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Stability Near Hydrostatic Equilibrium to the 2D Boussinesq Equations Without Thermal Diffusion

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Abstract

This paper furthers our studies on the stability problem for perturbations near hydrostatic equilibrium of the 2D Boussinesq equations without thermal diffusion and solves some of the problems left open in Doering et al. (Physica D 376(377):144–159, 2018). We focus on the periodic domain to avoid the complications due to the boundary. We present several results at two levels: the linear stability and the nonlinear stability levels. Our linear stability results state that the velocity field $\mathbf{u}$ associated with any initial perturbation converges uniformly to 0 and the temperature θ converges to an explicit function depending only on y as t tends to infinity. In addition, we obtain an explicit algebraic convergence rate for the velocity field in the L^2-sense. Our nonlinear stability results state that any initial velocity small in L^2 and any initial temperature small in L^2 lead to a stable solution of the full nonlinear perturbation equations in large time. Furthermore, we show that the temperature is eventually stratified and converges to a function depending only on y if we know it admits a certain uniform-in-time bound. An explicit decay rate for the velocity in L^2 is also ensured if we make assumption on the high-order norms of $\mathbf{u}$ and θ.

1. Introduction

1.1. Overview

This paper is concerned with the two-dimensional Boussinesq equations without thermal diffusion:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.1}$$

In [21], the authors studied the global well-posedness and large-time behavior of large-data classical solutions to (1.1) on 2D non-smooth domains subject to the stress-free boundary conditions. In particular, the global stability of the hydrostatic equilibrium associated with (1.1) was investigated (explained in more details later). The main purpose of this paper is to further develop the stability problem concerning (1.1) near the hydrostatic equilibrium through studying the explicit decay rate of the velocity field towards the zero equilibrium state and identifying the thermal structure of the final state.

1.2. Background and Literature Review

System (1.1) is a special (limiting) case of the 2D incompressible Boussinesq equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.2}$$

when $\kappa = 0$, which have a wide range of applications in geophysics and fluid mechanics, such as the modeling of large scale atmospheric and oceanic flows that are responsible for cold fronts and jet stream [25,43,45], and the study of Rayleigh–Bénard convection [15,20,24], just to mention a few. In (1.2), the unknown functions $\mathbf{u}$ and P denote the velocity field and pressure of the flow, respectively; θ is the deviation of density from the bottom density (which is taken to be 1 for simplicity) in the context of geophysical flows, or the temperature deviation in the study of Rayleigh–Bénard convection; $\nu \geqq 0$ and $\kappa \geqq 0$ stand for the kinematic viscosity and thermal (buoyancy) diffusivity, respectively; and $\mathbf{e}_2 = (0, 1)^{\mathrm{T}}$.

Besides physical applications, the 2D model (1.2) is also known to retain some key features of the 3D Euler and Navier–Stokes equations, such as the vortex stretching mechanism. Indeed, it has been commonly recognized that the growth of the vorticity associated with (1.2) depends on the temporal accumulation of $\nabla\mathbf{u}$, which is a scenario similar to the vortex stretching effect in 3D incompressible flows [44]. Another important feature of the 2D Boussinesq equations is that when $\nu = \kappa = 0$, the model can be identified with the 3D Euler equations for axisymmetric swirling flows when the radius $r > 0$ [44].

Collectively, the physical background and mathematical features of (1.2) make the model a rich area for mathematical investigations. Studies of the qualitative behavior of the model have been carried out for nearly half a century. Major concerns are oriented around the global well-posedness (GWP)/finite-time blowup (FTB) of large-data classical solutions (LDCS) under general initial and/or boundary conditions, which has a rather long history starting from the work of RABINOWITZ [46]. On one hand, when the dissipation coefficients, ν and κ, are all equal to zero, the GWP of LDCS to the model still largely remains open. We refer the reader to [5,12,13,16,22,31,48,50] for recent (analytical and numerical) studies concerning the local well-posedness and FTB of LDCS. On the other hand, when the parameters are not all equal to zero, the GWP of LDCS has been established in a systematic fashion by considering both the isotropic and anisotropic dissipations. We refer the reader to [1–4,8,9,11,14,17–19,27,28,30,32–34,37–39,41,42,58] for a non-exhaustive list of results in this direction. There are also works investigating the well-posedness and regularity of solutions to the model with critical and supercritical dissipation, and we refer the reader to [35,36,40,49,55–57] and the references therein.

Compared with the magnitude of research conducted on the GWP of the model, the large-time behavior (LTB) of solutions, especially the stability of physically relevant hydrostatic equilibria, has been studied relatively little. To the best of the authors' knowledge, the following results have been established in the literature:

- exponential decay of θ to constant states and uniform boundedness of kinetic energy of LDCS on bounded smooth domains when $\nu = 0,\ \kappa > 0$ [58],
- uniform boundedness of kinetic energy of LDCS on bounded smooth domains when $\nu > 0,\ \kappa = 0$ [37],
- algebraic decay of small-data classical solutions to constant ground states in $\mathbb{R}^3$ when $\nu > 0,\ \kappa > 0$ [7],
- long time averaged heat transport sustained by thermal boundary conditions, i.e., bounds for Rayleigh–Bénard convection [52,53],
- existence of a global attractor containing infinitely many invariant manifolds on periodic domains when $\nu > 0,\ \kappa = 0$ [6].

Nevertheless, we note that the above list does not provide any information about the global asymptotic stability of hydrostatic equilibria associated with (1.2), especially the partially dissipative systems. Until very recently, such an issue is partially resolved in [21], where the authors studied the large-time behavior of LDCS to an initial-boundary value problem of the partially dissipative system when $\kappa = 0$, which arises naturally as a relevant system in geophysics [29,47,51]. We briefly summarize the main results of [21].

First, [21] establishes the GWP of LDCS to the following IBVP:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}), \\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is either a rectangle or more general Lipschitz domain with minor constraints (see [21] for more details), $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, and $\omega = \partial_x v - \partial_y u$ is the 2D vorticity. Secondly, [21] obtains the global stability and large-time behavior of the perturbation near the hydrostatic equilibrium $[\mathbf{u}_{\text{he}}, P_{\text{he}}(y), \theta_{\text{he}}(y)]$ given by

$$\mathbf{u}_{\text{he}} = \mathbf{0}, \quad \theta_{\text{he}}(y) = \beta y + \bar{\theta}, \quad P_{\text{he}}(y) = \frac{1}{2}\beta y^2 + \bar{\theta} y.$$

More precisely, it is proven, for $\beta > 0$, that the L^2 norms of the velocity perturbation (not necessarily small) and its first order spatial and temporal derivatives converge to zero as $t \to \infty$. Consequently it is found that the pressure and temperature functions stratify in the vertical direction in a weak topology. Remarkably, the second order spatial derivatives of the velocity perturbation (not necessarily small) are shown to be bounded uniformly in time for all time. In addition, [21] contains extensive numerical simulations illustrating the analytic results and investigating unsolved problems. It is worth mentioning several closely related recent works [10,23,54]. [23] examined the stability and large-time behavior near the hydrostatic equilibrium of the inviscid Boussinesq system and obtained a sharp decay rate and stability results via dispersive type estimates. [54] studied the stability of special, stratified solutions of a 3D inviscid Boussinesq system and established that, as the strength of the gravity tends to infinity, the 3D system of equations tends to a stratified system of 2D Euler equations with stratified density. [10] investigated the stability of the 2D Boussinesq equations with a velocity damping term near the hydrostatic equilibrium and proved an asymptotic stability with explicit decay rates when the spatial domain is a strip.

1.3. Motivation and Goals

Now we would like to point out the facts that motivate the current work. Along with the aforementioned results established in [21], the authors proposed several open problems. The first is to find explicit decay rates for the velocity perturbation and its derivatives. The numerical simulation in [21] indicates that the velocity field might converge to zero as a power law. The second is to provide a precise description of the final buoyancy distribution in case of general initial conditions. The numerical test in [21] and an intuitive argument suggest that the final state of the relaxation problem should generically be the unique stably stratified distribution $\hat{\theta}(y)$ which is the inverse of a height function determined by the initial temperature [see (6.1) in [21]].

This paper intends to solve these open problems. To simplify the problem, we take the spatial domain Ω to be the periodic box

$$\Omega = \mathbb{T}^2 := [0, 2\pi] \times [0, 2\pi],$$

and consider the initial-value problem,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ (\mathbf{u}, \theta)(\mathbf{x}, 0) = (\mathbf{u}_0, \theta_0)(\mathbf{x}). \end{cases} \tag{1.4}$$

The corresponding vorticity $\omega = \nabla \times \mathbf{u}$ satisfies

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega + \partial_x \theta.$$

We take the hydrostatic equilibrium

$$\mathbf{u}_{\text{he}} = \mathbf{0}, \quad \theta_{\text{he}} = \beta y, \quad P_{\text{he}} = \frac{1}{2} \beta y^2,$$

and consider the perturbation

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{\rm he}, \quad \tilde{\theta} = \theta - \theta_{\rm he}, \quad \tilde{P} = P - P_{\rm he},$$

which satisfies

$$\begin{cases} \partial_t \tilde{u} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{u} = -\partial_x \tilde{p} + \nu \Delta \tilde{u}, \\ \partial_t \tilde{v} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{v} = -\partial_y \tilde{p} + \nu \Delta \tilde{v} + \tilde{\theta}, \\ \partial_t \tilde{\theta} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\theta} + \beta \tilde{v} = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0. \end{cases} \tag{1.5}$$

The corresponding perturbation in the vorticity $\tilde{\omega} = \nabla \times \tilde{\mathbf{u}}$ satisfies

$$\partial_t \tilde{\omega} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\omega} = \nu \Delta \tilde{\omega} + \partial_x \tilde{\theta}. \tag{1.6}$$

We separate the linear and the nonlinear parts in (1.5). To do so, we eliminate the pressure term. Taking the divergence of the velocity equation in (1.5), we find

$$-\Delta \tilde{p} = \nabla \cdot ((\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}) - \partial_y \tilde{\theta}$$

or

$$-\nabla \tilde{p} = \nabla \Delta^{-1} \nabla \cdot ((\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}) - \nabla \Delta^{-1} \partial_y \tilde{\theta}. \tag{1.7}$$

Inserting (1.7) in (1.5) yields

$$\begin{cases} \partial_t \tilde{u} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{u} - \partial_x \Delta^{-1} \nabla \cdot ((\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}) = \nu \Delta \tilde{u} - \Delta^{-1} \partial_{xy} \tilde{\theta}, \\ \partial_t \tilde{v} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{v} - \partial_y \Delta^{-1} \nabla \cdot ((\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}) = \nu \Delta \tilde{v} + \Delta^{-1} \partial_{xx} \tilde{\theta}, \\ \partial_t \tilde{\theta} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\theta} + \beta\, \tilde{v} = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0. \end{cases} \tag{1.8}$$

For notational convenience, we ignore the tilde and further write (1.8) as

$$\begin{cases} \partial_t u = \nu \Delta u - \Delta^{-1} \partial_{xy} \theta + N_1, \\ \partial_t v = \nu \Delta v + \Delta^{-1} \partial_{xx} \theta + N_2, \\ \partial_t \theta = -\beta\, v + N_3, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.9}$$

where N_1, N_2 and N_3 are the nonlinear terms

$$\begin{cases} N_1 = -(\mathbf{u} \cdot \nabla)u + \partial_x \Delta^{-1} \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}), \\ N_2 = -(\mathbf{u} \cdot \nabla)v + \partial_y \Delta^{-1} \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}), \\ N_3 = -(\mathbf{u} \cdot \nabla)\theta. \end{cases} \tag{1.10}$$

The goal of this paper is to study the large-time behavior of large-data classical solutions to (1.9) and its linearization on the 2D periodic domain $\mathbb{T}^2$ subject to various initial conditions. Specifically, we aim to identify the explicit decay rate of $\mathbf{u}$, and describe the profile of the equilibrium state of θ.

1.4. Challenges

We begin the investigation with the linearization of (1.4), which, according to (1.9), is given by

$$\begin{cases} \partial_t U = \nu \Delta U - \partial_{xy} \Delta^{-1} \Theta, \\ \partial_t V = \nu \Delta V + \partial_{xx} \Delta^{-1} \Theta, \\ \partial_t \Theta + \beta V = 0, \\ \partial_x U + \partial_y V = 0, \\ U(\mathbf{x}, 0) = U_0(\mathbf{x}), \quad V(\mathbf{x}, 0) = V_0(\mathbf{x}), \quad \Theta(\mathbf{x}, 0) = \Theta_0(\mathbf{x}). \end{cases} \tag{1.11}$$

We remark that the main results of [21] are proved by using pure energy methods. However, because of the nature of the energy methods, such an approach does not allow us to extract any decay rate out of the perturbation, even for the linearized system (1.11). Indeed, it is easy to check that (1.11) admits the following global and uniform Sobolev bound, for $s \geqq 0$,

$$\begin{aligned} &\|(U(t), V(t))\|_{H^s}^2 + \frac{1}{\beta}\|\Theta(t)\|_{H^s}^2 + 2\nu \int_0^t \|(\nabla U(\tau), \nabla V(\tau))\|_{H^s}^2 \, \mathrm{d}\tau \\ &\quad = \|(U_0, V_0)\|_{H^s}^2 + \frac{1}{\beta}\|\Theta_0\|_{H^s}^2. \end{aligned}$$

In Section 2 we show that the H^s norm of the linearized velocity field tends to zero and the temperature converges to a definite limit, as time goes to infinity. However, because of the absence of thermal dissipation and the coupling of the temperature with the velocity equation, it does not seem possible to derive any explicit decay rate of the velocity perturbation by using energy methods. On the other hand, by differentiating (1.11) with respect to t and making suitable substitutions, we can convert (1.11) into a system of degenerate wave type equations,

$$\begin{cases} \partial_{tt} U - \nu \Delta \partial_t U - \beta(-\Delta)^{-1} \partial_{xx} U = 0, \\ \partial_{tt} V - \nu \Delta \partial_t V - \beta(-\Delta)^{-1} \partial_{xx} V = 0, \\ \partial_{tt} \Theta - \nu \Delta \partial_t \Theta - \beta(-\Delta)^{-1} \partial_{xx} \Theta = 0, \end{cases} \tag{1.12}$$

which allows us to extract different global energy bounds, but explicit decay rates still do not follow from direct energy estimates.

In order to gain a better understanding of the stability problem, here we resort to the spectral method, that is to first solve the linearized system in the Fourier space, and then represent the solution of the full nonlinear system in an integral form via the Duhamel principle. These explicit representations, provided in Section 2, make it possible for the study of precise large-time behavior. First of all, by direct calculations, we can show that the Fourier transform of (1.9) is given by

$$\partial_t \psi = A\psi + F,$$

where

$$\psi = \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{\theta} \end{bmatrix}, \quad A = \begin{bmatrix} -\nu|\mathbf{k}|^2 & 0 & -\frac{k_1k_2}{|\mathbf{k}|^2} \\ 0 & -\nu|\mathbf{k}|^2 & \frac{k_1^2}{|\mathbf{k}|^2} \\ 0 & -\beta & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \widehat{N_1} \\ \widehat{N_2} \\ \widehat{N_3} \end{bmatrix}.$$

The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -\nu|\mathbf{k}|^2, \\ \lambda_2 &= -\frac{1}{2}\nu|\mathbf{k}|^2 \left(1 + \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}\right), \\ \lambda_3 &= -\frac{1}{2}\nu|\mathbf{k}|^2 \left(1 - \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}\right). \end{aligned}$$

Clearly, for $\beta > 0$ and $k_1 \neq 0$, the real parts of $\lambda_1, \lambda_2, \lambda_3$ are all negative. However, when $k_1 = 0$ or when $k_1^2 \ll |\mathbf{k}|^4$,

$$\lambda_3 = -\frac{\frac{2\beta k_1^2}{\nu|\mathbf{k}|^4}}{1 + \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}} \approx 0. \tag{1.13}$$

Although we are able to identify the explicit decay rates of the linearized velocity field, such a spectral property makes it considerably difficult to extract explicit convergence rates for solutions of the full nonlinear equations. We also observe that more regular initial perturbations here could lead to higher decay rates due to the fact that λ_3 behaves like $-\frac{2\beta k_1^2}{\nu|\mathbf{k}|^4}$ for some frequencies $\mathbf{k}$. This is reflected in the statement of Theorem 1.3. This phenomenon is different from the behavior of solutions to standard parabolic partial differential equations (PDEs). In general more regular perturbations generate slower decay rates in standard parabolic PDEs with the Laplacian operator or fractional Laplacian operator. This new phenomenon is due to the partial dissipation in the Boussinesq equations studied here.

