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Sharp decay estimates for Oldroyd-B model with only fractional stress tensor diffusion

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A B S T R A C T

Precise large-time behavior of physical quantities plays a crucial role in understanding many physical phenomena. For partial differential equation (PDE) models with full dissipation, powerful methods such as the Fourier-splitting technique have been developed. However, these methods may not be applied to PDE systems with only partial dissipation. This paper offers new ideas on how to obtain precise large-time decay estimates on a partially dissipated system. We examine the d-dimensional incompressible Oldroyd-B model without velocity dissipation and with only fractional diffusive stress. The discovery here is that the coupling and interaction of the velocity and the non-Newtonian stress actually enhances the regularity and the stability of the system. Without the stress, the Sobolev norms of the velocity could grow rather rapidly in time, let alone decay at explicit rates. Making use of the interaction, we deduce a system of damped wave equations obeyed by the velocity and the Leray projection of the divergence of the stress. By constructing a suitable Lyapunov functional, we are able to control the growth in the derivatives and extract explicit decay rates. The optimal

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decay rates are established by representing the wave equations in an integral form and applying a bootstrapping argument.

1. Introduction

Let $d \geq 2$ be an integer. Consider the initial-value problem for the d-dimensional Oldroyd-B model

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nabla \cdot \tau, \quad x \in \mathbb{R}^d, \ t > 0, \\ \partial_t \tau + u \cdot \nabla \tau + \eta(-\Delta)^\beta \tau + Q(\tau, \nabla u) = D(u), \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \tau(x,0) = \tau_0(x), \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), ..., u_d(x,t))$ denotes the velocity field of the fluid, $P = P(x,t)$ denotes the pressure, $\tau = \tau(x,t)$ denotes the non-Newtonian part of stress tensor (a symmetric matrix), and $\eta > 0$ and $\beta \geq 0$ are parameters. Here $D(u)$ is the symmetric part of ∇u,

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

and Q is the following bilinear form

$$Q(\tau, \nabla u) = \tau\Omega(u) - \Omega(u)\tau + b\left(D(u)\tau + \tau D(u)\right), \quad b \in [-1, 1]$$

with $\Omega(u)$ being the skew-symmetric part of ∇u, namely

$$\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

The fractional Laplacian operator $(-\Delta)^\beta$ is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\beta} f(\xi) \triangleq |\xi|^{2\beta} \widehat{f}(\xi), \qquad \widehat{f}(\xi) = \mathscr{F} f(\xi) \triangleq \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$

Sometimes we write $\Lambda = (-\Delta)^{\frac{1}{2}}$ for notational convenience.

The Oldroyd-B equations govern the motion of a class of complex fluids such as a solvent with particles suspended in it and have become one of the most studied models in viscoelastic flows (see, e.g., [1,27]). The Oldroyd-B model in (1.1) is a system coupling the forced Euler equations for the velocity with a kinetic description of the particles. When the fluid viscosity is small or Reynolds number is large, the Euler equations serve as a good approximation for the fluid motion. The equation of τ contains no damping but dissipation given by a fractional Laplacian operator. When the Weissenberg number is large, the damping is negligible. The fractional Laplacian dissipation $(-\Delta)^\beta$ with $\beta \geq 0$ includes the standard Laplacian as a special case and is physically relevant to nonlocal interactions. Mathematically the fractional dissipation allows the study on a family of equations simultaneously and gives us a broad view on how the behavior of solutions changes as the dissipation power varies.

The goal of this paper is to understand the large-time behavior of solutions to (1.1) and provide optimal estimates on the decay rates. The Oldroyd-B model examined here involves only partial dissipation with the velocity equation being inviscid. Large-time behavior plays a crucial role in understanding many physical phenomena and powerful tools have been created for PDE systems with full dissipation. The Fourier-splitting approach of Schonbek and her collaborators has been proven to be very useful for many fully dissipative systems such as the Navier-Stokes equations (see, e.g., [28,29]). However, when there is only partial dissipation, the large-time behavior problem is in general difficult. The Fourier-splitting method does not appear to work for our partially dissipated system. This paper presents new ideas on how to deal with the large-time behavior problem on an partially dissipated Oldroyd-B model. We take advantage of the fact that the coupling and interaction between the velocity u and the stress τ actually enhance the regularity and stability of the system. It is hoped that this work would pave a path for more discoveries on the large-time behavior of PDE models with only partial dissipation.

In addition, this paper serves as a continuation of a previous work of Constantin, Wu, Zhao and Zhu [6], which established the global stability of perturbations near the trivial solution of (1.1) in the case when $\beta \geq \frac{1}{2}$. More precisely, [6] shows that any sufficiently small initial data (u_0, τ_0) in the Sobolev space $H^r(\mathbb{R}^d)$ with $r > 1 + \frac{d}{2}$ leads to a unique global solution (u, τ) that remains comparable to the initial data. Since this result will be cited in our main results, we provide a precise statement of their result.

Theorem 1.1. *Consider (1.1) with $\eta > 0$ and $\frac{1}{2} \leq \beta \leq 1$. Let $d = 2, 3$ and $r > 1 + \frac{d}{2}$. Assume $(u_0, \tau_0) \in H^r(\mathbb{R}^d)$, $\nabla \cdot u_0 = 0$, and τ_0 is symmetric. Then there exists a small constant $\varepsilon > 0$ such that, if*

$$\|u_0\|_{H^r} + \|\tau_0\|_{H^r} \leq \varepsilon,$$

then (1.1) has a unique global solution (u, τ) satisfying,

$$E(t) \triangleq \|u(t)\|_{H^r}^2 + \|\tau(t)\|_{H^r}^2 + \int_0^t \left(\|\Lambda^\beta \tau(s)\|_{H^r}^2 + \|\nabla u(s)\|_{H^{r-\beta}}^2\right) ds \lesssim \varepsilon^2. \tag{1.2}$$

Here $A \lesssim B$ means that there exists a constant C such that $A \leq CB$. The positive constants C may be different in each case. This global stability result is not trivial. The velocity equation in (1.1) is the forced Euler equations. As demonstrated in several recent works [7,21,38], the Sobolev norms of the solutions to the Euler equations can grow rather quickly (even double exponentially) in time. [6] was able to prove the desired stability by making the new observation that the non-Newtonian stress tensor can actually smooth and stabilize the velocity field. Mathematically [6] observed that u and $\mathbb{P}\nabla \cdot \tau$ (the Leray projection of the divergence of τ) actually satisfy damped wave equations. By constructing a suitable Lyapunov functional as suggested by the wave structure, [6] was able to established the aforementioned global stability.

1.1. Main results

This paper focuses on the large-time behavior of the solutions obtained in [6]. We establish two main results. The first result assesses that any spatial derivative of order one or higher of the solution obtained in [6] actually decays at least at the rate of $(1+t)^{-\frac{1}{2}}$. More precisely, the following theorem holds.

Theorem 1.2. *Consider* (1.1) *with* $\eta > 0$ *and* $\frac{1}{2} \leq \beta \leq 1$. *Let* $d = 2, 3$ *and* $r > 1 + \frac{d}{2}$. *Assume* $(u_0, \tau_0) \in H^r(\mathbb{R}^d)$, $\nabla \cdot u_0 = 0$, *and* τ_0 *is symmetric. In addition,* (u_0, τ_0) *fulfills the smallness requirement of Theorem* 1.1, *namely*

$$\|u_0\|_{H^r} + \|\tau_0\|_{H^r} \leq \varepsilon$$

for sufficiently small $\varepsilon > 0$. *Let* (u, τ) *be the corresponding solution of* (1.1). *Then* (u, τ) *obeys the decay estimate, for any* $t \geq 0$,

$$\|\nabla u(t)\|_{H^{r-1}} + \|\nabla \tau(t)\|_{H^{r-1}} \lesssim \varepsilon (1+t)^{-\frac{1}{2}}. \tag{1.3}$$

Even though (1.1) is only a partially dissipated system, the decay result of Theorem 1.2 resemble those for fully dissipative PDE systems such as the heat equations and the Navier-Stokes equations. Due to the lack of dissipation in the velocity equation, this decay result is not trivial. One reason, as aforementioned, is the potential rapid growth in the Sobolev norms of the solution. In order to control the growth of the solution, we take advantage of the hidden wave structure for u and $\mathbb{P}\nabla \cdot \tau$,

$$\begin{cases} \partial_{tt} u + \eta(-\Delta)^{\beta} \partial_t u - \dfrac{1}{2}\Delta u = N_1, \quad x \in \mathbb{R}^d, \ t > 0, \\ \partial_{tt} \mathbb{P}\nabla \cdot \tau + \eta(-\Delta)^{\beta} \partial_t \mathbb{P}\nabla \cdot \tau - \dfrac{1}{2}\Delta \mathbb{P}\nabla \cdot \tau = N_2, \qquad \nabla \cdot u = 0, \end{cases} \tag{1.4}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denotes the Leray projection, and N_1 and N_2 are the nonlinear terms,

$$N_1 = -(\partial_t + \eta(-\Delta)^\beta)\mathbb{P}(u\cdot\nabla u) - \mathbb{P}\nabla\cdot(u\cdot\nabla\tau) - \mathbb{P}\nabla\cdot Q,$$
$$N_2 = -\frac{1}{2}\Delta\mathbb{P}(u\cdot\nabla u) - \partial_t\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) - \partial_t\mathbb{P}\nabla\cdot Q.$$

Since (1.4) plays a crucial role in our analysis, we provide the details of derivation. Applying the projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation and the operator $\mathbb{P}\nabla\cdot$ to the equation of τ in (1.1) lead to

$$\begin{cases} \partial_t u = \mathbb{P}\nabla\cdot\tau + \widetilde{N}_1, \\ \partial_t\mathbb{P}\nabla\cdot\tau + \eta(-\Delta)^\beta\mathbb{P}\nabla\cdot\tau - \frac{1}{2}\Delta u = \widetilde{N}_2, \end{cases} \tag{1.5}$$

where

$$\widetilde{N}_1 = -\mathbb{P}(u\cdot\nabla u) \quad \text{and} \quad \widetilde{N}_2 = -\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) - \mathbb{P}\nabla\cdot Q(\tau,\nabla u).$$

Differentiating the first equation of (1.5) in t yields

$$\partial_{tt}u = \partial_t\mathbb{P}\nabla\cdot\tau + \partial_t\widetilde{N}_1.$$

Replacing $\partial_t\mathbb{P}\nabla\cdot\tau$ by the second equation of (1.5), we obtain

$$\partial_{tt}u = -\eta(-\Delta)^\beta\mathbb{P}\nabla\cdot\tau + \frac{1}{2}\Delta u + \widetilde{N}_2 + \partial_t\widetilde{N}_1.$$

By further invoking $\mathbb{P}\nabla\cdot\tau = \partial_t u - \widetilde{N}_1$ via the first equation of (1.5), we have

$$\partial_{tt}u + \eta(-\Delta)^\beta\partial_t u - \frac{1}{2}\Delta u = \eta(-\Delta)^\beta\widetilde{N}_1 + \widetilde{N}_2 + \partial_t\widetilde{N}_1.$$

The terms on the right-hand side are the same as N_1,

$$\begin{aligned}&\eta(-\Delta)^\beta\widetilde{N}_1 + \widetilde{N}_2 + \partial_t\widetilde{N}_1\\ &= -(\partial_t + \eta(-\Delta)^\beta)\mathbb{P}(u\cdot\nabla u) - \mathbb{P}\nabla\cdot(u\cdot\nabla\tau) - \mathbb{P}\nabla\cdot Q = N_1.\end{aligned}$$

We have thus obtained the first equation in (1.4). The second equation in (1.4) can be derived very similarly.

