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GLOBAL SMALL SOLUTION TO THE 2D MHD SYSTEM WITH A VELOCITY DAMPING TERM*

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Abstract. This paper studies the global well-posedness of the incompressible magnetohydrodynamic (MHD) system with a velocity damping term. We establish the global existence and uniqueness of smooth solutions when the initial data is close to an equilibrium state. In addition, explicit large-time decay rates for various Sobolev norms of the solutions are also given.

Key words. MHD equations, global existence, velocity damping

AMS subject classifications. 35Q35, 35B65, 76D03

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1. Introduction.

This paper examines the global (in time) existence and uniqueness of solutions to the two-dimensional (2D) magnetohydrodynamic (MHD) system with a velocity damping term, namely

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \vec{u} + \nabla P = -\nabla\cdot(\nabla\phi\otimes\nabla\phi), \quad (t,\,x,\,y)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R},\\ \partial_t\phi + \vec{u}\cdot\nabla\phi = 0,\\ \nabla\cdot\vec{u} = 0,\\ \vec{u}|_{t=1} = \vec{u}_0(x,\,y), \quad \phi|_{t=1} = \phi_0(x,\,y), \end{cases} \tag{1.1}$$

where $\vec{u} = (u,\, v)$ represents the 2D velocity field, P the pressure and ϕ the magnetic stream function, and $\nabla\phi\otimes\nabla\phi$ denotes the tensor product. (1.1) is formally equivalent to the 2D MHD equations given by

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \vec{u} + \nabla P = -\frac{1}{2}\nabla\big(|\vec{b}|^2\big) + \vec{b}\cdot\nabla\vec{b},\\ \partial_t\vec{b} + \vec{u}\cdot\nabla\vec{b} = \vec{b}\cdot\nabla\vec{u},\\ \nabla\cdot\vec{b} = \nabla\cdot\vec{u} = 0. \end{cases} \tag{1.2}$$

In fact, $\nabla\cdot\vec{b} = 0$ implies that $\vec{b} = \nabla^{\perp}\phi \equiv \big(\partial_y\phi, -\partial_x\phi\big)$ for a scalar function ϕ and, with this substitution, (1.2) is reduced to (1.1). The MHD equations, modeling electrically conducting fluid in the presence of a magnetic field, consist essentially of the interaction between the fluid velocity and the magnetic field. Electric currents induced in the fluid as a result of its motion modify the field; at the same time their flow in the magnetic field leads to mechanical forces which modify the motion. The MHD equations underlie many phenomena, such as the geomagnetic dynamo in geophysics, and solar winds and solar flares in astrophysics (see, e.g., [2, 9, 20]).

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Mathematically, the MHD equations can be extremely difficult to analyze due to the nonlinear coupling between the forced Navier–Stokes equations and the induction equation. In fact, it remains an outstanding open problem whether solutions to the 2D MHD equations

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla P = -\frac{1}{2}\nabla\big(|\vec{b}|^2\big) + \vec{b} \cdot \nabla \vec{b}, \\ \partial_t \vec{b} + \vec{u} \cdot \nabla \vec{b} = \vec{b} \cdot \nabla \vec{u}, \\ \nabla \cdot \vec{b} = \nabla \cdot \vec{u} = 0 \end{cases} \tag{1.3}$$

exist for all time or blow up in a finite time. One main difficulty is the lack of global (in time) bounds for the Sobolev norms of the solutions. Adding a velocity damping term does not appear to be sufficient to overcome this difficulty and our aim here is for small global smooth solutions. Since the equation of ϕ in (1.1) is a transport equation without any damping or dissipation, it is a very involved problem to establish the global well-posedness of (1.1) even under the assumption that the initial data is small.

The global regularity problem on the 2D MHD equations with partial dissipation or partial damping has attracted considerable interest in the past few years and progress has been made in some cases. The anisotropic 2D MHD equations with horizontal dissipation and vertical magnetic diffusion were recently examined by Cao and Wu and were shown to possess global classical solutions for any sufficiently smooth data [5]. Advances have also been made for the case where the dissipation and the magnetic diffusion are both in the horizontal direction [3, 4]. Lin, Xu, and Zhang recently studied the MHD equations with the Laplacian dissipation in the velocity equation but without magnetic diffusion and, remarkably, they were able to establish the global existence of small solutions after translating the magnetic field by a constant vector ([18, 19, 27]). Their approach reformulates the system in Lagrangian coordinates and estimates the Lagrangian velocity through the anisotropic Littlewood–Paley theory and anisotropic Besov space techniques. The partial dissipation case, where only the magnetic diffusion is present, has also been examined and global H^1 weak solutions have been established (see, e.g., [5, 16]). In addition, if we increase the magnetic diffusion from the Laplacian operator to the fractional Laplacian operator $(-\Delta)^\beta$ with $\beta > 1$, then the resulting MHD equations do have global regular solutions [6, 15]. Many more recent results on the MHD equations with partial or fractional dissipation can be found in [7, 8, 11, 12, 14, 22, 23, 24, 25, 26, 28, 29, 30].

The contribution of this paper is the global existence and uniqueness of solutions of (1.1) with sufficiently smooth initial data (u_0, ϕ_0) close to the equilibrium state $(0, y)$. This work is partially inspired by [18]. Our approach here exploits the time decay properties of the solution kernels to a linear differential equation which, with suitable nonlinear forcing terms, governs the translated version of (1.1). We now give a more precise account of our ideas. Setting

$$\phi = \psi + y$$

converts (1.1) into the following equivalent system of equations for (u, v, ψ),

$$\begin{cases} \partial_t u + u\,\partial_x u + v\partial_y u + u + \partial_x \tilde{P} = -\Delta\psi\partial_x\psi, \\ \partial_t v + u\,\partial_x v + v\partial_y v + v + \partial_y \tilde{P} = -\Delta\psi - \Delta\psi\partial_y\psi, \\ \partial_t \psi + u\partial_x \psi + v\partial_y \psi + v = 0, \\ \partial_x u + \partial_y v = 0, \end{cases} \tag{1.4}$$

where $\tilde{P} = P + \frac{1}{2}|\nabla\phi|^2$. Applying $\nabla \cdot \vec{u} = 0$ to eliminate the pressure term yields

$$\partial_t u + u - \partial_{xy}\psi = \Pi_1, \tag{1.5}$$

$$\partial_t v + v + \partial_{xx}\psi = \Pi_2, \tag{1.6}$$

$$\partial_t \psi + u\partial_x\psi + v\partial_y\psi + v = 0, \tag{1.7}$$

where

$$\Pi_1 = -\vec{u}\cdot\nabla u + \partial_x\Delta^{-1}\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) - \Delta\psi\,\partial_x\psi + \partial_x\Delta^{-1}\nabla\cdot(\Delta\psi\,\nabla\psi), \tag{1.8}$$

$$\Pi_2 = -\vec{u}\cdot\nabla v + \partial_y\Delta^{-1}\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) - \Delta\psi\,\partial_y\psi + \partial_y\Delta^{-1}\nabla\cdot(\Delta\psi\,\nabla\psi). \tag{1.9}$$

Taking the time derivative on (1.5)–(1.7), we obtain

$$\begin{cases} \partial_{tt}u + \partial_t u - \partial_{xx}u = F_1, \\ \partial_{tt}v + \partial_t v - \partial_{xx}v = F_2, \\ \partial_{tt}\psi + \partial_t\psi - \partial_{xx}\psi = F_0, \\ \vec{u}|_{t=1} = \vec{u}_0(x, y), \quad \vec{u}_t|_{t=1} = \vec{u}_1(x, y), \\ \psi|_{t=1} = \psi_0(x, y), \quad \psi_t|_{t=1} = \psi_1(x, y), \end{cases} \tag{1.10}$$

where $\vec{u}_1 = (u_1(x, y), v_1(x, y))$, $\psi_0 = \phi_0 - y$, and

$$\begin{aligned} u_1 &= (-u + \partial_{xy}\psi + \Pi_1)|_{t=1}, \\ v_1 &= (-v - \partial_{xx}\psi + \Pi_2)|_{t=1}, \\ \psi_1 &= (-u\partial_x\psi - v\partial_y\psi - v)|_{t=1}, \end{aligned}$$

and

$$F_0 = -\vec{u}\cdot\nabla\psi - \partial_t(\vec{u}\cdot\nabla\psi) - \Pi_2, \tag{1.11}$$

$$F_1 = \partial_t\Pi_1 - \partial_{xy}(\vec{u}\cdot\nabla\psi), \tag{1.12}$$

$$F_2 = \partial_t\Pi_2 + \partial_{xx}(\vec{u}\cdot\nabla\psi). \tag{1.13}$$

The structure of the linear part in (1.10) plays a crucial role in ensuring the global existence of small solutions. In fact, the solution kernels of the linear equation decay in time in suitable spatial functional settings. Let us be more accurate. As detailed in section 2, the solution of the linear equation

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = 0$$

with the initial data

$$\Phi(1, x, y) = \Phi_0(x, y), \quad \Phi_t(1, x, y) = \Phi_1(x, y)$$

can be written as

$$\Phi(t, x, y) = K_0(t, \partial_x)\Phi_0 + K_1(t, \partial_x)\left(\frac{1}{2}\Phi_0 + \Phi_1\right),$$

where the solution operators K_1 and K_2 are explicitly derived in section 2. By Duhamel's principle, the solution of the inhomogeneous equation

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = F,$$

with initial data $\Phi(1,x) = \Phi_0$, $\partial_t\Phi(1,x) = \Phi_1$ is given by

$$\begin{aligned}\Phi(t,x,y) &= K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\Big(\frac{1}{2}\Phi_0 + \Phi_1\Big) \\ &\quad + \int_1^t K_1(t-s,\partial_x)F(s,x,y)\,ds.\end{aligned} \tag{1.14}$$

By letting $\Phi = (u,v,\psi)$ and $F = (F_0,F_1,F_2)$, (1.14) gives an integral representation of (1.10). Thanks to the time decay properties of K_1 and K_2 (established in section 2), the nonlinear parts in (1.10) remain small and the solution map is a contraction for all time. More details will be unfuoled in the subsequent sections.

To state our main result, we introduce the functional settings. Let X_0 be the Banach space defined by the norm

$$\|(\vec{u}_0,\,\psi_0)\|_{X_0} = \|\langle\nabla\rangle^N(\vec{u}_0,\,\nabla\psi_0)\|_{L^2_{xy}} + \|\langle\nabla\rangle^{6+}(\vec{u}_0,\,\psi_0)\|_{L^1_{xy}} + \|\langle\nabla\rangle^{6+}(\vec{u}_1,\,\psi_1)\|_{L^1_{xy}},$$

where $\langle\nabla\rangle = (I-\Delta)^{\frac{1}{2}}$ and $a+$ denotes $a+\epsilon$ for any small ϵ. For notational convenience, we also write

$$\|f\|_q = \|f\|_{L^q_{xy}}, \qquad 1 \le q \le \infty.$$

Now we define our working space as X with its norm given by

$$\begin{aligned}\|(\vec{u},\,\psi)\|_X = \sup_{t\ge 1}\Big\{ & t^{-\varepsilon}\|\langle\nabla\rangle^N(\vec{u}(t),\,\nabla\psi(t))\|_2 + t^{\frac{1}{4}}\|\langle\nabla\rangle^3\psi\|_2 \\ & + t^{\frac{3}{2}}\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle^3\partial_{xx}\psi\|_2 + t^{\frac{3}{2}}\|\partial_{xxx}\psi\|_2 \\ & + t^{\frac{3}{2}}\|\partial_t\vec{u}\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle\partial_t\vec{u}\|_2 + t\|\langle\nabla\rangle\partial_x\vec{u}\|_\infty + t^{\frac{3}{2}}\|\partial_x\partial_t v\|_2\Big\}.\end{aligned} \tag{1.15}$$

Here N is a big positive integer and $\varepsilon > 0$ is a small parameter. For the sake of clarity in our presentation, we intentionally avoid the tedious calculations needed to provide an accurate range of N. However, sufficiently large N and small $\varepsilon > 0$, say $N = 20$ and $\varepsilon = 0.01$, would serve our purpose.

Our main result can then be stated as follows. We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \le CB$ for some absolute constant $C > 0$.

THEOREM 1.1. *Let $\psi = \phi - y$ and $\psi_0 = \phi_0 - y$. Then there exists a small constant $\varepsilon_0 > 0$ such that, if the initial data $(\vec{u}_0,\,\phi_0)$ satisfies $\|(\vec{u}_0,\,\psi_0)\|_{X_0} \le \varepsilon_0$, then there exists a unique global solution $(u,\,v,\,\phi,\,P)$ to the system* (1.1) *with*

$$(u,\,v,\,\phi) \in X, \qquad P \in C\big([1,\infty); H^N(\mathbb{R}^2)\big).$$

Moreover, the following decay estimates hold

$$\|u(t)\|_{L^\infty_{xy}} \lesssim \varepsilon_0 t^{-1}; \quad \|v(t)\|_{L^\infty_{xy}} \lesssim \varepsilon_0 t^{-\frac{3}{2}}; \quad \|\psi(t)\|_{L^\infty_{xy}} \lesssim \varepsilon_0 t^{-\frac{1}{2}}; \quad \|P(t)\|_{L^\infty_{xy}} \lesssim \varepsilon_0 t^{-\frac{1}{2}}.$$

The proof of Theorem 1.1 relies on the following lemma, which can be deduced from a standard continuity argument (see, e.g., Theorem 4 in [1]).