1.5. Statement of Results

We present several results at two levels: level one for the linearized system (1.11) and level two for the full nonlinear system (1.9). We obtain three main results for the linearized system. The first result states that if the initial profile (U_0, V_0, Θ_0) is in the Sobolev space H^s for any $s \geqq 0$, then the Sobolev norm of the corresponding velocity field in H^s converges to zero and the H^s-norm of Θ converges to a definite limit. More precisely, we have the following theorem:

Theorem 1.1. *Let $s \geqq 0$. Assume that $(U_0, V_0, \Theta_0) \in H^s(\mathbb{T}^2)$ satisfies $\partial_x U_0 + \partial_y V_0 = 0$ and*

$$\int_{\mathbb{T}^2} U_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \textit{and} \quad \int_{\mathbb{T}^2} V_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0. \tag{1.14}$$

Let (U, V, Θ) be the corresponding solution of (1.11)*. Then, as $t \to \infty$,*

$$\|U(t)\|_{H^s} \to 0, \quad \|V(t)\|_{H^s} \to 0,$$
$$\|\Theta(t)\|_{H^s}^2 \to \|(U_0, V_0, \Theta_0)\|_{H^s}^2 - 2\nu \int_0^\infty \|(\nabla U, \nabla V)(\tau)\|_{H^s}^2 \, \mathrm{d}\tau.$$

Our second result for the linearized system (1.11) assesses that, if the Fourier series of the initial data is summable, then the velocity field (U, V) converges uniformly to zero and, more importantly, the temperature Θ converges pointwise to an explicit function that depends only on the vertical variable. This points to the stratification of the temperature.

Theorem 1.2. *Assume that (U_0, V_0, Θ_0) satisfies $\partial_x U_0 + \partial_y V_0 = 0$ and*

$$\sum_{\mathbf{k}} |\widehat{U_0}(\mathbf{k})| < \infty, \quad \sum_{\mathbf{k}} |k_2||\widehat{V_0}(\mathbf{k})| < \infty, \quad \sum_{\mathbf{k}} |\widehat{\Theta_0}(\mathbf{k})| < \infty. \tag{1.15}$$

Assume U_0 and V_0 satisfy the mean-zero condition (1.14)*. Let (U, V, Θ) be the corresponding solution of* (1.11)*. Then, as $t \to \infty$,*

$$\|U(t)\|_{L^\infty(\mathbb{T}^2)} \to 0, \quad \|V(t)\|_{L^\infty(\mathbb{T}^2)} \to 0, \tag{1.16}$$

$$\Theta(x, y, t) \to \widetilde{\Theta}(y) := \sum_{k_2} e^{ik_2 y} \left(\frac{\beta}{\nu k_2^2} \widehat{V_0}(0, k_2) + \widehat{\Theta_0}(0, k_2) \right). \tag{1.17}$$

The third result for the linearized system provides explicit bounds on the H^s-norm of the velocity field (U, V). In particular, these bounds give us the precise decay rates of the velocity perturbation.

Theorem 1.3. *Assume that U_0, V_0 and Θ_0 are in $L^2(\mathbb{T}^2)$, and satisfy $\partial_x U_0 + \partial_y V_0 = 0$ and the mean-zero condition* (1.14)*. Let (U, V, Θ) be the corresponding solution of* (1.11)*. Then the following L^2-estimates hold, for a pure constant $c_0 > 0$:*

$$\begin{aligned} \|U(t)\|_{L^2} \leqq\ & C\, e^{-c_0 \nu t} \|U_0\|_{L^2} + C \left(e^{-c_0 \nu t} + \frac{1}{(\nu t)^{3/4}} \right) \|V_0\|_{L^2} \\ & + C \left(e^{-c_0 \nu t} + \frac{1}{\sqrt{\nu t}} \right) \|\Theta_0\|_{L^2}, \end{aligned} \tag{1.18}$$

$$\|V(t)\|_{L^2} \leqq C \left(e^{-c_0 \nu t} + \frac{1}{\nu t} \right) \|V_0\|_{L^2} + C \left(e^{-c_0 \nu t} + \frac{1}{\sqrt{\nu t}} \right) \|\Theta_0\|_{L^2}, \tag{1.19}$$

where C is a constant independent of ν and t. If $\partial_y V_0 \in L^2(\mathbb{T}^2)$ instead of $V_0 \in L^2(\mathbb{T}^2)$, the decay rate in the second part of the bound for $\|U(t)\|_{L^2}$ can be improved,

$$\begin{aligned} \|U(t)\|_{L^2} \leqq\ & C\, e^{-c_0 \nu t} \|U_0\|_{L^2} + C \left(e^{-c_0 \nu t} + \frac{1}{\nu t} \right) \|\partial_y V_0\|_{L^2} \\ & + C \left(e^{-c_0 \nu t} + \frac{1}{\sqrt{\nu t}} \right) \|\Theta_0\|_{L^2}. \end{aligned} \tag{1.20}$$

For the full nonlinear system (1.9), we remark that the stability results of [21] remain valid in the periodic setting. Our first theorem presents stability and large-time behavior results similar to those in Theorem 1.2 of [21], but with a weakened assumption on θ_0. The results are obtained by combining Theorem 1.2 of [21] with a uniqueness result of [26,38] on the Boussinesq equations in a weak setting. We conclude that any initial data $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$ and $\theta_0 \in L^2(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ lead to a unique, global (in time) solution with the velocity, its time derivative, its first-order spatial derivatives all tend to zero, and its second-order spatial partials uniformly bounded. As a special consequence, if the L^2-norm of the initial data $(\mathbf{u}_0, \theta_0)$ is small, then the second-order spatial partials of the velocity becomes small in large time.

Theorem 1.4. *Assume $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$ is divergence-free and mean-zero, and $\theta_0 \in L^2(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. Then (1.9) has a unique global solution $(\mathbf{u}, \theta)$ satisfying,*

$$\mathbf{u} \in L^\infty(0, \infty; H^2), \quad \theta \in L^\infty(0, \infty; L^2).$$

More importantly, $\mathbf{u}$, θ and the corresponding pressure P satisfy, as $t \to \infty$,

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{L^2} \to 0, \quad \|\nabla \mathbf{u}(t)\|_{L^2} \to 0, \quad \|\partial_t \mathbf{u}(t)\|_{L^2} \to 0,\\
&\|\theta(t)\|_{L^2}^2 \to \|\mathbf{u}_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 - 2\nu \int_0^\infty \|\nabla \mathbf{u}(t)\|_{L^2}^2 \, \mathrm{d}t,\\
&\|\nabla P(t) - \theta(t)\mathbf{e}_2\|_{H^{-1}} \to 0.
\end{aligned}$$

In addition, the second-order spatial partials of $\mathbf{u}$ admit the uniform global bound, for an absolute constant C,

$$\|\Delta \mathbf{u}(t)\|_{L^2} \leqq C \left(\|\theta(t)\|_{L^2} + \|\partial_t \mathbf{u}(t)\|_{L^2} + \|\mathbf{u}(t)\|_{L^2} \|\nabla \mathbf{u}(t)\|_{L^2}^2\right)$$

for any $t > 0$. Especially, if the L^2-norm of the initial data is small, namely $\|\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{L^2}$ is small, then the L^2-norm of the temperature θ remains small and the H^2-norm of the velocity $\mathbf{u}$ becomes small in large time, namely,

$$\|\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{L^2} \leqq \varepsilon \quad \Longrightarrow \quad \begin{aligned} &\|\theta(t)\|_{L^2} \leqq \varepsilon, \quad \textit{for all } t > 0,\\ &\|\mathbf{u}(t)\|_{H^2} \leqq C\,\varepsilon, \quad \textit{when } t \textit{ is large}, \end{aligned}$$

where the constant C is independent of time.

We emphasize that the first part of Theorem 1.4 requires no smallness on the initial data and the global stability part follows as a special consequence. One interesting point about the global stability result is that the initial closeness to the hydrostatic equilibrium is only in the L^2-norm, but the velocity becomes close to the equilibrium in H^2-norm in large time.

The second result for the nonlinear system (1.9) assesses the large-time behavior of the Fourier frequencies of $\mathbf{u}$ and θ.

Theorem 1.5. *Assume that* $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$ *is divergence-free and mean-zero,*

$$\nabla \cdot \mathbf{u}_0 = 0, \quad \int_{\mathbb{T}^2} \mathbf{u}_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbf{0}.$$

Assume that θ_0 *satisfies*

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} |\widehat{\theta_0}(\mathbf{k})| < \infty.$$

Let $(\mathbf{u}, \theta)$ *be the solution of* (1.9)*. Then, for any* $\mathbf{k}$*,*

$$\widehat{\mathbf{u}}(\mathbf{k}, t) \to 0 \quad \text{as } t \to \infty,$$

and, for $\mathbf{k} = (k_1, k_2)$ *with* $k_1 \neq 0$*,*

$$\widehat{\theta}(\mathbf{k}, t) \to 0 \quad \text{as } t \to \infty.$$

Moreover, if there is a constant independent of t, such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} |\widehat{\theta}(\mathbf{k}, t)| \leqq C,$$

then $\theta(\mathbf{x}, t)$ *converges to a function depending on y only. More precisely, the large time asymptotics of* $\theta(\mathbf{x}, t)$ *is determined by* $S(y, t)$*, which satisfies*

$$S(y, t) = \overline{\theta_0}(y) - \beta(\nu\partial_{yy})^{-1}(e^{\nu t\,\partial_{yy}} - 1)\,\overline{v_0}(y) + \partial_y(\overline{v\theta})(y, t).$$

Here the bar denotes the horizontal average, namely

$$\overline{F}(y) = \frac{1}{2\pi}\int_{\mathbb{T}} F(x, y)\,\mathrm{d}x.$$

We remark that the aim of Theorem 1.5 has been to understand the large-time behavior and the eventual profile of the temperature. Theorem 1.5 indeed provides a large-time asymptotics that is independent of the horizontal variable. The earlier part of Theorem 1.5 is a special consequence of Theorem 1.4, which is based on energy estimates. However, the large-time asymptotics part is established using the explicit integral representation derived in Section 2.

Our third result for the nonlinear system (1.9) intends to provide an explicit decay rate for the velocity field. As we mentioned before, it is extremely difficult to obtain any decay rate, due to the fact that the third eigenvalue $\lambda_3(k, t)$ is of the order $-k_1^2/|\mathbf{k}|^4$ and is close to zero when $k_1^2 \ll |\mathbf{k}|^4$. It appears to be necessary to make some assumptions on the solution in order to obtain the desired decay rate. Our investigation indicates that no assumption on the decay of the temperature itself is needed. We find that if the difference between the temperature θ and its large-time asymptotics $S(y, t)$ decays at certain rate and if the large-time asymptotics obey some uniform bounds, then the L^2-norm of the velocity decays at the rate of $(1+t)^{-\frac{1}{2}}$ for large t.

Theorem 1.6. *Assume that* $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$ *is divergence-free and mean-zero,*

$$\nabla \cdot \mathbf{u}_0 = 0, \quad \int_{\mathbb{T}^2} \mathbf{u}_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbf{0}.$$

Assume $\theta_0 \in H^s(\mathbb{T}^2)$ *with* $s > 2$. *Let* $(\mathbf{u}, \theta)$ *be the corresponding solution of* (1.9) *and let* S *denote the large-time asymptotics defined in Theorem* 1.5. *If* $(\mathbf{u}, \theta)$ *obeys, for some small* $\varepsilon > 0$ *and a constant* $C > 0$,

$$\lim_{t\to\infty} t^{\varepsilon} \|\mathbf{u}(t)\|_{H^1} = 0,$$
$$\lim_{t\to\infty} t^{\frac{1}{4}} \|\nabla(\theta - S)(t)\|_{L^2} = 0, \quad \|\partial_y S(t)\|_{L^2} + \|\partial_{yy} S(t)\|_{L^2} \leqq C, \quad (1.21)$$

then

$$\|\mathbf{u}(t)\|_{L^2} \leqq \frac{C}{\sqrt{t+1}}, \quad (1.22)$$

for some constant C *which is independent of* t.

The rest of the paper is divided into three sections. The second section derives the integral representation of (1.9). The third section proves the three theorems for the linearized system (1.11), while the fourth section presents the proofs of three theorems for the nonlinear system (1.9). The paper is finished with concluding remarks.

2. Integral Representation

This section converts (1.9) into an integral form. The Fourier transform of (1.9) can be written as

$$\partial_t \psi = A\psi + F, \quad (2.1)$$

where

$$\psi = \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{\theta} \end{bmatrix}, \quad A = \begin{bmatrix} -\nu|\mathbf{k}|^2 & 0 & -\frac{k_1 k_2}{|\mathbf{k}|^2} \\ 0 & -\nu|\mathbf{k}|^2 & \frac{k_1^2}{|\mathbf{k}|^2} \\ 0 & -\beta & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \widehat{N_1} \\ \widehat{N_2} \\ \widehat{N_3} \end{bmatrix}. \quad (2.2)$$

Therefore, ψ can be represented as

$$\psi(t) = e^{At}\psi_0 + \int_0^t e^{A(t-\tau)} F(\tau)\,\mathrm{d}\tau. \quad (2.3)$$

In order to obtain a more explicit representation, we need to diagonalize A. To do so, we compute the eigenvalues and eigenvectors of A. The associated characteristic polynomial of A is given by

$$p(\lambda) = (\lambda + \nu|\mathbf{k}|^2)\left(\lambda^2 + \nu|\mathbf{k}|^2\lambda + \beta\,\frac{k_1^2}{|\mathbf{k}|^2}\right)$$

and the eigenvalues are

$$\lambda_1 = -\nu|\mathbf{k}|^2,$$
$$\lambda_2 = -\frac{1}{2}\nu|\mathbf{k}|^2 - \frac{1}{2}\sqrt{\nu^2\,|\mathbf{k}|^4 - \frac{4\beta k_1^2}{|\mathbf{k}|^2}} = -\frac{1}{2}\nu|\mathbf{k}|^2\left(1+\sqrt{1-\frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}\right), \tag{2.4}$$
$$\lambda_3 = -\frac{1}{2}\nu|\mathbf{k}|^2 + \frac{1}{2}\sqrt{\nu^2\,|\mathbf{k}|^4 - \frac{4\beta k_1^2}{|\mathbf{k}|^2}} = -\frac{1}{2}\nu|\mathbf{k}|^2\left(1-\sqrt{1-\frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}\right). \tag{2.5}$$

Clearly, for $\beta > 0$ and $k_1 \neq 0$, the real parts of $\lambda_1, \lambda_2, \lambda_3$ are all negative:

$$\lambda_1 < 0, \quad Re\lambda_2 < 0, \quad Re\lambda_3 < 0.$$

When $\lambda_2 \neq \lambda_3$ or $4\beta k_1^2 \neq \nu^2|\mathbf{k}|^6$, the eigenvectors corresponding to λ_1, λ_2 and λ_3 are given by

$$\eta_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} \frac{\beta k_1 k_2}{\lambda_3|\mathbf{k}|^2} \\ -\lambda_2 \\ \beta \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} \frac{\beta k_1 k_2}{\lambda_2|\mathbf{k}|^2} \\ -\lambda_3 \\ \beta \end{bmatrix}.$$

Consequently we can write

$$AW = WD \quad \text{or} \quad A = WDW^{-1},$$

where D is the diagonal matrix and W denotes the matrix with η_1, η_2 and η_3 being the column vectors, namely

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad W = [\eta_1, \eta_2, \eta_3] = \begin{bmatrix} 1 & \frac{\beta k_1 k_2}{\lambda_3|\mathbf{k}|^2} & \frac{\beta k_1 k_2}{\lambda_2|\mathbf{k}|^2} \\ 0 & -\lambda_2 & -\lambda_3 \\ 0 & \beta & \beta \end{bmatrix}.$$

For $k_1 \neq 0$, the inverse of W, denoted W^{-1}, is given by

$$W^{-1} = \begin{bmatrix} 1 & \frac{k_2}{k_1} & 0 \\ 0 & \frac{1}{\lambda_3-\lambda_2} & \frac{\lambda_3}{\beta(\lambda_3-\lambda_2)} \\ 0 & -\frac{1}{\lambda_3-\lambda_2} & -\frac{\lambda_2}{\beta(\lambda_3-\lambda_2)} \end{bmatrix},$$

where we have used $\lambda_2\lambda_3 = \frac{\beta k_1^2}{|\mathbf{k}|^2}$ to simplify the calculations. Therefore,

$$\psi(t) = W \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} W^{-1}\psi(0) + \int_0^t W \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} W^{-1}\,F(\tau)\,\mathrm{d}\tau.$$

More explicitly,

$$W\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} W^{-1} = \begin{bmatrix} e^{\lambda_1 t} & \frac{k_2}{k_1}(e^{\lambda_1 t} - G_1) & -\frac{k_1 k_2}{|\mathbf{k}|^2} G_2 \\ 0 & G_1 & \frac{k_1^2}{|\mathbf{k}|^2} G_2 \\ 0 & -\beta G_2 & G_3 \end{bmatrix}$$

where

$$G_1(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_3 e^{\lambda_3 t}}{\lambda_2 - \lambda_3}, \quad G_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_3 t}}{\lambda_2 - \lambda_3}, \quad G_3(t) = \frac{\lambda_3 e^{\lambda_2 t} - \lambda_2 e^{\lambda_3 t}}{\lambda_3 - \lambda_2}. \tag{2.6}$$