By exploiting the regularization due to the wave equations in (1.4), we are able to construct a suitable Lyapunov functional to control the growth of $\|\nabla u(t)\|_{H^{r-1}} + \|\nabla\tau(t)\|_{H^{r-1}}$. More detailed ideas on the proof of Theorem 1.2 will be presented in the later part of this introduction.

Theorem 1.2 does not provide the large-time behavior on the L^2-norm of the solution (u,τ) itself. This is not surprising. Even in the case of the heat equation, the L^2-norm of the solution is not known to decay in time if we only know that the initial data is in L^2. In order to make the behavior of L^2-norm of the solution definite, we need to make extra assumptions on the initial data (u_0, τ_0). The extra condition imposed here is $(u_0, \tau_0) \in L^1$. This type of functional setup or the Sobolev space with a negative index is usually required when dealing with large-time behavior of dissipative PDEs (see, e.g., [28,29]). Our second main result establishes the optimal decay rates for the L^2-norm of u and ∇u, the L^∞-norm of u and ∇u as well as the L^2-norm of $\mathbb{P}\nabla \cdot \tau$. The precise statement is provided in the following theorem. The following notation will be used throughout the rest of this paper

$$\|\cdot\|_{L^p} \triangleq \|\cdot\|_{L^p(\mathbb{R}^d)}, \quad \|\cdot\|_{H^r} \triangleq \|\cdot\|_{H^r(\mathbb{R}^d)}, \quad \langle t \rangle \triangleq 1 + |t|.$$

Theorem 1.3. *Let $d = 2, 3$. Consider (1.1) with $\eta > 0$ and $\frac{1}{2} \leq \beta \leq 1$. Assume $\frac{d+2}{4\beta} \neq 1$. Assume the initial data (u_0, τ_0) satisfies*

$$(u_0, \tau_0) \in L^1(\mathbb{R}^d) \cap H^r(\mathbb{R}^d) \quad \text{with} \quad r = 3 + \frac{d}{2} + \frac{2d+2}{\beta}, \quad \nabla \cdot u_0 = 0$$

and τ_0 is symmetric. Then there exists a sufficiently small parameter $\varepsilon > 0$ such that, if

$$\|(u_0, \tau_0)\|_{L^1 \cap H^r} \leq \varepsilon,$$

then (1.1) has a unique global solution (u, τ) that obeys the following decay properties

$$\begin{aligned} &\|u(t)\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-\frac{d}{4\beta}}, \quad \|u(t)\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-\frac{d}{2\beta}}, \quad \|\nabla u(t)\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-\frac{d+2}{4\beta}}, \\ &\|\nabla u(t)\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-\frac{d+1}{2\beta}}, \quad \|\mathbb{P}\nabla \cdot \tau(t)\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-\frac{d+2}{4\beta}}. \end{aligned}$$

The decay rates obtained in Theorem 1.3 are optimal. They are in line with those for the solutions of the generalized heat equation,

$$\begin{cases} \partial_t U + (-\Delta)^\beta U = 0, \quad x \in \mathbb{R}^d, \, t > 0, \\ U(x, 0) = U_0 \in L^1 \cap L^2. \end{cases}$$

This theorem and its proof offer a new approach on how to obtain sharp large-time behavior for systems of equations with only partial dissipation. The main discovery is that the coupling and interaction between the velocity equation and the equation of τ actually enhance the regularity and stability of the system. Due to the lack of dissipation in the equation of u, the decay rates can not be derived from the original system in (1.1). The idea here is to make use of the system of wave equations (1.4) satisfied by u and $\mathbb{P}\nabla \cdot \tau$. This system reflects the enhanced regularity and makes the desired decay possible. We will give more precise description on how we actually achieve the optimal rates later.

We remark that the Oldroyd-B models have attracted considerable interests and there are substantial recent developments. Significant progress has been made on many fundamental issues such as the well-posedness and stability problems (see, e.g., [2–6,8–14,16–20,22–26,31–37]).

1.2. Main ideas in the proofs of Theorem 1.2 and Theorem 1.3

We briefly describe the main ideas on how we prove Theorem 1.2 and Theorem 1.3. The proof of Theorem 1.2 is based on the following lemma, which provides a precise decay rate for a nonnegative integrable function when it decreases in a generalized sense.

Lemma 1.4. *Let $f = f(t)$ be a nonnegative function satisfying, for two constants $a_0 > 0$ and $a_1 > 0$,*

$$\int_0^\infty f(\tau)\, d\tau \leq a_0 < \infty \quad \text{and} \quad f(t) \leq a_1 f(s) \quad \text{for any } 0 \leq s < t. \tag{1.6}$$

Then, for $a_2 = \max\{2a_1 f(0), 2a_0 a_1\}$ and for any $t > 0$,

$$f(t) \leq a_2 (1+t)^{-1}.$$

By Lemma 1.4, to prove Theorem 1.2, it suffices to verify that

$$f(t) \triangleq \|\nabla u(t)\|_{H^{r-1}}^2 + \|\nabla \tau(t)\|_{H^{r-1}}^2$$

satisfies the two conditions in (1.6). The first condition of (1.6), namely the time integrability of $f(t)$, has been established in [6], as stated in Theorem 1.1. The second condition of (1.6), namely the generalized monotonicity of f, is the main focus of the proof of Theorem 1.2. Due to the lack of dissipation in the velocity equation, the Sobolev norms of the solution (u, τ) could potentially grow in time and the desired generalized monotonicity appears to be impossible. A key observation here is that the coupling and interaction between the velocity u and the non-Newtonian tensor τ helps smooth and stabilize the solution of (1.1). Mathematically the interaction allows us to derive the hidden wave equations obeyed by u and $\mathbb{P}\nabla \cdot \tau$, namely (1.4). The wave equations (1.4) decouple u from $\mathbb{P}\nabla \cdot \tau$ in the linearization, and are obtained by taking ∂_t of the equations for u and $\mathbb{P}\nabla \cdot \tau$,

$$\begin{cases} \partial_t u + \mathbb{P}(u \cdot \nabla u) = \mathbb{P}\nabla \cdot \tau, \\ \partial_t \mathbb{P}\nabla \cdot \tau + \mathbb{P}\nabla \cdot (u \cdot \nabla \tau) + \eta(-\Delta)^\beta \mathbb{P}\nabla \cdot \tau + \mathbb{P}\nabla \cdot Q(\tau, \nabla u) = \dfrac{1}{2}\Delta u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.7}$$

making several substitutions, and regrouping linear and nonlinear terms. The wave structure in (1.4) reveals that both u and $\mathbb{P}\nabla \cdot \tau$ are effectively dissipative and dispersive. In order to unearth the hidden regularization, we construct a suitable Lyapunov functional,

$$L(u, \tau) = \|\nabla u\|_{H^{r-1}}^2 + \|\nabla \tau\|_{H^{r-1}}^2 + 2k\, (\nabla u, \nabla \mathbb{P}\nabla \cdot \tau)_{H^{r-1-\beta}} \tag{1.8}$$

where $k > 0$ is a suitably selected parameter and $(f, g)_{H^\sigma}$ denotes the inner product in the Sobolev space H^σ. The inclusion of the inner product term will bring out the dissipation on u, which helps prevent any potential growth in the Sobolev norm of u. This is how we obtain the aforementioned generalized monotonicity, for any $0 \le s \le t$

$$\|\Lambda u(t)\|^2_{H^{r-1}} + \|\Lambda \tau(t)\|^2_{H^{r-1}} \le C \left(\|\Lambda u(s)\|^2_{H^{r-1}} + \|\Lambda \tau(s)\|^2_{H^{r-1}}\right),$$

which, together with the time integrability of $f(t)$ guaranteed by (1.2), leads to the desired decay rates stated in Theorem 1.2. We leave more technical details to the proof of Theorem 1.2 in Section 3.

Due to the lack of dissipation in the velocity equation in (1.1), the optimal decay rates stated in Theorem 1.3 do not follow from classical approaches designed for fully dissipative systems. Our idea here is to exploit the regularization and damping effects created by the wave equations satisfied by u and $\mathcal{A} \triangleq \mathbb{P}\nabla\cdot\tau$. We represent the equations of u and $\mathcal{A}$ in an integral form via the spectral analysis,

$$\begin{aligned}
\widehat{u}(\xi, t) =& M_1\widehat{u_0} + M_2\widehat{\mathcal{A}_0} + \int_0^t M_1(t-s)\widehat{G}(s)ds \\
&+ \int_0^t M_2(t-s)\left(\widehat{F}(s) + \widehat{H}(s)\right)ds, \\
\widehat{\mathcal{A}}(\xi, t) =& -\frac{|\xi|^2}{2}M_2\widehat{u_0} + M_3\widehat{\mathcal{A}_0} - \int_0^t \frac{|\xi|^2}{2}M_2(t-s)\widehat{G}(s)ds \\
&+ \int_0^t M_3(t-s)\left(\widehat{F}(s) + \widehat{H}(s)\right)ds.
\end{aligned} \tag{1.9}$$

The derivation of (1.9) is obtained in Lemma 4.1. The explicit formulas of the kernel operators M_1, M_2 and M_3 are also specified there. The framework of the proof is to apply the bootstrapping argument to (1.9). A direct applicable form of the bootstrapping argument can be found in [30, p. 21]). As a preparation, we need to derive optimal explicit upper bounds on the kernel functions M_1, M_2 and M_3. These kernel functions are nonhomogeneous and frequency dependent. To achieve the sharp upper bounds, we divide the whole frequency space into suitable subdomains and derive definite upper bounds for these kernels in each subdomain. The precise division of the frequency space and the explicit upper bounds are presented in Proposition 4.2 in Section 4.

To apply the bootstrapping argument, we need to define a suitable functional setting. We introduce the following time-weighted norm

$$X(t) = \sup_{0\le s\le t}\left\{\langle s\rangle^{\frac{d}{4\beta}}\|u(s)\|_{L^2(\mathbb{R}^d)} + \langle s\rangle^{\frac{d}{2\beta}}\|\widehat{u}(s)\|_{L^1(\mathbb{R}^d)}\right\}, \tag{1.10}$$

$$Y(t) = \sup_{0\le s\le t} \Big\{ \langle s\rangle^{\frac{d+2}{4\beta}} \left(\|\nabla u(s)\|_{L^2(\mathbb{R}^d)} + \|\mathcal{A}(s)\|_{L^2(\mathbb{R}^d)}\right) \\ + \langle s\rangle^{\frac{d+1}{2\beta}} \||\xi|\widehat{u}(s)\|_{L^1(\mathbb{R}^d)}\Big\}. \tag{1.11}$$

Our main efforts are devoted to establishing

$$X(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^r} + X(t)Y(t) + (X(t)+Y(t))\|\tau\|_{L^\infty H^r}, \tag{1.12}$$

$$Y(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^r} + \|(u,\tau)\|^2_{L^\infty H^r} \\ + (X(t)+Y(t))\,(Y(t)+\|(u,\tau)\|_{L^\infty H^r})\,. \tag{1.13}$$

If we take the initial data (u_0,τ_0) to be sufficiently small,

$$\|(u_0,\tau_0)\|_{L^1\cap H^r} \le \varepsilon$$

for a suitable small $\varepsilon > 0$, Theorem 1.1 assesses that the corresponding solution remains small for all time,

$$\|(u(t),\tau(t))\|_{H^r} \le C\,\varepsilon.$$

Then (1.12) and (1.13) imply

$$X(t) + Y(t) \le C\,\varepsilon + C\,(X(t)+Y(t))^2. \tag{1.14}$$

Then a simple application of the bootstrapping argument to (1.14) would lead to

$$X(t) + Y(t) \le C\,\varepsilon,$$

which yields the desired result of Theorem 1.3. The proof of (1.12) and (1.13) is a lengthy and technical process. We apply various new techniques such as sharp decay rates for generalized heat operators associated with fractional Laplacian, and fractional derivative identities and commutators to facilitate the shifting of derivatives. The details are provided in Section 5.

