_{1,0}}$ and $\Phi_{\theta_{2,0}}$ are positive definite, we combine this bound with the equivalence of norms to obtain the desired decay bounds for $\|\widehat{\omega}\|_{L^2}$ and $\|\partial_y\widehat{\omega}\|_{L^2}$. One remark we would like to make is that we have deliberately left a $\frac{1}{4}\|\widehat{\omega}\|^2_{L^2}$ gap in this estimate, i.e. coefficient of $\frac{5}{4}$ instead of 1. The purpose is to make room for the induction argument in Subsection 2.3.

Since $\partial_x^i\omega$ satisfies a similar equation as ω,

$$\partial_t\partial_x^i\omega=-y\partial_x\partial_x^i\omega+\nu\partial_{yy}\partial_x^i\omega+\partial_x\partial_x^i\theta,$$

the same method can be applied to these equations for $\partial_x^i\omega$. This generalization concludes that

$$\|\widehat{\partial_x^i\omega}(t)\|_{L^2}+\alpha_\nu\|\partial_y\widehat{\partial_x^i\omega}(t)\|_{L^2}\leqslant C\,\Phi_{Combo,\omega_{i,0}}(0)\;e^{-\frac{2}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}},$$

where C is a pure constant and $\Phi_{Combo,\omega_{i,0}}$ is defined as before. This completes the proof of Proposition 2.2. □

2.3. Decay involving higher derivatives in y

This subsection proves the exponential decay for higher y-derivatives of θ and ω, as stated in Theorem 1.2.

Proof of Theorem 1.2. We first estimate $\|\partial_y^j\partial_x^i\theta(t)\|_{L^2}$. This is done by induction on j. For all i and $j=0$, the case has been proved in Subsection 2.1. Suppose that $\|\partial_y^m\partial_x^i\theta(t)\|_{L^2}$ satisfies the decay estimates of Theorem 1.2 for all i and for all $0\leqslant m\leqslant j-1$, we prove the exponential decay for $\|\partial_y^j\partial_x^i\theta(t)\|_{L^2}$. We start with the equation of $\partial_y^j\partial_x^i\theta$

$$\partial_t(\partial_y^j\partial_x^i\theta)+j\partial_x^{i+1}(\partial_y^{j-1})\theta+y\partial_x(\partial_y^j\partial_x^i\theta)=\eta\partial_{yy}(\partial_y^j\partial_x^i\theta).$$

Due to the presence of the extra term $j\partial_x^{i+1}(\partial_y^{j-1})\theta$, we need to construct a suitable combination of functionals. To shed light on the combination, we define

$$\Phi_{\theta_{i,j}} = \|\partial_y^j \widehat{\partial_x^i \theta}\|_{L^2}^2 + \alpha_\eta \|\partial_y^{j+1} \widehat{\partial_x^i \theta}\|_{L^2}^2 + k\beta_\eta Re\langle \mathbf{i}\, \partial_y^j \widehat{\partial_x^i \theta}, \partial_y^{j+1} \widehat{\partial_x^i \theta}\rangle$$