Therefore, for $\lambda_2 \neq \lambda_3$ and $k_1 \neq 0$,

$$\begin{aligned} \widehat{u}(\mathbf{k}, t) &= e^{\lambda_1 t}\, \widehat{u_0}(\mathbf{k}) + \frac{k_2}{k_1}(e^{\lambda_1 t} - G_1(t))\, \widehat{v_0}(\mathbf{k}) - \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t)\, \widehat{\theta_0}(\mathbf{k}) \\ &\quad + \int_0^t \Big(e^{\lambda_1 (t-\tau)}\, \widehat{N}_1(\mathbf{k}, \tau) + \frac{k_2}{k_1}(e^{\lambda_1 (t-\tau)} - G_1(t-\tau))\, \widehat{N}_2(\mathbf{k}, \tau) \\ &\quad - \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t-\tau)\, \widehat{N}_3(\mathbf{k}, \tau) \Big)\, \mathrm{d}\tau, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \widehat{v}(\mathbf{k}, t) &= G_1(t)\, \widehat{v_0}(\mathbf{k}) + \frac{k_1^2}{|\mathbf{k}|^2} G_2(t)\, \widehat{\theta_0}(\mathbf{k}) \\ &\quad + \int_0^t \Big(G_1(t-\tau)\; \widehat{N}_2(\mathbf{k}, \tau) + \frac{k_1^2}{|\mathbf{k}|^2} G_2(t-\tau)\, \widehat{N}_3(\mathbf{k}, \tau) \Big)\, \mathrm{d}\tau, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \widehat{\theta}(\mathbf{k}, t) &= -\beta G_2(t)\, \widehat{v_0}(\mathbf{k}) + G_3(t)\, \widehat{\theta_0}(\mathbf{k}) \\ &\quad + \int_0^t \Big(- \beta G_2(t-\tau)\; \widehat{N}_2(\mathbf{k}, \tau) + G_3(t-\tau)\, \widehat{N}_3(\mathbf{k}, \tau) \Big)\, \mathrm{d}\tau. \end{aligned} \tag{2.9}$$

For $k_1 = 0$,

$$\lambda_2 = -\nu |\mathbf{k}|^2, \quad \lambda_3 = 0, \quad G_1 = e^{\lambda_2 t}, \quad G_2 = \frac{1}{\lambda_2}(e^{\lambda_2 t} - 1), \quad G_3 = 1$$

and

$$W^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\lambda_3 - \lambda_2} & -\frac{\lambda_3}{\beta(\lambda_3 - \lambda_2)} \\ 0 & \frac{1}{\lambda_3 - \lambda_2} & \frac{\lambda_2}{\beta(\lambda_3 - \lambda_2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\lambda_2} & 0 \\ 0 & -\frac{1}{\lambda_2} & \frac{1}{\beta} \end{bmatrix}$$

and

$$W\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} W^{-1} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & -\beta G_2 & G_3 \end{bmatrix}.$$

Therefore, for $k_1 = 0$, the integral representation of (1.9) is given by

$$\widehat{u}(\mathbf{k}, t) = e^{\lambda_1 t}\, \widehat{u_0}(\mathbf{k}) + \int_0^t e^{\lambda_1 (t-\tau)}\, \widehat{N}_1(\mathbf{k}, \tau)\, \mathrm{d}\tau, \tag{2.10}$$

$$\widehat{v}(\mathbf{k},t) = G_1(t)\,\widehat{v_0}(\mathbf{k}) + \int_0^t G_1(t-\tau)\ \widehat{N_2}(\mathbf{k},\tau)\,\mathrm{d}\tau, \tag{2.11}$$

$$\begin{aligned}\widehat{\theta}(\mathbf{k},t) &= -\beta G_2(t)\,\widehat{v_0}(\mathbf{k}) + \widehat{\theta_0}(\mathbf{k})\\ &\quad + \int_0^t \Big(-\beta G_2(t-\tau)\ \widehat{N_2}(\mathbf{k},\tau) + \widehat{N_3}(\mathbf{k},\tau)\Big)\,\mathrm{d}\tau. \end{aligned}\tag{2.12}$$

We remark this representation is actually the limit of (2.7), (2.8) and (2.9) as $k_1 \to 0$, due to the fact that

$$\lim_{k_1\to 0} \frac{e^{\lambda_1 t} - G_1(t)}{k_1} = 0.$$

For the sake of conciseness, we sometimes still use the representation in (2.7), (2.8) and (2.9) even for $k_1 = 0$.

In the case when $\lambda_2 = \lambda_3$, the eigenvectors associated with the eigenvalues are different from those for $\lambda_2 \neq \lambda_3$. Fortunately the representation formula in (2.7), (2.8) and (2.9) remain valid if G_1, G_2 and G_3 in (2.6) are interpreted as their corresponding limits,

$$G_1 = \lim_{\lambda_2\to\lambda_3} \frac{\lambda_2 e^{\lambda_2 t} - \lambda_3 e^{\lambda_3 t}}{\lambda_2 - \lambda_3} = (1+\lambda_2 t)e^{\lambda_2 t}, \tag{2.13}$$

$$G_2 = \lim_{\lambda_2\to\lambda_3} \frac{e^{\lambda_2 t} - e^{\lambda_3 t}}{\lambda_2 - \lambda_3} = t e^{\lambda_2 t}, \tag{2.14}$$

$$G_3 = \lim_{\lambda_2\to\lambda_3} \frac{\lambda_3 e^{\lambda_2 t} - \lambda_2 e^{\lambda_3 t}}{\lambda_3 - \lambda_2} = (1-\lambda_2 t)e^{\lambda_2 t}. \tag{2.15}$$

That is, when $\lambda_2 = \lambda_3$ or $4\beta k_1^2 = \nu^2|\mathbf{k}|^6$, the integral representation of (1.9) is given by (2.7), (2.8) and (2.9) with G_1, G_2 and G_3 being specified in (2.13), (2.14) and (2.15).

To prepare for the proofs in the subsequent sections, we provide some preliminary bounds on G_1, G_2 and G_3. They admit different bounds for different $\mathbf{k}$'s. When $\mathbf{k} = (k_1, k_2)$ satisfies

$$4\beta k_1^2 > \nu|\mathbf{k}|^6, \tag{2.16}$$

$\sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}$ is a pure imaginary number and λ_2 given by (2.4) and λ_3 given by (2.5) behave like their real parts $-\frac{1}{2}\nu|\mathbf{k}|^2$. In order to make our presentation concise, we shall ignore the case (2.16) since G_1, G_2 and G_3 admit very similar bounds as those for the case $\mathbf{k} \in S_1$, as provided in the following lemma.

Lemma 2.1. *Let S_1 and S_2 be subsets of $\mathbb{Z}^2$ (the set of all pairs of integers),*

$$S_1 := \left\{ \mathbf{k} \in \mathbb{Z}^2 :\ k_1^2 \geqq \frac{3\nu^2}{16\beta}|\mathbf{k}|^6 \ \ \mathit{or}\ \ \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}} \leqq \frac{1}{2} \right\}, \tag{2.17}$$

$$S_2 := \left\{ \mathbf{k} \in \mathbb{Z}^2 :\ k_1^2 < \frac{3\nu^2}{16\beta}|\mathbf{k}|^6 \ \ \mathit{or}\ \ \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}} > \frac{1}{2} \right\}. \tag{2.18}$$

Then the following estimates hold:

(1) *for any* $\mathbf{k} \in S_1$,

$$\begin{aligned}
\lambda_2 &\leqq -\frac{1}{2}\nu|\mathbf{k}|^2, \quad \lambda_3 \leqq -\frac{1}{4}\nu|\mathbf{k}|^2,\\
|G_1(t)| &\leqq e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + \frac{1}{2}\nu|\mathbf{k}|^2 t\, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t},\\
|G_2(t)| &\leqq t\, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} \leqq \frac{C}{\nu|\mathbf{k}|^2} \quad \textit{for a constant } C \textit{ independent of } \mathbf{k} \textit{ and } t,\\
|G_3(t)| &\leqq e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + \nu|\mathbf{k}|^2 t\, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t};
\end{aligned}$$

(2) *for any* $\mathbf{k} \in S_2$,

$$\begin{aligned}
\lambda_2 &< -\frac{1}{2}\nu|\mathbf{k}|^2, \quad \lambda_3 \leqq -\frac{4\beta k_1^2}{3\nu|\mathbf{k}|^4}, \quad \lambda_3 - \lambda_2 \geqq \frac{1}{2}\nu|\mathbf{k}|^2,\\
|G_1(t)| &\leqq \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t} + 2e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} \leqq C,\\
|G_2(t)| &\leqq \frac{2}{\nu|\mathbf{k}|^2} e^{\lambda_2 t} + \frac{2}{\nu|\mathbf{k}|^2} e^{\lambda_3 t} \leqq \frac{C}{\nu|\mathbf{k}|^2},\\
|G_3(t)| &\leqq 2e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t} + \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6} e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} \leqq C.
\end{aligned}$$

Proof. We start with the first case, $\mathbf{k} \in S_1$. As we remarked before the statement of this lemma, we shall always assume $\sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}$ is real-valued, without loss of generality. For $\mathbf{k} \in S_1$, λ_2 given by (2.4) and λ_3 given by (2.5) obviously satisfy

$$\lambda_2 \leqq -\frac{1}{2}\nu|\mathbf{k}|^2, \quad \lambda_3 \leqq -\frac{1}{4}\nu|\mathbf{k}|^2.$$

By the mean-value theorem, there is $\rho \in (\lambda_2, \lambda_3)$ such that

$$G_1 = e^{\lambda_2 t} + \lambda_3 t e^{\rho t} \leqq e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + \frac{1}{2}\nu|\mathbf{k}|^2 t\, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t}.$$

The bounds for G_2 and G_3 are similarly obtained. We now turn to the case $\mathbf{k} \in S_2$. Obviously, $\lambda_2 < -\frac{1}{2}\nu|\mathbf{k}|^2$. We write λ_3 as

$$\lambda_3 = -\frac{1}{2}\nu|\mathbf{k}|^2 \left(1 - \sqrt{1 - \frac{4k_1^2}{\nu^2|\mathbf{k}|^6}}\right) = -\frac{\frac{2\beta k_1^2}{\nu|\mathbf{k}|^4}}{1 + \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}}} \leqq -\frac{4\beta k_1^2}{3\nu|\mathbf{k}|^4}.$$

We have the difference

$$\lambda_3 - \lambda_2 = \nu|\mathbf{k}|^2 \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}} \geqq \frac{1}{2}\nu|\mathbf{k}|^2.$$

The bound for G_2 follows directly from the lower bound of this difference. To bound G_1, we have

$$|G_1(t)| \leqq \frac{|\lambda_3|}{|\lambda_3-\lambda_2|}e^{\lambda_3 t} + \frac{|\lambda_2|}{|\lambda_3-\lambda_2|}e^{\lambda_2 t} \leqq \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t} + 2e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t}.$$

The estimate for G_3 is similar. This completes the proof of Lemma 2.1. □

3. Proofs for the Linear Stability Results

This section proves Theorems 1.1, 1.2 and 1.3 stated in the introduction. For the convenience of the reader, we recall the linearized system (1.11):

$$\begin{cases} \partial_t U = \nu\Delta U - \partial_{xy}\Delta^{-1}\Theta, \\ \partial_t V = \nu\Delta V + \partial_{xx}\Delta^{-1}\Theta, \\ \partial_t\Theta + \beta V = 0, \\ \partial_x U + \partial_y V = 0, \\ U(\mathbf{x},0) = U_0(\mathbf{x}), \quad V(\mathbf{x},0) = V_0(\mathbf{x}), \quad \Theta(\mathbf{x},0) = \Theta_0(\mathbf{x}), \end{cases} \tag{3.1}$$

and its explicit representation in the Fourier space given by the linearization of (2.7), (2.8) and (2.9):

$$\begin{cases} \widehat{U}(\mathbf{k},t) = e^{\lambda_1 t}\,\widehat{U_0}(\mathbf{k}) + \frac{k_2}{k_1}(G_1(t) - e^{\lambda_1 t})\,\widehat{V_0}(\mathbf{k}) + \frac{k_1 k_2}{|\mathbf{k}|^2}G_2(t)\,\widehat{\Theta_0}(\mathbf{k}), \\ \widehat{V}(\mathbf{k},t) = G_1(t)\,\widehat{V_0}(\mathbf{k}) + \frac{k_1^2}{|\mathbf{k}|^2}G_2(t)\,\widehat{\Theta_0}(\mathbf{k}), \\ \widehat{\Theta}(\mathbf{k},t) = -\beta G_2(t)\,\widehat{V_0}(\mathbf{k}) + G_3(t)\,\widehat{\Theta_0}(\mathbf{k}). \end{cases} \tag{3.2}$$

To prove Theorem 1.1, we recall the following lemma (see [21]). It assesses that a uniformly continuous and integrable function must vanish at infinity. A proof of this simple fact is provided in [21].

Lemma 3.1. *Assume $f \in L^1(0,\infty)$ is a nonnegative and uniformly continuous function. Then,*

$$f(t) \to 0 \quad \text{as } t \to \infty.$$

Especially, if $f \in L^1(0,\infty)$ is nonnegative and satisfies, for a constant C and any $0 \leqq s < t < \infty$,

$$|f(t) - f(s)| \leqq C\,|t-s|,$$

then $f(t) \to 0$ as $t \to \infty$.

For the conciseness of the presentation, we set $\beta = 1$ from now on. We start with the proof of Theorem 1.1.

Proof of Theorem 1.1. Due to the linearity of (3.1), it suffices to prove the result for $s = 0$. Dotting (3.1) with (U, V, Θ) and integrating by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(U, V, \Theta)\|_{L^2}^2 + 2\nu\|(\nabla U, \nabla V)\|_{L^2}^2 = 0,$$

which implies, for any $0 \leqq s \leqq t$,

$$\|(U, V, \Theta)(t)\|_{L^2}^2 + 2\nu \int_s^t \|(\nabla U, \nabla V)(\tau)\|_{L^2}^2 \,\mathrm{d}\tau = \|(U, V, \Theta)(s)\|_{L^2}^2. \quad (3.3)$$

Therefore, $\|(U, V, \Theta)(t)\|_{L^2}$ is a decreasing function of t and it must have a limit as $t \to \infty$. In fact, as $t \to \infty$,

$$\|(U, V, \Theta)(t)\|_{L^2}^2 \quad \to \quad \|(U_0, V_0, \Theta_0)\|_{L^2}^2 - 2\nu \int_0^\infty \|(\nabla U, \nabla V)(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \quad (3.4)$$

Next we show that, as $t \to \infty$,

$$\|(U(t), V(t))\|_{L^2} \to 0.$$

Taking the inner product of (U, V) with the first two equations in (3.1) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(U, V)\|_{L^2}^2 + 2\nu\|(\nabla U, \nabla V)\|_{L^2}^2 &= 2\int \Theta\, V \,\mathrm{d}x \\ &\leqq \|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2 \leqq \|(U_0, V_0, \Theta_0)\|_{L^2}^2,\end{aligned}$$

which implies

$$\begin{aligned}&\left|\|(U(t), V(t))\|_{L^2}^2 - \|(U(s), V(s))\|_{L^2}^2\right| \\ &\quad \leqq 2\nu \int_s^t \|(\nabla U, \nabla V)(\tau)\|_{L^2}^2 d\tau + \|(U_0, V_0, \Theta_0)\|_{L^2}^2 |t - s|. \quad (3.5)\end{aligned}$$

Note that (3.3) implies $\|(\nabla U, \nabla V)(t)\|_{L^2}^2 \in L^1(0, \infty)$. Hence, (3.5) implies that $\|(U(t), V(t))\|_{L^2}^2$ is absolutely (and so is uniformly) continuous with respect to time. Moreover, the periodic setting and the mean-zero condition (1.14) allow the Poincaré type inequality

$$\|(U, V)\|_{L^2} \leqq C_0\, \|(\nabla U, \nabla V)\|_{L^2}.$$

It then follows from (3.3) that

$$\int_0^\infty \|(U(t), V(t))\|_{L^2}^2 \,\mathrm{d}t < \infty.$$

Lemma 3.1 then implies, as $t \to \infty$, that

$$\|U(t)\|_{L^2} \to 0, \quad \|V(t)\|_{L^2} \to 0,$$

which, together with (3.4), implies the desired limits. This completes the proof of Theorem 1.1. □

The key components of the proof of Theorem 1.2 are stated in the following two lemmas. The first lemma provides the limit of $\widehat{U}(\mathbf{k},t)$, $\widehat{V}(\mathbf{k},t)$ and $\widehat{\Theta}(\mathbf{k},t)$ as $t\to\infty$ while the second lemma establishes the uniform summability of $\widehat{U}(\mathbf{k},t)$, $\widehat{V}(\mathbf{k},t)$ and $\widehat{\Theta}(\mathbf{k},t)$.