LEMMA 1.2. *Assume the initial data $(\vec{u}_0,\,\psi_0) \in X_0$. Suppose that $(\vec{u},\,\psi)$ given by* (1.14)*, namely the integral representation of* (1.10)*, satisfies*

$$\|(\vec{u},\,\psi)\|_X \lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0} + Q(\|\vec{u},\,\psi)\|_X), \tag{1.16}$$

where $Q(a) \geq Ca^{\beta}$ *for* $a \lesssim 1$ *and* $\beta > 1$. *Then there exists* $r_0 > 0$ *such that, if*

$$\|(\vec{u}_0, \psi_0)\|_{X_0} \leq r_0,$$

then (1.10) *has a unique global solution* $(\vec{u}, \psi) \in X$ *and*

$$\|(\vec{u}, \psi)\|_X \lesssim r_0.$$

In addition, to facilitate the proof, we introduce an auxiliary functional space, in which more terms with explicit time decay estimates are included. Let

$$\|(\vec{u}, \psi)\|_Y = \|(\vec{u}, \psi)\|_X + \sup_{t \geq 1} \Big\{ t^{\frac{1}{2}} \|\langle\nabla\rangle^2 \psi\|_\infty + t^{\frac{3}{4}} \|\langle\nabla\rangle \partial_x \psi\|_2 + t\|\partial_x \langle\nabla\rangle^3 \psi\|_\infty \\ + t\|u\|_\infty + t^{\frac{3}{2}} \|v\|_\infty + t^{\frac{3}{4}} \|u\|_2 + t^{\frac{5}{4}} \|\partial_x u\|_2 + t^{\frac{5}{4}} \|v\|_2 \Big\}.$$

Roughly speaking, the decay rates of the extra terms in the Y-norm obey the following rules:

$$L^\infty \sim t^{-\frac{1}{2}}; \quad L^2 \sim t^{-\frac{1}{4}}; \quad \partial_x \sim t^{-\frac{1}{2}}; \quad \partial_t \sim \partial_{xx}; \quad u \sim \partial_x \sim t^{-\frac{1}{2}}; \quad v \sim \partial_{xx} \sim t^{-1}.$$

As we show in section 3, the norms $\|(\vec{u}, \psi)\|_Y$ and $\|(\vec{u}, \psi)\|_X$ are related through the following lemma.

LEMMA 1.3. *Let the spaces* X, Y *and their norms be defined as above. Then*

$$\|(\vec{u}, \psi)\|_Y \lesssim \|(\vec{u}, \psi)\|_X + Q(\|(\vec{u}, \psi)\|_X).$$

As a consequence of Lemma 1.3, to prove (1.16), it is enough to verify

$$\|(\vec{u}, \psi)\|_X \lesssim \|(\vec{u}_0, \psi_0)\|_{X_0} + Q\big(\|(\vec{u}, \psi)\|_Y\big). \tag{1.17}$$

Therefore, the proof of Theorem 1.1 is then reduced to establishing (1.17) and our main effort is devoted to achieving this goal. This is a long process and involves several major components. The first consists of crucial and sharp decay estimates and four tool lemmas on the kernels. The second are the bounds on the nonlinearities F_0, F_1, and F_2 due to treating the lower and higher frequencies of their terms differently. The third involves the estimates of each member in the X-norm in (1.15) through the first two components.

The rest of the paper is divided into six sections and an appendix. The second section derives the solution kernel of the linear equation and represents the solution of (1.10) in an integral form through the Duhamel formula. Crucial decay estimates for the solution kernels and several tool lemmas to be used repeatedly are also presented in this section. Section 3 proves Lemma 1.3. The rest of the sections are devoted to proving (1.17). Section 4 bounds $t^{-\varepsilon}\|\langle\nabla\rangle^N(\vec{u}(t), \nabla\psi(t))\|_2$ through energy estimates, which give a control of the first term in the definition of the norm of X. Section 5 provides suitable estimates for the nonlinear terms F_0, F_1 and F_2. To obtain these estimates, we decompose the terms involved to high and low frequencies and apply the results from the second section and an inequality involving the Riesz transform (Lemma 5.1). With the estimates for F_0, F_1 and F_2 at our disposal, sections 6 and 7 continue the proof of (1.17) by repeatedly applying the tool lemmas in the second section and the estimates for F_0, F_1, and F_2. The appendix serves four purposes. It gives an explicit representation of Π_1 and Π_2. The key point of this representation is that each term is written in such a way that it possesses as many directives in the x-direction as possible. As seen from section 2, the more x-derivatives a term has, the faster it decays in time. This point has played an important role in the estimates of F_0, F_1, and F_2 in section 5. It also provides the proofs of Lemmas 5.1 and 5.5. Finally, the properties of the pressure are also given here.

2. Preliminary. This section is divided into two subsections. The first subsection derives the integral formulation (1.14) with explicit representations for K_1 and K_2. In addition, key decay estimates for K_1 and K_2 are also obtained here. The second subsection proves several tool lemmas to be used in the proof of Theorem 1.1.

2.1. Linear operators. We consider the linear equation

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = 0 \tag{2.1}$$

with the initial data

$$\Phi(1,x,y) = \Phi_0(x,y),\ \ \Phi_t(1,x,y) = \Phi_1(x,y).$$

Taking the Fourier transform of (2.1) yields

$$\partial_{tt}\hat{\Phi} + \partial_t\hat{\Phi} + \xi^2\hat{\Phi} = 0, \tag{2.2}$$

where the Fourier transform $\hat{\Phi}$ is defined as

$$\hat{\Phi}(t,\xi,\eta) = \int_{\mathbb{R}^2} e^{-ix\xi - iy\eta}\Phi(t,x,y)\,dxdy.$$

Solving (2.2) by a simple ODE theory, we have

$$\begin{aligned}\hat{\Phi}(t,\xi,\eta) = &\frac{1}{2}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\widehat{\Phi_0}(\xi,\eta)\\ &+ \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\Big(e^{(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2})t} - e^{(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2})t}\Big)\Big(\frac{1}{2}\widehat{\Phi_0}(\xi,\eta) + \widehat{\Phi_1}(\xi,\eta)\Big).\end{aligned}$$

DEFINITION 2.1. *Let the operators* $K_0(t,\partial_x)$ *and* $K_1(t,\partial_x)$ *be defined as*

$$\widehat{K_0(t,\partial_x)}f(t,\xi,\eta) = \frac{1}{2}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\hat{f}(t,\xi,\eta);$$

and

$$\widehat{K_1(t,\partial_x)}f(t,\xi,\eta) = \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} - e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\hat{f}(t,\xi,\eta),$$

where $\sqrt{-1} = i$. By Definition 2.1, the solution Φ of (2.1) is written as

$$\Phi(t,x,y) = K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\left(\frac{1}{2}\Phi_0 + \Phi_1\right).$$

Moreover, consider the inhomogeneous equation,

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = F, \tag{2.3}$$

with initial data $\Phi(1,x) = \Phi_0$, $\partial_t\Phi(1,x) = \Phi_1$. Then we have the following standard Duhamel formula,

$$\begin{aligned}\Phi(t,x,y) = &K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\left(\frac{1}{2}\Phi_0 + \Phi_1\right)\\ &+ \int_1^t K_1(t-s,\partial_x)F(s,x,y)\,ds.\end{aligned} \tag{2.4}$$

In the following, we present some decay estimates on K_0, K_1.

LEMMA 2.2. *Let* K_0, K_1 *be defined in Definition* 2.1. *Then for any* $\alpha \geq 0$, $1 \leq q \leq \infty$, $i = 0, 1$,

(1) $\big\||\xi|^\alpha \widehat{K_i}(t,\cdot)\big\|_{L^q_\xi(|\xi|\le\frac12)} \lesssim \langle t\rangle^{-\frac12(\frac1q+\alpha)}$;

(2) $\big\|\partial_t \widehat{K_i}(t,\cdot)\big\|_{L^q_\xi(|\xi|\le\frac12)} \lesssim \langle t\rangle^{-1-\frac{1}{2q}}$;

(3) $\big|\widehat{K_i}(t,\xi)\big| \lesssim e^{-\frac12 t}$ *for any* $|\xi|\ge\frac12$;

(4) $\big|\langle\xi\rangle^{-1}\partial_t\widehat{K_0}(t,\xi)\big|, \big|\partial_t\widehat{K_1}(t,\xi)\big| \lesssim e^{-\frac12 t}$ *for any* $|\xi|\ge\frac12$.

Proof. Since the decay properties of the operators K_0, K_1 are distinct between low and high frequencies, we will split the frequencies into the following two parts:

$$|\xi|\le\frac12; \qquad |\xi|>\frac12.$$

In the following, we will analyze the two parts separately.

Case I: $|\xi|\le\frac12$.

According to the expressions of K_0, K_1 as follows

$$\widehat{K_0}(t,\xi)=\frac12\Big(e^{\left(-\frac12+\sqrt{\frac14-\xi^2}\right)t}+e^{\left(-\frac12-\sqrt{\frac14-\xi^2}\right)t}\Big)$$

and

$$\widehat{K_1}(t,\xi)=\frac{1}{2\sqrt{\frac14-\xi^2}}\Big(e^{(-\frac12+\sqrt{\frac14-\xi^2})t}-e^{(-\frac12-\sqrt{\frac14-\xi^2})t}\Big),$$

we obtain, for $|\xi|\le\frac12$,

$$-\frac12+\sqrt{\frac14-\xi^2}=\frac{-\xi^2}{\frac12+\sqrt{\frac14-\xi^2}}\le-\xi^2; \quad -\frac12-\sqrt{\frac14-\xi^2}\le-\xi^2.$$

Then we get

$$0\le\widehat{K_0}(t,\xi)\le e^{-t\xi^2}.$$

Therefore,

$$|\xi|^\alpha\widehat{K_0}(t,\xi)\le|\xi|^\alpha e^{-t\xi^2}\lesssim\langle t\rangle^{-\frac\alpha2};$$

$$\big\||\xi|^\alpha\widehat{K_0}(t,\xi)\big\|_{L^1_\xi(|\xi|\le\frac12)}\le\int_{|\xi|\le\frac12}|\xi|^\alpha\widehat{K_0}(t,\xi)\,d\xi\le\int_{|\xi|\le\frac12}|\xi|^\alpha e^{-t\xi^2}\,d\xi\lesssim\langle t\rangle^{-\frac12(1+\alpha)}.$$

So estimate (1) of Lemma 2.2 for K_0 follows from interpolation. Moreover,

$$\dot K_0(t)=-\frac12K_0(t)+\left(\frac14+\partial_{xx}\right)K_1(t); \tag{2.5}$$

$$\dot K_1(t)=K_0(t)-\frac12K_1(t). \tag{2.6}$$

Thus,

$$\begin{aligned}\partial_t\widehat{K_0}(t,\xi)&=-\frac12\widehat{K_0}(t,\xi)+\left(\frac14-\xi^2\right)\widehat{K_1}(t,\xi)\\ &=\frac14\big(\sqrt{1-4\xi^2}-1\big)e^{(-\frac12+\sqrt{\frac14-\xi^2})t}-\frac14\big(\sqrt{1-4\xi^2}+1\big)e^{(-\frac12-\sqrt{\frac14-\xi^2})t}. \qquad (2.7)\end{aligned}$$

Since $1-\sqrt{1-4\xi^2}\le 4\xi^2$ when $|\xi|\le\frac{1}{2}$, we have

$$\big|\partial_t\widehat{K_0}(t,\xi)\big|\lesssim|\xi|^2e^{-t\xi^2}+e^{-\frac{1}{2}t}.$$

Roughly speaking, $\partial_t\widehat{K_0}(t,\xi)\sim\xi^2\widehat{K_0}(t,\xi)$, and thus we have

$$\big\|\partial_t\widehat{K_0}(t,\cdot)\big\|_{L^q_\xi(|\xi|\le\frac{1}{2})}\lesssim\langle t\rangle^{-1-\frac{1}{2q}}.$$

Similarly, we can obtain

$$0\le\widehat{K_1}(t,\xi)\le e^{-t\xi^2};\quad \big|\partial_t\widehat{K_1}(t,\xi)\big|\lesssim|\xi|^2e^{-t\xi^2}+e^{-\frac{1}{2}t}.$$

Thus for K_1, we have the same estimates as K_0. Hence, we deduce estimates (1) and (2) of Lemma 2.2.

Case II: $|\xi|>\frac{1}{2}$.

For this case, the expressions of K_0, K_1 can be written as

$$\widehat{K_0}(t,\xi)=\frac{1}{2}\Big(e^{\big(-\frac{1}{2}+i\sqrt{\xi^2-\frac{1}{4}}\big)t}+e^{\big(-\frac{1}{2}-i\sqrt{\xi^2-\frac{1}{4}}\big)t}\Big)$$

and

$$\widehat{K_1}(t,\xi)=\frac{1}{2i\sqrt{\xi^2-\frac{1}{4}}}\Big(e^{(-\frac{1}{2}+i\sqrt{\xi^2-\frac{1}{4}})t}-e^{(-\frac{1}{2}-i\sqrt{\xi^2-\frac{1}{4}})t}\Big).$$

By virtue of the expression of K_0, K_1, we then get, for any $|\xi|>\frac{1}{2}$,

$$\big|\widehat{K_0}(t,\xi)\big|,\ \big|\widehat{K_1}(t,\xi)\big|\lesssim e^{-\frac{1}{2}t},$$

Further, by (2.6) and (2.7), we also have

$$\big|\langle\xi\rangle^{-1}\partial_t\widehat{K_0}(t,\xi)\big|,\big|\partial_t\widehat{K_1}(t,\xi)\big|\lesssim e^{-\frac{1}{2}t}.$$

Hence we complete the proof of Lemma 2.2. ∎

2.2. Tool lemmas. To prove (1.17), we need several lemmas. In the following, $\mathcal{S}$ denotes the Schwartz class and $\|f\|_{L^p_xL^q_y}\equiv\big\|\|f\|_{L^q_y}\big\|_{L^p_x}$.