In [15] Yan Guo and Yanjin Wang proposed and applied a powerful and effective approach for the problem of optimal decay rates on dissipative equations in the whole space. We discuss the possibility of implementing their approach for the decay problem solved in this paper. We briefly outline the mechanism of their approach. It is based on the energy estimates. The initial data is assumed to be in the standard Sobolev setting $H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$ as well as in a Sobolev space of negative index, $H^\sigma(\mathbb{R}^d)$ with $\sigma < 0$. Generally the initial data is also assumed to be small in $H^s(\mathbb{R}^d)$ in order to obtain the global uniform bound on the H^s-norm of the solution. A crucial feature of their approach is to show that the negative Sobolev norm of the solution is preserved along time evolution. To obtain the decay rate for the $H^{s'}$-norm with $s' < s$, one interpolates the $H^{s'}$-norm in terms of the H^s-norm and the H^σ-norm in the energy estimates. In order to apply their approach to our problem here, we need to prove that the negative Sobolev norm is preserved in time. For the Oldroyd-B model considered here, the lack of the velocity dissipation and the involvement of the nonlinear term Q in the equation of τ make this task difficult. Clearly we need to construct suitable Lyapunov functional involving the negative Sobolev norm and carefully crafted inner product terms of u and $\mathbb{P}\nabla\cdot\tau$. We will pursue this approach in our future research.

The rest of this paper is divided into four sections. Section 2 provides several tool lemmas to be used in the proofs of subsequent sections. Section 3 proves Theorem 1.2. Section 4 serves as a preparation for the proof of Theorem 1.3. It converts the equations of u and $\mathcal{A}$ into an integral form in terms of the kernel functions M_1, M_2 and M_3. Sharp upper bounds on these kernel functions are also derived in this section. Section 5 provides the proof of Theorem 1.3. The proof is very long and is divided into six subsections.

2. Preparation

This section serves as a preparation. It lists several tools to be used in the proofs of Theorem 1.2 and Theorem 1.3. The first provides an upper bound on an convolution integral (see, e.g., [31]).

Lemma 2.1. *If $0 < s_1 \le s_2$, then*

$$\int_0^t \langle t-s\rangle^{-s_1}\langle s\rangle^{-s_2}ds \le \begin{cases} C\,\langle t\rangle^{-s_1}, & \text{if} \quad s_2>1,\\ C\,\langle t\rangle^{-s_1}\ln(1+t), & \text{if} \quad s_2=1,\\ C\,\langle t\rangle^{1-s_1-s_2}, & \text{if} \quad s_2<1.\end{cases}$$

The next lemma provides an exact decay estimate for the heat operator associated with a fractional Laplacian.

Lemma 2.2. *Let $\nu > 0$ and $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Then,*

$$\|e^{-\nu(-\Delta)^\alpha t} f\|_{L^2(\mathbb{R}^d)} \le C\ \langle t\rangle^{-\frac{d}{4\alpha}}\ \|f\|_{L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)}. \tag{2.1}$$

Proof. We provide an elementary proof. For $0 \le t \le 1$,

$$\|e^{-\nu(-\Delta)^\alpha t} f\|_{L^2(\mathbb{R}^d)} = \|e^{-\nu|\xi|^{2\alpha} t}\,\widehat{f}\|_{L^2(\mathbb{R}^d)} \le \|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

For $t > 1$,

$$\begin{aligned}\|e^{-\nu(-\Delta)^\alpha t} f\|_{L^2(\mathbb{R}^d)} &= \|e^{-\nu|\xi|^{2\alpha} t}\,\widehat{f}\|_{L^2(\mathbb{R}^d)} \le \|e^{-\nu|\xi|^{2\alpha} t}\|_{L^2(\mathbb{R}^d)}\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)}\\ &\le \|e^{-\nu|\xi|^{2\alpha} t}\|_{L^2(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)} = C(\nu)t^{-\frac{d}{4\alpha}}\|f\|_{L^1(\mathbb{R}^d)}.\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) leads to (2.1). □

The following lemma rewrites the nonlinear term $\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)$ into three terms with one of them containing $\mathcal{A}\triangleq\mathbb{P}\nabla\cdot\tau$ and the other two containing ∇u. This identity is useful when we try to prove (1.12) and (1.13) in Section 5. It is derived in [36, Proposition 3.1].

Lemma 2.3. *For any sufficiently smooth d-dimensional vector u and any tensor $\tau = [\tau_{ij}]_{d\times d}$,*

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) = \mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau) + \mathbb{P}(\nabla u\cdot\nabla\tau) - \mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau). \tag{2.4}$$

Finally we state a commutator estimate that can be found in [31, Lemma 4.1]. This estimate will be useful in the proof of Theorem 1.3.

Lemma 2.4. *Let v be a scalar function and τ be a tensor. Then*

$$\|[\mathbb{P}\nabla\cdot, v]\tau\|_{L^1} \lesssim \|\nabla v\|_{L^2}\|\tau\|_{L^2} + \|v\|_{L^2}\|\mathcal{A}\|_{L^2}. \tag{2.5}$$

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. We set

$$f(t) \triangleq \|\nabla u(t)\|^2_{H^{r-1}} + \|\nabla\tau(t)\|^2_{H^{r-1}}, \quad r > 1 + \frac{d}{2},$$

and apply Lemma 1.4. According to (1.2) in Theorem 1.1,

$$\int_0^\infty f(t)\,dt \le C\,\varepsilon^2,$$

where $C > 0$ is a pure constant. This verifies the first condition in (1.6). If we can establish the second condition in (1.6), namely

$$f(t) \le C\,f(s) \quad \text{for any } 0 \le s < t, \tag{3.1}$$

then Lemma 1.4 would conclude that

$$f(t) \le C\,\varepsilon^2\,(1+t)^{-1},$$

which is exactly (1.3) in Theorem 1.2. The rest of the proof shows (3.1). As we have explained in the introduction, it is not trivial to verify (3.1) due to the lack of velocity dissipation. We need to work with the Lyapunov functional L defined in (1.8) in order to materialize the dissipative effect revealed by the wave structure in (1.4).

The attention will be focused on the evolution of L. Recall that $\Lambda = (-\Delta)^{\frac{1}{2}}$. Due to the equivalence of the norms

$$\|\nabla g\|_{H^{r-1}} \quad \sim \quad \|\Lambda g\|_{L^2} + \|\Lambda^r g\|_{L^2},$$

we use the norm on the right in the estimates. Applying Λ to (1.1) and dotting by $(\Lambda u, \Lambda\tau)$, and applying Λ^r to (1.1) and dotting by $(\Lambda^r u, \Lambda^r\tau)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda u\|_{H^{r-1}}^2 + \|\Lambda\tau\|_{H^{r-1}}^2) + \eta\|\Lambda^{1+\beta}\tau\|_{H^{r-1}}^2\\ &= -(\Lambda(u\cdot\nabla u), \Lambda u)_{H^{r-1}} - (\Lambda(u\cdot\nabla\tau), \Lambda\tau)_{H^{r-1}} - (\Lambda Q(\tau,\nabla u), \Lambda\tau)_{H^{r-1}}, \end{aligned}\tag{3.2}$$

where we have used, for $l = 1$ and r,

$$\int_{\mathbb{R}^d}(\Lambda^l u\cdot(\Lambda^l\nabla\cdot\tau) + \Lambda^l D(u)\cdot\Lambda^l\tau)dx = 0.$$

It is not difficult to use (1.7) to verify that

$$\begin{aligned}&\frac{d}{dt}(\Lambda u, \Lambda\mathbb{P}\nabla\cdot\tau) + \frac{1}{2}\|\Delta u\|_{L^2}^2 - \|\Lambda\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\ &= -(\Lambda(u\cdot\nabla u), \Lambda\mathbb{P}\nabla\cdot\tau) - (\Lambda\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \Lambda u)\\ &\quad - (\Lambda\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \Lambda u) - \eta(\Lambda^{2\beta+1}\mathbb{P}\nabla\cdot\tau, \Lambda u).\end{aligned}\tag{3.3}$$

Similarly we can check that

$$\begin{aligned}&\frac{d}{dt}(\Lambda^{r-\beta}u, \Lambda^{r-\beta}\mathbb{P}\nabla\cdot\tau) + \frac{1}{2}\|\Lambda^{r-1-\beta}\Delta u\|_{L^2}^2 - \|\Lambda^{r-\beta}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\ &= -(\Lambda^{r-\beta}(u\cdot\nabla u), \Lambda^{r-\beta}\mathbb{P}\nabla\cdot\tau) - (\Lambda^{r-\beta}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \Lambda^{r-\beta}u)\\ &\quad - (\Lambda^{r-\beta}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \Lambda^{r-\beta}u) - \eta(\Lambda^{r+\beta}\mathbb{P}\nabla\cdot\tau, \Lambda^{r-\beta}u).\end{aligned}\tag{3.4}$$

For a constant $k > 0$, (3.2)+k(3.3)+k(3.4) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda u\|_{H^{r-1}}^2 + \|\Lambda\tau\|_{H^{r-1}}^2 + 2k(\Lambda u, \Lambda\mathbb{P}\nabla\cdot\tau)_{H^{r-1-\beta}}\right) + \eta\|\Lambda^{1+\beta}\tau\|_{H^{r-1}}^2\\ &+ \frac{k}{2}\|\Delta u\|_{H^{r-1-\beta}}^2 - k\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{H^{r-1-\beta}}^2 = \sum_{i=1}^{7} Z_i,\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}Z_1 &= -k(\Lambda(u\cdot\nabla u), \Lambda\mathbb{P}\nabla\cdot\tau)_{H^{r-1-\beta}},\\ Z_2 &= -k(\Lambda\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \Lambda u)_{H^{r-1-\beta}},\end{aligned}$$

$$
\begin{aligned}
Z_3 &= -k(\Lambda\mathbb{P}\nabla\cdot Q(\tau,\nabla u),\Lambda u)_{H^{r-1-\beta}},\\
Z_4 &= -k\eta(\Lambda^{2\beta+1}\mathbb{P}\nabla\cdot\tau,\Lambda u)_{H^{r-1-\beta}},\\
Z_5 &= -(\Lambda(u\cdot\nabla u),\Lambda u)_{H^{r-1}},\\
Z_6 &= -(\Lambda(u\cdot\nabla\tau),\Lambda\tau)_{H^{r-1}},\\
Z_7 &= -(\Lambda Q(\tau,\nabla u),\Lambda\tau)_{H^{r-1}}.
\end{aligned}
$$