Lemma 3.2. *Under the assumptions of Theorem 1.2, $\widehat{U}(\mathbf{k},t)$, $\widehat{V}(\mathbf{k},t)$ and $\widehat{\Theta}(\mathbf{k},t)$ obey the following large-time behavior:*

$$\begin{aligned}
&\textit{for any } \mathbf{k},\quad \widehat{U}(\mathbf{k},t),\quad \widehat{V}(\mathbf{k},t)\to 0 \quad \textit{as } t\to\infty,\\
&\textit{for any } \mathbf{k}=(k_1,k_2) \textit{ with } k_1\neq 0,\quad \widehat{\Theta}(\mathbf{k},t)\to 0 \quad \textit{as } t\to\infty,\\
&\textit{for any } \mathbf{k}=(0,k_2),\quad \widehat{\Theta}(\mathbf{k},t)\to \frac{1}{\nu k_2^2}\widehat{V_0}(0,k_2)+\widehat{\Theta_0}(0,k_2) \quad \textit{as } t\to\infty.
\end{aligned}$$

Lemma 3.3. *Under the assumptions of Theorem 1.2, $\widehat{U}(\mathbf{k},t)$, $\widehat{V}(\mathbf{k},t)$ and $\widehat{\Theta}(\mathbf{k},t)$ are uniformly summable, in the sense that the series converge uniformly in time $t\in(0,\infty)$,*

$$\sum_{\mathbf{k}}|\widehat{U}(\mathbf{k},t)|<\infty,\quad \sum_{\mathbf{k}}|\widehat{V}(\mathbf{k},t)|<\infty,\quad \sum_{\mathbf{k}}|\widehat{\Theta}(\mathbf{k},t)|<\infty.$$

Proof of Theorem 1.2. With the preparations of the two lemmas above, we can easily prove Theorem 1.2. Lemmas 3.2 and 3.3 allow us to use the Dominated Convergence Theorem. Therefore,

$$\begin{aligned}
\lim_{t\to\infty}U(x,y,t)&=\lim_{t\to\infty}\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\widehat{U}(\mathbf{k},t)=\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\lim_{t\to\infty}\widehat{U}(\mathbf{k},t)=0,\\
\lim_{t\to\infty}V(x,y,t)&=\lim_{t\to\infty}\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\widehat{V}(\mathbf{k},t)=\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\lim_{t\to\infty}\widehat{V}(\mathbf{k},t)=0,\\
\lim_{t\to\infty}\Theta(x,y,t)&=\lim_{t\to\infty}\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\widehat{\Theta}(\mathbf{k},t)=\sum_{\mathbf{k}}e^{i(k_1x+k_2y)}\lim_{t\to\infty}\widehat{\Theta}(\mathbf{k},t)\\
&=\sum_{k_2}e^{ik_2y}\left(\frac{1}{\nu k_2^2}\widehat{V_0}(0,k_2)+\widehat{\Theta_0}(0,k_2)\right).
\end{aligned}$$

This completes the proof of Theorem 1.2. □

We now prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. We invoke the representation of $\widehat{U}(\mathbf{k},t)$, $\widehat{V}(\mathbf{k},t)$ and $\widehat{\Theta}(\mathbf{k},t)$ in (3.2). For each $\mathbf{k}=(k_1,k_2)$ with $k_1\neq 0$, the eigenvalues all have negative real parts,

$$\begin{aligned}
&\lambda_1=-\nu|\mathbf{k}|^2<0,\quad \lambda_2=-\frac{1}{2}\nu|\mathbf{k}|^2\left(1+\sqrt{1-\frac{4k_1^2}{\nu^2|\mathbf{k}|^6}}\right)<0,\\
&\lambda_3=-\frac{1}{2}\nu|\mathbf{k}|^2\left(1-\sqrt{1-\frac{4k_1^2}{\nu^2|\mathbf{k}|^6}}\right)<0
\end{aligned}$$

and, for $\lambda_2 \neq \lambda_3$ or $4k_1^2 \neq \nu^2|\mathbf{k}|^6$, as $t \to \infty$,

$$G_1(t) = \frac{\lambda_3 e^{\lambda_3 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_3 - \lambda_2} \to 0, \quad G_2(t) = \frac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} \to 0,$$
$$G_3(t) = \frac{\lambda_3 e^{\lambda_2 t} - \lambda_2 e^{\lambda_3 t}}{\lambda_3 - \lambda_2} \to 0.$$

In the case when $4k_1^2 = \nu^2|\mathbf{k}|^6$, we have $\lambda_2 = \lambda_3$. Then G_1, G_2 and G_3 are given by the limit form and, as $t \to \infty$,

$$G_1(t) = \lim_{\lambda_2 \to \lambda_3} \frac{\lambda_3 e^{\lambda_3 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_3 - \lambda_2} = (1 + \lambda_2 t) e^{\lambda_2 t} \to 0,$$
$$G_2(t) = \lim_{\lambda_2 \to \lambda_3} \frac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} = t e^{\lambda_2 t} \to 0,$$
$$G_3(t) = \lim_{\lambda_2 \to \lambda_3} \frac{\lambda_3 e^{\lambda_2 t} - \lambda_2 e^{\lambda_3 t}}{\lambda_3 - \lambda_2} = (1 - \lambda_2 t) e^{\lambda_2 t} \to 0.$$

Therefore, for $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$, as $t \to \infty$,

$$\widehat{U}(\mathbf{k}, t) = e^{\lambda_1 t}\, \widehat{U_0}(\mathbf{k}) + \frac{k_2}{k_1}(G_1(t) - e^{\lambda_1 t})\, \widehat{V_0}(\mathbf{k}) + \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t)\, \widehat{\Theta_0}(\mathbf{k}) \to 0,$$
$$\widehat{V}(\mathbf{k}, t) = G_1(t)\, \widehat{V_0}(\mathbf{k}) + \frac{k_1^2}{|\mathbf{k}|^2} G_2(t)\, \widehat{\Theta_0}(\mathbf{k}) \to 0,$$
$$\widehat{\Theta}(\mathbf{k}, t) = -G_2(t)\, \widehat{V_0}(\mathbf{k}) + G_3(t)\, \widehat{\Theta_0}(\mathbf{k}) \to 0.$$

When $k_1 = 0$, or $\mathbf{k} = (0, k_2)$ with $k_2 \neq 0$,

$$\lambda_1 = -\nu k_2^2 < 0, \quad \lambda_2 = -\nu k_2^2 < 0, \quad \lambda_3 = 0$$

and

$$G_1(t) = e^{\lambda_2 t}, \quad G_2(t) = \frac{1}{\lambda_2}(e^{\lambda_2 t} - 1), \quad G_3(t) = 1.$$

According to the representation for the case $k_1 = 0$, namely in (2.10), (2.11) and (2.12), we have, as $t \to \infty$,

$$\widehat{U}(\mathbf{k}, t) = e^{\lambda_1 t}\, \widehat{U_0}(\mathbf{k}) \quad \to \quad 0,$$
$$\widehat{V}(\mathbf{k}, t) = G_1(t)\, \widehat{V_0}(\mathbf{k}) \quad \to \quad 0,$$
$$\widehat{\Theta}(\mathbf{k}, t) = -G_2(t)\, \widehat{V_0}(\mathbf{k}) + G_3(t)\, \widehat{\Theta_0}(\mathbf{k}) \quad \to \quad \frac{1}{\nu k_2^2} \widehat{V_0}(0, k_2) + \widehat{\Theta_0}(0, k_2).$$

This completes the proof of Lemma 3.2. $\square$

We now turn to the proof of Lemma 3.3.

Proof of Lemma 3.3. The proof is devoted to establishing the following uniform-in-time bounds, for $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$,

$$|\widehat{U}(\mathbf{k}, t)| \leqq |\widehat{U_0}(\mathbf{k})| + C\,|k_2||\widehat{V_0}(\mathbf{k})| + \frac{C}{\nu|\mathbf{k}|^2}\,|\widehat{\Theta_0}(\mathbf{k})|,$$
$$|\widehat{V}(\mathbf{k}, t)| \leqq C\,|\widehat{V_0}(\mathbf{k})| + \frac{C}{\nu|\mathbf{k}|^2}\,|\widehat{\Theta_0}(\mathbf{k})|,$$
$$|\widehat{\Theta}(\mathbf{k}, t)| \leqq \frac{C}{\nu|\mathbf{k}|^2}\,|\widehat{V_0}(\mathbf{k})| + C\,|\widehat{\Theta_0}(\mathbf{k})|$$

and, for $\mathbf{k} = (0, k_2)$,

$$|\widehat{U}(\mathbf{k}, t)| \leqq |\widehat{U_0}(\mathbf{k})|,$$
$$|\widehat{V}(\mathbf{k}, t)| \leqq C\,|\widehat{V_0}(\mathbf{k})|,$$
$$|\widehat{\Theta}(\mathbf{k}, t)| \leqq \frac{C}{\nu|\mathbf{k}|^2}\,|\widehat{V_0}(\mathbf{k})| + C\,|\widehat{\Theta_0}(\mathbf{k})|,$$

where C is a pure constant. As a consequence, for U_0, V_0 and Θ_0 satisfying (1.15),

$$\sum_{\mathbf{k}} |\widehat{U}(\mathbf{k}, t)|, \quad \sum_{\mathbf{k}} |\widehat{V}(\mathbf{k}, t)|, \quad \sum_{\mathbf{k}} |\widehat{\Theta}(\mathbf{k}, t)|$$
$$\leqq C \sum_{\mathbf{k}} (|\widehat{U_0}(\mathbf{k})| + |k_2||\widehat{V_0}(\mathbf{k})| + |\widehat{\Theta_0}(\mathbf{k})|) < \infty.$$

The rest of this proof shows the aforementioned uniform bounds. As our first step, we prove the following bounds for G_1, G_2 and G_3:

$$|G_1(t)| \leqq C, \quad |G_2(t)| \leqq \frac{C}{\nu|\mathbf{k}|^2}, \quad |G_3(t)| \leqq C, \tag{3.6}$$

where C is a pure constant. For $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$, by the Mean-Value Theorem, there exists A satisfying $\lambda_2 \leqq A \leqq \lambda_3 < 0$ such that

$$G_1(t) = (1 + At)e^{At} \leqq C.$$

For $\mathbf{k} = (0, k_2)$, $\lambda_2 = -\nu|\mathbf{k}|^2$ and $\lambda_3 = 0$, and

$$G_1(t) = e^{\lambda_2 t} \leqq 1.$$

Furthermore, for $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$,

$$\left|\frac{k_2}{k_1}(G_1(t) - e^{\lambda_1 t})\right| \leqq \frac{1}{|k_1|}\left(|k_2||G_1(t) - e^{\lambda_1 t}|\right) \leqq C|k_2|, \tag{3.7}$$

where we have used the fact that, for $k_1 \neq 0$, $\frac{1}{|k_1|} \leqq C$. In the case when $k_1 = 0$, as we have explained before, $\frac{G_1(t) - e^{\lambda_1 t}}{k_1}$ is defined by the limit

$$\frac{G_1(t) - e^{\lambda_1 t}}{k_1} = \lim_{k_1 \to 0} \frac{G_1(t) - e^{\lambda_1 t}}{k_1} = 0.$$

Now we turn to bounding $G_2(t)$. For $k_1 = 0$ and $k = (0, k_2)$, $\lambda_2 = -\nu|\mathbf{k}|^2$ and $\lambda_3 = 0$, and

$$G_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_3 t}}{\lambda_2 - \lambda_3} = \frac{1}{\lambda_2}(e^{\lambda_2 t} - 1) \leqq \frac{1}{\nu|\mathbf{k}|^2}.$$

We consider $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$. We invoke the bounds from Lemma 2.1. By Lemma 2.1,

$$G_2(t) \leqq \frac{C}{\nu|\mathbf{k}|^2}.$$

Due to

$$G_3(t) = \frac{\lambda_3 e^{\lambda_2 t} - \lambda_2 e^{\lambda_3 t}}{\lambda_3 - \lambda_2} = e^{\lambda_2 t} - \lambda_2\, G_2(t),$$

G_3 is bounded by

$$|G_3(t)| \leqq 1 + \nu|\mathbf{k}|^2 \cdot \frac{C}{\nu|\mathbf{k}|^2} \leqq 1 + C.$$

We thus have established the bounds in (3.6). Inserting these bounds in (3.2) yields the desired bounds for $\widehat{U}$, $\widehat{V}$ and $\widehat{\Theta}$. This completes the proof of Lemma 3.3. □

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Since U and V are mean zero,

$$\widehat{U}(0, t) = 0, \quad \widehat{V}(0, t) = 0.$$

By Plancherel's theorem,

$$\begin{aligned}
&\|U(t)\|_{L^2}^2 = \sum_{\mathbf{k}\neq 0} |\widehat{U}(\mathbf{k}, t)|^2 \\
&\leqq 3 \sum_{\mathbf{k}\neq 0} e^{2\lambda_1 t}|\widehat{U}_0(\mathbf{k})|^2 + 3 \sum_{k_1\neq 0} \frac{k_2^2}{k_1^2}(G_1(t) - e^{\lambda_1 t})^2|\widehat{V}_0(\mathbf{k})|^2 + 3 \sum_{k_1\neq 0, k_2\neq 0} \frac{k_1^2 k_2^2}{|\mathbf{k}|^4} G_2^2|\widehat{\Theta}_0|^2 \\
&:= I_1 + I_2 + I_3. \qquad (3.8)
\end{aligned}$$

Since $\lambda_1 = -\nu|\mathbf{k}|^2$, there is $c_0 > 0$ such that

$$I_1 \leqq 3\, e^{-c_0\, \nu t}\, \|U_0\|_{L^2}^2.$$

We now estimate I_3. The key is to bound G_2 and we invoke the bounds in Lemma 2.1. According to Lemma 2.1,

$$\text{for } \mathbf{k}=(k_1, k_2) \in S_1 \text{ or } k_1^2 \geqq \frac{3\nu^2}{16}|\mathbf{k}|^6, \quad G_2(t)=t\, e^{\rho t}, \quad -\frac{1}{2}\nu|\mathbf{k}|^2 \leqq \rho \leqq -\frac{1}{4}\nu|\mathbf{k}|^2$$

and

$$\text{for } \mathbf{k} = (k_1, k_2) \in S_2 \text{ or } k_1^2 < \frac{3\nu^2}{16}|\mathbf{k}|^6, \quad |G_2(t)| \leqq \frac{2}{\nu|\mathbf{k}|^2} e^{\lambda_2 t} + \frac{2}{\nu|\mathbf{k}|^2} e^{\lambda_3 t}.$$

The summation in I_3 is naturally divided into two summations:

$$
\begin{aligned}
I_3 &= 3\sum_{\mathbf{k}\in S_1}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}G_2^2\,|\widehat{\Theta_0}|^2 + 3\sum_{\mathbf{k}\in S_2}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}G_2^2\,|\widehat{\Theta_0}|^2\\
&\leqq 3\sum_{\mathbf{k}\in S_1}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}\,t^2e^{2\rho t}\,|\widehat{\Theta_0}(\mathbf{k})|^2 + C\sum_{\mathbf{k}\in S_2}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}\,\frac{1}{\nu^2|\mathbf{k}|^4}e^{2\lambda_2 t}\,|\widehat{\Theta_0}(\mathbf{k})|^2\\
&\quad + C\sum_{\mathbf{k}\in S_2}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}\,\frac{1}{\nu^2|\mathbf{k}|^4}e^{2\lambda_3 t}\,|\widehat{\Theta_0}(\mathbf{k})|^2\\
&\leqq C\,(t^2+1)\,e^{-c_0\nu t}\,\|\Theta_0\|_{L^2}^2 + C\sum_{\mathbf{k}\in S_2}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}\,\frac{1}{\nu^2|\mathbf{k}|^4}e^{2\lambda_3 t}\,|\widehat{\Theta_0}(\mathbf{k})|^2.
\end{aligned}
$$

The estimate of the last term in I_3 is slightly more complex. As in Lemma 2.1, for $\mathbf{k}\in S_2$ and $k_1\neq 0$,

$$
\lambda_3 = -\frac{\frac{2k_1^2}{\nu|\mathbf{k}|^4}}{1+\sqrt{1-\frac{4k_1^2}{\nu^2|\mathbf{k}|^6}}} \leqq -\frac{4k_1^2}{3\nu|\mathbf{k}|^4},
$$

and thus,

$$
\begin{aligned}
\sum_{\mathbf{k}\in S_2}\frac{k_1^2k_2^2}{|\mathbf{k}|^4}\,\frac{1}{\nu^2|\mathbf{k}|^4}e^{2\lambda_3 t}\,|\widehat{\Theta_0}(\mathbf{k})|^2 &\leqq \sum_{\mathbf{k}\in S_2}\frac{1}{\nu^2|\mathbf{k}|^4}e^{-\frac{8k_1^2}{3\nu|\mathbf{k}|^4}t}\,|\widehat{\Theta_0}(\mathbf{k})|^2\\
&\leqq \sum_{\mathbf{k}\in S_2}\frac{1}{\nu^2|\mathbf{k}|^4}e^{-\frac{8}{3\nu|\mathbf{k}|^4}t}\,|\widehat{\Theta_0}(\mathbf{k})|^2\\
&\leqq \frac{1}{\nu t}\sum_{\mathbf{k}\in S_2}\frac{t}{\nu|\mathbf{k}|^4}e^{-\frac{4t}{3\nu|\mathbf{k}|^4}}\,|\widehat{\Theta_0}(\mathbf{k})|^2\\
&\leqq \frac{C}{\nu t}\|\Theta_0\|_{L^2}^2,
\end{aligned}
$$

where we have used $k_1\neq 0$ and the simple fact $xe^{-x}\leqq C$ for any $x\geqq 0$. We now turn to I_2 in (3.8). The key is to bound $G_1(t)-e^{\lambda_1 t}$. Again we split the consideration into two cases: $\mathbf{k}\in S_1$ and $\mathbf{k}\in S_2$. We invoke the bounds for G_1 in Lemma 2.1. For $\mathbf{k}\in S_1$,

$$
|G_1(t)| = |(1+\rho t)\,e^{\rho t}| \leqq \left(1+\frac{1}{2}\nu|\mathbf{k}|^2 t\right)\,e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t}.
$$

For $\mathbf{k}\in S_2$,

$$
|G_1(t)| \leqq \frac{4k_1^2}{\nu^2|\mathbf{k}|^6}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t} + 2e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t}.
$$