LEMMA 2.3. *Let $K(t,\partial_x)$ denote a Fourier multiplier operator with*

$$\big\|\widehat{K}(t,\xi)\big\|_{L^1_\xi}<\infty.$$

Then, for any space-time Schwartz function f,

$$\big\|K(t,\partial_x)f\big\|_{L^\infty_{xy}}\lesssim\big\|\widehat{K}(t,\xi)\big\|_{L^1_\xi}\big\|\partial_yf\big\|_{L^1_{xy}}. \tag{2.8}$$

Proof. For any $g\in\mathcal{S}(\mathbb{R})$, we have

$$\|g\|_{L^\infty(\mathbb{R})}\lesssim\|g'\|_{L^1(\mathbb{R})}\text{ or }\|\hat{g}\|_{L^1(\mathbb{R})}. \tag{2.9}$$

Then, from the inequalities above, we get

$$\begin{aligned}\big\|K(t,\partial_x)f\big\|_{L^\infty_{xy}} &\lesssim \big\|K(t,\partial_x)\partial_y f\big\|_{L^\infty_x L^1_y}\\ &\lesssim \big\|K(t,\partial_x)\partial_y f\big\|_{L^1_y L^\infty_x}\\ &\lesssim \Big\|\big\|\widehat{K}(t,\xi)\mathcal{F}_\xi\big(\partial_y f\big)(t,\xi,y)\big\|_{L^1_\xi}\Big\|_{L^1_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^1_\xi}\Big\|\big\|\mathcal{F}_\xi\big(\partial_y f\big)(t,\xi,y)\big\|_{L^\infty_\xi}\Big\|_{L^1_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^1_\xi}\Big\|\big\|\partial_y f(t,x,y)\big\|_{L^1_x}\Big\|_{L^1_y}\\ &= \big\|\widehat{K}(t,\xi)\big\|_{L^1_\xi}\big\|\partial_y f\big\|_{L^1_{xy}}.\end{aligned}$$

This proves Lemma 2.3. $\square$

LEMMA 2.4. *Assume that* $\big\|\widehat{K}(t,\cdot)\big\|_{L^\infty}$ *is bounded. Then, for any space-time Schwartz function* f, *and any* $\epsilon > 0$,

$$\big\|K(t,\partial_x)f\big\|_{L^\infty_{xy}} \lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\big\|\partial_y\langle\nabla\rangle^{1+\epsilon}f\big\|_{L^1_{xy}}. \tag{2.10}$$

Proof. By (2.9), Sobolev's inequality and Plancherel's identity, we have

$$\begin{aligned}\big\|K(t,\partial_x)f\big\|_{L^\infty_{xy}} &\lesssim \big\|K(t,\partial_x)\partial_y f\big\|_{L^\infty_x L^1_y}\\ &\lesssim \big\|K(t,\partial_x)\partial_y\langle\nabla\rangle^{\frac12+\frac{\epsilon}{2}}f\big\|_{L^1_y L^2_x}\\ &= \Big\|\big\|\widehat{K}(t,\xi)\mathcal{F}_\xi\big(\partial_y\langle\nabla\rangle^{\frac12+\frac{\epsilon}{2}}f\big)(t,\xi,y)\big\|_{L^2_\xi}\Big\|_{L^1_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\Big\|\big\|\mathcal{F}_\xi\big(\partial_y\langle\nabla\rangle^{\frac12+\frac{\epsilon}{2}}f\big)(t,\xi,y)\big\|_{L^2_\xi}\Big\|_{L^1_y}\\ &= \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\Big\|\big\|\partial_y\langle\nabla\rangle^{\frac12+\frac{\epsilon}{2}}f(t,x,y)\big\|_{L^2_x}\Big\|_{L^1_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\big\|\partial_y\langle\nabla\rangle^{1+\epsilon}f\big\|_{L^1_{xy}}.\end{aligned}$$

This proves Lemma 2.4. $\square$

As a special consequence of Lemmas 2.3 and 2.4, we have the following corollary.

COROLLARY 2.5. *Let* $K(t,\partial_x)$ *be a Fourier multiplier operator satisfying*

$$\big\|\widehat{\partial_x^\alpha K}(t,\xi)\big\|_{L^1_\xi(|\xi|\le\frac12)} < \infty, \quad \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi(|\xi|\ge\frac12)} < \infty, \quad \alpha \ge 0.$$

Then, for any space-time Schwartz function f,

$$\begin{aligned}\big\|\partial_x^\alpha K(t,\partial_x)f\big\|_{L^\infty_{xy}} &\lesssim \Big(\big\|\widehat{\partial_x^\alpha K}(t,\xi)\big\|_{L^1_\xi(|\xi|\le\frac12)} + \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi(|\xi|\ge\frac12)}\Big)\\ &\quad\times\big\|\langle\nabla\rangle^{\alpha+1+\epsilon}\partial_y f\big\|_{L^1_{xy}}.\end{aligned} \tag{2.11}$$

LEMMA 2.6. *Assume that* $\big\|\widehat{K}(t,\cdot)\big\|_{L^2}$ *is bounded. Then, for any space-time Schwartz function* f, *and any* $\epsilon > 0$,

$$\big\|K(t,\partial_x)f\big\|_{L^2_{xy}} \lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^2_\xi}\big\||\nabla|^{\frac12-\epsilon}\langle\nabla\rangle^{2\epsilon}f\big\|_{L^1_{xy}}. \tag{2.12}$$

Proof. By a similar manner, we have

$$\begin{aligned}\big\|K(t,\partial_x)f\big\|_{L^2_{xy}} &= \Big\|\big\|\widehat{K}(t,\xi)\mathcal{F}_\xi f(t,\xi,y)\big\|_{L^2_\xi}\Big\|_{L^2_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^2_\xi}\Big\|\big\|\mathcal{F}_\xi f(t,\xi,y)\big\|_{L^\infty_\xi}\Big\|_{L^2_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^2_\xi}\Big\|\big\|f(t,x,y)\big\|_{L^1_x}\Big\|_{L^2_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^2_\xi}\Big\|\big\|f(t,x,y)\big\|_{L^2_y}\Big\|_{L^1_x}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^2_\xi}\big\||\nabla|^{\frac12-\epsilon}\langle\nabla\rangle^{2\epsilon}f\big\|_{L^1_{xy}}.\end{aligned}$$

This proves the lemma. ∎

LEMMA 2.7. *Assume that $\big\|\widehat{K}(t,\cdot)\big\|_{L^\infty}$ is bounded. Then, for any space-time Schwartz function f, and any $\epsilon>0$,*

$$\big\|K(t,\partial_x)f\big\|_{L^2_{xy}} \lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\big\||\nabla|^{\frac12-\epsilon}\langle\nabla\rangle^{\frac12+2\epsilon}f\big\|_{L^1_{xy}}. \tag{2.13}$$

Proof. By a similar manner, we have the following.

$$\begin{aligned}\big\|K(t,\partial_x)f\big\|_{L^2_{xy}} &= \Big\|\big\|\widehat{K}(t,\xi)\mathcal{F}_\xi f(t,\xi,y)\big\|_{L^2_\xi}\Big\|_{L^2_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\Big\|\big\|\mathcal{F}_\xi f(t,\xi,y)\big\|_{L^2_\xi}\Big\|_{L^2_y}\\ &= \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\Big\|\big\|f(t,x,y)\big\|_{L^2_x}\Big\|_{L^2_y}\\ &\lesssim \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi}\big\||\nabla|^{\frac12-\epsilon}\langle\nabla\rangle^{\frac12+2\epsilon}f\big\|_{L^1_{xy}}.\end{aligned}$$

This proves the lemma. ∎

Combining Lemmas 2.6 and 2.7, we have the following.

COROLLARY 2.8. *Assume the Fourier multiplier operator $K(t,\partial_x)$ satisfies*

$$\big\|\widehat{\partial_x^\alpha K}(t,\xi)\big\|_{L^2_\xi(|\xi|\le\frac12)}<\infty,\quad \big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi(|\xi|\ge\frac12)}<\infty,\quad \alpha\ge0.$$

Then, for any space-time Schwartz function f, and any $\epsilon>0$,

$$\begin{aligned}\big\|\partial_x^\alpha K(t,\partial_x)f\big\|_{L^2_{xy}} \lesssim &\Big(\big\|\widehat{\partial_x^\alpha K}(t,\xi)\big\|_{L^2_\xi(|\xi|\le\frac12)}+\big\|\widehat{K}(t,\xi)\big\|_{L^\infty_\xi(|\xi|\ge\frac12)}\Big)\\ &\cdot\big\|\langle\nabla\rangle^{\alpha+\frac12+2\epsilon}|\nabla|^{\frac12-\epsilon}f\big\|_{L^1_{xy}}.\end{aligned} \tag{2.14}$$

3. Proof of Lemma 1.3.

This section provides the proof of Lemma 1.3. More precisely, we show that

$$\|(\vec u,\psi)\|_Y \lesssim \|(\vec u,\psi)\|_X + Q(\|(\vec u,\psi)\|_X),$$

where X, Y, and Q are defined as in the introduction.

Proof of Lemma 1.3. First, we recall the basic inequality

$$\begin{aligned}\|g(x,y)\|_{L^\infty_{xy}} &\lesssim \|g\|_{L^2}^{\frac14}\|\partial_x g\|_{L^2}^{\frac14}\|\partial_y g\|_{L^2}^{\frac14}\|\partial_x\partial_y g\|_{L^2}^{\frac14}\\ &\lesssim \|\langle\nabla\rangle\partial_x g\|_{L^2_{xy}}^{\frac12}\|\langle\nabla\rangle g\|_{L^2_{xy}}^{\frac12}.\end{aligned} \tag{3.1}$$

We now estimate each term in Y and start with the terms related to ψ. By interpolation,

$$\begin{aligned}\|\langle\nabla\rangle^3\partial_x\psi(s,\,x,\,y)\|_{L^2_{xy}} &\lesssim \|\langle\nabla\rangle^3\partial_{xx}\psi\|^{\frac12}_{L^2_{xy}}\|\langle\nabla\rangle^3\psi\|^{\frac12}_{L^2_{xy}}\\ &\lesssim s^{-\frac58}\|(\vec u,\,\psi)\|^{\frac12}_X\, s^{-\frac18}\|(\vec u,\,\psi)\|^{\frac12}_X\\ &\lesssim s^{-\frac34}\|(\vec u,\,\psi)\|_X. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\|\langle\nabla\rangle^2\partial_x\psi(s,\,x,\,y)\|_{L^\infty_{xy}} \lesssim \|\langle\nabla\rangle^3\partial_{xx}\psi\|^{\frac12}_{L^2_{xy}}\|\langle\nabla\rangle^3\partial_x\psi\|^{\frac12}_{L^2_{xy}} \lesssim s^{-1}\|(\vec u,\,\psi)\|_X \tag{3.3}$$

and

$$\|\langle\nabla\rangle^2\psi(s,\,x,\,y)\|_{L^\infty_{xy}} \lesssim \|\langle\nabla\rangle^3\partial_x\psi\|^{\frac12}_{L^2_{xy}}\|\langle\nabla\rangle^3\psi\|^{\frac12}_{L^2_{xy}} \lesssim s^{-\frac12}\|(\vec u,\,\psi)\|_X. \tag{3.4}$$

The other terms in Y are a little tricky. We first construct the following two inequalities, for $j = 1, 2$ and some $0 < \alpha < 1$,

$$\|\langle\nabla\rangle\Pi_j\|_2 \lesssim s^{-\frac54}\|(\vec u,\,\psi)\|^\alpha_Y\,\|(\vec u,\,\psi)\|^{2-\alpha}_X; \tag{3.5}$$

$$\|\Pi_j\|_\infty \lesssim s^{-\frac32}\|(\vec u,\,\psi)\|^\alpha_Y\,\|(\vec u,\,\psi)\|^{2-\alpha}_X. \tag{3.6}$$

Since the cases $j = 1$ and $j = 2$ can be treated in the same way, we only deal with the case $j = 1$. According to the expression of Π_1 in (A.1),

$$\begin{aligned}\|\langle\nabla\rangle\Pi_1\|_2 \lesssim{}& \|\langle\nabla\rangle v\|_2\,\|\langle\nabla\rangle\partial_x u\|_\infty + \|\langle\nabla\rangle v\|_\infty\,\|\langle\nabla\rangle^2 u\|_2 + \|\langle\nabla\rangle^3\psi\|_2\,\|\langle\nabla\rangle^2\partial_x\psi\|_\infty\\ &+ \|\langle\nabla\rangle u\|_2\,\|\langle\nabla\rangle\partial_x u\|_\infty + \|\langle\nabla\rangle u\|_2\,\|\langle\nabla\rangle\partial_x v\|_\infty + \|\langle\nabla\rangle\partial_{xx}\psi\|_\infty\,\|\langle\nabla\rangle^2\psi\|_2\\ ={}& I_1 + I_2 + I_3 + \text{remainder terms}.\end{aligned}$$