Next we estimate Z_1 through Z_7. The following Sobolev inequalities will be used frequently,

$$
\|g\|_{L^4(\mathbb{R}^2)} \le C\|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{3.6}
$$

$$
\|g\|_{L^6(\mathbb{R}^3)} \le C\|\nabla g\|_{L^2(\mathbb{R}^3)}. \tag{3.7}
$$

The case $d=2$ is treated differently from the case $d=3$. We start with $d=2$. By Hölder's inequality, (3.6) and some basic embedding inequalities,

$$
\begin{aligned}
|Z_1| &\lesssim \|\nabla u\|_{L^4(\mathbb{R}^2)}^2\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)}\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{L^2(\mathbb{R}^2)}\\
&\quad + \|\Lambda^{r+1-\beta}\tau\|_{L^2(\mathbb{R}^2)}\|u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^2)}\\
&\lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\\
&\quad + \|u\|_{H^r(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\\
&\lesssim \|u\|_{H^r(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\\
&\lesssim \|u\|_{H^r(\mathbb{R}^2)}\left(\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2 + \|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|Z_2| &\lesssim \|\nabla u\|_{L^4(\mathbb{R}^2)}\|\nabla\tau\|_{L^4(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)}\\
&\quad + \|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^2)}\left(\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^2)}\|\tau\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^{r+1-\beta}\tau\|_{L^2(\mathbb{R}^2)}\right)\\
&\lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla\tau\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\Delta u\|_{L^2(\mathbb{R}^2)}^{\frac{3}{2}} + \|\tau\|_{H^r(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2\\
&\quad + \|u\|_{H^r(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\\
&\lesssim \left(\|u\|_{H^r(\mathbb{R}^2)} + \|\tau\|_{H^r(\mathbb{R}^2)}\right)\left(\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2 + \|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|Z_3| &\lesssim \|\nabla\tau\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^4(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|\tau\|_{L^\infty(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)}^2\\
&\quad + \|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^2)}\left(\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^2)}\|\tau\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^{r-\beta}\tau\|_{L^2(\mathbb{R}^2)}\right)\\
&\lesssim \left(\|u\|_{H^r(\mathbb{R}^2)} + \|\tau\|_{H^r(\mathbb{R}^2)}\right)\left(\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2 + \|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2\right).
\end{aligned}
$$

Z_4 is bounded by

$$\begin{aligned}|Z_4| &\le k\eta\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}\\ &\le \frac{\eta}{4}\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^2)} + k^2\eta\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)}.\end{aligned} \tag{3.8}$$

By $\nabla\cdot u = 0$, $\frac{1}{2}\le\beta\le 1$ and $r > 1+\frac{d}{2}$, we have

$$\begin{aligned}|Z_5| &= \left|\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot\Delta u\,dx - \int_{\mathbb{R}^2}(\Lambda^r(u\cdot\nabla u) - u\cdot\nabla\Lambda^r u)\cdot\Lambda^r u\,dx\right|\\ &\lesssim \|u\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^4(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^r u\|^2_{L^2(\mathbb{R}^2)}\\ &\lesssim \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\Delta u\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)} + \|u\|_{H^r(\mathbb{R}^2)}\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)}\\ &\le \frac{k}{8}\|\Delta u\|^2_{L^2(\mathbb{R}^2)} + C\|u\|^2_{L^2(\mathbb{R}^2)}\|\nabla u\|^4_{L^2(\mathbb{R}^2)} + C\|u\|_{H^r(\mathbb{R}^2)}\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)}.\end{aligned}$$

Similarly,

$$\begin{aligned}|Z_6| &= \left|-\int_{\mathbb{R}^2}\nabla u\cdot\nabla\tau\cdot\nabla\tau\,dx - \int_{\mathbb{R}^2}(\Lambda^r(u\cdot\nabla\tau) - u\cdot\nabla\Lambda^r\tau)\,\Lambda^r\tau\,dx\right|\\ &\lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla\tau\|^2_{L^4(\mathbb{R}^2)}\\ &\quad + \|\Lambda^r\tau\|_{L^2(\mathbb{R}^2)}(\|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^r\tau\|_{L^2(\mathbb{R}^2)} + \|\Lambda^r u\|_{L^2(\mathbb{R}^2)}\|\nabla\tau\|_{L^\infty(\mathbb{R}^2)})\\ &\lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla\tau\|_{L^2(\mathbb{R}^2)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)} + \left(\|u\|_{H^r(\mathbb{R}^2)} + \|\tau\|_{H^r(\mathbb{R}^2)}\right)\\ &\quad\times\left(\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)} + \|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^2)}\right)\\ &\le \frac{\eta}{8}\|\Lambda^2\tau\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\|\nabla\tau\|^2_{L^2(\mathbb{R}^2)}\\ &\quad + C\left(\|u\|_{H^r(\mathbb{R}^2)} + \|\tau\|_{H^r(\mathbb{R}^2)}\right)\left(\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^2)} + \|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)}\right).\end{aligned}$$

Z_7 is bounded by

$$\begin{aligned}|Z_7| &= \left|(Q(\tau,\nabla u),\Lambda^2\tau) + (\Lambda^{r-\beta}Q(\tau,\nabla u),\Lambda^{r+\beta}\tau)\right|\\ &\lesssim \|\tau\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^4(\mathbb{R}^2)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)}\\ &\quad + \|\Lambda^{r+\beta}\tau\|_{L^2(\mathbb{R}^2)}\left(\|\Lambda^{r-\beta}\nabla u\|_{L^2(\mathbb{R}^2)}\|\tau\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\Lambda^{r-\beta}\tau\|_{L^2(\mathbb{R}^2)}\right)\\ &\lesssim \|\tau\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla\tau\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)}\\ &\quad + \|\tau\|_{H^r(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)} + \|u\|_{H^r(\mathbb{R}^2)}\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^2)}\end{aligned}$$

$$\le \frac{k}{8}\|\Delta u\|_{L^2(\mathbb{R}^2)}^2+\frac{\eta}{8}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^2)}^2+C\|\tau\|_{L^2(\mathbb{R}^2)}^2\|\nabla\tau\|_{L^2(\mathbb{R}^2)}^2\|\nabla u\|_{L^2(\mathbb{R}^2)}^2$$
$$+C\left(\|u\|_{H^r(\mathbb{R}^2)}+\|\tau\|_{H^r(\mathbb{R}^2)}\right)\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2\right).$$

In addition, for $\frac{1}{2}\le\beta\le 1$, we have

$$k\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{H^{r-1-\beta}(\mathbb{R}^d)}^2\le k\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^d)}^2.$$

Inserting the estimates for Z_1 through Z_7 in (3.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda u\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^2)}^2+2k(\Lambda u,\Lambda\mathbb{P}\nabla\cdot\tau)_{H^{r-1-\beta}(\mathbb{R}^2)}\right)\\
&\quad+\left(\frac{\eta}{2}-k\right)\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2+\left(\frac{k}{4}-k^2\eta\right)\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2\\
&\le C\left(\|u\|_{H^r(\mathbb{R}^2)}+\|\tau\|_{H^r(\mathbb{R}^2)}\right)\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2\right)\\
&\quad+C\left(\|u\|_{L^2(\mathbb{R}^2)}^2\|\nabla u\|_{L^2(\mathbb{R}^2)}^2+\|\nabla\tau\|_{L^2(\mathbb{R}^2)}^2+\|\tau\|_{L^2(\mathbb{R}^2)}^2\|\nabla\tau\|_{L^2(\mathbb{R}^2)}^2\right)\|\nabla u\|_{L^2(\mathbb{R}^2)}^2\\
&\le C\varepsilon\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^2)}^2\right)\\
&\quad+C\left(\|(u,\tau)\|_{H^r(\mathbb{R}^2)}^2+1\right)\left(\|\nabla u\|_{L^2(\mathbb{R}^2)}^2+\|\nabla\tau\|_{L^2(\mathbb{R}^2)}^2\right)\\
&\qquad\times\left(\|\Lambda u\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^2)}^2\right).
\end{aligned}$$

To proceed, we set

$$F_1(t)=\|\Lambda u(t)\|_{H^{r-1}(\mathbb{R}^2)}^2+\|\Lambda\tau(t)\|_{H^{r-1}(\mathbb{R}^2)}^2+2k(\Lambda u(t),\Lambda\mathbb{P}\nabla\cdot\tau(t))_{H^{r-1-\beta}(\mathbb{R}^2)}.$$

Since

$$\begin{aligned}
|2k(\Lambda u,\Lambda\mathbb{P}\nabla\cdot\tau)_{H^{r-1-\beta}(\mathbb{R}^d)}|&\le 2k\|\Lambda u\|_{H^{r-1}(\mathbb{R}^d)}\|\Lambda\tau\|_{H^{r-2\beta}(\mathbb{R}^d)}\\
&\le 2k\|\Lambda u\|_{H^{r-1}(\mathbb{R}^d)}\|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^d)}\\
&\le \frac{1}{2}\|\Lambda u\|_{H^{r-1}(\mathbb{R}^d)}^2+2k^2\|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^d)}^2,
\end{aligned}\tag{3.9}$$

we have

$$F_1(t)\ge\frac{1}{2}\|\Lambda u\|_{H^{r-1}(\mathbb{R}^d)}+\left(1-2k^2\right)\|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^d)}^2.$$

If we choose $k>0$ small enough, say

$$k\le\min\left\{\frac{1}{2},\ \frac{\eta}{4},\ \frac{1}{8\eta}\right\},$$

then we have

$$\begin{aligned}&\frac{d}{dt}F_1(t)+(C(\eta)-C_1\varepsilon)\left(\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^2)}+\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^2)}\right)\\ &\le C\left(\|(u,\tau)\|^2_{H^r(\mathbb{R}^2)}+1\right)\left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tau\|^2_{L^2(\mathbb{R}^2)}\right)F_1(t).\end{aligned}$$

Here $\varepsilon>0$ is taken to be sufficiently small such that $C(\eta)-C_1\varepsilon\ge 0$. Applying Gronwall's inequality and (1.2) yields, for any $0\le s\le t<\infty$,

$$F_1(t)\le CF_1(s)e^{C\left(\sup\|(u,\tau)\|^2_{H^r(\mathbb{R}^2)}+1\right)\int_s^t\left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tau\|^2_{L^2(\mathbb{R}^2)}\right)dt'}\le CF_1(s),$$

which, together with (3.9), implies that, for any $0\le s\le t<\infty$,

$$\|\Lambda u(t)\|^2_{H^{r-1}(\mathbb{R}^2)}+\|\Lambda\tau(t)\|^2_{H^{r-1}(\mathbb{R}^2)}\le C\left(\|\Lambda u(s)\|^2_{H^{r-1}(\mathbb{R}^2)}+\|\Lambda\tau(s)\|^2_{H^{r-1}(\mathbb{R}^2)}\right).$$

This is exactly the desired inequality in (3.1). It then follows from Lemma 1.4 that

$$\|\Lambda u(t)\|^2_{H^{r-1}(\mathbb{R}^2)}+\|\Lambda\tau(t)\|^2_{H^{r-1}(\mathbb{R}^2)}\le C\,\varepsilon^2\,\langle t\rangle^{-1}\quad\text{with}\quad r>2. \tag{3.10}$$

Next we consider the case when $d=3$. Some of the terms are estimated differently. By Hölder's inequality and (3.7),

$$\begin{aligned}|Z_1|&\lesssim\|\nabla u\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{L^2(\mathbb{R}^3)}\\ &\quad+\|u\|_{L^\infty(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)}\|\Lambda\mathbb{P}\nabla\cdot\tau\|_{L^2(\mathbb{R}^3)}\\ &\quad+\|\Lambda^{r+1-\beta}\tau\|_{L^2(\mathbb{R}^3)}\|u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)}\\ &\lesssim\|u\|_{H^r(\mathbb{R}^3)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}\\ &\lesssim\|u\|_{H^r(\mathbb{R}^3)}\left(\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^3)}+\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^3)}\right).\end{aligned}$$