To bound I_2, we split the summation in I_2 into two pieces and use the bounds above for G_1. We emphasize that the summation does not involve $k_1 = 0$ and $\frac{1}{|k_1|}$ is bounded above:

$$I_2 \leqq C \sum_{\mathbf{k}\in S_1} k_2^2 \,(|G_1(t)|^2 + e^{2\lambda_1 t})|\widehat{V}_0(\mathbf{k})|^2 + C \sum_{\mathbf{k}\in S_2} k_2^2 \,(|G_1(t)|^2 + e^{2\lambda_1 t})|\widehat{V}_0(\mathbf{k})|^2$$
$$\leqq C \sum_{\mathbf{k}\in S_1} k_2^2 \left((1+\frac{1}{2}\nu|\mathbf{k}|^2 t)^2\, e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + e^{-2\nu|\mathbf{k}|^2 t}\right) |\widehat{V}_0(\mathbf{k})|^2$$
$$+ C \sum_{\mathbf{k}\in S_2} k_2^2 \left(\left(\frac{4k_1^2}{\nu^2|\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t} + 2e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t}\right)^2 + e^{-2\nu|\mathbf{k}|^2 t}\right) |\widehat{V}_0(\mathbf{k})|^2.$$

For $V_0 \in L^2(\mathbb{T}^2)$, we further bound I_2 as follows:

$$I_2 \leqq C\, e^{-c_0\nu t}\|V_0\|_{L^2}^2 + C \sum_{\mathbf{k}\in S_2} \left(\frac{4k_1^2 k_2}{\nu^2|\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t}\right)^2 |\widehat{V}_0(\mathbf{k})|^2$$
$$\leqq C\, e^{-c_0\nu t}\|V_0\|_{L^2}^2 + C\, \frac{1}{(\nu t)^{3/2}} \sum_{\mathbf{k}\in S_2} \left(\frac{t^{\frac{3}{4}}}{\nu|\mathbf{k}|^3} e^{-\frac{4}{3\nu|\mathbf{k}|^4}t}\right)^2 |\widehat{V}_0(\mathbf{k})|^2$$
$$\leqq C\, e^{-c_0\nu t}\|V_0\|_{L^2}^2 + C\, \frac{1}{(\nu t)^{3/2}}\, \|V_0\|_{L^2}^2,$$

where again we have used the fact that $xe^{-x} \leqq C$ for all $x \geqq 0$. If we have $\partial_y V_0 \in L^2$ instead of $V_0 \in L^2$, the decay rate in this part can be improved. For any $t > 0$, we have

$$I_2 \leqq C\, e^{-c_0\nu t}\|\partial_y V_0\|_{L^2}^2 + C \sum_{\mathbf{k}\in S_2} \left(\frac{4k_1^2}{\nu^2|\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}t}\right)^2 k_2^2|\widehat{V}_0(\mathbf{k})|^2$$
$$\leqq C\, e^{-c_0\nu t}\|\partial_y V_0\|_{L^2}^2 + C\, \frac{1}{(\nu t)^2} \sum_{\mathbf{k}\in S_2} \left(\frac{t}{\nu|\mathbf{k}|^4} e^{-\frac{4}{3\nu|\mathbf{k}|^4}t}\right)^2 k_2^2|\widehat{V}_0(\mathbf{k})|^2$$
$$\leqq C\, e^{-c_0\nu t}\|\partial_y V_0\|_{L^2}^2 + C\, \frac{1}{(\nu t)^2}\, \|\partial_y V_0\|_{L^2}^2.$$

Combining the bounds for I_1, I_2 and I_3 leads to the desired bound for $\|U(t)\|_{L^2}$ in (1.18) and (1.20). The bound for $\|V(t)\|_{L^2}$ in (1.19) can be similarly obtained. This completes the proof of Theorem 1.3. □

4. Proofs of the Theorems for the Nonlinear System (1.9)

This section proves the three theorems concerning the nonlinear system (1.9).

Proof of Theorem 1.4. Theorem 1.4 is very close to the statement of Theorem 1.2 in [21]. The main difference here is that the assumption on θ_0 is weaker than in Theorem 1.2 in [21]. The weaker setting makes the proof for the uniqueness harder. By adopting the approach of [26,38], we can still prove the uniqueness when $\theta_0 \in L^2$ (no need for $\theta_0 \in L^\infty$). [26,38] introduced the new unknown η satisfying $\Delta\eta = \theta$ and proved the uniqueness by considering the difference $\|\nabla\eta_1 - \nabla\eta_2\|_{L^2}$. This approach still works here and more details can be found in [26,38].

The proof for the large-time behavior, as $t \to \infty$,

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{L^2} \to 0, \quad \|\nabla\mathbf{u}(t)\|_{L^2} \to 0, \quad \|\partial_t\mathbf{u}(t)\|_{L^2} \to 0,\\
&\|\theta(t)\|_{L^2}^2 \to \|\mathbf{u}_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 - 2\nu\int_0^\infty \|\nabla\mathbf{u}(t)\|_{L^2}^2\,\mathrm{d}t,\\
&\|\nabla P(t) - \theta(t)\mathbf{e}_2\|_{H^{-1}} \to 0
\end{aligned}$$

is very similar to the proof of Theorem 1.2 in [21]. We now provide a proof for the global bound on the second-order spatial partials of $\mathbf{u}$, for $t > 0$:

$$\|\Delta\mathbf{u}(t)\|_{L^2} \leqq C\,(\|\theta(t)\|_{L^2} + \|\partial_t\mathbf{u}(t)\|_{L^2} + \|\mathbf{u}(t)\|_{L^2}\,\|\nabla\mathbf{u}(t)\|_{L^2}^2). \tag{4.1}$$

Recall that $\mathbf{u}$ satisfies (1.9). We rewrite the velocity equation in (1.9) as

$$\nu\Delta\mathbf{u} - \nabla\big(\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \Delta^{-1}\partial_y\theta\big) = \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \theta e_2. \tag{4.2}$$

Taking the L^2-norm each side yields

$$\begin{aligned}
&\nu^2\|\Delta\mathbf{u}\|_{L^2}^2 + \|\nabla\big(\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \Delta^{-1}\partial_y\theta\big)\|_{L^2}^2\\
&\quad\leqq C\,\|\partial_t\mathbf{u}\|_{L^2}^2 + C\,\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\theta\|_{L^2}^2,
\end{aligned} \tag{4.3}$$

where, due to $\nabla\cdot\mathbf{u} = 0$, we have used the fact that $\Delta\mathbf{u}$ and $\nabla\big(\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \Delta^{-1}\partial_y\theta\big)$ are perpendicular in L^2, or

$$\int \Delta\mathbf{u}\cdot\nabla\big(\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \Delta^{-1}\partial_y\theta\big)\,\mathrm{d}x = 0.$$

For the nonlinear term on the right-hand side of (4.3), we can show that

$$\begin{aligned}
C\,\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 &\leqq C\,\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\
&\leqq C\,\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\|\Delta\mathbf{u}\|_{L^2}\\
&\leqq \frac{\nu^2}{2}\|\Delta\mathbf{u}\|_{L^2}^2 + C\,\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) leads to (4.1).

In particular, since $\|\theta(t)\|_{L^2} \leqq \|(\mathbf{u}_0, \theta_0)\|_{L^2}$, when $\|(\mathbf{u}_0, \theta_0)\|_{L^2}$ is small, taking into account of the large-time behavior of $\|\mathbf{u}(t)\|_{H^1}$ and $\|\partial_t\mathbf{u}(t)\|_{L^2}$, we conclude from (4.1) that $\|\mathbf{u}\|_{H^2}$ becomes small in large time. This completes the proof of Theorem 1.4. □

We now turn to the proof of Theorem 1.5. We make use of the representation formula derived in Section 2.

Proof of Theorem 1.5. We recall the equation of $\widehat{\theta}(\mathbf{k},t)$ in (2.9),

$$\widehat{\theta}(\mathbf{k},t) = \widehat{\Theta}(\mathbf{k},t) + \int_0^t \Big(- G_2(t-\tau)\ \widehat{N}_2(\mathbf{k},\tau) + G_3(t-\tau)\,\widehat{N}_3(\mathbf{k},\tau)\Big)\,\mathrm{d}\tau, \tag{4.5}$$

where $\widehat{\Theta}(\mathbf{k},t)$ denotes the corresponding linear part, namely

$$\widehat{\Theta}(\mathbf{k},t) = -G_2(t)\,\widehat{v_0}(\mathbf{k}) + G_3(t)\,\widehat{\theta_0}(\mathbf{k}).$$

As shown in Lemma 3.2, for $\mathbf{k}=(k_1,k_2)$ with $k_1\neq 0$,

$$\widehat{\Theta}(\mathbf{k},t)\to 0 \quad \text{as } t\to\infty.$$

We focus on the last two terms in (4.5),

$$I_1 = \int_0^t (-G_2(t-\tau)\ \widehat{N}_2(\mathbf{k},\tau))\,\mathrm{d}\tau, \quad I_2 = \int_0^t G_3(t-\tau)\,\widehat{N}_3(\mathbf{k},\tau\,\mathrm{d}\tau.$$

We recall the bounds for G_2 and G_3 obtained Lemma 2.1. For $\mathbf{k}\in S_1$,

$$\begin{aligned}
\lambda_2 &\leqq -\frac{1}{2}\nu|\mathbf{k}|^2, \quad \lambda_3 \leqq -\frac{1}{4}\nu|\mathbf{k}|^2,\\
|G_2(t)| &\leqq t\,e^{\rho t}, \quad -\frac{1}{2}\nu|\mathbf{k}|^2 \leqq \rho \leqq -\frac{1}{4}\nu|\mathbf{k}|^2;\\
G_3(t) &= e^{\lambda_2 t} - \lambda_2 G_2(t), \quad |G_3(t)| \leqq e^{\lambda_2 t} + |\lambda_2| t e^{bt}.
\end{aligned} \tag{4.6}$$

For $\mathbf{k}\in S_2$,

$$\lambda_2 \leqq -\frac{1}{2}\nu|\mathbf{k}|^2, \quad \lambda_3 = -\frac{\frac{2k_1^2}{\nu|\mathbf{k}|^4}}{1+\sqrt{1-\frac{4k_1^2}{\nu^2|\mathbf{k}|^6}}} \leqq -\frac{4k_1^2}{3\nu|\mathbf{k}|^4},$$

$$|G_2(t)| \leqq \frac{2}{\nu|\mathbf{k}|^2}e^{\lambda_2 t} + \frac{2}{\nu|\mathbf{k}|^2}e^{\lambda_3 t}, \tag{4.7}$$

$$|G_3(t)| \leqq 3e^{\lambda_2 t} + 2e^{\lambda_3 t}. \tag{4.8}$$

Recalling the definitions of N_2 and N_3 in (1.10), we have, for any $|\mathbf{k}|\neq 0$,

$$|\widehat{N}_2| \leqq 2\,|\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(k,t)|, \quad |\widehat{N}_3| \leqq |\widehat{(\mathbf{u}\cdot\nabla)\theta}(k,t)|.$$

Assume $\mathbf{k}=(k_1,k_2)$ with $k_1\neq 0$. We now estimate I_1 and I_2 for $\mathbf{k}\in S_1$. We split the time integral into two parts:

$$\begin{aligned}
|I_1| &\leqq 2\int_0^t (t-\tau)\,e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)}|\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\mathbf{k},\tau)|\,d\tau\\
&= 2\int_0^{\frac{t}{2}} (t-\tau)\,e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)}|\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\mathbf{k},\tau)|\,d\tau
\end{aligned}$$

$$+2\int_{\frac{t}{2}}^{t} (t-\tau)\, e^{-\frac14 \nu |\mathbf{k}|^2 (t-\tau)} |\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\mathbf{k},\tau)|\, d\tau$$
$$:= I_{11} + I_{12}. \tag{4.9}$$

By Hölder's inequality and Poincaré's inequality, for a pure constant C,

$$\begin{aligned}
|I_{11}| &\leqq C\, t\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \int_0^{\frac{t}{2}} \|(\mathbf{u}\cdot\nabla)\mathbf{u}(\tau)\|_{L^1}\, \mathrm{d}\tau \\
&\leqq C\, t\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \int_0^{\frac{t}{2}} \|\mathbf{u}(\tau)\|_{L^2}\, \|\nabla \mathbf{u}(\tau)\|_{L^2}\, \mathrm{d}\tau \\
&\leqq C\, t\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \int_0^{\infty} \|\nabla \mathbf{u}(\tau)\|_{L^2}^2\, \mathrm{d}\tau,
\end{aligned} \tag{4.10}$$

where we have used the simple fact that $\|\widehat{f}(\mathbf{k})\|_{l^\infty} \leqq \|f\|_{L^1}$ with l^∞ denoting the space of bounded sequences. Therefore, $I_{11} \to 0$ as $t \to \infty$, and we have

$$\begin{aligned}
|I_{12}| &\leqq C \int_{\frac{t}{2}}^{t} |\mathbf{k}|\, (t-\tau)\, e^{-\frac14 \nu |\mathbf{k}|^2 (t-\tau)}\, \|\mathbf{u}(\tau)\|_{L^2}^2\, d\tau \\
&\leqq C \sup_{\frac{t}{2}\leqq \tau \leqq t} \|\mathbf{u}(\tau)\|_{L^2}^2 \int_{\frac{t}{2}}^{t} |\mathbf{k}|\, (t-\tau)\, e^{-\frac14 \nu |\mathbf{k}|^2 (t-\tau)}\, \mathrm{d}\tau \\
&\leqq C \sup_{\frac{t}{2}\leqq \tau \leqq t} \|\mathbf{u}(\tau)\|_{L^2}^2\, (\nu |\mathbf{k}|)^{-1} (1 - e^{-\frac18 \nu |\mathbf{k}|^2 t}).
\end{aligned} \tag{4.11}$$

Using the fact that

$$\lim_{t\to\infty} \sup_{\frac{t}{2}\leqq \tau \leqq t} \|\mathbf{u}(\tau)\|_{L^2}^2 = 0,$$

we conclude that $I_{12} \to 0$ as $t \to \infty$. Therefore,

$$I_1 \to 0 \quad \text{as } t \to \infty.$$

I_2 can be similarly estimated. In fact, by the bound for G_3 in (4.6),

$$\begin{aligned}
|I_2| &\leqq \int_0^t (1 + \nu |\mathbf{k}|^2 (t-\tau)) e^{-\frac14 \nu |\mathbf{k}|^2 (t-\tau)}\, |\mathbf{k}| \|\mathbf{u}(\tau)\|_{L^2}\, \|\theta(\tau)\|_{L^2}\, d\tau \\
&\leqq C\, |\mathbf{k}| (1 + \nu |\mathbf{k}|^2 t)\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \int_0^{\frac{t}{2}} \|\mathbf{u}(\tau)\|_{L^2}\, \|\theta(\tau)\|_{L^2}\, \mathrm{d}\tau \\
&\quad + C \sup_{\frac{t}{2}\leqq \tau \leqq t} \|\mathbf{u}(\tau)\|_{L^2}\, \|\theta(\tau)\|_{L^2} \int_{\frac{t}{2}}^{t} (1 + \nu |\mathbf{k}|^2 (t-\tau))\, e^{-\frac18 \nu |\mathbf{k}|^2 (t-\tau)}\, \mathrm{d}\tau \\
&\leqq C\, \|(\mathbf{u}_0, \theta_0)\|_{L^2} |\mathbf{k}| (1 + \nu |\mathbf{k}|^2 t)\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \sqrt{t} \left(\int_0^{\frac{t}{2}} \|\nabla \mathbf{u}\|_{L^2}^2\, \mathrm{d}\tau \right)^{\frac12} \\
&\quad + C\, \|(\mathbf{u}_0, \theta_0)\|_{L^2} \sup_{\frac{t}{2}\leqq \tau \leqq t} \|\mathbf{u}(\tau)\|_{L^2} \left(\frac{C}{\nu |\mathbf{k}|} (1 - e^{-\frac18 \nu |\mathbf{k}|^2 t}) + C\, |\mathbf{k}|\, t\, e^{-\frac18 \nu |\mathbf{k}|^2 t} \right),
\end{aligned}$$

where we have invoked Poincaré's inequality and the global bound

$$\|\theta(t)\|_{L^2} \leqq \|(\mathbf{u}_0, \theta_0)\|_{L^2}.$$

Due to the facts that

$$\sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2} \to 0, \quad 2\nu \int_0^\infty \|\nabla\mathbf{u}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leqq \|(\mathbf{u}_0, \theta_0)\|_{L^2}^2,$$

it is easy to see from the bound for I_2 that, as $t \to \infty$,

$$I_2 \to 0.$$

We now turn to the case $\mathbf{k} \in S_2$ and use the bounds in (4.7) and (4.8) to bound I_1 and I_2. For any $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$ and $\mathbf{k} \in S_2$,

$$\begin{aligned}
|I_1| &\leqq \frac{1}{\nu|\mathbf{k}|^2} \int_0^t \left(e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \right) \|\mathbf{u}(\tau)\cdot\nabla\mathbf{u}(\tau)\|_{L^2} \, \mathrm{d}\tau \\
&\leqq \frac{C}{\nu|\mathbf{k}|^2} \left(e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} + e^{-\frac{2k_1^2}{3\nu|\mathbf{k}|^4}t} \right) \int_0^{\frac{t}{2}} \|\nabla\mathbf{u}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \\
&\quad + \frac{C}{\nu|\mathbf{k}|} \sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2}^2 \int_{\frac{t}{2}}^t \left(e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \right) \mathrm{d}\tau \\
&\leqq \frac{C}{\nu|\mathbf{k}|^2} \left(e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} + e^{-\frac{2k_1^2}{3\nu|\mathbf{k}|^4}t} \right) \int_0^{\frac{t}{2}} \|\nabla\mathbf{u}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \\
&\quad + \frac{C}{\nu|\mathbf{k}|} \sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2}^2 \left(\frac{1}{\nu|\mathbf{k}|^2} + \frac{3\nu|\mathbf{k}|^4}{4k_1^2} \right).
\end{aligned}$$