The first three terms I_1, I_2, and I_3 are typical of the terms on the right, and the estimates of the remainder terms are similar to them. Therefore, we shall only present their estimates.

$$\begin{aligned}I_1 &= \|\langle\nabla\rangle v\|_2\,\|\langle\nabla\rangle\partial_x u\|_\infty\\ &\lesssim \|\langle\nabla\rangle^N v\|_2^{\frac1N}\|v\|_2^{1-\frac1N}\,\|\langle\nabla\rangle\partial_x u\|_\infty\\ &\lesssim s^{\frac{\varepsilon}{N}}\|(\vec u,\psi)\|_X^{\frac1N}\cdot s^{-\frac54(1-\frac1N)}\|(\vec u,\psi)\|_Y^{1-\frac1N}\cdot s^{-1}\|(\vec u,\psi)\|_X\\ &\lesssim s^{-\frac94+\frac54\cdot\frac1N+\frac{\varepsilon}{N}}\|(\vec u,\psi)\|_Y^{1-\frac1N}\cdot\|(\vec u,\psi)\|_X^{1+\frac1N}\\ &\lesssim s^{-\frac54}\|(\vec u,\psi)\|_Y^{1-\frac1N}\cdot\|(\vec u,\psi)\|_X^{1+\frac1N},\end{aligned}$$

where $\varepsilon > 0$ is a small parameter and N has been chosen large enough. For the second term, by interpolation we obtain

$$\begin{aligned}I_2 &= \|\langle\nabla\rangle v\|_\infty\,\|\langle\nabla\rangle^2 u\|_2\\ &\lesssim \|\langle\nabla\rangle^N v\|_2^{1+\frac{1-\varepsilon}{N-1}}\|v\|_\infty^{1-\frac{1-\varepsilon}{N-1}}\\ &\lesssim s^{\frac{1-\varepsilon}{N-1}\cdot\varepsilon}\|(\vec u,\,\psi)\|_X^{\frac{1-\varepsilon}{N-1}}\,s^{-\frac32(1-\frac{1-\varepsilon}{N-1})}\|(\vec u,\,\psi)\|_Y^{1-\frac{1-\varepsilon}{N-1}}\,s^{\varepsilon}\|(\vec u,\,\psi)\|_X\\ &\lesssim s^{-\frac54}\|(\vec u,\,\psi)\|_Y^{1-\frac{1-\varepsilon}{N-1}}\cdot\|(\vec u,\,\psi)\|_X^{1+\frac{1-\varepsilon}{N-1}}.\end{aligned}$$

By (3.3), we have

$$I_3 = \|\langle\nabla\rangle^3\psi\|_2\, \|\langle\nabla\rangle^2\partial_x\psi\|_\infty \lesssim s^{-\frac{1}{4}}\|(\vec{u},\, \psi)\|_X \cdot s^{-1}\|(\vec{u},\, \psi)\|_X \lesssim s^{-\frac{5}{4}}\|(\vec{u},\, \psi)\|_X^2.$$

Thus, we have obtained that, for some $\alpha_1 \in (0,\, 1)$,

$$\|\langle\nabla\rangle\Pi_1\|_2 \lesssim s^{-\frac{5}{4}}\|(\vec{u},\, \psi)\|_X^{2-\alpha_1}\|(\vec{u},\, \psi)\|_Y^{\alpha_1}.$$

Through similar computations, for some $\alpha_2 \in (0,\, 1)$, we can get

$$\|\langle\nabla\rangle\partial_x\Pi_1\|_2 \lesssim s^{-\frac{7}{4}}\|(\vec{u},\, \psi)\|_X^{2-\alpha_2}\|(\vec{u},\, \psi)\|_Y^{\alpha_2}.$$

Therefore, by Nash's inequality, we obtain, for some $\alpha \in (0,\, 1)$,

$$\|\Pi_1\|_\infty \lesssim \|\langle\nabla\rangle\partial_x\Pi_1\|_2^{\frac{1}{2}}\, \|\langle\nabla\rangle\Pi_1\|_2^{\frac{1}{2}} \lesssim s^{-\frac{3}{2}}\|(\vec{u},\, \psi)\|_X^{2-\alpha}\|(\vec{u},\, \psi)\|_Y^{\alpha}.$$

Thus we complete the proof of (3.5) and (3.6).

Now, using the inequalities (3.2), (3.3), (3.5), and (3.6) with $j = 1$ and (1.5), we have for small $\eta_0 > 0$, $s \geq 1$,

$$\begin{aligned}
\|u\|_\infty &\lesssim \|\partial_t u\|_\infty + \|\langle\nabla\rangle\partial_x\psi\|_\infty + \|\Pi_1\|_\infty \\
&\lesssim s^{-\frac{3}{2}}\|(\vec{u},\, \psi)\|_X + s^{-1}\|(\vec{u},\, \psi)\|_X + s^{-\frac{3}{2}}\|(\vec{u},\, \psi)\|_Y^{\alpha}\|(\vec{u},\, \psi)\|_X^{2-\alpha} \\
&\lesssim s^{-1}(\|(\vec{u},\, \psi)\|_X + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}) + s^{-1}\eta_0\|(\vec{u},\, \psi)\|_Y.
\end{aligned}$$

$$\begin{aligned}
\|u\|_2 &\lesssim \|\partial_t u\|_2 + \|\langle\nabla\rangle\partial_x\psi\|_2 + \|\Pi_1\|_2 \\
&\lesssim s^{-\frac{5}{4}}\|(\vec{u},\, \psi)\|_X + s^{-\frac{3}{4}}\|(\vec{u},\, \psi)\|_X + s^{-\frac{5}{4}}\|(\vec{u},\, \psi)\|_Y^{\alpha}\|(\vec{u},\, \psi)\|_X^{2-\alpha} \\
&\lesssim s^{-\frac{3}{4}}(\|(\vec{u},\, \psi)\|_X + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}) + s^{-\frac{3}{4}}\eta_0\|(\vec{u},\, \psi)\|_Y.
\end{aligned}$$

$$\begin{aligned}
\|\partial_x u\|_2 &\lesssim \|\partial_x\partial_t u\|_2 + \|\langle\nabla\rangle\partial_{xx}\psi\|_2 + \|\partial_x\Pi_1\|_2 \\
&\lesssim \|\langle\nabla\rangle\partial_t u\|_2 + \|\langle\nabla\rangle\partial_{xx}\psi\|_2 + \|\langle\nabla\rangle\Pi_1\|_2 \\
&\lesssim \|\langle\nabla\rangle\partial_t u\|_2 + \|\langle\nabla\rangle\partial_{xx}\psi\|_2 + s^{-\frac{5}{4}}\|(\vec{u},\, \psi)\|_Y^{\alpha}\|(\vec{u},\, \psi)\|_X^{2-\alpha} \\
&\lesssim s^{-\frac{5}{4}}(\|(\vec{u},\, \psi)\|_X + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}) + s^{-\frac{5}{4}}\eta_0\|(\vec{u},\, \psi)\|_Y.
\end{aligned}$$

Similarly, by equation (1.6) and inequalities (3.3), (3.5), and (3.6) with $j = 2$, we have

$$\begin{aligned}
\|v\|_\infty &\lesssim s^{-\frac{3}{2}}(\|(\vec{u},\, \psi)\|_X + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}) + s^{-\frac{3}{2}}\eta_0\|(\vec{u},\, \psi)\|_Y. \\
\|v\|_2 &\lesssim s^{-\frac{5}{4}}(\|(\vec{u},\, \psi)\|_X + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}) + s^{-\frac{5}{4}}\eta_0\|(\vec{u},\, \psi)\|_Y.
\end{aligned}$$

Therefore, collecting the estimates above, we prove that

$$\|(\vec{u},\, \psi)\|_Y \lesssim \|(\vec{u},\, \psi)\|_X + \eta_0\|(\vec{u},\, \psi)\|_Y + C(\eta_0)\,\|(\vec{u},\, \psi)\|_X^{\frac{2-\alpha}{1-\alpha}}.$$

Choosing η_0 small enough, and noting that $\frac{2-\alpha}{1-\alpha} > 1$, we get

$$\|(\vec{u},\, \psi)\|_Y \lesssim \|(\vec{u},\, \psi)\|_X + Q\big(\|(\vec{u},\, \psi)\|_X\big).$$

This completes the proof of Lemma 1.3. ∎

4. Energy estimates. The rest of the sections are devoted to proving (1.17), namely,

$$\|(\vec{u},\,\psi)\|_X \lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0} + Q\big(\|(\vec{u},\,\psi)\|_Y\big).$$

This section shows that the first term in the definition of the norm of X obeys this inequality.

For this purpose, we first show that, for any real number $\sigma > 0$,

$$\begin{aligned} &\frac{d}{dt}\Big(\big\|\langle\nabla\rangle^\sigma \vec{u}\big\|_{L^2}^2 + \big\|\langle\nabla\rangle^\sigma \vec{b}\big\|_{L^2}^2\Big) \\ &\lesssim \Big(\|\nabla\vec{u}\|_{L^\infty} + \|\nabla\vec{b}\|_{L^\infty}^2\Big)\Big(\|\langle\nabla\rangle^\sigma\vec{u}\|_{L^2}^2 + \|\langle\nabla\rangle^\sigma\vec{b}\|_{L^2}^2\Big). \end{aligned} \tag{4.1}$$

To do so, we take advantage of (1.2), which is equivalent to (1.1). Applying $\langle\nabla\rangle^\sigma$ to (1.2) and taking the inner product with $(\langle\nabla\rangle^\sigma\vec{u}, \langle\nabla\rangle^\sigma\vec{b})$, we obtain, after integrating by parts and invoking $\nabla\cdot\vec{u} = 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\big\|\langle\nabla\rangle^\sigma \vec{u}\big\|_{L^2}^2 + \big\|\langle\nabla\rangle^\sigma \vec{b}\big\|_{L^2}^2\Big) + \big\|\langle\nabla\rangle^\sigma \vec{u}\big\|_{L^2}^2 \\ &\quad = -\int [\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{u}\cdot\langle\nabla\rangle^\sigma\vec{u}\,dxdy - \int [\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{b}\cdot\langle\nabla\rangle^\sigma\vec{b}\,dxdy \\ &\qquad + \int [\langle\nabla\rangle^\sigma, \vec{b}\cdot\nabla]\vec{b}\cdot\langle\nabla\rangle^\sigma\vec{u}\,dxdy + \int [\langle\nabla\rangle^\sigma, \vec{b}\cdot\nabla]\vec{u}\cdot\langle\nabla\rangle^\sigma\vec{b}\,dxdy, \end{aligned} \tag{4.2}$$

where we have used the standard commutator notation

$$[\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{u} = \langle\nabla\rangle^\sigma(\vec{u}\cdot\nabla\vec{u}) - \vec{u}\cdot\nabla\langle\nabla\rangle^\sigma\vec{u}.$$

By Hölder's inequality and a standard commutator estimate, we have

$$\begin{aligned} \left|\int [\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{u}\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx\right| &\le \big\|[\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{u}\big\|_{L^2}\,\big\|\langle\nabla\rangle^\sigma\vec{u}\big\|_{L^2} \\ &\lesssim \big\|\langle\nabla\rangle^\sigma\vec{u}\big\|_{L^2}\,\|\nabla\vec{u}\|_{L^\infty}\,\big\|\langle\nabla\rangle^\sigma\vec{u}\big\|_{L^2}, \end{aligned}$$

moreover, for some constant $C > 0$,

$$\begin{aligned} \left|\int [\langle\nabla\rangle^\sigma, \vec{b}\cdot\nabla]\vec{u}\cdot\langle\nabla\rangle^\sigma\vec{b}\,dx\right| &\le \big\|[\langle\nabla\rangle^\sigma, \vec{b}\cdot\nabla]\vec{u}\big\|_{L^2}\,\big\|\langle\nabla\rangle^\sigma\vec{b}\big\|_{L^2} \\ &\lesssim \Big(\big\|\langle\nabla\rangle^\sigma\vec{u}\big\|_{L^2}\,\|\nabla\vec{b}\|_{L^\infty} + \big\|\langle\nabla\rangle^\sigma\vec{b}\big\|_{L^2}\,\|\nabla\vec{u}\|_{L^\infty}\Big)\,\big\|\langle\nabla\rangle^\sigma\vec{b}\big\|_{L^2} \\ &\le \frac{1}{4}\|\langle\nabla\rangle^\sigma\vec{u}\|_{L^2}^2 + C\Big(\|\nabla\vec{b}\|_{L^\infty}^2 + \|\nabla\vec{u}\|_{L^\infty}\Big)\,\big\|\langle\nabla\rangle^\sigma\vec{b}\big\|_{L^2}^2. \end{aligned}$$

The other two terms can be similarly bounded. We obtain (4.1) after we insert the estimates above in (4.2). Setting $\sigma = N$ and integrating (4.1) in time, we obtain

$$\big\|\langle\nabla\rangle^N(\vec{u},\vec{b})\big\|_{L^2}^2 \lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}^2 + \int_1^t \Big(\|\nabla\vec{u}\|_{L^\infty} + \|\nabla\vec{b}\|_{L^\infty}^2\Big)\cdot\big\|\langle\nabla\rangle^N(\vec{u},\vec{b})\big\|_{L^2}^2\,ds.$$

To bound $\|\nabla\vec{b}\|_{L^\infty}$, by the definition of Y,

$$\|\nabla\vec{b}(s)\|_{L^\infty}^2 \lesssim \|\langle\nabla\rangle^2\psi(s)\|_{L^\infty}^2 \lesssim s^{-1}\,\|(\vec{u},\,\psi)\|_Y^2.$$

Also, we have

$$\|\nabla\vec{u}(s)\|_{L^\infty} \le \|\nabla u(s)\|_{L^\infty} + \|\nabla v(s)\|_{L^\infty} \le s^{-1}\|(\vec{u},\,\psi)\|_Y.$$

Therefore,

$$\begin{aligned}\big\|\langle\nabla\rangle^N(\vec{u},\vec{b})\big\|_{L^2}^2 &\lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}^2 + \int_1^t \langle s\rangle^{-1+2\varepsilon}\,ds\,(\|(\vec{u},\,\psi)\|_Y + \|(\vec{u},\,\psi)\|_Y^2)\,\|(\vec{u},\,\psi)\|_Y^2\\ &\lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}^2 + t^{2\varepsilon}\,(\|(\vec{u},\,\psi)\|_Y^3 + \|(\vec{u},\,\psi)\|_Y^4).\end{aligned}$$

This proves that

$$\sup_{t\ge1}\big(t^{-\varepsilon}\big\|\langle\nabla\rangle^N\big(\vec{u}(t),\,\nabla\psi(t)\big)\big\|_2\big) \lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0} + (\|(\vec{u},\,\psi)\|_Y^3 + \|(\vec{u},\,\psi)\|_Y^4)^{\frac{1}{2}}.$$

5. Estimates of nonlinearities. This section estimates the nonlinear terms F_0, F_1, and F_2, defined in (1.10). These bounds will be used in the proof of (1.17) given in the subsequent sections.