Similarly, Z_2 and Z_3 are bounded by

$$\begin{aligned}|Z_2|&\lesssim\|\nabla\tau\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)}\\ &\quad+\|u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)}\\ &\quad+\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)}\,\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)}\|\tau\|_{L^\infty(\mathbb{R}^3)}\\ &\quad+\|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)}\,\|u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{r+1-\beta}\tau\|_{L^2(\mathbb{R}^3)}\\ &\lesssim\left(\|u\|_{H^r(\mathbb{R}^3)}+\|\tau\|_{H^r(\mathbb{R}^3)}\right)\left(\|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^3)}+\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^3)}\right)\end{aligned}$$

and

$$
\begin{aligned}
|Z_3| &\lesssim \|\nabla\tau\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)} + \|\tau\|_{L^\infty(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)}^2 \\
&\quad + \|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)} \|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)}\|\tau\|_{L^\infty(\mathbb{R}^3)} \\
&\quad + \|\Lambda^{r+1-\beta}u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{r-\beta}\tau\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \left(\|u\|_{H^r(\mathbb{R}^3)} + \|\tau\|_{H^r(\mathbb{R}^3)}\right)\left(\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2 + \|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2\right).
\end{aligned}
$$

Z_4 is bounded the same as (3.8). By $\nabla\cdot u = 0$, $\frac{1}{2}\le\beta\le 1$ and $r > 1+\frac{d}{2}$,

$$
\begin{aligned}
|Z_5| &= \left|\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u\,dx - \int_{\mathbb{R}^3}\left(\Lambda^r(u\cdot\nabla u) - u\cdot\nabla\Lambda^r u\right)\cdot\Lambda^r u\,dx\right| \\
&\lesssim \|u\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)} + \|\nabla u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^r u\|_{L^2(\mathbb{R}^3)}^2 \\
&\lesssim \|u\|_{H^r(\mathbb{R}^3)}\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|Z_6| &= \left|\int_{\mathbb{R}^3} u\cdot\nabla\tau\cdot\Lambda^2\tau\,dx + \int_{\mathbb{R}^3}\left(\Lambda^r(u\cdot\nabla\tau) - u\cdot\nabla\Lambda^r\tau\right)\cdot\Lambda^r\tau\,dx\right| \\
&\lesssim \|u\|_{L^3(\mathbb{R}^3)}\|\nabla\tau\|_{L^6(\mathbb{R}^3)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^3)} + \|\Lambda^r\tau\|_{L^2(\mathbb{R}^3)} \\
&\quad \times \left(\|\nabla u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^r\tau\|_{L^2(\mathbb{R}^3)} + \|\Lambda^r u\|_{L^2(\mathbb{R}^3)}\|\nabla\tau\|_{L^\infty(\mathbb{R}^3)}\right) \\
&\lesssim \left(\|u\|_{H^r(\mathbb{R}^3)} + \|\tau\|_{H^r(\mathbb{R}^3)}\right)\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + \|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2\right).
\end{aligned}
$$

Z_7 is bounded by

$$
\begin{aligned}
|Z_7| &= \left|(Q(\tau,\nabla u),\Lambda^2\tau) + (\Lambda^{r-\beta}Q(\tau,\nabla u),\Lambda^{r+\beta}\tau)\right| \\
&\lesssim \|\tau\|_{L^3(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\|\Lambda^2\tau\|_{L^2(\mathbb{R}^3)} \\
&\quad + \|\Lambda^{r+\beta}\tau\|_{L^2(\mathbb{R}^3)}\left(\|\Lambda^{r-\beta}\nabla u\|_{L^2(\mathbb{R}^3)}\|\tau\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{r-\beta}\tau\|_{L^2(\mathbb{R}^3)}\right) \\
&\lesssim\ C\left(\|u\|_{H^r(\mathbb{R}^3)} + \|\tau\|_{H^r(\mathbb{R}^3)}\right)\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + \|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2\right).
\end{aligned}
$$

Collecting the estimates for Z_1 to Z_7, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda u\|_{H^{r-1}(\mathbb{R}^3)}^2 + \|\Lambda\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + 2k(\Lambda u,\Lambda\mathbb{P}\nabla\cdot\tau)_{H^{r-1-\beta}(\mathbb{R}^3)}\right) \\
&\quad + \left(\frac{3}{4}\eta - k\right)\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + \left(\frac{k}{2} - k^2\eta\right)\|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2 \\
&\le C\left(\|u\|_{H^r(\mathbb{R}^3)} + \|\tau\|_{H^r(\mathbb{R}^3)}\right)\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + \|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2\right) \\
&\le C_2\,\varepsilon\left(\|\Lambda^{1+\beta}\tau\|_{H^{r-1}(\mathbb{R}^3)}^2 + \|\Delta u\|_{H^{r-1-\beta}(\mathbb{R}^3)}^2\right).
\end{aligned}
$$

By choosing $k > 0$ to be sufficiently small and writing

$$F_2(t) = \|\Lambda u\|^2_{H^{r-1}(\mathbb{R}^3)} + \|\Lambda \tau\|^2_{H^{r-1}(\mathbb{R}^3)} + 2k(\Lambda u, \Lambda \mathbb{P}\nabla \cdot \tau)_{H^{r-1-\beta}(\mathbb{R}^3)},$$

we have

$$\frac{d}{dt}F_2(t) + (C(\eta) - C_2\,\varepsilon)\left(\|\Lambda^{1+\beta}\tau\|^2_{H^{r-1}(\mathbb{R}^3)} + \|\Delta u\|^2_{H^{r-1-\beta}(\mathbb{R}^3)}\right) \le 0.$$

Here $\varepsilon > 0$ is taken to be small such that $C(\eta) - C_2\,\varepsilon \ge 0$. Therefore, for any $0 \le s \le t < \infty$,

$$F_2(t) \le F_2(s).$$

Invoking (3.9) and Lemma 1.4, we have

$$\|\Lambda u(t)\|^2_{H^{r-1}(\mathbb{R}^3)} + \|\Lambda \tau(t)\|^2_{H^{r-1}(\mathbb{R}^3)} \lesssim \varepsilon^2\, \langle t\rangle^{-1} \quad \text{with} \quad r > \frac{5}{2}. \tag{3.11}$$

(3.10) for $d = 2$ and (3.11) for $d = 3$ yield

$$\|\Lambda u(t)\|^2_{H^{r-1}(\mathbb{R}^d)} + \|\Lambda \tau(t)\|^2_{H^{r-1}(\mathbb{R}^d)} \lesssim \varepsilon^2\, \langle t\rangle^{-1} \quad \text{with} \quad r > 1 + \frac{d}{2}.$$

This completes the proof of Theorem 1.2. $\square$

4. Spectral analysis

This section serves as a preparation for the proof of Theorem 1.3. We present an integral representation of (1.1) via the spectral analysis. The key components of this representation are several kernel operators given by the Fourier multipliers. These operators are anisotropic and inhomogeneous. The second main result of this section provides sharp upper bounds for the symbols of these operators.

Recall that $\mathcal{A} = \mathbb{P}\nabla \cdot \tau$. Clearly, any solution (u, τ) of (1.1) also solves (1.7), namely,

$$\begin{cases} \partial_t u = \mathcal{A} + G, \quad G = -\mathbb{P}(u \cdot \nabla u), \\ \partial_t \mathcal{A} = -\eta(-\Delta)^\beta \mathcal{A} + \frac{1}{2}\Delta u + F + H, \\ F = -\mathbb{P}\nabla \cdot (u \cdot \nabla \tau), \\ H = -\mathbb{P}\nabla \cdot Q(\tau, \nabla u). \end{cases} \tag{4.1}$$

(4.1) can be converted into an equivalent integral form given in the following lemma.

Lemma 4.1. *Assume $(u, \mathcal{A})$ solves (4.1). Then $(u, \mathcal{A})$ satisfies the following integral representation,*

$$
\begin{aligned}
\widehat{u}(\xi,t) =& M_1\widehat{u_0} + M_2\widehat{\mathcal{A}_0} + \int_0^t M_1(t-s)\widehat{G}(s)\,ds \\
&+ \int_0^t M_2(t-s)\left(\widehat{F}(s)+\widehat{H}(s)\right)\,ds, \\
\widehat{\mathcal{A}}(\xi,t) =& -\frac{|\xi|^2}{2}M_2\widehat{u_0} + M_3\widehat{\mathcal{A}_0} - \int_0^t \frac{|\xi|^2}{2}M_2(t-s)\widehat{G}(s)\,ds \\
&+ \int_0^t M_3(t-s)\left(\widehat{F}(s)+\widehat{H}(s)\right)\,ds,
\end{aligned}
\tag{4.2}
$$

where the kernel operators M_1, M_2 and M_3 are given by

$$
M_1(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \quad M_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad M_3(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \tag{4.3}
$$

with λ_1 and λ_2 being the roots of

$$
\lambda^2 + \eta|\xi|^{2\beta}\lambda + \frac{1}{2}|\xi|^2 = 0
$$

or

$$
\lambda_1 = -\frac{1}{2}\eta|\xi|^{2\beta}\left(1+\sqrt{1-\frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right), \tag{4.4}
$$

$$
\lambda_2 = -\frac{1}{2}\eta|\xi|^{2\beta}\left(1-\sqrt{1-\frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right). \tag{4.5}
$$

When $\lambda_1 = \lambda_2$, the kernel functions M_1, M_2 and M_3 in (4.3) are replaced by their corresponding limits as $\lambda_2 \to \lambda_1$ (see (4.12), (4.13) and (4.14) in the proof).

In order to understand the large-time behavior of u and $\mathcal{A}$ in (4.2), we need precise estimates on the kernels M_1, M_2 and M_3. The behavior of these kernels is inhomogeneous and depends on the frequency ξ. This suggests that we divide the frequency space into subdomains to obtain definite controls on the kernels. The following proposition provides optimal upper bounds for M_1, M_2 and M_3.

Proposition 4.2. *Let D_1 and D_2 be subsets of $\mathbb{R}^d$,*

$$
D_1 \triangleq \left\{\xi \in \mathbb{R}^d : \ |\xi| < \eta^{-\frac{1}{2\beta-1}}\right\}, \tag{4.6}
$$

$$
D_2 \triangleq \left\{\xi \in \mathbb{R}^d : \ |\xi| \geq \eta^{-\frac{1}{2\beta-1}}\right\}. \tag{4.7}
$$

Then M_1, M_2 and M_3 satisfy the following estimates:

(1) *When $\xi \in D_1$,*

$$|M_1(\xi,t)|, |M_3(\xi,t)| \lesssim e^{-\frac{\eta}{2}|\xi|^{2\beta}t}, \quad |M_2(\xi,t)| \lesssim |\xi|^{-1} e^{-\frac{\eta}{2}|\xi|^{2\beta}t}. \tag{4.8}$$

(2) *When $\xi \in D_2$,*

$$\begin{aligned} &|M_1(\xi,t)| \lesssim e^{-ct}, \ |M_2(\xi,t)| \lesssim |\xi|^{-2\beta} e^{-c|\xi|^{2-2\beta}t}, \\ &|M_3(\xi,t)| \lesssim e^{-c\,|\xi|^{2\beta}t} + |\xi|^{2-4\beta}\, e^{-c|\xi|^{2-2\beta}t}, \end{aligned} \tag{4.9}$$

where $c > 0$ is a constant and depends on η and β. Especially, for $\xi \in D_2$,

$$|M_1(\xi,t)|, \ |M_3(\xi,t)| \lesssim e^{-ct}, \ |M_2(\xi,t)| \lesssim |\xi|^{-2\beta} e^{-ct}.$$

The rest of this section proves Lemma 4.1 and then Proposition 4.2.