It is then clear that, as $t \to \infty$,

$$I_1 \to 0.$$

For any $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$ and $\mathbf{k} \in S_2$, the bound for G_3 in (4.8) implies

$$\begin{aligned}
|I_2| &\leqq \int_0^t \left(e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \right) |\mathbf{k}| \|\mathbf{u}(\tau)\|_{L^2} \, \|\theta(\tau)\|_{L^2} \, \mathrm{d}\tau \\
&\leqq |\mathbf{k}| \left(e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} + e^{-\frac{2k_1^2}{3\nu|\mathbf{k}|^4}t} \right) \int_0^{\frac{t}{2}} \|\mathbf{u}(\tau)\|_{L^2} \, \|\theta(\tau)\|_{L^2} \, \mathrm{d}\tau \\
&\quad + \sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2} \|\theta(\tau)\|_{L^2} \int_{\frac{t}{2}}^t \left(e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \right) \mathrm{d}\tau \\
&\leqq |\mathbf{k}| \, \|(\mathbf{u}_0, \theta_0)\|_{L^2} \left(e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} + e^{-\frac{2k_1^2}{3\nu|\mathbf{k}|^4}t} \right) \sqrt{t} \left(\int_0^\infty \|\mathbf{u}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad + C\, |\mathbf{k}| \, \|(\mathbf{u}_0, \theta_0)\|_{L^2} \sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2} \left(\frac{1}{\nu|\mathbf{k}|^2} + \frac{3\nu|\mathbf{k}|^4}{4k_1^2} \right).
\end{aligned}$$

Therefore, as $t \to \infty$,

$$I_2 \to 0.$$

In summary, we have shown in either cases that, as $t \to \infty$, I_1 and I_2 both converge to 0. As a consequence, for any $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$,

$$\widehat{\theta}(\mathbf{k}, t) \quad \to \quad 0$$

as $t \to \infty$. Therefore, for large time $t > 0$, $\theta(x, y, t)$ is mainly determined by

$$S(y,t) = \sum_{k_2} e^{ik_2 y} \widehat{\theta}(0, k_2, t) = \frac{1}{2\pi} \int_{\mathbb{T}} \theta(x, y, t)\, \mathrm{d}x.$$

We derive an equation for $S(y,t)$. Recall from (2.12) that, for $\mathbf{k} = (0, k_2)$,

$$\begin{aligned} \widehat{\theta}(\mathbf{k}, t) &= -\beta G_2(t)\, \widehat{v_0}(\mathbf{k}) + \widehat{\theta_0}(\mathbf{k}) \\ &\quad + \int_0^t \Big(-\beta G_2(t-\tau)\ \widehat{N}_2(\mathbf{k}, \tau) + \widehat{N}_3(\mathbf{k}, \tau) \Big)\, \mathrm{d}\tau. \end{aligned} \tag{4.12}$$

Multiplying each side of (4.12) by $e^{ik_2 y}$ and summing over k_2 yields

$$\begin{aligned} S(y,t) = S(y,0) &- \beta \sum_{k_2} e^{ik_2 y} G_2(t)\, \widehat{v_0}(0, k_2) \\ &- \beta \int_0^t \sum_{k_2} e^{ik_2 y} G_2(t-\tau)\ \widehat{N}_2(0, k_2, \tau)\, \mathrm{d}\tau \\ &+ \int_0^t \sum_{k_2} e^{ik_2 y} \widehat{N}_3(0, k_2, \tau)\, \mathrm{d}\tau. \end{aligned}$$

Recall the definition of N_2 in (1.10):

$$N_2 = -(\mathbf{u} \cdot \nabla) v + \partial_y \Delta^{-1} \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}).$$

We find, by a direct calculation, that the identity holds, for any k_2 and τ, such that

$$\widehat{N}_2(0, k_2, \tau) = 0.$$

Invoking the definitions of G_2 and N_3 and identifying

$$\sum_{k_2} e^{ik_2 y} \widehat{F}(0, k_2) = \frac{1}{2\pi} \int_{\mathbb{T}} F(x, y)\, \mathrm{d}x,$$

we have

$$\begin{aligned} S(y,t) = S(y,0) &- \frac{\beta}{2\pi} (\nu \partial_{yy})^{-1} (e^{\nu t\, \partial_{yy}} - 1) \int_{\mathbb{T}} v_0(x, y)\, \mathrm{d}x \\ &+ \frac{1}{2\pi} \int_{\mathbb{T}} \mathbf{u} \cdot \nabla \theta(x, y, t)\, \mathrm{d}x. \end{aligned} \tag{4.13}$$

Writing $\mathbf{u}\cdot\nabla\theta = u\partial_x\theta + v\partial_y\theta$ and applying the periodic boundary condition, we find

$$\int_{\mathbb{T}} \mathbf{u}\cdot\nabla\theta(x,y,t)\,\mathrm{d}x = \partial_y \int_{\mathbb{T}} v(x,y,t)\,\theta(x,y,t)\,\mathrm{d}x.$$

We introduce the notation

$$\overline{F}(y) = \frac{1}{2\pi}\int_{\mathbb{T}} F(x,y)\,\mathrm{d}x.$$

Then (4.13) becomes

$$S(y,t) = \overline{\theta_0}(y) - \beta(\nu\partial_{yy})^{-1}(e^{\nu t\,\partial_{yy}} - 1)\,\overline{v_0}(y) + \partial_y(\overline{v\theta})(y,t).$$

This completes the proof of Theorem 1.5. □

We now prove Theorem 1.6.

Proof of Theorem 1.6. Let $\mathbf{k} = (k_1, k_2)$ with $k_1 \neq 0$. Taking the l^2-norm of the sequences on each side of (2.7) yields

$$\begin{aligned}
\|\widehat{u}(\mathbf{k},t)\|_{l^2} &\leqq \|\widehat{U}(\mathbf{k},t)\|_{l^2} + \left\| \int_0^t e^{\lambda_1(t-\tau)}\widehat{N}_1(\mathbf{k},\tau)\,\mathrm{d}\tau \right\|_{l^2} \\
&\quad + \left\| \int_0^t \frac{k_2}{k_1}(G_1(t-\tau) - e^{\lambda_1(t-\tau)})\widehat{N}_2(\mathbf{k},\tau)\,\mathrm{d}\tau \right\|_{l^2} \\
&\quad + \left\| \int_0^t \frac{k_1k_2}{|\mathbf{k}|^2} G_2(t-\tau)\widehat{N}_3(\mathbf{k},\tau)\,\mathrm{d}\tau \right\|_{l^2} \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.14}$$

where $\widehat{U}(\mathbf{k},t)$ denotes the linear part,

$$\widehat{U}(\mathbf{k},t) = e^{\lambda_1 t}\,\widehat{u_0}(\mathbf{k}) + \frac{k_2}{k_1}(G_1(t) - e^{\lambda_1 t})\,\widehat{v_0}(\mathbf{k}) + \frac{k_1k_2}{|\mathbf{k}|^2}G_2(t)\,\widehat{\theta_0}(\mathbf{k}).$$

We can directly use the result of Theorem 1.3 to obtain

$$\begin{aligned}
I_1 &= \|U(t)\|_{L^2} \\
&\leqq C\,e^{-c_0\,\nu t}\,\|U_0\|_{L^2} + C\left(e^{-c_0\,\nu t} + \frac{1}{(\nu t)^{3/4}}\right)\|V_0\|_{L^2} \\
&\quad + C\left(e^{-c_0\,\nu t} + \frac{1}{\sqrt{\nu t}}\right)\|\Theta_0\|_{L^2},
\end{aligned}$$

which clearly has the desired decay rate $t^{-\frac{1}{2}}$. To estimate I_2, we split the time integral into two parts:

$$\begin{aligned}
I_2 &\leqq \left\| \int_0^{\frac{t}{2}} e^{\lambda_1(t-\tau)}\widehat{N}_1(\mathbf{k},\tau)\,\mathrm{d}\tau \right\|_{l^2} + \left\| \int_{\frac{t}{2}}^t e^{\lambda_1(t-\tau)}\widehat{N}_1(\mathbf{k},\tau)\,\mathrm{d}\tau \right\|_{l^2} \\
&:= I_{21} + I_{22}.
\end{aligned}$$

By the definition of N_1 in (1.10), we have

$$|\widehat{N_1}(\mathbf{k})| \leqq |\widehat{\mathbf{u}\cdot\nabla u}(\mathbf{k},t)| + \frac{|k\otimes k|}{|\mathbf{k}|^2}|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}(\mathbf{k},t)|$$
$$\leqq 2|\mathbf{k}|\,|\widehat{\mathbf{u}\otimes\mathbf{u}}(\mathbf{k},t)|.$$

Therefore,

$$|I_{21}| \leqq \int_0^{\frac{t}{2}} \|e^{\lambda_1(t-\tau)}\widehat{N}_1(\mathbf{k},\tau)\|_{l^2}\,\mathrm{d}\tau$$
$$\leqq \int_0^{\frac{t}{2}} \||\mathbf{k}|e^{-\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\widehat{\mathbf{u}\otimes\mathbf{u}}(\mathbf{k},\tau)\|_{l^\infty}\,d\tau.$$

Bounding the l^2-norm in terms of its corresponding integral, we have

$$\begin{aligned}\||\mathbf{k}|e^{-\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2} &= \left(\sum_{k\neq 0}|\mathbf{k}|^2 e^{-2\nu|\mathbf{k}|^2(t-\tau)}\right)^{\frac{1}{2}}\\ &\leqq \left(\int_{\mathbb{R}^2}|x|^2 e^{-2\nu|x|^2(t-\tau)}\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &= \left(2\pi\int_0^\infty r^2\,e^{-2\nu r^2(t-\tau)}\,r dr\right)^{\frac{1}{2}}\\ &= C\,(t-\tau)^{-1}.\end{aligned} \tag{4.15}$$

In addition,

$$\|\widehat{\mathbf{u}\otimes\mathbf{u}}((\mathbf{k},\tau)\|_{l^\infty} \leqq \|\mathbf{u}\otimes\mathbf{u}\|_{L^1} \leqq \|\mathbf{u}\|_{L^2}^2.$$

Therefore,

$$|I_{21}| \leqq C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\,\|\mathbf{u}(\tau)\|_{L^2}^2\,d\tau \leqq C\,t^{-1},$$

where we have used the fact that

$$\int_0^\infty \|\mathbf{u}(\tau)\|_{L^2}^2\,d\tau < \infty.$$

To bound I_{22}, we fix $\varepsilon > 0$ (a positive small parameter) and proceed as in the estimate of I_{21},

$$\begin{aligned}|I_{22}| &\leqq \int_{\frac{t}{2}}^t \|e^{\lambda_1(t-\tau)}\widehat{N_1}(\mathbf{k},\tau)\|_{l^2}\,\mathrm{d}\tau\\ &\leqq \int_{\frac{t}{2}}^t \||\mathbf{k}|^{1-2\varepsilon}e^{-\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\widehat{\Lambda^{2\varepsilon}(\mathbf{u}\otimes\mathbf{u})}(\mathbf{k},t)\|_{l^\infty}\,\mathrm{d}\tau\\ &\leqq \int_{\frac{t}{2}}^t (t-\tau)^{-1+\varepsilon}\,\|\Lambda^{2\varepsilon}(\mathbf{u}\otimes\mathbf{u})\|_{L^1}\,\mathrm{d}\tau\end{aligned}$$

$$\begin{aligned}
&\leqq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon} \, \|\Lambda^{2\varepsilon}\mathbf{u}\|_{L^2} \, \|\mathbf{u}\|_{L^2} \, \mathrm{d}\tau \\
&\leqq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon} \, \|\mathbf{u}\|_{L^2}^{2-2\varepsilon} \, \|\nabla\mathbf{u}\|_{L^2}^{2\varepsilon} \, d\tau \\
&\leqq C \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon} \|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon} \, \|\nabla\mathbf{u}(\tau)\|_{L^2}^{2\varepsilon} \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon} \tau^{-\frac{1}{2}} \, \tau^{-\varepsilon} \, d\tau \\
&\leqq C \, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon} \|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon} \, \|\nabla\mathbf{u}(\tau)\|_{L^2}^{2\varepsilon},
\end{aligned}$$

where

$$M(t) = t^{\frac{1}{2}} \, \|\mathbf{u}(t)\|_{L^2}. \tag{4.16}$$

Here we have used the fact that, for a constant $C > 0$,

$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon} \tau^{-\frac{1}{2}-\varepsilon} \, \mathrm{d}\tau = C \, t^{-\frac{1}{2}}.$$

We now turn to I_3. We again split the time integral into two parts:

$$\begin{aligned}
I_3 \leqq & \left\| \int_0^{\frac{t}{2}} \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1(t-\tau)}) \widehat{N}_2(\mathbf{k},\tau) \, \mathrm{d}\tau \right\|_{l^2} \\
& + \left\| \int_{\frac{t}{2}}^{t} \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1(t-\tau)}) \widehat{N}_2(\mathbf{k},\tau) \, \mathrm{d}\tau \right\|_{l^2} \\
:= & \; I_{31} + I_{32}.
\end{aligned}$$

Clearly,

$$|\widehat{N}_2(\mathbf{k},\tau)| \leqq 2|\mathbf{k}| \, |\widehat{(\mathbf{u}\otimes\mathbf{u})}(\mathbf{k},\tau)|.$$

Therefore,

$$I_{31} \leqq \int_0^{\frac{t}{2}} \||\mathbf{k}| \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1(t-\tau)})\|_{l^2} \|\widehat{(\mathbf{u}\otimes\mathbf{u})}(\mathbf{k},\tau)\|_{l^\infty} \, \mathrm{d}\tau. \tag{4.17}$$

As pointed out in Lemma 2.1, $G_1(\mathbf{k}, t)$ obeys different bounds for $\mathbf{k}$ in different ranges. More precisely,

$$\begin{aligned}
|G_1(\mathbf{k},t)| &\leqq e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + \frac{1}{2}\nu|\mathbf{k}|^2 t \, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} \quad \text{if } \mathbf{k} \in S_1, \\
|G_1(\mathbf{k},t)| &\leqq \frac{4k_1^2}{\nu^2|\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4} t} + 2e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} \quad \text{if } \mathbf{k} \in S_2,
\end{aligned}$$

where S_1 and S_2 are defined by (2.17) and (2.18), namely

$$S_1 := \left\{ \mathbf{k} \in \mathbb{Z}^2 : \; k_1^2 \geqq \frac{3\nu^2}{16\beta} |\mathbf{k}|^6 \quad \text{or} \quad \sqrt{1 - \frac{4\beta k_1^2}{\nu^2|\mathbf{k}|^6}} \leqq \frac{1}{2} \right\},$$

$$S_2 := \left\{ \mathbf{k} \in \mathbb{Z}^2 : \; k_1^2 < \frac{3\nu^2}{16\beta} |\mathbf{k}|^6 \quad \text{or} \quad \sqrt{1 - \frac{4\beta k_1^2}{\nu^2 |\mathbf{k}|^6}} > \frac{1}{2} \right\}.$$

Correspondingly, $\| |\mathbf{k}| \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1(t-\tau)}) \|_{l^2}$ is split into two parts

$$\left\| |\mathbf{k}| \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1(t-\tau)}) \right\|_{l^2} \leqq I_{311} + I_{312}, \tag{4.18}$$

where

$$I_{311} := \left(\sum_{\mathbf{k} \in S_1} |\mathbf{k}|^2 \frac{k_2^2}{k_1^2} |G_1(t-\tau) - e^{\lambda_1(t-\tau)}|^2 \right)^{\frac{1}{2}},$$

$$I_{312} := \left(\sum_{\mathbf{k} \in S_2} |\mathbf{k}|^2 \frac{k_2^2}{k_1^2} |G_1(t-\tau) - e^{\lambda_1(t-\tau)}|^2 \right)^{\frac{1}{2}}.$$

We note that $k_1 \neq 0$ in the summations above. As we explained in Section 2, $I_3 = 0$ when $k_1 = 0$. By the definition of S_1 in (2.17), $\mathbf{k} \in S_1$ implies

$$k_1^2 \geqq \frac{3\nu^2}{16\beta} |\mathbf{k}|^6,$$

which further yields, for any $k_1 \neq 0$ and a constant C (independent of $\mathbf{k}$), that

$$\left| \frac{k_2}{k_1} \right| \leqq C.$$

Invoking the bound for $G_1(\mathbf{k}, \tau)$ in the case $\mathbf{k} \in S_1$, we find

$$I_{311} \leqq C \left(\sum_{\mathbf{k} \in S_1} |\mathbf{k}|^2 \left(1 + \frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)\right)^2 e^{-\frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)} \right)^{\frac{1}{2}}.$$