and compute its time-derivative,

$$\begin{aligned}
\frac{d}{dt}\Phi_{\theta_{i,j}}(t) \leqslant -\eta^{1/3}k^{2/3}\Bigg(&\|\partial_y^j \widehat{\partial_x^i \theta}\|_{L^2}^2 + \frac{3}{2}\alpha_\eta \|\partial_y^{j+1} \widehat{\partial_x^i \theta}\|_{L^2}^2 \\
&+ \frac{2}{3}k\beta_\eta Re\langle \mathbf{i}\, \partial_y^j \widehat{\partial_x^i \theta}, \partial_y^{j+1} \widehat{\partial_x^i \theta}\rangle \Bigg) \\
&+\eta^{1/3}k^{2/3}\left(12\, j^2\, \eta^{-1/3}k^{-2/3}\|\partial_y^{j-1} \widehat{\partial_x^{i+1} \theta}\|_{L^2}^2 + \frac{1}{12}\|\partial_y^j \widehat{\partial_x^i \theta}\|_{L^2}^2\right) \\
&+\eta^{1/3}k^{2/3}\left(12\, j^2\, \eta^{-1/3}k^{-2/3}\alpha\|\partial_y^{j} \widehat{\partial_x^{i+1} \theta}\|_{L^2}^2 + \frac{1}{12}\alpha_\eta\|\partial_y^{j+1} \widehat{\partial_x^i \theta}\|_{L^2}^2\right) \\
&+\eta^{1/3}k^{2/3}\left(9\, j^2\, \eta^{-2/3}k^{-10/3}\|\partial_y^{j-1} \widehat{\partial_x^{i+2} \theta}\|_{L^2}^2 + \frac{1}{12}\alpha_\eta\|\partial_y^{j+1} \widehat{\partial_x^i \theta}\|_{L^2}^2\right) \\
&+\eta^{1/3}k^{2/3}\left(9\, j^2\, \eta^{-2/3}k^{-10/3}\alpha\|\partial_y^{j} \widehat{\partial_x^{i+2} \theta}\|_{L^2}^2 + \frac{1}{12}\|\partial_y^j \widehat{\partial_x^i \theta}\|_{L^2}^2\right).
\end{aligned}$$

This calculation indicates that we should work with the combination

$$\begin{aligned}
\Phi_{Combo,\theta_{i,j}} = \Phi_{\theta_{i,j}} &+ 72\, j^2\, \eta^{-1/3}k^{-2/3}\, \Phi_{Combo,\theta_{i+1,j-1}} \\
&+ 54\, j^2\, \eta^{-2/3}k^{-10/3}\, \Phi_{Combo,\theta_{i+2,j-1}},
\end{aligned}$$

where the coefficients 72 and 54 are chosen to be 6 times the coefficient of the third and the fourth terms above (or 12) and 6 times the coefficient of the fifth and the sixth terms above (or 9), respectively. The magic number 6 comes from the fact that the time derivatives of $\Phi_{Combo,\theta_{i+1,j-1}}$ and $\Phi_{Combo,\theta_{i+2,j-1}}$ will each be able to spare $\frac{1}{6}$ of the negative parts to cancel the corresponding positive parts in the time derivatives of $\Phi_{\theta_{i,j}}(t)$. We then differentiate $\Phi_{Combo,\theta_{i,j}}$ in time and make use of the cancelations we just mentioned to obtain

$$\frac{d}{dt}\Phi_{Combo,\theta_{i,j}} \leq -\frac{2}{3}\eta^{1/3}k^{2/3}\Phi_{Combo,\theta_{i,j}} + E, \tag{2.6}$$

where E contains terms of the form $\|\partial_y^m \partial_x^l \theta(t)\|_{L^2}$ with $j-3 \leq m \leq j-1$ and $i+1 \leq l \leq i+6$. Integrating (2.6) in time and invoking the inductive assumption, we obtain

$$\begin{aligned}
\Phi_{Combo,\theta_{i,j}}(t) &\leq \Phi_{Combo,\theta_{i,j}}(0)\, e^{-\frac{2}{3}\eta^{1/3}k^{2/3}t} + \int_0^t e^{-\frac{2}{3}\eta^{1/3}k^{2/3}(t-\tau)}\, E(\tau)\, d\tau \\
&= P\left(t, k^{-\frac{2}{3}}, k^{\frac{2}{3}}\right) e^{-\frac{2}{3}\eta^{1/3}k^{2/3}t},
\end{aligned}$$

where P denotes a polynomial of its variables and $\Phi_{Combo,\theta_{i,j}}(0)$ can be figured out by the induction on the index:

$$\Phi_{Combo,\theta_{i,j}}(0) = \Phi_{\theta_{i,j}}(0) + \sum_{n=1}^{j}\Big[\Big(\Pi_{l=1}^{n}(j+1-l)^2\Big)\sum_{m=0}^{n} C_n^m\,(72\eta^{-1/3}k^{-2/3})^m(54\eta^{-2/3}k^{-10/3})^{n-m}\Phi_{\theta_{i+m+n,j-n}}(0)\Big].$$

Consequently,

$$\|\partial_y^j\widehat{\partial_x^i\theta}\|_{L^2}^2 + \alpha_\eta\|\partial_y^{j+1}\widehat{\partial_x^i\theta}\|_{L^2}^2 \le C\,\Phi_{\theta_{i,j}}(t) \le C\,\Phi_{Combo,\theta_{i,j}}(t),$$

where leads to the desired estimate in (1.6).