We will use the Littlewood–Paley projection operators. Let $\phi(\xi)$ be a smooth bump function supported in the ball $|\xi| \le 2$ and equal to one on the ball $|\xi| \le 1$. For any real number $M > 0$ and $f \in \mathcal{S}'$ (tempered distributions), the projection operators can be defined as follows:

$$\begin{aligned}\widehat{P_{\le M}f}(\xi) &:= \phi(\xi/M)\,\hat{f}(\xi),\\ \widehat{P_{>M}f}(\xi) &:= (1-\phi(\xi/M))\;\hat{f}(\xi),\\ \widehat{P_M f}(\xi) &:= (\phi(\xi/M)-\phi(2\xi/M))\;\hat{f}(\xi).\end{aligned}$$

We also need the following estimate involving the Riesz transform $\mathcal{R}$.

Lemma 5.1. *For any $\epsilon > 0$,*

$$\big\|\mathcal{R}f\big\|_{L^1_{xy}} \lesssim \big\||\nabla|^{-\epsilon}\langle\nabla\rangle^{2\epsilon}f\big\|_{L^1_{xy}}. \tag{5.1}$$

The proof of Lemma 5.1 is presented in Appendix A.2.

Lemma 5.2. *For any $s \ge 1$,*

$$\big\|\langle\nabla\rangle^5|\nabla|^{\frac{1}{2}-\epsilon}F_1(s,\cdot)\big\|_{L^1_{xy}} \lesssim s^{-\frac{3}{2}-\varepsilon}Q\big(\|(\vec{u},\,\psi)\|_Y\big), \tag{5.2}$$

where ϵ is same as in Corollary 2.8*, and ε is same as in* (1.15)*.*

Proof. By definition (1.12),

$$F_1 = -\,\partial_x\partial_y\big(u\,\partial_x\psi\big) - \partial_x\partial_y\big(v\,\partial_y\psi\big) + \partial_t\Pi_1, \tag{5.3}$$

where Π_1 is explicitly given in Appendix A.1. Since F_1 is a quadratic nonlinearity, we write

$$F_1 = F_{11}(u,v) + F_{12}(u,\psi) + F_{13}(v,\psi) + F_{14}(u,u) + F_{15}(v,v) + F_{16}(\psi,\psi),$$

where $F_{11}(u,v)$ is a collection of the terms which contain the unknown function (u,v), and $F_{12},\cdots,F_{16}$ are similarly defined. To bound these terms, wesplit F_1 into low frequency and high frequency parts. More precisely, for small $\delta > 0$ to be specified later, we write

$$F_1 := F_{1,high} + F_{1,low}, \tag{5.4}$$

where

$$\begin{aligned} F_{1,high} &= F_{11}\big((1-P_{\le s^\delta})u, v\big) + F_{11}\big(P_{\le s^\delta}u, (1-P_{\le s^\delta})v\big) + \cdots, \\ F_{1,low} &= F_{11}\big(P_{\le s^\delta}u, P_{\le s^\delta}v\big) + \cdots, \end{aligned}$$

that is, each term in $F_{1,high}$ contains at least one high frequency part and the terms in $F_{1,low}$ involve only low frequencies.

Although the number of the terms in F_1 is large, they can be treated similarly. For the sake of clarity, we shall only present the estimates for a representative term. That is, we write

$$F_1 = \frac{\partial_y\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y\psi_t\big) + \text{similar terms}. \tag{5.5}$$

We now focus on the representative term $\frac{\partial_y\partial_y}{\Delta}\big(\partial_y\psi\partial_x\partial_y\psi_t\big)$. As in (5.4), we split it into high and low frequency terms. First, we deal with those involving high frequencies, which can be treated in a standard way (see [17] for some related analysis). We focus on $\frac{\partial_y\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big)$ and by (5.1),

$$\begin{aligned} &\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_y\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big)\Big\|_{L^1_{xy}} \\ &\lesssim \Big\|\langle\nabla\rangle^{5+2\epsilon}|\nabla|^{\frac12-2\epsilon}\big(\partial_y\psi\,\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big)\Big\|_{L^1_{xy}} \\ &\lesssim \Big\|\langle\nabla\rangle^{6}\big(\partial_y\psi\,\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big)\Big\|_{L^1_{xy}} \\ &\lesssim \Big\|\langle\nabla\rangle^{6}\partial_y\psi\Big\|_{L^2_{xy}}\big\|\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big\|_{L^2_{xy}} + \big\|\partial_y\psi\big\|_{L^2_{xy}}\Big\|\langle\nabla\rangle^{6}\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\Big\|_{L^2_{xy}} \\ &\lesssim \Big\|\langle\nabla\rangle^{7}\psi\Big\|_{L^2_{xy}}\Big\|\langle\nabla\rangle^{8} P_{\gtrsim s^\delta}\psi_t\Big\|_{L^2_{xy}}. \end{aligned}$$

Since $\psi_t = -v - \vec{u}\cdot\nabla\psi$, we have, for large enough N,

$$\begin{aligned} \Big\|\langle\nabla\rangle^{8} P_{\gtrsim s^\delta}\psi_t\Big\|_{L^2_{xy}} &\lesssim \Big\|\langle\nabla\rangle^{8} P_{\gtrsim s^\delta}v\Big\|_{L^2_{xy}} + \Big\|\langle\nabla\rangle^{8} P_{\gtrsim s^\delta}\big(\vec{u}\cdot\nabla\psi\big)\Big\|_{L^2_{xy}} \\ &\lesssim s^{-(N-8)\delta}\Big(\big\|\langle\nabla\rangle^N v\big\|_{L^2_{xy}} + \big\|\langle\nabla\rangle^N\big(\vec{u}\cdot\nabla\psi\big)\big\|_{L^2_{xy}}\Big) \\ &\lesssim s^{-(N-8)\delta}\Big(\big\|\langle\nabla\rangle^N v\big\|_{L^2_{xy}} + \big\|\langle\nabla\rangle^N \vec{u}\big\|_{L^2_{xy}}\cdot\big\|\langle\nabla\rangle^{N+1}\psi\big\|_{L^2_{xy}}\Big) \\ &\lesssim s^{-\frac32-2\varepsilon}\Big(\|(\vec{u},\psi)\|_Y + \|(\vec{u},\psi)\|_Y^2\Big). \end{aligned}$$

Therefore, combining the above two estimates, we obtain that

$$\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_y\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y P_{\gtrsim s^\delta}\psi_t\big)\Big\|_{L^1_{xy}} \lesssim s^{-\frac32-\varepsilon}Q\big(\|(\vec{u},\,\psi)\|_Y\big).$$

Now we turn to $F_{1,low}$. Again, using (5.1), we have

$$\begin{aligned}
&\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_y\partial_y}{\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi_t\big)\Big\|_{L^1_{xy}}\\
&=\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_y\partial_y}{\Delta}P_{\le 4s^\delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi_t\big)\Big\|_{L^1_{xy}}\\
&\lesssim\Big\|\langle\nabla\rangle^{5+2\epsilon}|\nabla|^{\frac12-2\epsilon}P_{\le 4s^\delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi_t\big)\Big\|_{L^1_{xy}}\\
&\lesssim s^{6\delta}\big\|\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi_t\big\|_{L^1_{xy}}\\
&\lesssim s^{6\delta}\big\|\partial_y P_{\le s^\delta}\psi\big\|_{L^2_{xy}}\big\|\partial_x\partial_y P_{\le s^\delta}\psi_t\big\|_{L^2_{xy}}\\
&\lesssim s^{8\delta}\big\|\psi\big\|_{L^2_{xy}}\big\|\partial_x\psi_t\big\|_{L^2_{xy}}. \qquad (5.6)
\end{aligned}$$

Now we need the following estimate:

$$\big\|\partial_x\psi_t\big\|_{L^2_{xy}}\lesssim s^{-\frac32}\big(\|(\vec u,\psi)\|_Y+Q\big(\|(\vec u,\,\psi)\|_Y\big)\big). \qquad (5.7)$$

Indeed, by equations (1.6) and (1.7), we have

$$\begin{aligned}
\big\|\partial_x\psi_t\big\|_{L^2_{xy}}&\lesssim\|\partial_x v\|_2+\|\partial_x(\vec u\cdot\nabla\psi)\|_2\\
&\lesssim\|\partial_x\partial_t v\|_2+\|\partial_x^3\psi\|_2+\big(\|\partial_x u\|_2+\|\partial_x v\|_2\big)\|\langle\nabla\rangle\psi\|_\infty\\
&\quad+\big(\|u\|_\infty+\|v\|_\infty\big)\|\langle\nabla\rangle\partial_x\psi\|_2+\|\partial_x\Pi_2\|_2. \qquad (5.8)
\end{aligned}$$

As in the proof of (3.6), we have

$$\|\partial_x\Pi_2\|_2\lesssim s^{-\frac32}Q\big(\|(\vec u,\,\psi)\|_Y\big).$$

Then, (5.7) follows from (5.8). By (5.6) and (5.7), for any $8\delta\le\frac14-\varepsilon$, we get

$$\begin{aligned}
\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_y\partial_y}{-\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi_t\big)\Big\|_{L^1_{xy}}&\lesssim s^{8\delta}s^{-\frac74}\|(\vec u,\psi)\|_Y^2\\
&\lesssim s^{-\frac32-\varepsilon}Q\big(\|(\vec u,\,\psi)\|_Y\big).
\end{aligned}$$

This proves the lemma. ▯

LEMMA 5.3. *Let ϵ,ε be the same as in Lemma 5.2. Then for any $s\ge1$,*

$$\big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}F_2\big\|_{L^1_{xy}}\lesssim s^{-\frac32-\varepsilon}Q\big(\|(\vec u,\,\psi)\|_Y\big). \qquad (5.9)$$

Proof. As before, we write

$$\begin{aligned}
F_2&=\partial_t\Pi_2+\partial_x\partial_x\big(\vec u\,\nabla\psi\big)\\
&=-2\frac{\partial_x\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y\psi_t\big)+\text{similar terms}. \qquad (5.10)
\end{aligned}$$

Since (5.10) has a similar form to (5.3), we have the same estimate as the one for F_1. The details are omitted here. ▯

The nonlinear term F_0 behaves quite differently from F_1, F_2, indeed, we have the following.

LEMMA 5.4. *Let ϵ, ε be the same as in Lemma 5.2. Then for any $s \geq 1$,*

$$\big\|\langle\nabla\rangle^{\frac{11}{2}+2\epsilon}|\nabla|^{\frac12-\epsilon}F_0\big\|_{L^1_{xy}} \lesssim s^{-1-\varepsilon}Q\big(\|(\vec{u},\,\psi)\|_Y\big); \tag{5.11}$$

$$\big\|\langle\nabla\rangle^{5}\nabla F_0\big\|_{L^1_{xy}} \lesssim s^{-\frac32+\varepsilon}Q\big(\|(\vec{u},\,\psi)\|_Y\big); \tag{5.12}$$

$$\big\|\langle\nabla\rangle^{5}\nabla\partial_x F_0\big\|_{L^1_{xy}} \lesssim s^{-\frac32-\varepsilon}Q\big(\|(\vec{u},\,\psi)\|_Y\big). \tag{5.13}$$

Proof. By (1.11), combining with (1.5) and (1.6), we write

$$\begin{aligned} F_0 &= -\,(\vec{u}+\vec{u}_t)\cdot\nabla\phi - \Pi_2 - \vec{u}\cdot\nabla\psi_t\\ &= -\,\Pi_2 - \partial_x\partial_y\psi\,\partial_x\psi - \Pi_1\,\partial_x\psi + \partial_x^2\psi\,\partial_y\psi - \Pi_2\,\partial_y\psi - u\,\partial_x\psi_t - v\,\partial_y\psi_t\\ &= -\,\frac{\partial_x\partial_y}{\Delta}\big(\partial_y\psi\,\partial_x\partial_y\psi\big) + \text{similar terms}. \end{aligned} \tag{5.14}$$

We also write

$$F_0 := F_{0,low} + F_{0,high},$$

where

$$F_{0,low} = F_0\big(P_{\leq s^\delta}u, P_{\leq s^\delta}v, P_{\leq s^\delta}\psi\big).$$

We only consider $F_{0,low}$, since the part $F_{0,high}$ can be treated the same way as in the proof of Lemma 5.2. We need the following lemma.