We further bound I_{311} as in (4.15) to obtain

$$I_{311} \leqq C\,(t-\tau)^{-1}. \tag{4.19}$$

To bound I_{312}, we invoke the bound for $G_1(\mathbf{k}, \tau)$ in the case $\mathbf{k} \in S_2$ and use the facts $\frac{1}{k_1^2} \leqq 1$ and $|k_2| \leqq |\mathbf{k}|$ to obtain

$$\begin{aligned} I_{312} &\leqq \left(\sum_{\mathbf{k} \in S_2} |\mathbf{k}|^4 \left(2e^{-\frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)} + \frac{4k_1^2}{\nu^2 |\mathbf{k}|^6} e^{-\frac{4k_1^2}{3\nu |\mathbf{k}|^4}(t-\tau)} \right)^2 \right)^{\frac{1}{2}} \\ &\leqq I_{3121} + I_{3122}, \end{aligned} \tag{4.20}$$

where

$$I_{3121} := C \left(\sum_{\mathbf{k} \in S_2} |\mathbf{k}|^4 \, e^{-\frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)} \right)^{\frac{1}{2}},$$

$$I_{3122} := C \left(\sum_{\mathbf{k} \in S_2} \frac{k_1^4}{|\mathbf{k}|^8} \, e^{-\frac{8k_1^2}{3\nu |\mathbf{k}|^4} (t-\tau)} \right)^{\frac{1}{2}}.$$

Clearly, I_{3121} can be similarly estimated as I_{311} and

$$I_{3121} \leqq C\,(t-\tau)^{-\frac{3}{2}}. \tag{4.21}$$

Estimating I_{3122} is slightly more complex. Noting that the summation is for $|\mathbf{k}| \geqq 1$, we can bound it by an integral

$$I_{3122}^2 \leqq C \int_{|\mathbf{x}| \geqq 1} \frac{x^4}{|\mathbf{x}|^8} e^{-8 \frac{x^2}{|\mathbf{x}|^4} (t-\tau)} \, \mathrm{d}\mathbf{x}.$$

Using polar coordinates and then changing variables, we have

$$\begin{aligned} I_{3122}^2 &\leqq C \int_0^{2\pi} \int_1^{\infty} \frac{1}{r^4} \cos^4\theta \, e^{-8 \frac{1}{r^2} \cos^2\theta \, (t-\tau)} \, r dr d\theta \\ &= C \int_0^{2\pi} \int_0^1 \rho \cos^4\theta \, e^{-8\rho^2 \cos^2\theta \, (t-\tau)} \, d\rho d\theta. \end{aligned}$$

To further bound this integral, we convert it back into Cartesian coordinates as follows:

$$\begin{aligned} I_{3122}^2 &\leqq C \int_{|\mathbf{x}| \leqq 1} \frac{x^4}{|\mathbf{x}|^4} \, e^{-8x^2 \, (t-\tau)} \, \mathrm{d}\mathbf{x} \\ &= C \int_{-1}^{1} x^4 \, e^{-8x^2 \, (t-\tau)} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{-2} \, \mathrm{d}y \mathrm{d}x \\ &= 2C \int_{-1}^{1} x^4 \, e^{-8x^2 \, (t-\tau)} \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{-2} \, \mathrm{d}y \mathrm{d}x \\ &= C \int_{-1}^{1} x^4 \, e^{-8x^2 \, (t-\tau)} \left(\frac{1}{x^3} \arctan \frac{\sqrt{1-x^2}}{x} + \frac{\sqrt{1-x^2}}{x^2} \right) \mathrm{d}x \end{aligned}$$

Using the basic facts that

$$-\frac{\pi}{2} < \arctan \frac{\sqrt{1-x^2}}{x} < \frac{\pi}{2}, \quad \sqrt{1-x^2} \leqq 1,$$

we find that

$$I_{3122}^2 \leqq C \int_{-1}^{1} |x|\, e^{-8x^2\,(t-\tau)} \mathrm{d}x + C \int_{-1}^{1} x^2\, e^{-8x^2\,(t-\tau)} \mathrm{d}x$$
$$\leqq C\,(t-\tau)^{-1} + C\,(t-\tau)^{-\frac{3}{2}}. \tag{4.22}$$

(4.21) and (4.22) together imply

$$I_{312} \leqq C\,(t-\tau)^{-\frac{1}{2}}. \tag{4.23}$$

Combining (4.17), (4.18), (4.19) and (4.23), we obtain

$$I_{31} \leqq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}}\, \|\mathbf{u}\|_{L^2}^2\, \mathrm{d}\tau \leqq C\, t^{-\frac{1}{2}}.$$

We now turn to I_{32}. We split the l^2-norm into two parts:

$$I_{32} \leqq I_{321} + I_{322},$$

where I_{321} contains the summation over $\mathbf{k} \in S_1$ and I_{322} over $\mathbf{k} \in S_2$, namely

$$I_{321} := \int_{\frac{t}{2}}^{t} \left\| \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1 (t-\tau)}) \widehat{N_2}(\mathbf{k},\tau) \right\|_{l^2(S_1)} d\tau,$$
$$I_{322} := \left\| \int_{\frac{t}{2}}^{t} \frac{k_2}{k_1} (G_1(t-\tau) - e^{\lambda_1 (t-\tau)}) \widehat{N_2}(\mathbf{k},\tau)\, d\tau \right\|_{l^2(S_2)}.$$

We use the fact that, for $\mathbf{k} \in S_1$,

$$\left| \frac{k_2}{k_1} \right| \leqq C.$$

As in the estimate of I_{22},

$$I_{321} \leqq \int_{\frac{t}{2}}^{t} \left(\sum_{\mathbf{k}\in S_1} |\mathbf{k}|^{2-4\varepsilon} \left(1 + \frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)\right)^2 e^{-\frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)} \right)^{\frac{1}{2}} \|\Lambda^{2\varepsilon} (\mathbf{u}\otimes\mathbf{u})\|_{L^1}\, \mathrm{d}\tau.$$

As in the estimate of I_{311} (see (4.19)), we have

$$\left(\sum_{\mathbf{k}\in S_1} |\mathbf{k}|^{2-4\varepsilon} \left(1 + \frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)\right)^2 e^{-\frac{1}{2}\nu |\mathbf{k}|^2 (t-\tau)} \right)^{\frac{1}{2}} \leqq C\,(t-\tau)^{-1+\varepsilon}.$$

Therefore,

$$I_{321} \leqq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon}\, \|\Lambda^{2\varepsilon} (\mathbf{u}\otimes\mathbf{u})\|_{L^1}\, \mathrm{d}\tau$$
$$\leqq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon}\, \|\Lambda^{2\varepsilon} \mathbf{u}\|_{L^2}\, \|\mathbf{u}\|_{L^2}\, \mathrm{d}\tau$$
$$\leqq C \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\varepsilon}\, \|\mathbf{u}\|_{L^2}^{2-2\varepsilon}\, \|\nabla\mathbf{u}\|_{L^2}^{2\varepsilon}\, d\tau$$

$$\leqq C \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon}\|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon}\,\|\nabla\mathbf{u}(\tau)\|_{L^2}^{2\varepsilon}\int_{\frac{t}{2}}^{t}(t-\tau)^{-1+\varepsilon}\tau^{-\frac{1}{2}}\,\tau^{-\varepsilon}\,d\tau$$
$$\leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon}\|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon}\,\|\nabla\mathbf{u}(\tau)\|_{L^2}^{2\varepsilon}.$$

I_{322} is estimated differently from I_{321}. For $\mathbf{k}\in S_2$, we use the simple fact $|k_1|\geqq 1$ due to $k_1\neq 0$, and thus

$$\left|\frac{k_2}{k_1}\right| \leqq |k_2| \leqq |\mathbf{k}|.$$

In addition, we use the bound

$$|\widehat{N}_2(\mathbf{k},\tau)| \leqq 2|\mathbf{k}|\,|\widehat{(\mathbf{u}\otimes\mathbf{u})}(\mathbf{k},\tau)|.$$

Then I_{322} is bounded by

$$I_{322} \leqq \int_{\frac{t}{2}}^{t}\left\| |\mathbf{k}|^{2-2\varepsilon}\left(2e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + \frac{4k_1^2}{\nu^2|\mathbf{k}|^6}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\right)\right\|_{l^\infty}\|\Lambda^{2\varepsilon}(\mathbf{u}\otimes\mathbf{u})\|_{L^2}\,\mathrm{d}\tau.$$

It is clear that

$$\left\| |\mathbf{k}|^{2-2\varepsilon}\left(2e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} + \frac{4k_1^2}{\nu^2|\mathbf{k}|^6}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\right)\right\|_{l^\infty} \leqq C(t-\tau)^{-1+\varepsilon}.$$

By Hölder's inequality and Sobolev's inequality,

$$\|\Lambda^{2\varepsilon}(\mathbf{u}\otimes\mathbf{u})\|_{L^2} \leqq C\,\|\mathbf{u}\|_{L^2}^{1-\varepsilon}\,\|\nabla\mathbf{u}\|_{L^2}^{1+\varepsilon}.$$

Therefore,

$$I_{322} \leqq C\int_{\frac{t}{2}}^{t}(t-\tau)^{-1+\varepsilon}\,\|\mathbf{u}\|_{L^2}^{1-\varepsilon}\,\|\nabla\mathbf{u}\|_{L^2}^{1+\varepsilon}\,\mathrm{d}\tau$$
$$\leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M^{1-\varepsilon}(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\frac{3}{2}\varepsilon}\|\nabla\mathbf{u}(\tau)\|_{L^2}^{1+\varepsilon}.$$

Therefore,

$$I_{32} \leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon}\|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon}\,\|\nabla\mathbf{u}(\tau)\|_{L^2}^{2\varepsilon}$$
$$+C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M^{1-\varepsilon}(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\frac{3}{2}\varepsilon}\|\nabla\mathbf{u}(\tau)\|_{L^2}^{1+\varepsilon}.$$

To estimate I_4, we split it into four parts,

$$I_4 = I_{41} + I_{42} + I_{43} + I_{44},$$

where

$$I_{41} = \int_0^{\frac{t}{2}}\left\|\frac{k_1k_2}{|\mathbf{k}|^2}G_2(t-\tau)\widehat{N}_3(\mathbf{k},\tau)\right\|_{l^2(S_1)}\mathrm{d}\tau,$$

$$I_{42} = \int_0^{\frac{t}{2}} \left\| \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t-\tau) \widehat{N}_3(\mathbf{k},\tau) \right\|_{l^2(S_2)} \mathrm{d}\tau,$$
$$I_{43} = \int_{\frac{t}{2}}^{t} \left\| \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t-\tau) \widehat{N}_3(\mathbf{k},\tau) \right\|_{l^2(S_1)} \mathrm{d}\tau,$$
$$I_{44} = \int_{\frac{t}{2}}^{t} \left\| \frac{k_1 k_2}{|\mathbf{k}|^2} G_2(t-\tau) \widehat{N}_3(\mathbf{k},\tau) \right\|_{l^2(S_2)} \mathrm{d}\tau.$$

We recall that G_2 obeys the following bounds, according to Lemma 2.1,

$$|G_2(t)| \leqq t\, e^{-\frac{1}{4}\nu|\mathbf{k}|^2 t} \quad \text{if } \mathbf{k} \in S_1, \tag{4.24}$$
$$|G_2(t)| \leqq \frac{2}{\nu|\mathbf{k}|^2} e^{-\frac{1}{2}\nu|\mathbf{k}|^2 t} + \frac{2}{\nu|\mathbf{k}|^2} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4} t} \quad \text{if } \mathbf{k} \in S_2. \tag{4.25}$$

Applying the bound for G_2 and invoking the bound for $\widehat{N_3}$,

$$|\widehat{N_3}| \leqq |\mathbf{k}||\widehat{\mathbf{u}\theta}(\mathbf{k},\tau)|, \tag{4.26}$$

we have

$$\begin{aligned} |I_{41}| &\leqq \int_0^{\frac{t}{2}} \big\| |\mathbf{k}| G_2(t-\tau) |\widehat{\mathbf{u}\theta}(\mathbf{k},\tau)| \big\|_{l^2(S_1)} \mathrm{d}\tau \\ &\leqq \int_0^{\frac{t}{2}} \left\| |\mathbf{k}|(t-\tau) e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)} \right\|_{l^2(S_1)} \| |\widehat{\mathbf{u}\theta}(\mathbf{k},\tau)| \|_{l^\infty} \mathrm{d}\tau. \end{aligned}$$

By the simple fact that $|\mathbf{k}| \geqq 1$ for $\mathbf{k} \neq 0$,

$$\begin{aligned} |I_{41}| &\leqq \int_0^{\frac{t}{2}} \left\| |\mathbf{k}|^4 (t-\tau) e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)} | \right\|_{l^2} \|\mathbf{u}(\tau)\|_{L^2} \|\theta(\tau)\|_{L^2} \mathrm{d}\tau \\ &\leqq \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}} \|\mathbf{u}(\tau)\|_{L^2} \|\theta(\tau)\|_{L^2} \mathrm{d}\tau \\ &\leqq C\, t^{-1} \|(\mathbf{u}_0, \theta_0)\|_{L^2} \left(\int_0^{\frac{t}{2}} \|\mathbf{u}(\tau)\|_{L^2}^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \leqq C\, t^{-1}. \end{aligned}$$

To bound I_{42}, we obtain by applying (4.26) to bound $\widehat{N_3}$ and (4.25) to bound G_2

$$\begin{aligned} |I_{42}| &\leqq C \int_0^{\frac{t}{2}} \left\| \frac{1}{|\mathbf{k}|} e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)} \right\|_{l^2(S_2)} \| |\widehat{\mathbf{u}\theta}(\mathbf{k},\tau)| \|_{l^\infty(S_2)} \mathrm{d}\tau \\ &\quad + C \int_0^{\frac{t}{2}} \left\| \frac{k_1 k_2}{|\mathbf{k}|^4} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},\tau) \right\|_{l^2(S_2)} \mathrm{d}\tau \\ &:= I_{421} + I_{422}. \end{aligned}$$

To bound I_{421}, we again use the simple fact that $|\mathbf{k}| \geqq 1$ for $\mathbf{k} \neq 0$ to obtain

$$\begin{aligned}
I_{421} &\leqq C \int_0^{\frac{t}{2}} \||\mathbf{k}|^2 e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2} \|\mathbf{u}(\tau)\|_{L^2} \|\theta(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\leqq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}} \|\mathbf{u}(\tau)\|_{L^2} \|\theta(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\leqq C\, t^{-1} \|(\mathbf{u}_0,\theta_0)\|_{L^2} \left(\int_0^{\frac{t}{2}} \|\mathbf{u}(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

For $k_1 = 0$ or $k_2 = 0$, we have $I_{422} = 0$. It suffices to consider the case when $k_1 \neq 0$ and $k_2 \neq 0$. Recall that $S(y,t)$ denotes the horizontal average of θ. We write

$$\widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},t) = \mathbf{u}\cdot\widehat{\nabla(\theta-S)}(\mathbf{k},t) + \widehat{v\partial_y S}(\mathbf{k},t).$$

In addition,

$$\begin{aligned}
\widehat{v\partial_y S}(\mathbf{k},t) &= \sum_{k_2'+k_2''=k_2} \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t) \\
&= \sum_{|k_2'|\geqq|k_2''|} \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t) + \sum_{|k_2'|<|k_2''|} \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t).
\end{aligned}$$

For $|k_2'| \geqq |k_2''|$, we have $|k_2| \leqq |k_2'| + |k_2''| \leqq 2|k_2'|$ and

$$|\mathbf{k}| = \sqrt{k_1^2 + k_2^2} \leqq 2\sqrt{k_1^2 + (k_2')^2}.$$

Therefore,

$$\begin{aligned}
&\sum_{|k_2'|\geqq|k_2''|} \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t) \\
&= \sum_{|k_2'|\geqq|k_2''|} \frac{1}{\sqrt{k_1^2+(k_2')^2}} \sqrt{k_1^2+(k_2')^2}\, \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t) \\
&\leqq \frac{2}{|\mathbf{k}|} \sum_{|k_2'|\geqq|k_2''|} |\widehat{|\nabla| v}(k_1,k_2',t)\, \widehat{\partial_y S}(k_2'',t)|.
\end{aligned}$$

For $|k_2'| < |k_2''|$, we have $|k_2| \leqq |k_2'| + |k_2''| \leqq 2|k_2''|$. Thus,

$$\begin{aligned}
&\sum_{|k_2'|<|k_2''|} \widehat{v}(k_1,k_2',t)\, k_2''\widehat{S}(k_2'',t) \\
&= \sum_{|k_2'|<|k_2''|} \frac{1}{|k_2''|} \widehat{v}(k_1,k_2',t)\, (k_2'')^2\widehat{S}(k_2'',t) \\
&\leqq \frac{2}{|k_2|} \sum_{|k_2'|<|k_2''|} |\widehat{v}(k_1,k_2',t)|\, |\widehat{\partial_{yy} S}(k_2'',t)|.
\end{aligned}$$

Thus we have written $\widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},t)$ into three pieces. Correspondingly the estimate of I_{422} is split into three parts I_{4221}, I_{4222} and I_{4223}. To bound the first part, we use the simple fact that, for $|k_1| \geqq 1$ and $|k_2| \leqq |\mathbf{k}|$, we have

$$\left|\frac{k_1 k_2}{|\mathbf{k}|^4}\right| \leqq C\left(\frac{4k_1^2}{3\nu|\mathbf{k}|^4}\right)^{\frac{3}{4}}. \tag{4.27}$$