Proof of Lemma 4.1. Taking the Fourier transform of (4.1) leads to

$$\partial_t \varphi = B\varphi + R,$$

where

$$\varphi = \begin{pmatrix} \widehat{u} \\ \widehat{\mathcal{A}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -\frac{|\xi|^2}{2} & -\eta|\xi|^{2\beta} \end{pmatrix}, \quad R = \begin{pmatrix} \widehat{G} \\ \widehat{F} + \widehat{H} \end{pmatrix}.$$

Therefore, according to the ODE theory φ can be represented as

$$\varphi(t) = e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)} R(s)ds.$$

To obtain a more explicit representation, we need to diagonalize B. First, we compute the eigenvalues and eigenvectors of B. The characteristic polynomial of B is given by

$$p(\lambda) = \lambda(\lambda + \eta|\xi|^{2\beta}) + \frac{|\xi|^2}{2}$$

and its roots are given by (4.4) and (4.5), namely

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}\eta|\xi|^{2\beta}\left(1 + \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right), \\ \lambda_2 &= -\frac{1}{2}\eta|\xi|^{2\beta}\left(1 - \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right). \end{aligned}$$

When

$$\eta^2|\xi|^{4\beta} \geq 2|\xi|^2,$$

λ_1 and λ_2 are real numbers. When $\eta^2|\xi|^{4\beta} \neq 2|\xi|^2$ or $\lambda_1 \neq \lambda_2$, the eigenvectors corresponding to λ_1 and λ_2 are given by

$$V_1 = \begin{pmatrix} \lambda_2 \\ \frac{|\xi|^2}{2} \end{pmatrix}, \qquad V_2 = \begin{pmatrix} \lambda_1 \\ \frac{|\xi|^2}{2} \end{pmatrix},$$

respectively. Consequently we can write

$$BW = WD \quad \text{or} \quad B = WDW^{-1},$$

where D is the diagonal matrix and W denotes the matrix with the eigenvectors as columns, namely

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad W = \begin{pmatrix} \lambda_2 & \lambda_1 \\ \frac{|\xi|^2}{2} & \frac{|\xi|^2}{2} \end{pmatrix}.$$

For $|\xi| \neq 0$, the inverse of W, denoted W^{-1}, is given by

$$W^{-1} = \begin{pmatrix} \frac{1}{\lambda_2-\lambda_1} & -\frac{2}{|\xi|^2}\frac{\lambda_1}{\lambda_2-\lambda_1} \\ -\frac{1}{\lambda_2-\lambda_1} & \frac{2}{|\xi|^2}\frac{\lambda_2}{\lambda_2-\lambda_1} \end{pmatrix}.$$

Therefore,

$$\varphi(t) = W \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} W^{-1}\varphi_0 + \int_0^t W \begin{pmatrix} e^{\lambda_1 (t-s)} & 0 \\ 0 & e^{\lambda_2 (t-s)} \end{pmatrix} W^{-1} R(s) ds.$$

More explicitly,

$$W \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} W^{-1} = \begin{pmatrix} M_1 & M_2 \\ -\frac{|\xi|^2}{2} M_2 & M_3 \end{pmatrix},$$

where

$$M_1(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \quad M_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad M_3(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1}. \tag{4.10}$$

Therefore, for $\lambda_1 \neq \lambda_2$ or $\eta^2|\xi|^{4\beta} > 2|\xi|^2$,

$$\begin{aligned}
\widehat{u}(\xi,t) =& M_1\widehat{u_0} + M_2\widehat{\mathcal{A}_0} + \int_0^t M_1(t-s)\widehat{G}(s)\,ds \\
&+ \int_0^t M_2(t-s)\left(\widehat{F}(s)+\widehat{H}(s)\right)\,ds, \\
\widehat{\mathcal{A}}(\xi,t) =& -\frac{|\xi|^2}{2}M_2\widehat{u_0} + M_3\widehat{\mathcal{A}_0} - \int_0^t \frac{|\xi|^2}{2}M_2(t-s)\widehat{G}(s)\,ds \\
&+ \int_0^t M_3(t-s)\left(\widehat{F}(s)+\widehat{H}(s)\right)\,ds.
\end{aligned} \tag{4.11}$$

For $\eta^2|\xi|^{4\beta} = 2|\xi|^2$ or $\lambda_1 = \lambda_2$, the eigenvectors associated with eigenvalues are different from those for the case when $\lambda_1 \neq \lambda_2$. However the representation formula in (4.11) remains valid if M_1, M_2 and M_3 in (4.10) are interpreted as their corresponding limits, namely

$$M_1 = \lim_{\lambda_2\to\lambda_1} \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = (1-\lambda_1 t)e^{\lambda_1 t}, \tag{4.12}$$

$$M_2 = \lim_{\lambda_2\to\lambda_1} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = te^{\lambda_1 t}, \tag{4.13}$$

$$M_3 = \lim_{\lambda_2\to\lambda_1} \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = (1+\lambda_1 t)e^{\lambda_1 t}. \tag{4.14}$$

When

$$\eta^2|\xi|^{4\beta} < 2|\xi|^2,$$

$\sqrt{1-\frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}$ is a pure imaginary number, and λ_1 and λ_2 are given by

$$\lambda_1 = -\frac{1}{2}\eta|\xi|^{2\beta}\left(1 + i\sqrt{\frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}} - 1}\right),$$

$$\lambda_2 = -\frac{1}{2}\eta|\xi|^{2\beta}\left(1 - i\sqrt{\frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}} - 1}\right).$$

By going through the same process, $\widehat{u}(\xi,t)$ and $\widehat{\mathcal{A}}(\xi,t)$ can also be represented by (4.11), which is the desired formula (4.2). This completes the proof of Lemma 4.1. □

We now prove Proposition 4.2.

Proof of Proposition 4.2. Let D_1 and D_2 be the subdomains defined as in (4.6) and (4.7). We first prove (1). For $\xi \in D_1$, we have $\eta^2|\xi|^{4\beta} < |\xi|^2$, and thus λ_1 and λ_2 are complex numbers. Then

$$|\lambda_2 - \lambda_1| = |\xi|\sqrt{2 - \eta^2|\xi|^{4\beta-2}} > |\xi|$$

and

$$|\lambda_1| = |\lambda_2| = \frac{\sqrt{2}}{2}|\xi|, \quad |e^{\lambda_1 t}| = |e^{\lambda_2 t}| = e^{-\frac{\eta}{2}|\xi|^{2\beta}t}.$$

Therefore,

$$\begin{aligned} |M_1(\xi,t)| &\le \frac{|\lambda_2|}{|\lambda_2-\lambda_1|}|e^{\lambda_1 t}| + \frac{|\lambda_1|}{|\lambda_2-\lambda_1|}|e^{\lambda_2 t}| \lesssim e^{-\frac{\eta}{2}|\xi|^{2\beta}t}, \\ |M_2(\xi,t) &\le \frac{1}{|\lambda_2-\lambda_1|}|e^{\lambda_2 t}| + \frac{1}{|\lambda_2-\lambda_1|}|e^{\lambda_1 t}| \lesssim |\xi|^{-1}e^{-\frac{\eta}{2}|\xi|^{2\beta}t}. \end{aligned}$$

M_3 obeys the same bound as that for M_1. We now turn to (2). In order to analyze the property of λ_1 and λ_2 more accurately, we further split D_2 into the following three regions:

$$\begin{aligned} D_{21} &\triangleq \left\{ \xi \in \mathbb{R}^d : \ \eta^{-\frac{1}{2\beta-1}} \le |\xi| < \left(\frac{2}{\eta^2}\right)^{\frac{1}{4\beta-2}} \right\}, \\ D_{22} &\triangleq \left\{ \xi \in \mathbb{R}^d : \ \left(\frac{2}{\eta^2}\right)^{\frac{1}{4\beta-2}} \le |\xi| < \left(\frac{8}{3\eta^2}\right)^{\frac{1}{4\beta-2}} \right\}, \\ D_{23} &\triangleq \left\{ \xi \in \mathbb{R}^d : \ |\xi| \ge \left(\frac{8}{3\eta^2}\right)^{\frac{1}{4\beta-2}} \right\}. \end{aligned}$$

It is clear that λ_1 and λ_2 given by (4.4) and (4.5) are complex numbers in D_{21} and real numbers in $D_{22} \cup D_{23}$. Our consideration is split into three cases.

(i) $\xi \in D_{21}$. The difference $|\lambda_2 - \lambda_1|$ can get really close to zero when $|\xi|$ is close to $\left(\frac{2}{\eta^2}\right)^{\frac{1}{4\beta-2}}$. We need to make use of the difference $e^{\lambda_1 t} - e^{\lambda_2 t}$. Using the simple fact that $|\sin x| \le |x|$, we have

$$\begin{aligned} |M_2(\xi,t)| &= \left| e^{-\frac{\eta}{2}|\xi|^{2\beta}t} \left(\frac{2\sin\left(\frac{|\xi|\sqrt{2-\eta^2|\xi|^{4\beta-2}}}{2}t\right)}{|\xi|\sqrt{2-\eta^2|\xi|^{4\beta-2}}} \right) \right| \\ &\lesssim t e^{-\frac{\eta}{2}|\xi|^{2\beta}t} \lesssim |\xi|^{-2\beta} e^{-\frac{\eta}{4}|\xi|^{2\beta}t} \lesssim |\xi|^{-2\beta} e^{-c|\xi|^{2\beta}t}. \end{aligned}$$

Since $M_1 = e^{\lambda_1 t} - \lambda_1 M_2$ and $|\lambda_1| = |\lambda_2| = \frac{\sqrt{2}}{2}|\xi|$,

$$|M_1(\xi,t)| \le |e^{\lambda_1 t}| + |\lambda_1||M_2| \le e^{-\frac{\eta}{2}|\xi|^{2\beta}t} + |\xi||\xi|^{-2\beta}e^{-c|\xi|^{2\beta}t} \lesssim e^{-ct}.$$

Similarly,

$$|M_3(\xi,t)| = |e^{\lambda_1 t} + \lambda_2 M_2| \le |e^{\lambda_1 t}| + |\lambda_2||M_2| \lesssim e^{-c|\xi|^{2\beta}t}.$$

(ii) $\xi \in D_{22}$. When $\xi \in D_{22}$, $\lambda_1 \le -\frac{\eta}{2}|\xi|^{2\beta}$ and $\lambda_2 \le -\frac{\eta}{4}|\xi|^{2\beta}$. By the mean-value theorem, there is $\rho \in (\lambda_1, \lambda_2)$ such that

$$M_2 = te^{\rho t} \le te^{-\frac{\eta}{4}|\xi|^{2\beta}t} \lesssim |\xi|^{-2\beta}e^{-\frac{\eta}{8}|\xi|^{2\beta}t} \lesssim |\xi|^{-2\beta}e^{-c|\xi|^{2\beta}t}.$$