LEMMA 5.5. *Let $\alpha > 0, N_0 \geq 0$, then for any $\epsilon > 0, f \in \mathcal{S}$,*

$$\Big\||\nabla|^{\alpha}\langle\nabla\rangle^{N_0}\frac{\partial_x\partial_y}{-\Delta}f\Big\|_{L^1_{xy}} \lesssim \|f\|_{L^1_{xy}}^{1-\alpha+\epsilon}\|\partial_x f\|_{L^1_{xy}}^{\alpha-\epsilon} + \big\|\langle\nabla\rangle^{N_0-1+\alpha+\epsilon}\partial_x f\big\|_{L^1_{xy}}. \tag{5.15}$$

The proof of Lemma 5.5 is presented in Appendix A.3.

To prove (5.11), we use Lemma 5.5 for $\alpha = \frac12 - \epsilon$ to get

$$\begin{aligned} &\Big\|\langle\nabla\rangle^{\frac{11}{2}+2\epsilon}|\nabla|^{\frac12-\epsilon}\frac{\partial_x\partial_y}{-\Delta}\big(\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\ &\lesssim \Big\|\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\Big\|_{L^1_{xy}}^{\frac12+2\epsilon}\Big\|\partial_x\big(\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\big)\Big\|_{L^1_{xy}}^{\frac12-2\epsilon}\\ &\quad+\Big\|\langle\nabla\rangle^{6+2\epsilon}\partial_x\big(\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\big)\Big\|_{L^1_{xy}}. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} \Big\|\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\Big\|_{L^1_{xy}} &\lesssim \big\|\partial_y P_{\leq s^\delta}\psi\big\|_{L^2_{xy}}\big\|\partial_x\partial_y P_{\leq s^\delta}\psi\big\|_{L^2_{xy}}\\ &\lesssim s^{2\delta}\big\|\psi\big\|_{L^2_{xy}}\big\|\partial_x\psi\big\|_{L^2_{xy}} \lesssim s^{2\delta-1}\big\|(\vec{u},\psi)\big\|_Y^2. \end{aligned} \tag{5.16}$$

Moreover,

$$\begin{aligned} &\Big\|\partial_x\big(\partial_y P_{\leq s^\delta}\psi\,\partial_x\partial_y P_{\leq s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\ &\lesssim \big\|\partial_x\partial_y P_{\leq s^\delta}\psi\big\|_{L^2_{xy}}^2 + \big\|\partial_y P_{\leq s^\delta}\psi\big\|_{L^2_{xy}}\big\|\partial_x^2\partial_y P_{\leq s^\delta}\psi\big\|_{L^2_{xy}}\\ &\lesssim s^{2\delta}\big\|\partial_x\psi\big\|_{L^2_{xy}}^2 + s^{2\delta}\big\|\psi\big\|_{L^2_{xy}}\big\|\partial_x^2\psi\big\|_{L^2_{xy}}\\ &\lesssim s^{2\delta-\frac32}\big\|(\vec{u},\psi)\big\|_Y^2, \end{aligned} \tag{5.17}$$

and using (5.17),

$$\big\|\langle\nabla\rangle^{6+2\epsilon}\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\big\|_{L^1_{xy}} \lesssim s^{(6+2\epsilon)\delta}\Big\|\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}} \lesssim s^{9\delta-\frac32}\big\|(\vec u,\psi)\big\|_Y^2. \tag{5.18}$$

Therefore, combining (5.16)–(5.18), we deduce that, for any $\delta>0$ satisfying $9\delta+\varepsilon\le\frac12$,

$$\Big\|\langle\nabla\rangle^5|\nabla|^{\frac12-\epsilon}\frac{\partial_x\partial_y}{-\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}\big\|(\vec u,\psi)\big\|_Y^2.$$

For (5.12), we only need to prove

$$\Big\|\langle\nabla\rangle^5\nabla\frac{\partial_x\partial_y}{-\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\lesssim s^{-\frac32+\varepsilon}Q\big(\|(\vec u,\,\psi)\|_Y\big).$$

By Lemmas 5.1 and 5.5 for $\alpha=1-\epsilon$, and by (5.16)–(5.18) we have

$$\begin{aligned}
&\Big\|\langle\nabla\rangle^5\nabla\frac{\partial_x\partial_y}{-\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\Big\|\langle\nabla\rangle^{5+2\epsilon}|\nabla|^{1-\epsilon}\frac{\partial_x\partial_y}{-\Delta}\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\Big\|\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\Big\|_{L^1_{xy}}^{2\epsilon}\Big\|\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}^{1-2\epsilon}\\
&\quad+\Big\|\langle\nabla\rangle^{6+2\epsilon}\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\big(s^{2\epsilon(2\delta-1)+(1-2\epsilon)(2\delta-\frac32)}+s^{9\delta-\frac32}\big)\big\|(\vec u,\psi)\big\|_Y^2\\
&\lesssim s^{-\frac32+\varepsilon}\big\|(\vec u,\psi)\big\|_Y^2.
\end{aligned}$$

This gives (5.12). To prove (5.13), we only need to prove

$$\Big\|\langle\nabla\rangle^5\nabla\frac{\partial_x\partial_y}{-\Delta}\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\lesssim s^{-\frac32-\varepsilon}Q\big(\|(\vec u,\,\psi)\|_Y\big).$$

Again, similarly as for (5.17), we have

$$\Big\|\partial_x^2\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\lesssim s^{2\delta-\frac74}\big\|(\vec u,\psi)\big\|_Y^2.$$

Hence, using this estimate and (5.17),

$$\begin{aligned}
&\Big\|\langle\nabla\rangle^5\nabla\frac{\partial_x\partial_y}{-\Delta}\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\Big\|\langle\nabla\rangle^{5+2\epsilon}|\nabla|^{1-\epsilon}\frac{\partial_x\partial_y}{-\Delta}\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\Big\|\partial_x\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}^{2\epsilon}\Big\|\partial_x^2\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}^{1-2\epsilon}\\
&\quad+\Big\|\langle\nabla\rangle^{6+2\epsilon}\partial_x^2\big(\partial_y P_{\le s^\delta}\psi\,\partial_x\partial_y P_{\le s^\delta}\psi\big)\Big\|_{L^1_{xy}}\\
&\lesssim\big(s^{2\epsilon(2\delta-\frac32)+(1-2\epsilon)(2\delta-\frac74)}+s^{9\delta-\frac74}\big)\big\|(\vec u,\psi)\big\|_Y^2\\
&\lesssim s^{-\frac32-\varepsilon}\big\|(\vec u,\psi)\big\|_Y^2,
\end{aligned}$$

where $\delta>0$ and satisfies $9\delta+\varepsilon\le\frac14$. This gives (5.13). This completes the proof of Lemma 5.4. ∎

6. Estimates of $\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty$, $\|\partial_t\vec{u}\|_\infty$, $\|\langle\nabla\rangle\partial_x\vec{u}\|_\infty$. This section continues the proof for (1.17). For the sake of clarity, we divide this section into subsections with each devoted to one of the terms. The tool lemmas in section 2 will be used extensively here.

6.1. Estimate of $\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty$. Using the Duhamel formula, namely (2.4),

$$\psi(t,\,x,y) = K_0(t,\,\partial_x)\psi_0 + K_1(t,\,\partial_x)\left(\frac{1}{2}\psi_0+\psi_1\right) + \int_1^t K_1(t-s,\,\partial_x)F_0(s)\,ds.$$

For notational convenience, we may sometimes write $K_0(t)$ for $K_0(t,\,\partial_x)$ and $K_1(t)$ for $K_1(t,\,\partial_x)$. Therefore,

$$\begin{aligned}\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty \lesssim\ & \|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty + \|\langle\nabla\rangle\partial_{xx}K_1(t)\left(\frac{1}{2}\psi_0+\psi_1\right)\|_\infty\\ & + \|\int_1^t\langle\nabla\rangle\partial_{xx}K_1(t-s)F_0(s)ds\|_\infty.\end{aligned}$$

By Corollary 2.5 and Lemma 2.2,

$$\begin{aligned}&\|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty\\ &\lesssim \big(\|\widehat{\partial_{xx}K_0}(t,\,\xi)\|_{L^1_\xi(|\xi|\le\frac12)} + \|\widehat{K_0}(t,\,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)}\big)\,\|\langle\nabla\rangle^{2+\varepsilon}\partial_{xx}\partial_y\psi_0\|_{L^1_{xy}}\\ &\lesssim \big(t^{-\frac32}+e^{-\frac t2}\big)\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{L^1_{xy}} \lesssim t^{-\frac32}\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{X_0}.\end{aligned}$$

Since the estimates for K_0 and K_1 are the same, we also have

$$\|\langle\nabla\rangle\partial_{xx}K_1(t)\left(\frac12\psi_0+\psi_1\right)\|_\infty \lesssim t^{-\frac32}\big\|\langle\nabla\rangle^{5+\varepsilon}\left(\frac12\psi_0+\psi_1\right)\big\|_{X_0}.$$

Moreover,

$$\begin{aligned}&\left\|\int_1^t\langle\nabla\rangle\partial_{xx}K_1(t-s)\,F_0(s)\,ds\right\|_\infty\\ &\lesssim \int_1^t\|\partial_{xx}K_1(t-s)\,\langle\nabla\rangle F_0(s)\|_\infty\,ds\\ &\lesssim \int_1^{\frac t2}\|\partial_{xx}K_1(t-s)\,\langle\nabla\rangle F_0(s)\|_\infty\,ds + \int_{\frac t2}^t\|\partial_xK_1(t-s)\,\langle\nabla\rangle\partial_xF_0(s)\|_\infty\,ds.\end{aligned}\tag{6.1}$$

Now we consider these two parts separately. By Corollary 2.5, and Lemmas 2.2 and 5.4,

$$\begin{aligned}&\int_1^{\frac t2}\|\partial_{xx}K_1(t-s)\,\langle\nabla\rangle F_0(s)\|_\infty\,ds\\ &\lesssim \int_1^{\frac t2}(\|\widehat{\partial_{xx}K_1}(t-s,\,\xi)\|_{L^1_\xi(|\xi|\le\frac12)} + \|\widehat{K_1}(t-s,\,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)})\|\nabla\langle\nabla\rangle^{4+\varepsilon}F_0(s)\|_{L^1_{xy}}\,ds\\ &\lesssim \int_1^{\frac t2}\langle t-s\rangle^{-\frac32}\|\,\nabla\langle\nabla\rangle^{4+\varepsilon}F_0(s)\|_{L^1_{xy}}\,ds\\ &\lesssim \int_1^{\frac t2}\langle t-s\rangle^{-\frac32}s^{-\frac32+\varepsilon}\,ds\cdot Q\big(\|(\vec u,\,\psi)\|_Y\big)\\ &\lesssim t^{-\frac32}Q\big(\|(\vec u,\,\psi)\|_Y\big),\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} \left\|\partial_x K_1(t-s)\,\langle\nabla\rangle \partial_x F_0(s)\right\|_\infty ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \left(\left\|\widehat{\partial_x K_1}(t-s,\,\xi)\right\|_{L^1_\xi(|\xi|\le\frac12)} + \left\|\widehat{K_1}(t-s,\,\xi)\right\|_{L^\infty_\xi(|\xi|\ge\frac12)}\right)\|\langle\nabla\rangle^{3+\varepsilon}\nabla\partial_x F_0(s)\|_{L^1_{xy}}\,ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-1} s^{-\frac32-\varepsilon}\,ds\, Q\big(\|(\vec{u},\,\psi)\|_Y\big) \\
&\lesssim t^{-\frac32}\, Q\big(\|(\vec{u},\,\psi)\|_Y\big).
\end{aligned}
$$

Combining the estimates above, we obtain

$$
\sup_{t\ge1}\left(t^{\frac32}\left\|\langle\nabla\rangle\partial_{xx}\psi(t)\right\|_\infty\right) \lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0} + Q\big(\|(\vec{u},\,\psi)\|_Y\big).
$$

6.2. Estimate of $\|\langle\nabla\rangle\partial_t\vec{u}\|_\infty$. Using the Duhamel formula and noting that $K_1(0)=0$, we obtain

$$
\partial_t u(t,\,x) = \dot{K}_0(t)u_0 + \dot{K}_1(t)\left(\frac12 u_0 + u_1\right) + \int_1^t \dot{K}_1(t-s)\,F_1(s)\,ds,
$$

then we have

$$
\begin{aligned}
\|\partial_t\langle\nabla\rangle u(t)\|_\infty \le& \|\dot{K}_0(t)\langle\nabla\rangle u_0\|_\infty + \|\dot{K}_1(t)\langle\nabla\rangle\left(\frac12 u_0+u_1\right)\|_\infty \\
&+ \|\int_1^t \dot{K}_1(t-s)\,\langle\nabla\rangle F_1(s)\,ds\|_\infty.
\end{aligned}
$$