We now turn to the proof for the exponential decay of $\partial_y^j\partial_x^i\omega$ by an induction argument on j. Subsection 2.2 shows the desired decay for all i and $j=0$, which initiates the induction. We assume that (1.7) is true for all i and for $0 \leqslant m \leqslant j-1$. Applying $\partial_y^j\partial_x^i$ to the original ω equation, we have

$$\partial_t(\partial_y^j\partial_x^i\omega) + j\partial_y^{j-1}\partial_x^{i+1}\omega + y\partial_x(\partial_y^j\partial_x^i\omega) = \nu\partial_{yy}(\partial_y^j\partial_x^i\omega) + \partial_y^j\partial_x^{i+1}\theta.$$

We seek a combined functional in terms of several levels of derivatives of ω and θ. To determine a suitable combination, we define

$$\Phi_{\omega_{i,j}} = \|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2 + \alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2 + \kappa\,\beta_\nu Re\langle\mathbf{i}\,\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle$$

and find its time derivative. It is clear that

$$\begin{aligned}
&\langle\partial_t\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^j\widehat{\partial_x^i\omega}\rangle + \langle\partial_y^j\widehat{\partial_x^i\omega},\,\partial_t\partial_y^j\widehat{\partial_x^i\omega}\rangle\\
&\quad= -2\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2 + 2Re\langle\partial_y^i\widehat{\partial_x^{j+1}\theta},\,\partial_y^j\widehat{\partial_x^i\omega}\rangle + 2\,j\,Re\langle\partial_y^{j-1}\widehat{\partial_x^{i+1}\omega},\,\partial_y^j\widehat{\partial_x^i\omega}\rangle,\\
&\quad\langle\partial_t\partial_y^{j+1}\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle + \langle\partial_y^{j+1}\widehat{\partial_x^i\omega},\,\partial_t\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle\\
&\qquad= -2\nu\|\partial_y^{j+2}\widehat{\partial_x^i\omega}\|_{L^2}^2 - 2kRe\langle\mathbf{i}\,\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle\\
&\qquad\quad+2Re\langle\partial_y^{j+1}\widehat{\partial_x^{i+1}\theta},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle + 2\,j\,Re\langle\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^{i+1}\omega}\rangle
\end{aligned}$$

and

$$\begin{aligned}
&kRe\langle\mathbf{i}\,\partial_t\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle + k\,Re\langle\mathbf{i}\,\partial_y^j\widehat{\partial_x^i\omega},\,\partial_t\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle\\
&\quad= 2Re\langle\mathbf{i}\,k\nu\partial_y^{j+2}\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle - k^2\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2\\
&\quad\quad+Re\langle -k^2\partial_y^j\widehat{\partial_x^i\theta},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle + Re\langle\mathbf{i}\,k\partial_y^j\widehat{\partial_x^i\omega},\,\partial_y^{j+1}\widehat{\partial_x^{i+1}\theta}\rangle\\
&\quad\quad+Re\langle\mathbf{i}\,kj\partial_y^{j-1}\widehat{\partial_x^{i+1}\omega},\,\partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle + Re\langle\mathbf{i}\,k\partial_y^j\widehat{\partial_x^i\omega},\,(j+1)\,\partial_y^j\widehat{\partial_x^{i+1}\omega}\rangle\,.
\end{aligned}$$