It is easy to check that the upper bounds for $|\lambda_1|$ and $|\lambda_2|$ in D_{22} which yield for any $\xi \in D_{22}$

$$|\lambda_1|, |\lambda_2| \le \frac{1}{2}\eta|\xi|^{2\beta}\left(1 + \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right) \le \frac{3}{4}\eta|\xi|^{2\beta}.$$

Then

$$\begin{aligned}
|M_1| &= |e^{\lambda_1 t} - \lambda_1 M_2| = |e^{\lambda_1 t} - \lambda_1 te^{\rho t}| \lesssim e^{-\frac{\eta}{2}|\xi|^{2\beta}t} + \eta|\xi|^{2\beta}te^{-\frac{\eta}{2}|\xi|^{2\beta}t} \lesssim e^{-ct}, \\
|M_3| &= |e^{\lambda_1 t} + \lambda_2 M_2| = |e^{\lambda_1 t} + \lambda_2 te^{\rho t}| \lesssim e^{-c|\xi|^{2\beta}t}.
\end{aligned}$$

(iii) $\xi \in D_{23}$. When $\xi \in D_{23}$, λ_1 given by (4.4) obvious satisfies

$$\lambda_1 \le -\frac{\eta}{2}|\xi|^{2\beta},\ |\lambda_1| \le \eta|\xi|^{2\beta}.$$

Next we estimate λ_2, we rewrite λ_2 as

$$\lambda_2 = -\frac{1}{2}\eta|\xi|^{2\beta}\left(1 - \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}\right) = -\frac{1}{2}\frac{\frac{2|\xi|^2}{\eta|\xi|^{2\beta}}}{1 + \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}} \le -\frac{1}{2\eta}|\xi|^{2-2\beta}.$$

Furthermore,

$$|\lambda_2| = \left|\frac{1}{2}\frac{\frac{2|\xi|^2}{\eta|\xi|^{2\beta}}}{1 + \sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}}}\right| \le \frac{2}{3\eta}|\xi|^{2-2\beta}.$$

Noticing the difference

$$\lambda_2 - \lambda_1 = \eta|\xi|^{2\beta}\sqrt{1 - \frac{2|\xi|^2}{\eta^2|\xi|^{4\beta}}} \ge \frac{1}{2}\eta|\xi|^{2\beta}.$$

Therefore, for $\frac{1}{2} \leq \beta \leq 1$ and $\xi \in D_{23}$,

$$|M_1(\xi,t)| \leq \frac{|\lambda_2|}{|\lambda_2 - \lambda_1|} e^{\lambda_1 t} + \frac{|\lambda_1|}{|\lambda_2 - \lambda_1|} e^{\lambda_2 t} \lesssim |\xi|^{2-4\beta} e^{-\frac{1}{2}\eta|\xi|^{2\beta} t} + e^{-\frac{1}{2\eta}|\xi|^{2-2\beta} t} \lesssim e^{-ct}.$$

Similarly,

$$\begin{aligned}
|M_2(\xi,t)| &\leq \frac{1}{|\lambda_2 - \lambda_1|} e^{\lambda_2 t} + \frac{1}{|\lambda_2 - \lambda_1|} e^{\lambda_1 t} \\
&\lesssim |\xi|^{-2\beta} e^{-c|\xi|^{2-2\beta} t} + |\xi|^{-2\beta} e^{-c|\xi|^{2\beta} t} \\
&\lesssim |\xi|^{-2\beta} e^{-c|\xi|^{2-2\beta} t}, \\
|M_3(\xi,t)| &\leq \frac{|\lambda_2|}{|\lambda_2 - \lambda_1|} e^{\lambda_2 t} + \frac{|\lambda_1|}{|\lambda_2 - \lambda_1|} e^{\lambda_1 t} \\
&\lesssim |\xi|^{2-4\beta} e^{-c|\xi|^{2-2\beta} t} + e^{-c|\xi|^{2\beta} t} \lesssim e^{-ct}.
\end{aligned}$$

This completes the proof of Proposition 4.2. □

5. Proof of Theorem 1.3

This section completes the proof of Theorem 1.3. We have made some preparations in the previous section.

Proof of Theorem 1.3. Recall the definitions of X and Y given by (1.10) and (1.11), respectively. As we have described in the introduction, to complete the proof of Theorem 1.3, it suffices to show that X and Y satisfy (1.12) and (1.13), namely

$$X(t) \lesssim \|(u_0, \tau_0)\|_{L^1 \cap H^r} + X(t)Y(t) + (X(t) + Y(t))\|\tau\|_{L^\infty H^r}, \tag{5.1}$$

$$\begin{aligned}
Y(t) \lesssim\ & \|(u_0, \tau_0)\|_{L^1 \cap H^r} + \|(u, \tau)\|_{L^\infty H^r}^2 \\
& + (X(t) + Y(t))\,(Y(t) + \|(u, \tau)\|_{L^\infty H^r}).
\end{aligned} \tag{5.2}$$

By taking the initial norm $\|(u_0, \tau_0)\|_{L^1 \cap H^r}$ to be sufficiently small, namely

$$\|(u_0, \tau_0)\|_{L^1 \cap H^r} \leq \varepsilon$$

for a suitable small $\varepsilon > 0$, Theorem 1.1 assesses that the corresponding solution remains small for all time,

$$\|(u(t), \tau(t))\|_{H^r} \leq C\,\varepsilon.$$

Then (5.1) and (5.2) imply

$$X(t) + Y(t) \leq C\,\varepsilon + C\,(X(t) + Y(t))^2 \tag{5.3}$$

and a simple application of the bootstrapping argument to (5.3) would lead to

$$X(t) + Y(t) \le C\,\varepsilon,$$

which yields the desired result of Theorem 1.3.

The rest of the proof focuses on (5.1) and (5.2). For the sake of clarity, we split the proof into several subsections with each devoted to one of the norms in the definitions of X and Y. More precisely, the rest consists of five subsections estimating the norms $\|u\|_{L^2}$, $\|\nabla u\|_{L^2}$, $\|\mathcal{A}\|_{L^2}$, $\|\widehat{u}\|_{L^1}$ and $\||\xi|\widehat{u}\|_{L^1}$, respectively. Subsection six is a summary and completes the proof.

5.1. The estimate of $\|u\|_{L^2}$

Thanks to $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$ and (4.11), we have

$$\begin{aligned}
\|u\|_{L^2} = \|\widehat{u}\|_{L^2} &\lesssim \|M_1\widehat{u_0}\|_{L^2} + \|M_2\widehat{\mathcal{A}_0}\|_{L^2} + \int_0^t \|M_1(t-s)\widehat{G}(s)\|_{L^2}\,ds \\
&\quad + \int_0^t \|M_2(t-s)\widehat{F}(s)\|_{L^2}\,ds + \int_0^t \|M_2(t-s)\widehat{H}(s)\|_{L^2}\,ds \\
&\triangleq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By (4.8), (4.9) and Lemma 2.2,

$$\begin{aligned}
I_1 = \|M_1\widehat{u_0}\|_{L^2} &= \|M_1\widehat{u_0}\|_{L^2(D_1)} + \|M_1\widehat{u_0}\|_{L^2(D_2)} \\
&\lesssim \|e^{-\frac{\eta}{2}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^2(D_1)} + \|e^{-ct}\widehat{u_0}\|_{L^2(D_2)} \\
&\lesssim \left(\langle t\rangle^{-\frac{d}{4\beta}} + e^{-ct}\right)\|u_0\|_{L^1\cap L^2} \lesssim \langle t\rangle^{-\frac{d}{4\beta}}\|u_0\|_{L^1\cap L^2}.
\end{aligned}$$

I_2 can be bounded similarly,

$$I_2 \lesssim \langle t\rangle^{-\frac{d}{4\beta}}\|\tau_0\|_{L^1\cap L^2}.$$

By (4.8), (4.9), Lemma 2.2 and then Lemma 2.1,

$$\begin{aligned}
I_3 &= \int_0^t \|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_1)}\,ds \\
&\quad + \int_0^t \|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_2)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_1)}\,ds \\
&\quad + \int_0^t \|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}}\,\|u\cdot\nabla u\|_{L^1\cap L^2} + e^{-c(t-s)}\|u\cdot\nabla u\|_{L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}}\,\|\nabla u\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty}) + e^{-c(t-s)}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}}\left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}}\right) + e^{-c(t-s)}\,\langle s\rangle^{-\frac{3d+2}{4\beta}}\;ds\,X(t)\,Y(t) \\
&\lesssim \langle t\rangle^{-\frac{d}{4\beta}}\;X(t)\,Y(t).
\end{aligned}
$$

Due to $x^n e^{-x} \le C$ for any $n \ge 0$ and its variant

$$
|\xi|^k e^{-\frac{\eta}{2}|\xi|^{2\beta}t} \lesssim \langle t\rangle^{-\frac{k}{2\beta}}\,e^{-\frac{\eta}{4}|\xi|^{2\beta}t} \qquad \text{for any } k\ge 0 \text{ and } \ \xi\in D_1, \tag{5.4}
$$

we have

$$
\begin{aligned}
I_4 &= \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^2(D_1)}\,ds \\
&\quad + \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\widehat{u\otimes\tau}\|_{L^2(D_1)}\,ds + \int_0^t \||\xi|^{1-2\beta}e^{-c(t-s)}\widehat{u\cdot\nabla\tau}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\,\|u\otimes\tau\|_{L^1\cap L^2} + e^{-c(t-s)}\|u\cdot\nabla\tau\|_{L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\,(\|u\|_{L^2}\|\tau\|_{L^2} + \|u\|_{L^\infty}\|\tau\|_{L^2}) + e^{-c(t-s)}\|u\|_{L^\infty}\|\nabla\tau\|_{L^2}\,ds
\end{aligned}
$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \left(\langle s\rangle^{-\frac{d}{4\beta}} + \langle s\rangle^{-\frac{d}{2\beta}}\right) + e^{-c(t-s)} \langle s\rangle^{-\frac{d}{2\beta}} \, ds \, X(t) \, \|\tau\|_{L^\infty H^1}$$

$$\lesssim \langle t\rangle^{-\frac{d}{4\beta}} \, X(t) \, \|\tau\|_{L^\infty H^1},$$

where we have used the following inequality in the last step, for $\frac{d+2}{4\beta} > 1$,

$$\int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \, \langle s\rangle^{-\frac{d}{4\beta}} \, ds \lesssim \langle t\rangle^{-\frac{d}{4\beta}} \, . \tag{5.5}$$

This explains why we have assumed that $\frac{d+2}{4\beta} \neq 1$ in Theorem 1.3. When $\frac{d+2}{4\beta} = 1$ or when $d = 2$ and $\beta = 1$, the upper bound in (5.5) is no longer valid and would need an extra logarithm. We shall no longer mention this when we encounter similar situations. We now estimate I_5,

$$\begin{aligned}
I_5 &= \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_1)}\, ds \\
&\quad + \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\, ds \\
&\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds \\
&\quad + \int_0^t \||\xi|^{1-2\beta}e^{-c(t-s)}\widehat{Q(\tau,\nabla u)}\|_{L^2(D_2)}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}} \|Q(\tau,\nabla u)\|_{L^1\cap L^2} + e^{-c(t-s)}\|Q(\tau,\nabla u)\|_{L^2}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}} \|\tau\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty}) + e^{-c(t-s)}\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{4\beta}} \left(\langle s\rangle^{-\frac{d+2}{4\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right) + e^{-c(t-s)} \langle s\rangle^{-\frac{d+1}{2\beta}} \, ds \, Y(t) \, \|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d}{4\beta}} \, Y(t) \, \|\tau\|_{L^\infty L^2}.
\end{aligned}$$