For the linear parts, by Lemmas 2.2, 2.3, and 2.4, we have

$$
\begin{aligned}
\|\dot{K}_0(t)\langle\nabla\rangle u_0\|_{L^\infty_\xi} \lesssim& \left\|\widehat{\dot{K}_0}(t,\xi)\right\|_{L^1_\xi(|\xi|\le\frac12)}\left\|\partial_y\langle\nabla\rangle u_0\right\|_{L^1_{xy}} \\
&+ \left\|\langle\xi\rangle^{-1}\widehat{\dot{K}_0}(t,\xi)\right\|_{L^\infty_\xi(|\xi|\ge\frac12)}\left\|\partial_y\langle\nabla\rangle^{3+\epsilon}u_0\right\|_{L^1_{xy}} \\
\lesssim& t^{-\frac32}\left\|\langle\nabla\rangle^{4+\epsilon}u_0\right\|_{L^1_{xy}}.
\end{aligned}
$$

Also, we have

$$
\|\dot{K}_1(t)\langle\nabla\rangle\left(\frac12 u_0+u_1\right)\|_{L^\infty_{xy}} \lesssim t^{-\frac32}\left\|\langle\nabla\rangle^{3+\epsilon}\left(\frac12 u_0+u_1\right)\right\|_{L^1_{xy}}.
$$

For the nonlinear part, by Lemma 5.2 we have

$$
\begin{aligned}
\|\int_1^t \dot{K}_1(t-s)\langle\nabla\rangle F_1(s)\,ds\|_\infty \lesssim& \int_1^t \Big(\|\widehat{\dot{K}_1}(t-s,\,\xi)\|_{L^1_\xi(|\xi|\le\frac12)} \\
&+ \|\widehat{\dot{K}_1}(t-s,\,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)}\Big)\cdot\|\langle\nabla\rangle^{2+\varepsilon}\nabla F_1(s)\|_{L^1_{xy}}\,ds \\
\lesssim& \int_1^t \langle t-s\rangle^{-\frac32}\,s^{-\frac32-\varepsilon}ds\cdot Q(\|(\vec{u},\,\psi)\|_Y) \\
\lesssim& t^{-\frac32}Q(\|(\vec{u},\,\psi)\|_Y).
\end{aligned}
$$

Combining the estimates above, we deduce that

$$\sup_{t\geq 1}\Big(t^{\frac{3}{2}}\big\|\langle\nabla\rangle\partial_t u(t)\big\|_\infty\Big)\lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}+Q\big(\|(\vec{u},\,\psi)\|_Y\big).$$

Similarly, by the Duhamel formula, we have

$$\partial_t v(t,\,x)=\dot{K}_0(t)v_0+\dot{K}_1(t)\left(\frac{1}{2}v_0+v_1\right)+\int_1^t\dot{K}_1(t-s)\,F_2(s)\,ds.$$

Using Lemma 5.3 instead, it obeys a similar estimate as the one for u. Therefore,

$$\sup_{t\geq 1}\Big(t^{\frac{3}{2}}\big\|\langle\nabla\rangle\partial_t \vec{u}(t)\big\|_\infty\Big)\lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}+Q\big(\|(\vec{u},\,\psi)\|_Y\big).$$

6.3. Estimate of $\|\partial_x\langle\nabla\rangle\vec{u}(t)\|_\infty$. The estimate for $\|\partial_x\langle\nabla\rangle u(t)\|_\infty$ is similar to that of $\|\partial_t\langle\nabla\rangle u(t)\|_\infty$:

$$\begin{aligned}\|\partial_x\langle\nabla\rangle u(t)\|_\infty &\lesssim \|\partial_x K_0(t)\langle\nabla\rangle u_0\|_\infty+\|\partial_x K_1(t)\langle\nabla\rangle\left(\frac{1}{2}u_0+u_1\right)\|_\infty\\ &\quad+\|\int_1^t\partial_x K_1(t-s)\,\langle\nabla\rangle F_1(s)\,ds\|_\infty\\ &\lesssim t^{-1}(\|\langle\nabla\rangle^{3+\varepsilon}u_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{3+\varepsilon}u_1\|_{L^1_{xy}})\\ &\quad+\int_1^t\langle t-s\rangle^{-\frac{3}{2}}\,s^{-\frac{3}{2}-\varepsilon}ds\cdot Q(\|(\vec{u},\,\psi)\|_Y)\\ &\lesssim t^{-1}\big(\|(\vec{u}_0,\,\psi_0)\|_{X_0}+Q\big(\|(\vec{u},\,\psi)\|_Y\big)\big).\end{aligned}$$

$\|\partial_x\langle\nabla\rangle v(t)\|_\infty$ can be bounded similarly as the one for $\|\partial_x\langle\nabla\rangle u(t)\|_\infty$. We omit the details. Thus,

$$\sup_{t\geq 1}\Big(t\,\big\|\langle\nabla\rangle\partial_x \vec{u}(t)\big\|_\infty\Big)\lesssim \|(\vec{u}_0,\,\psi_0)\|_{X_0}+Q\big(\|(\vec{u},\,\psi)\|_Y\big).$$

7. Estimates of $\|\langle\nabla\rangle^3\psi\|_2$, $\|\langle\nabla\rangle^3\partial_x^2\psi\|_2$, $\|\partial_x^3\psi\|_2$, $\|\langle\nabla\rangle\partial_t\vec{u}\|_2$, and $\|\partial_x\partial_t v\|_2$.

The estimates of $\|\langle\nabla\rangle^3\psi\|_2$, $\|\langle\nabla\rangle^2\partial_x^2\psi\|_2$, $\|\partial_x^3\psi\|_2$, $\|\langle\nabla\rangle\partial_t\vec{u}\|_2$, and $\|\partial_x\partial_t v\|_2$ can be similarly obtained as in section 6.

By Lemma 2.2, Corollary 2.8, and (5.11),

$$\begin{aligned}\|\langle\nabla\rangle^3\psi\|_2 &\lesssim \|K_0(t)\langle\nabla\rangle^3\psi_0\|_2+\|K_1(t)\langle\nabla\rangle^3\left(\frac{1}{2}\psi_0+\psi_1\right)\|_2\\ &\quad+\|\int_1^t K_1(t-s)\langle\nabla\rangle^3F_0(s)\,ds\|_{L^2}\\ &\lesssim t^{-\frac{1}{4}}(\|\langle\nabla\rangle^{4+\varepsilon}u_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{4+\varepsilon}u_1\|_{L^1_{xy}})+\int_1^t(\|\widehat{K_1}(t-s,\,\xi)\|_{L^2_\xi(|\xi|\leq\frac{1}{2})}\\ &\quad+\|\widehat{K_1}(t-s,\,\xi)\|_{L^\infty_\xi(|\xi|\geq\frac{1}{2})})\|\,|\nabla|^{\frac{1}{2}-\varepsilon}\langle\nabla\rangle^{\frac{7}{2}+2\varepsilon}F_0(s)\|_{L^1_{xy}}\,ds\\ &\lesssim t^{-\frac{1}{4}}\|(\vec{u}_0,\,\psi_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{1}{4}}\,s^{-1-\varepsilon}ds\cdot Q(\|(\vec{u},\,\psi)\|_Y)\\ &\lesssim t^{-\frac{1}{4}}(\|(\vec{u}_0,\,\psi_0)\|_{X_0}+Q(\|(\vec{u},\,\psi)\|_Y)).\end{aligned}$$

We define a Fourier multiplier operator $|\partial_x|^\alpha, \alpha \in \mathbb{R}$ as

$$|\partial_x|^\alpha f(x,y) = \int e^{ix\xi+iy\eta} |\xi|^\alpha \widehat{f}(\xi,\eta)\, d\xi\, d\eta.$$

Then as before, by Lemma 2.2, Corollary 2.8, and Lemma 5.4, we have

$$\begin{aligned}
\|\langle\nabla\rangle^3 \partial_x^2 \psi\|_2 \lesssim\ & \|\partial_x^2 K_0(t)\langle\nabla\rangle^3\psi_0\|_2 + \|\partial_x^2 K_1(t)\langle\nabla\rangle^3 \left(\frac{1}{2}\psi_0+\psi_1\right)\psi_0\|_2 \\
& + \|\int_1^t \partial_x^2 K_1(t-s)\langle\nabla\rangle^2 F_0(s)\, ds\|_{L^2} \\
\lesssim\ & t^{-\frac{5}{4}}(\|\langle\nabla\rangle^{6+\varepsilon}u_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{6+\varepsilon}u_1\|_{L^1_{xy}}) \\
& + \int_1^{\frac{t}{2}} \|\partial_x^2 K_1(t-s)\langle\nabla\rangle^2 F_0(s)\|_{L^2}\, ds \\
& + \int_{\frac{t}{2}}^t \big\||\partial_x|^{\frac{3}{2}-2\varepsilon} K_1(t-s)\langle\nabla\rangle^2 |\partial_x|^{\frac{1}{2}+2\varepsilon} F_0(s)\big\|_{L^2}\, ds \\
\lesssim\ & t^{-\frac{5}{4}}\|(\vec{u}_0,\, \psi_0)\|_{X_0} + \int_1^{\frac{t}{2}} \Big(\|\widehat{\partial_x^2 K_1}(t-s,\, \xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})} \\
& + \|\widehat{K_1}(t-s,\,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})}\Big)\|\,|\nabla|^{\frac{1}{2}-\varepsilon}\langle\nabla\rangle^{\frac{11}{2}+2\varepsilon} F_0(s)\|_{L^1_{xy}}\, ds \\
& + \int_{\frac{t}{2}}^t \Big(\big\||\xi|^{\frac{3}{2}-2\varepsilon}\widehat{K_1}(t-s,\,\xi)\big\|_{L^2_\xi(|\xi|\le\frac{1}{2})} \\
& + \big\|\widehat{K_1}(t-s,\,\xi)\big\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})}\Big)\|\,\langle\nabla\rangle^{4+2\varepsilon}\nabla F_0(s)\|_{L^1_{xy}}\, ds \\
\lesssim\ & t^{-\frac{5}{4}}\|(\vec{u}_0,\, \psi_0)\|_{X_0} + \int_1^{\frac{t}{2}} \langle t-s\rangle^{-\frac{5}{4}}\, s^{-1-\varepsilon} ds \cdot Q\big(\|(\vec{u},\, \psi)\|_Y\big) \\
& + \int_{\frac{t}{2}}^t \langle t-s\rangle^{-1+\varepsilon}\, s^{-\frac{3}{2}+\varepsilon} ds \cdot Q\big(\|(\vec{u},\, \psi)\|_Y\big) \\
\lesssim\ & t^{-\frac{5}{4}}\big(\|(\vec{u}_0,\, \psi_0)\|_{X_0} + Q\big(\|(\vec{u},\, \psi)\|_Y\big)\big).
\end{aligned}$$

Now we consider $\|\partial_x^3\psi\|_2$. Similarly as above,

$$\begin{aligned}
\|\partial_x^3\psi\|_2 \lesssim\ & \|\partial_x^3 K_0(t)\psi_0\|_2 + \|\partial_x^3 K_1(t)\left(\frac{1}{2}\psi_0+\psi_1\right)\|_2 \\
& + \|\int_1^t \partial_x^3 K_1(t-s) F_0(s)\, ds\|_{L^2} \\
\lesssim\ & t^{-\frac{7}{4}}(\|\langle\nabla\rangle^{4+\varepsilon}u_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{4+\varepsilon}u_1\|_{L^1_{xy}}) + \int_1^{\frac{t}{2}} \|\partial_x^3 K_1(t-s)F_0(s)\|_{L^2}\, ds \\
& + \int_{\frac{t}{2}}^t \|\,|\partial_x|^{\frac{3}{2}-2\varepsilon} K_1(t-s)|\partial_x|^{\frac{3}{2}+2\varepsilon} F_0(s)\|_{L^2}\, ds.
\end{aligned}$$

We proceed in the same way as in the previous estimate:

$$
\begin{aligned}
\|\partial_x^3\psi\|_2 \lesssim\ & t^{-\frac{7}{4}}\|(\vec{u}_0,\, \psi_0)\|_{X_0} + \int_1^{\frac{t}{2}} \langle t-s\rangle^{-\frac{7}{4}} \big\| |\nabla|^{\frac12-\varepsilon}\langle\nabla\rangle^{\frac72+2\varepsilon} F_0(s)\big\|_{L^1_{xy}} ds \\
& + \int_{\frac{t}{2}}^t \langle t-s\rangle^{-1+\varepsilon}\, \|\nabla\partial_x\langle\nabla\rangle^{2+2\varepsilon}F_0\|_{L^1_{xy}} ds \\
\lesssim\ & t^{-\frac{7}{4}}\|(\vec{u}_0,\, \psi_0)\|_{X_0} + \int_1^{\frac{t}{2}} \langle t-s\rangle^{-\frac{7}{4}}\, s^{-1-\varepsilon} ds \cdot Q\big(\|(\vec{u},\, \psi)\|_Y\big) \\
& + \int_{\frac{t}{2}}^t \langle t-s\rangle^{-1+\varepsilon}\, s^{-\frac32-\varepsilon} ds \cdot Q\big(\|(\vec{u},\, \psi)\|_Y\big) \\
\lesssim\ & t^{-\frac32}\big(\|(\vec{u}_0,\, \psi_0)\|_{X_0} + Q\big(\|(\vec{u},\, \psi)\|_Y\big)\big).
\end{aligned}
$$

We now bound $\|\langle\nabla\rangle\partial_t\vec{u}\|_2$. By the Duhamel formula,

$$
\begin{aligned}
\|\langle\nabla\rangle\partial_t u\|_2 \lesssim\ & \|\dot{K}_0(t)\langle\nabla\rangle u_0\|_2 + \|\dot{K}_1(t)\langle\nabla\rangle\left(\frac12 u_0 + u_1\right)\|_2 \\
& + \Big\|\int_1^t \dot{K}_1(t-s)\,\langle\nabla\rangle F_1(s)\,ds\Big\|_2.
\end{aligned}
$$