Collecting the terms and estimating them suitably, we have

$$\begin{aligned}\frac{d}{dt}\Phi_{\omega_{i,j}}(t) \leqslant -\nu^{1/3}k^{2/3}\Big(&\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2 + \frac{3}{2}\alpha\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2 \\ &+\frac{2}{3}k\beta Re\langle \mathbf{i}\,\partial_y^j\widehat{\partial_x^i\omega}, \partial_y^{j+1}\widehat{\partial_x^i\omega}\rangle\Big) \\ &+\nu^{1/3}k^{2/3}\Big(24\,j^2\,\nu^{-2/3}k^{-4/3}\|\partial_y^{j-1}\widehat{\partial_x^{i+1}\omega}\|_{L^2}^2 + \frac{1}{24}\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(24\,j^2\,\nu^{-2/3}k^{-4/3}\alpha_\nu\|\partial_y^j\widehat{\partial_x^{i+1}\omega}\|_{L^2}^2 + \frac{1}{24}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(18\,j^2\,\nu^{-2/3}k^{-10/3}\|\partial_y^{j-1}\widehat{\partial_x^{i+2}\omega}\|_{L^2}^2 + \frac{1}{24}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(18\,(j+1)^2\,\nu^{-2/3}k^{-10/3}\alpha_\nu\|\partial_y^j\widehat{\partial_x^{i+2}\omega}\|_{L^2}^2 + \frac{1}{24}\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(24\,\nu^{-2/3}k^{-4/3}\|\partial_y^j\widehat{\partial_x^{i+1}\theta}\|_{L^2}^2 + \frac{1}{24}\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(24\,\nu^{-2/3}k^{-4/3}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^{i+1}\theta}\|_{L^2}^2 + \frac{1}{24}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(18\,\nu^{-2/3}k^{-10/3}\|\partial_y^j\widehat{\partial_x^{i+2}\theta}\|_{L^2}^2 + \frac{1}{24}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2\Big) \\ &+\nu^{1/3}k^{2/3}\Big(18\,\nu^{-2/3}k^{-10/3}\alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^{i+2}\theta}\|_{L^2}^2 + \frac{1}{24}\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2\Big).\end{aligned}$$

Thus, we choose the combination

$$\begin{aligned}\Phi_{Combo,\omega_{i,j}} = \Phi_{\omega_{i,j}} &+ 144\,j^2\,\nu^{-2/3}k^{-4/3}\,\Phi_{Combo,\omega_{i+1,j-1}} \\ &+108\,(j+1)^2\,\nu^{-2/3}k^{-10/3}\,\Phi_{Combo,\omega_{i+2,j-1}} \\ &+(144\,\nu^{-1/3}\eta^{-1/3}k^{-4/3} + 72\nu^{1/3}\eta^{-1}k^{-4/3})\,\Phi_{Combo,\theta_{i+1,j}} \\ &+(108\,\nu^{-1/3}\eta^{-1/3}k^{-10/3} + 54\nu^{1/3}\eta^{-1}k^{-10/3})\,\Phi_{Combo,\theta_{i+2,j}}\,.\end{aligned}$$

Using the estimates on the time derivative of $\Phi_{\omega_{i,j}}$ and previous combinations, we can reach the inequality

$$\frac{d}{dt}\Phi_{Combo,\omega_{i,j}} \leqslant -\frac{2}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}\Phi_{Combo,\omega_{i,j}} + F,$$

where F collects all the terms with of $\|\partial_y^k\partial_x^l\omega\|_{L_y^2}$ and $\|\partial_y^k\partial_x^l\theta\|_{L_y^2}$ for $k \le j-1$. Integrating in time, invoking the inductive assumption and the decay estimates for θ in (1.6), we obtain the desired exponential decay in (1.7),

$$\|\partial_y^j\widehat{\partial_x^i\omega}\|_{L^2}^2 + \alpha_\nu\|\partial_y^{j+1}\widehat{\partial_x^i\omega}\|_{L^2}^2 \leqslant Q\left(t, k^{-\frac{2}{3}}, k^{\frac{2}{3}}\right)\, e^{-\frac{1}{3}[\min(\nu,\eta)]^{1/3}k^{2/3}t},$$

where Q is a polynomial of its variables and depends on the L^2-norms of ω_0 and θ_0. This completes the proof of Theorem 1.2. □

Acknowledgments

J. Wu was partially supported by the National Science Foundation grant DMS 1614246 and by the AT&T Foundation at Oklahoma State University.

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