Thus,

$$\begin{aligned} I_{4221} &\leqq C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{4}}\left\|\left(\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)\right)^{\frac{3}{4}} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\right\|_{l^2} \\ &\quad\times\|\mathbf{u}\cdot\widehat{\nabla(\theta-}S)(\mathbf{k},\tau)\|_{l^\infty}\,\mathrm{d}\tau \\ &\leqq C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{4}}\,\|\mathbf{u}\cdot\widehat{\nabla(\theta-}S)(\mathbf{k},\tau)\|_{l^\infty}\,\mathrm{d}\tau. \end{aligned}$$

By Young's inequality for sequence convolutions,

$$I_{4221} \leqq C\,t^{-\frac{1}{2}}\sup_{0\leqq\tau\leqq\frac{t}{2}}\tau^{\frac{1}{4}}\|\nabla(\theta-S)(\tau)\|_{L^2},$$

where we have used the simple fact that

$$\int_0^{\frac{t}{2}}\|\mathbf{u}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqq C.$$

Now we bound I_{4222}:

$$I_{4222} \leqq C\int_0^{\frac{t}{2}}\left\|\frac{k_1k_2}{|\mathbf{k}|^4}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\frac{2}{|\mathbf{k}|}\sum_{|k_2'|\geqq|k_2''|}|\widehat{|\nabla|v}(k_1,k_2',\tau)\,\widehat{\partial_y S}(k_2'',\tau)|\right\|_{l^2} d\tau.$$

Clearly, for $k_1 \neq 0$,

$$\frac{k_1k_2}{|\mathbf{k}|^4}e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\frac{2}{|\mathbf{k}|} \leqq (t-\tau)^{-1}\left(\frac{k_1^2}{|\mathbf{k}|^4}(t-\tau)\,e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\right).$$

Therefore,

$$\begin{aligned} I_{4222} &\leqq C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\left\|\sum_{|k_2'|\geqq|k_2''|}|\widehat{|\nabla|v}(k_1,k_2',\tau)\,\widehat{\partial_y S}(k_2'',\tau)|\right\|_{l^\infty} d\tau \\ &= C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\left\|\widehat{\nabla v\,\partial_y S}(\mathbf{k},\tau)\right\|_{l^\infty} d\tau \\ &= C\,t^{-\frac{1}{2}}\sup_{0\leqq\tau\leqq\frac{t}{2}}\|\partial_y S(\tau)\|_{L^2}, \end{aligned}$$

We now bound I_{4223}:

$$I_{4223} \leqq C \int_0^{\frac{t}{2}} \left\| \frac{k_1 k_2}{|\mathbf{k}|^4} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)} \frac{2}{|k_2|} \sum_{|k_2'|<|k_2''|} |\widehat{v}(k_1, k_2', t)|\, |\widehat{\partial_{yy} S}(k_2'', t)| \right\|_{l^2} d\tau.$$

The process is similar to that for I_{4222}; in fact,

$$\begin{aligned} I_{4223} &\leqq C \int_0^{\frac{t}{2}} (t-\tau)^{-1} \left\| \widehat{v\, \partial_{yy} S}(\mathbf{k}, \tau) \right\|_{l^\infty} d\tau \\ &\leqq C\, t^{-\frac{1}{2}} \sup_{0 \leqq \tau \leqq \frac{t}{2}} \|\partial_{yy} S(\tau)\|_{L^2}. \end{aligned}$$

In summary, we have obtained the bound for I_{42}:

$$|I_{422}| \leqq C\, t^{-\frac{1}{2}} \sup_{0 \leqq \tau \leqq \frac{t}{2}} \left(\|\partial_y S(\tau)\|_{L^2} + \|\partial_{yy} S(\tau)\|_{L^2} + \tau^{\frac{1}{4}} \|\nabla(\theta - S)(\tau)\|_{L^2} \right).$$

To estimate I_{43}, we recall the bound (4.24) for $G_2(t)$ with $\mathbf{k} \in S_1$ to obtain

$$I_{43} \leqq C \int_{\frac{t}{2}}^{t} \|(t-\tau) e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)} \mathbf{k} \cdot \widehat{\mathbf{u}\,\theta}(\mathbf{k}, \tau)\|_{l^2}\, \mathrm{d}\tau.$$

We use the equation of v to write

$$\theta = \partial_t v + \mathbf{u} \cdot \nabla v + \partial_y p - \nu \Delta v.$$

Then

$$\mathbf{k} \cdot \widehat{\mathbf{u}\,\theta} = \mathbf{k} \cdot \widehat{\mathbf{u}\partial_t v} + \mathbf{k} \cdot \mathbf{u}\widehat{(\mathbf{u} \cdot \nabla v)} + \mathbf{k} \cdot \widehat{\mathbf{u}\,\partial_y p} - \nu \mathbf{k} \cdot \widehat{\mathbf{u}\Delta v}.$$

We further write $\widehat{\mathbf{u}\partial_y p}$ as

$$\begin{aligned} \mathbf{k} \cdot \widehat{\mathbf{u}\,\partial_y p}(k, t) &= \mathbf{k} \cdot \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \widehat{\mathbf{u}}(k_1', k_2', t)\, k_2'' \widehat{p}(k_1'', k_2'', t) \\ &= \mathbf{k} \cdot \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \widehat{\mathbf{u}}(k_1', k_2', t)\, \frac{k_2''}{|\mathbf{k}''|^2} |\mathbf{k}''|^2 \widehat{p}(k_1'', k_2'', t). \end{aligned}$$

Thus

$$\left| \mathbf{k} \cdot \widehat{\mathbf{u}\,\partial_y p}(k, t) \right| \leqq C\, |\mathbf{k}|^3\, |\mathbf{u} \cdot \widehat{(-\Delta)^{-1}} \partial_y p(\mathbf{k}, t)|.$$

Similarly,

$$\mathbf{k} \cdot \widehat{\mathbf{u}\,\Delta v} \leqq C\, |\mathbf{k}|^2\, |\widehat{\mathbf{u}\,\nabla \mathbf{u}}(\mathbf{k}, t)|.$$

Therefore,

$$\begin{aligned} \|\widehat{\mathbf{u} \cdot \nabla\theta}(\mathbf{k}, \tau)\|_{l^\infty} \leqq\; & |\mathbf{k}|\, \|\mathbf{u}\|_{L^2}\, \|\partial_t v\|_{L^2} + |\mathbf{k}|\, \|\mathbf{u}\|_{L^4}^2\, \|\nabla v\|_{L^2} \\ & + C\, |\mathbf{k}|^3\, \|\mathbf{u}\|_{L^2}\, \|(-\Delta)^{-1} \partial_y p\|_{L^2} + C\, |\mathbf{k}|^2\, \|\mathbf{u}\|_{L^2}\, \|\nabla \mathbf{u}\|_{L^2} \end{aligned}$$

$$\leqq C\,|\mathbf{k}|^3\,\|\mathbf{u}\|_{L^2}\Big(\|\partial_t v\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^2 + \|(-\Delta)^{-1}\partial_y p\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\Big) = C\,|\mathbf{k}|^3\,\|\mathbf{u}\|_{L^2}\,A(t),$$

where, for notational convenience, we have written

$$A(t) := \|\partial_t v\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^2+\|(-\Delta)^{-1}\partial_y p\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}. \tag{4.28}$$

According to Theorem 1.4, as $t\to\infty$,

$$\|\partial_x p\|_{H^{-1}} \quad\to\quad 0 \quad\text{or}\quad \sum_{\mathbf{k}}\frac{k_1^2}{|\mathbf{k}|^2}|\widehat{p}(\mathbf{k},t)|^2 \quad\to\quad 0.$$

Since

$$\|(-\Delta)^{-1}\partial_y p\|_{L^2}^2=\sum_{\mathbf{k}}\frac{k_2^2}{|\mathbf{k}|^4}|\widehat{p}(\mathbf{k},t)|^2\leqq\sum_{\mathbf{k}}\frac{1}{|\mathbf{k}|^2}|\widehat{p}(\mathbf{k},t)|^2\leqq\sum_{\mathbf{k}}\frac{k_1^2}{|\mathbf{k}|^2}|\widehat{p}(\mathbf{k},t)|^2,$$

we have, as $t\to\infty$,

$$\|(-\Delta)^{-1}\partial_y p\|_{L^2}\quad\to\quad 0.$$

Therefore, as $t\to\infty$,

$$A(t)\quad\to\quad 0.$$

We are now ready to estimate I_{43}:

$$\begin{aligned}
I_{43} &\leqq C\int_{\frac{t}{2}}^{t}\|(t-\tau)e^{-\frac14\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},\tau)\|_{l^\infty}\,\mathrm{d}\tau\\
&\leqq C\int_{\frac{t}{2}}^{t}\||\mathbf{k}|^3(t-\tau)e^{-\frac14\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\mathbf{u}(\tau)\|_{L^2}\,A(\tau)\,\mathrm{d}\tau\\
&= C\int_{\frac{t}{2}}^{t-\delta}\||\mathbf{k}|^3(t-\tau)e^{-\frac14\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\mathbf{u}(\tau)\|_{L^2}\,A(\tau)\,\mathrm{d}\tau\\
&\quad+C\int_{t-\delta}^{t}\||\mathbf{k}|^3(t-\tau)e^{-\frac14\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\mathbf{u}(\tau)\|_{L^2}\,A(\tau)\,\mathrm{d}\tau,\\
&:= I_{431}+I_{432},
\end{aligned}$$

where the small number $\delta>0$ is to be specified later. Using the simple fact that $|\mathbf{k}|\geqq 1$ for $\mathbf{k}\neq 0$, we have, for any $m>0$,

$$\begin{aligned}
I_{431} &\leqq C\int_{\frac{t}{2}}^{t-\delta}\||\mathbf{k}|^{2m+4}(t-\tau)e^{-\frac14\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\,\|\mathbf{u}(\tau)\|_{L^2}\,A(\tau)\,\mathrm{d}\tau\\
&\leqq C\sup_{\frac{t}{2}\leqq\tau\leqq t}\|\mathbf{u}(\tau)\|_{L^2}\sup_{\frac{t}{2}\leqq\tau\leqq t}A(\tau)\int_{\frac{t}{2}}^{t-\delta}(t-\tau)^{-m-1}\,d\tau
\end{aligned}$$

$$\leqq \frac{C}{m}\,\delta^{-m} \sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2} \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau). \tag{4.29}$$

I_{432} is estimated slightly differently. For small number $\varepsilon > 0$,

$$\begin{aligned}
I_{432} &\leqq C \int_{t-\delta}^{t} \||\mathbf{k}|^{4-2\varepsilon}\,(t-\tau)e^{-\frac{1}{4}\nu|\mathbf{k}|^2(t-\tau)}\|_{l^2}\, \|\mathbf{u}(\tau)\|_{L^2}\, A(\tau)\,\mathrm{d}\tau \\
&\leqq C \int_{t-\delta}^{t} (t-\tau)^{-1+\varepsilon} \|\mathbf{u}(\tau)\|_{L^2}\, A(\tau)\,d\tau \\
&\leqq C \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau) \int_{t-\delta}^{t} (t-\tau)^{-1+\varepsilon}\,\tau^{-\frac{1}{2}}\,d\tau \\
&\leqq C \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau)\,(t-\delta)^{-\frac{1}{2}}\,(\frac{1}{\varepsilon}\,\delta^{\varepsilon}).
\end{aligned} \tag{4.30}$$

We can choose a small $\delta > 0$ such that the two bounds in (4.29) and (4.30) are equal. In fact, if we set

$$\delta = t^{\frac{1}{2(m+\varepsilon)}} \left(\frac{\varepsilon}{m}\right)^{\frac{1}{m+\varepsilon}} \left(\frac{\sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2}}{\sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau)}\right)^{\frac{1}{m+\varepsilon}},$$

then the two bounds become the same and

$$\begin{aligned}
|I_{43}| \leqq |I_{431}| + |I_{432}| &\leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau) \\
&\times \frac{1}{\varepsilon}\,t^{\frac{\varepsilon}{2(m+\varepsilon)}} \left(\frac{\varepsilon}{m}\right)^{\frac{\varepsilon}{m+\varepsilon}} \left(\frac{\sup_{\frac{t}{2}\leqq\tau\leqq t} \|\mathbf{u}(\tau)\|_{L^2}}{\sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau)}\right)^{\frac{\varepsilon}{m+\varepsilon}},
\end{aligned}$$

which holds for any $m > 0$. By letting $m \to \infty$, we find

$$|I_{43}| \leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau).$$

Invoking the bound for G_2 in (4.25), we have

$$\begin{aligned}
|I_{44}| &\leqq C \int_{\frac{t}{2}}^{t} \left\| \frac{1}{|\mathbf{k}|^2} e^{-\frac{1}{2}\nu|\mathbf{k}|^2(t-\tau)}\, |\widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},\tau)| \right\|_{l^2} \mathrm{d}\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} \left\| \frac{k_1 k_2}{|\mathbf{k}|^4} e^{-\frac{4k_1^2}{3\nu|\mathbf{k}|^4}(t-\tau)}\, \widehat{\mathbf{u}\cdot\nabla\theta}(\mathbf{k},\tau) \right\|_{l^2} \mathrm{d}\tau \\
&:= I_{441} + I_{442}.
\end{aligned}$$

I_{441} can be estimated similarly as I_{43}. Without repeating the details, we find

$$I_{441} \leqq C\,t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau).$$

The estimate of I_{442} is close to that for I_{422}. The bound is

$$I_{442} \leqq C\, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} \Big(\|\partial_y S(\tau)\|_{L^2} + \|\partial_{yy} S(\tau)\|_{L^2} + \tau^{\frac{1}{4}} \|\nabla(\theta - S)(\tau)\|_{L^2} \Big).$$

We have finished bounding all the terms in (4.14). Collecting all the estimates above leads to

$$\begin{aligned}
\|u(t)\|_{L^2} \leqq\ & C\, t^{-\frac{1}{2}} + C\, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\varepsilon} \|\mathbf{u}(\tau)\|_{L^2}^{1-2\varepsilon} \|\nabla \mathbf{u}(\tau)\|_{L^2}^{2\varepsilon} \\
& + C\, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M^{1-\varepsilon}(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} \tau^{\frac{3}{2}\varepsilon} \|\nabla \mathbf{u}(\tau)\|_{L^2}^{1+\varepsilon} \\
& + C\, t^{-\frac{1}{2}} \sup_{0\leqq\tau\leqq \frac{t}{2}} \Big(\|\partial_y S(\tau)\|_{L^2} + \|\partial_{yy} S(\tau)\|_{L^2} + \tau^{\frac{1}{4}} \|\nabla(\theta - S)(\tau)\|_{L^2} \Big) \\
& + C\, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} M(\tau) \sup_{\frac{t}{2}\leqq\tau\leqq t} A(\tau) \\
& + C\, t^{-\frac{1}{2}} \sup_{\frac{t}{2}\leqq\tau\leqq t} \Big(\|\partial_y S(\tau)\|_{L^2} + \|\partial_{yy} S(\tau)\|_{L^2} + \tau^{\frac{1}{4}} \|\nabla(\theta - S)(\tau)\|_{L^2} \Big),
\end{aligned}$$

where $A(t)$ is defined in (4.28) and $A(t) \to 0$ as $t \to \infty$. The estimates for $\|\widehat{v}(\mathbf{k}, t)\|_{l^2}$ are very similar and we shall omit the details. Multiplying each term by $t^{\frac{1}{2}}$, recalling the definition of $M(t)$ in (4.16) and making use of the conditions in (1.21), we find that, for $C_1 < 1$,

$$\sup_{t\leqq T} M(t) \leqq C + C \left(\sup_{t\leqq T} M(t) \right)^{1-\varepsilon} + C_1 \sup_{t\leqq T} M(t), \tag{4.31}$$

where C is a constant depending on the initial data only. The decay rate in (1.22) follows directly from (4.31). This completes the proof of Theorem 1.6. □

4.1. Conclusion and Discussion

We have studied the large-time behavior of large-data classical solutions to the initial value problems of the 2D Boussinesq equations without thermal diffusion on the periodic domain $\mathbb{T}^2$. By utilizing spectral method, we established several stability results regarding the global stability of the hydrostatic equilibria associated with the model at both the linear and nonlinear levels. For the linearized system, we identified the explicit decay rate of the velocity field towards the zero steady state, and gave a precise description of the thermal structure of the final state of the temperature. For the full nonlinear system, we first obtained a similar result regrading the global stability of hydrostatic equilibria as in [21], but under a weakened condition on the initial data. Then similar results as in the linear case are proved under certain assumptions on the solution.

Collectively, the results reported in this paper give partial answers to the open questions proposed in the recent study [21] regarding the large-time behavior of large-data classical solutions to the 2D Boussinesq equations without thermal diffusion. However, it should be emphasized that our results on the full nonlinear system, especially the explicit decay rate and description of final thermal state, are still not satisfactory, due to they are obtained under certain assumptions on the solution, which can hardly be verified. This is largely caused by the degeneracy in the third eigenvalue associated with the linearized system (see (1.13)). We leave the further investigation in a forthcoming paper.

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