By Lemma 2.2 and Corollary 2.8,

$$
\begin{aligned}
& \|\dot{K}_0(t)\langle\nabla\rangle u_0\|_2 + \|\dot{K}_1(t)\langle\nabla\rangle\left(\frac12 u_0 + u_1\right)\|_2 \\
& \qquad \lesssim t^{-\frac54}\left(\|\langle\nabla\rangle^{3+\varepsilon}u_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{2+\varepsilon}u_1\|_{L^1_{xy}}\right).
\end{aligned}
$$

By Lemmas 2.2, Corollary 2.8, and Lemma 5.2,

$$
\begin{aligned}
& \Big\|\int_1^t \dot{K}_1(t-s)\,\langle\nabla\rangle F_1(s)\,ds\Big\|_2 \\
\lesssim\ & \int_1^t \Big(\|\widehat{\dot{K}_1}(t-s)\|_{L^2_\xi(|\xi|\le\frac12)} + \|\widehat{\dot{K}_1}(t-s)\|_{L^\infty_\xi(|\xi|\ge\frac12)}\Big)\big\|\langle\nabla\rangle^{\frac32+\varepsilon}|\nabla|^{\frac12-\varepsilon}F_1(s)\big\|_2 \\
\lesssim\ & \int_1^t \langle t-s\rangle^{-\frac54}\, s^{-\frac32-\varepsilon}\, ds\, Q\big(\|(\vec{u},\, \psi)\|_Y\big) \\
\lesssim\ & t^{-\frac54}\, Q\big(\|(\vec{u},\, \psi)\|_Y\big).
\end{aligned}
$$

Therefore,

$$
\|\langle\nabla\rangle\partial_t u\|_2 \lesssim t^{-\frac54}\big(\|(\vec{u}_0,\, \psi_0)\|_{X_0} + Q\big(\|(\vec{u},\, \psi)\|_Y\big)\big).
$$

Similarly,

$$
\begin{aligned}
\|\langle\nabla\rangle\partial_t v\|_2 \lesssim\ & \|\dot{K}_0(t)\langle\nabla\rangle v_0\|_2 + \|\dot{K}_1(t)\langle\nabla\rangle\left(\frac12 v_0 + v_1\right)\|_2 \\
& + \Big\|\int_1^t \dot{K}_1(t-s)\,\langle\nabla\rangle F_2(s)\,ds\Big\|_2 \\
\lesssim\ & t^{-\frac54}\big(\|(\vec{u}_0,\, \psi_0)\|_{X_0} + Q(\|(\vec{u},\, \psi)\|_Y)\big).
\end{aligned}
$$

Moreover,

$$\begin{aligned}\|\partial_x\partial_t v\|_2 &\lesssim \|\dot{K}_0(t)\partial_x v_0\|_2 + \|\dot{K}_1(t)\partial_x\left(\frac{1}{2}v_0+v_1\right)\|_2 \\ &\quad + \|\int_1^t \partial_x \dot{K}_1(t-s)\,F_2(s)\,ds\|_2 \\ &\lesssim t^{-\frac{3}{2}}(\|\langle\nabla\rangle^{3+\varepsilon}v_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{2+\varepsilon}v_1\|_{L^1_{xy}}) \\ &\quad + \int_1^t \langle t-s\rangle^{-\frac{3}{2}}\big\|\,|\nabla|^{\frac{1}{2}-\varepsilon}\langle\nabla\rangle^{\frac{3}{2}+2\varepsilon}F_2\big\|_{L^1_{xy}}ds \\ &\lesssim t^{-\frac{3}{2}}\|(\vec{u}_0,\,\psi_0)\|_{X_0} + \int_1^t \langle t-s\rangle^{-\frac{3}{2}}\,s^{-\frac{3}{2}-\varepsilon}ds\cdot Q(\|(\vec{u},\,\psi)\|_Y) \\ &\lesssim t^{-\frac{3}{2}}\big(\|(\vec{u}_0,\,\psi_0)\|_{X_0} + Q(\|(\vec{u},\,\psi)\|_Y)\big).\end{aligned}$$

Appendix A.

This appendix serves four purposes. It gives an explicit representation of Π_1 and Π_2, which has been used in the estimates of F_0, F_1 and F_2. It also provides the proofs of Lemmas 5.1 and 5.5. Finally, the properties of the pressure are also given here.

A.1. Another expression of Π_j, $j = 1, 2$. This subsection writes out each term of Π_1 and Π_2 explicitly. Π_1 and Π_2 were previously represented in vector form in (1.8) and (1.9), respectively. Our key point here is that each term is written in a way as to possess as many directives in the x-direction as possible. As seen from Lemma 2.2 in section 2, the more x-derivatives a term has, the faster it decays in time. This point has played an important role in the estimates of F_0, F_1, and F_2 in section 5.

$$\begin{aligned}\Pi_1 = &- u\partial_x u - v\partial_y u + \frac{\partial_{xy}}{\Delta}(u\partial_x v) - \frac{\partial_{xy}}{\Delta}(v\partial_x u) \\ &+ \frac{\partial_{xx}}{\Delta}(u\partial_x u) + \frac{\partial_{xx}}{\Delta}(v\partial_y u) - \partial_x\psi\partial_{xx}\psi - \partial_x\psi\partial_{yy}\psi \\ &+ \frac{\partial_{xy}}{\Delta}(\partial_y\psi\partial_{xx}\psi) + \frac{\partial_{yy}}{\Delta}(\partial_y\psi\partial_{xy}\psi) + \frac{\partial_{xx}}{\Delta}(\partial_x\psi\partial_{xx}\psi) + \frac{\partial_{xx}}{\Delta}(\partial_x\psi\partial_{yy}\psi),\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}\Pi_2 &= -u\partial_x v - v\partial_y v + \frac{\partial_{xy}}{\Delta}(u\partial_x u) + \frac{\partial_{xy}}{\Delta}(v\partial_y u) \\ &\quad + \frac{\partial_{yy}}{\Delta}(u\partial_x v) + \frac{\partial_{yy}}{\Delta}(v\partial_y v) - \partial_{yy}\psi\partial_y\psi - \partial_{xx}\psi\partial_y\psi \\ &\quad + \frac{\partial_{yy}}{\Delta}(\partial_{xx}\psi\partial_y\psi) + \frac{\partial_{yy}}{\Delta}(\partial_{yy}\psi\partial_y\psi) + \frac{\partial_{xy}}{\Delta}(\partial_{xx}\psi\partial_x\psi) + \frac{\partial_{xy}}{\Delta}(\partial_{yy}\psi\partial_x\psi) \\ &= -\frac{\partial_{xx}}{\Delta}(u\partial_x v) - \frac{\partial_{xx}}{\Delta}(v\partial_y v) + \frac{\partial_{xxy}}{\Delta}(uu) + \frac{\partial_{xyy}}{\Delta}(uv) - \frac{\partial_{xx}}{\Delta}(\partial_{xx}\psi\partial_y\psi) \\ &\quad - \frac{\partial_{xx}}{\Delta}(\partial_{yy}\psi\partial_y\psi) + \frac{1}{2}\frac{\partial_{xxy}}{\Delta}(\partial_x\psi\partial_x\psi) + \frac{\partial_{xyy}}{\Delta}(\partial_x\psi\partial_y\psi) - \frac{\partial_{xy}}{\Delta}(\partial_y\psi\partial_{xy}\psi) \\ &= -2\frac{\partial_{xx}}{\Delta}(u\partial_x v) + \partial_x(uv) + 2\frac{\partial_{xy}}{\Delta}(u\partial_x u) - \frac{\partial_{xx}}{\Delta}(\partial_{xx}\psi\partial_y\psi) \\ &\quad - 2\frac{\partial_{xy}}{\Delta}(\partial_y\psi\partial_{xy}\psi) + \frac{\partial_{xy}}{\Delta}(\partial_x\psi\partial_{xx}\psi) + \frac{\partial_{yy}}{\Delta}(\partial_y\psi\partial_{xx}\psi) + \frac{\partial_{yy}}{\Delta}(\partial_x\psi\partial_{xy}\psi).\end{aligned} \tag{A.2}$$

A.2. Proof of Lemma 5.1. It is easily followed by the Littlewood–Paley decomposition. Indeed,

$$\begin{aligned}\big\|\mathcal{R}f\big\|_{L^1_{xy}} &\le \sum_{M\le 1}\big\|P_M\mathcal{R}f\big\|_{L^1_{xy}}+\sum_{M\ge 1}\big\|P_M\mathcal{R}f\big\|_{L^1_{xy}}\\ &\le \sum_{M\le 1}\big\|P_Mf\big\|_{L^1_{xy}}+\sum_{M\ge 1}\big\|P_Mf\big\|_{L^1_{xy}}\\ &\lesssim \sum_{M\le 1}M^{\epsilon}\big\|P_M|\nabla|^{-\epsilon}f\big\|_{L^1_{xy}}+\sum_{M\ge 1}M^{-\epsilon}\big\|P_M\langle\nabla\rangle^{\epsilon}f\big\|_{L^1_{xy}}\\ &\lesssim \big\||\nabla|^{-\epsilon}\langle\nabla\rangle^{2\epsilon}f\big\|_{L^1_{xy}}.\end{aligned}$$

This proves Lemma 5.1. ▯

A.3. Proof of Lemma 5.5. By the Littlewood–Paley decomposition,

$$\begin{aligned}\Big\||\nabla|^{\alpha}\langle\nabla\rangle^{N_0}\frac{\partial_x\partial_y}{-\Delta}f\Big\|_{L^1_{xy}} &\lesssim \sum_{M\le 1}M^{\alpha-1}\big\|\partial_xP_Mf\big\|_{L^1_{xy}}+\sum_{M\ge 1}M^{N_0+\alpha-1}\big\|\partial_xP_Mf\big\|_{L^1_{xy}}\\ &\lesssim \sum_{M\le 1}M^{\epsilon}M^{-1}\big\|\nabla\partial_xP_Mf\big\|_{L^1_{xy}}^{\alpha-\epsilon}\big\|\partial_xP_Mf\big\|_{L^1_{xy}}^{1-\alpha+\epsilon}\\ &\quad+\sum_{M\ge 1}M^{-\epsilon}\big\|\langle\nabla\rangle^{N_0-1+\alpha+\epsilon}\partial_xP_Mf\big\|_{L^1_{xy}}\\ &\lesssim \sum_{M\le 1}M^{\epsilon}\big\|\partial_xP_Mf\big\|_{L^1_{xy}}^{\alpha-\epsilon}\big\|P_Mf\big\|_{L^1_{xy}}^{1-\alpha+\epsilon}+\big\|\langle\nabla\rangle^{N_0-1+\alpha+\epsilon}\partial_xP_Mf\big\|_{L^1_{xy}}\\ &\lesssim \|f\|_{L^1_{xy}}^{1-\alpha+\epsilon}\|\partial_xf\|_{L^1_{xy}}^{\alpha-\epsilon}+\big\|\langle\nabla\rangle^{N_0-1+\alpha+\epsilon}\partial_xf\big\|_{L^1_{xy}}.\end{aligned}$$

This proves Lemma 5.5. ▯

A.4. Properties of the pressure *P*. First of all, by applying $\nabla\cdot\vec{u}=0$, we have

$$\begin{aligned}P &= \frac{\nabla\cdot\big[\nabla\cdot(\nabla\phi\otimes\nabla\phi+\vec{u}\otimes\vec{u})\big]}{-\Delta}\\ &= -2\partial_y\psi+\frac{\nabla\cdot\big[\nabla\cdot(\nabla\psi\otimes\nabla\psi+\vec{u}\otimes\vec{u})\big]}{-\Delta}.\end{aligned}$$

Therefore, for any $t\ge 1$, by Sobolev's inequality,

$$\begin{aligned}\|P(t)\|_{H^N} &\lesssim \|\partial_y\psi\|_{H^N}+\|\nabla\psi\otimes\nabla\psi\|_{H^N}+\|\vec{u}\otimes\vec{u}\|_{H^N}\\ &\lesssim \|\nabla\psi\|_{H^N}+\|\nabla\psi\|_\infty\|\nabla\psi\|_{H^N}+\|\vec{u}\|_\infty\|\vec{u}\|_{H^N}\\ &\lesssim t^{\varepsilon}\big(\|(\vec{u},\psi)\|_Y+\|(\vec{u},\psi)\|_Y^2\big)\\ &\lesssim \varepsilon_0\, t^{\varepsilon}.\end{aligned}$$

This implies that $P\in C\big([1,\infty);H^N(\mathbb{R}^2)\big)$. Furthermore,

$$\begin{aligned}\|P(t)\|_\infty &\lesssim \|\partial_y\psi\|_\infty+\|\langle\nabla\rangle^2(\nabla\psi\otimes\nabla\psi)\|_2+\|\langle\nabla\rangle^2(\vec{u}\otimes\vec{u})\|_2\\ &\lesssim \|\nabla\psi\|_\infty+\|\langle\nabla\rangle^3\psi\|_2\|\nabla\psi\|_\infty+\|\langle\nabla\rangle^2\vec{u}\|_2\|\vec{u}\|_\infty\\ &\lesssim t^{-\frac12}\big(\|(\vec{u},\psi)\|_Y+\|(\vec{u},\psi)\|_Y^2\big)\\ &\lesssim \varepsilon_0\, t^{-\frac12}.\end{aligned}$$

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