Collecting the bounds above for I_1 through I_5, we find

$$\|\widehat{u}\|_{L^2} \lesssim \langle t\rangle^{-\frac{d}{4\beta}} \|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{d}{4\beta}} X(t)Y(t) \\ + \langle t\rangle^{-\frac{d}{4\beta}} (X(t)+Y(t))\|\tau\|_{L^\infty H^1}. \tag{5.6}$$

5.2. The estimate of $\|\nabla u\|_{L^2}$

We compute the L^2-norm of ∇u via (4.11),

$$\begin{aligned}\|\nabla u\|_{L^2} = \||\xi|\widehat{u}\|_{L^2} &\lesssim \||\xi|M_1\widehat{u_0}\|_{L^2} + \||\xi|M_2\widehat{\mathcal{A}_0}\|_{L^2} + \int_0^t \||\xi|M_1(t-s)\widehat{G}(s)\|_{L^2}\,ds \\ &+ \int_0^t \||\xi|M_2(t-s)\widehat{F}(s)\|_{L^2}\,ds + \int_0^t \||\xi|M_2(t-s)\widehat{H}(s)\|_{L^2}\,ds \\ &\triangleq J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

By (4.8), (4.9) and Lemma 2.2,

$$\begin{aligned}J_1 &= \||\xi|M_1\widehat{u_0}\|_{L^2(D_1)} + \||\xi|M_1\widehat{u_0}\|_{L^2(D_2)} \\ &\lesssim \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^2(D_1)} + \||\xi|e^{-ct}\widehat{u_0}\|_{L^2(D_2)} \\ &\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \|u_0\|_{L^1\cap L^2} + e^{-ct}\|u\|_{H^1} \\ &\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \|u_0\|_{L^1\cap H^1}.\end{aligned}$$

Similarly,

$$J_2 \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \|\tau_0\|_{L^1\cap H^1}.$$

Again, by (4.8), (4.9) and Lemma 2.2,

$$\begin{aligned}J_3 &= \int_0^t \||\xi|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_1)}\,ds \\ &\quad+ \int_0^t \||\xi|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\lesssim \int_0^t \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_1)}\,ds \\ &\quad+ \int_0^t \||\xi|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_2)}\,ds\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|u\cdot\nabla u\|_{L^1\cap L^2} + e^{-c(t-s)}\||\xi|\widehat{u\cdot\nabla u}\|_{L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\nabla u\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty})\,ds \\
&\quad + \int_0^t e^{-c(t-s)}(\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{u}\|_{L^2} + \|\widehat{u}\|_{L^1}\||\xi|^2\widehat{u}\|_{L^2})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}}\right) ds\, X(t)\, Y(t) \\
&\quad + \int_0^t e^{-c(t-s)} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{d}{2\beta}}\right) ds\,(X(t)+Y(t))\,\|u\|_{L^\infty H^2} \\
&\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t)+Y(t))\,\|u\|_{L^\infty H^2}\right).
\end{aligned}
$$

To estimate J_4, we split it into two parts,

$$
\begin{aligned}
J_4 &= \int_0^t \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^2(D_1)}\,ds \\
&\quad + \int_0^t \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^2(D_2)}\,ds \\
&\triangleq J_{41} + J_{42}.
\end{aligned}
$$

We first compute J_{42}. By (4.9),

$$
\begin{aligned}
J_{42} &= \int_0^t \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_0^t \|e^{-c(t-s)}|\xi|\widehat{u\cdot\nabla\tau}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_0^t e^{-c(t-s)} \left(\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{\tau}\|_{L^2} + \|\widehat{u}\|_{L^1}\||\xi|^2\widehat{\tau}\|_{L^2}\right) ds \\
&\lesssim \int_0^t e^{-c(t-s)} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{d}{2\beta}}\right) ds\,(X(t)+Y(t))\,\|\tau\|_{L^\infty H^2}
\end{aligned}
$$

$$\lesssim \langle t\rangle^{-\frac{d}{2\beta}}\ (X(t)+Y(t))\,\|\tau\|_{L^\infty H^2}$$
$$\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\ (X(t)+Y(t))\,\|\tau\|_{L^\infty H^2},$$

where the last step needs $d \geq 2$. A new approach is needed in order to obtain a suitable upper bound for J_{41}. If we estimate J_{41} similarly as before, we would end up with an integral of the form

$$\int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\ \|u\|_{L^2}\|\nabla\tau\|_{L^2}\,ds \lesssim \langle t\rangle^{-\frac{d}{4\beta}}\ X(t)\,\|\tau\|_{L^\infty H^2},$$

which does not have the desired decay rate $\langle t\rangle^{-\frac{d+2}{4\beta}}$. In order to generate enough decay to close the estimates, we write

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) = \sum_{i=1}^d \partial_i\mathbb{P}\nabla\cdot(u_i\tau) = \sum_{i=1}^d(\partial_i[\mathbb{P}\nabla\cdot, u_i]\tau + \partial_i(u_i\mathcal{A})). \tag{5.7}$$

By (4.8), Lemma 2.2 and (2.5),

$$\begin{aligned}
J_{41} &\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{u\otimes\mathcal{A}\}\|_{L^2(D_1)}\,ds \\
&\quad + \sum_{i=1}^d\int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{[\mathbb{P}\nabla\cdot, u_i]\tau\}\|_{L^2(D_1)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\ \big(\|u\otimes\mathcal{A}\|_{L^1} + \|u\otimes\mathcal{A}\|_{L^2} \\
&\qquad\qquad + \|[\mathbb{P}\nabla\cdot, u_i]\tau\|_{L^1} + \|[\mathbb{P}\nabla\cdot, u_i]\tau\|_{L^2}\big)\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\ \big(\|u\|_{L^2}\|\mathcal{A}\|_{L^2} + \|u\|_{L^\infty}\|\mathcal{A}\|_{L^2} \\
&\qquad\qquad + \|\nabla u\|_{L^2}\|\tau\|_{L^2} + \||\xi|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^2}\big)\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\ \Big(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}} + \langle s\rangle^{-\frac{d+2}{4\beta}}\Big)\ ds\, Y(t)\,(X(t) + \|\tau\|_{L^\infty L^2}) \\
&\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\ Y(t)\,(X(t) + \|\tau\|_{L^\infty L^2}),
\end{aligned}$$

where we have used the fact (see [31, Appendix])

$$\|[\mathbb{P}\nabla\cdot, u_i]\tau\|_{L^2} \lesssim \||\xi|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^2}. \tag{5.8}$$

Thus

$$J_4 \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t)+Y(t))\|\tau\|_{L^\infty H^2}\right).$$

Next we estimate J_5. The process is more elaborate. First, it is naturally split into two parts,

$$\begin{aligned}
J_5 = &\int_0^t \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_1)}\,ds \\
&+ \int_0^t \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\
&\triangleq J_{51} + J_{52}.
\end{aligned}$$

J_{51} can be treated in a similar fashion as before, but J_{52} requires some new approaches. By (4.8) and Lemma 2.2,

$$\begin{aligned}
J_{51} &\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_1)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|Q(\tau,\nabla u)\|_{L^1\cap L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\tau\|_{L^2}\left(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty}\right)\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \left(\langle s\rangle^{-\frac{d+2}{4\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right)\,ds\, Y(t)\,\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\; Y(t)\,\|\tau\|_{L^\infty L^2}.
\end{aligned} \tag{5.9}$$

To estimate J_{52}, we use the upper bound for M_2 with $\xi \in D_2$,

$$|M_2(t)| \lesssim |\xi|^{-2\beta}\, e^{-c\,|\xi|^{2-2\beta}t}, \qquad c = c(\eta) > 0.$$

Therefore,

$$J_{52} \lesssim \int_0^t \||\xi|^{2-2\beta}\, e^{-c\,|\xi|^{2-2\beta}(t-s)}\,\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds. \tag{5.10}$$

We need to distinguish between two cases: $\beta = 1$ and $\frac{1}{2} \leq \beta < 1$. When $\beta = 1$,

$$\begin{aligned} J_{52} &\lesssim \int_0^t \|e^{-c\,(t-s)}\,\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\lesssim \int_0^t e^{-c\,(t-s)}\,\|\tau(s)\|_{L^2}\,\|\nabla u(s)\|_{L^\infty}\,ds \\ &\lesssim \int_0^t e^{-c\,(t-s)}\,\|\tau(s)\|_{L^2}\,\||\xi|\widehat{u}(s)\|_{L^1}\,ds \\ &\lesssim \int_0^t e^{-c\,(t-s)}\,\langle s\rangle^{-\frac{d+1}{2\beta}}\,ds\,Y(t)\,\|\tau\|_{L^\infty L^2} \\ &\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\,Y(t)\,\|\tau\|_{L^\infty L^2}. \end{aligned} \tag{5.11}$$

When $\frac{1}{2} \leq \beta < 1$, we need to split the time integral in (5.10) into two parts,

$$\begin{aligned} J_{52} &\lesssim \int_0^{t/2} \||\xi|^{2-2\beta}\,e^{-c\,|\xi|^{2-2\beta}(t-s)}\,\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\quad + \int_{t/2}^t \||\xi|^{2-2\beta}\,e^{-c\,|\xi|^{2-2\beta}(t-s)}\,\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\triangleq J_{521} + J_{522}. \end{aligned} \tag{5.12}$$

Thanks to $|\xi|^{1-2\beta} \leq C$ for any $\xi \in D_2$ and Sobolev embedding,

$$\begin{aligned} J_{521} &\lesssim \int_0^{t/2} e^{-c(t-s)}\,\||\xi|\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2}\,ds \\ &\lesssim \int_0^{t/2} e^{-c(t-s)}\,\big(\|\nabla\tau\|_{L^4}\|\nabla u\|_{L^4} + \|\tau\|_{L^\infty}\|\nabla^2 u\|_{L^2}\big)\,ds \\ &\lesssim \int_0^{t/2} e^{-c(t-s)}\,\|u\|_{H^2}\,\|\tau\|_{H^2}\,ds \\ &\lesssim e^{-\frac{c}{2}t}\,\frac{t}{2}\,\|u\|_{L^\infty H^2}\|\tau\|_{L^\infty H^2} \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\,\|u\|_{L^\infty H^2}\|\tau\|_{L^\infty H^2}. \end{aligned} \tag{5.13}$$

To estimate J_{522}, we take $0 < \sigma < 1$ to be a small number and write

$$
\begin{aligned}
J_{522} &= \int_{t/2}^{t} \||\xi|^{(2-2\beta)(1-\sigma)}\ e^{-c\,|\xi|^{2-2\beta}(t-s)}\,|\xi|^{(2-2\beta)\sigma}\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\
&\lesssim \int_{t/2}^{t} (t-s)^{-(1-\sigma)}\,\||\xi|^{(2-2\beta)\sigma}\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2}\,ds \\
&\lesssim t^{\sigma} \sup_{t/2\le s\le t} \|\Lambda^{(2-2\beta)\sigma}Q(\tau,\nabla u)\|_{L^2} \\
&\lesssim t^{\sigma} \sup_{t/2\le s\le t} \Big(\|\Lambda^{(2-2\beta)\sigma}\tau\|_{L^2}\,\|\nabla u\|_{L^\infty} + \|\tau\|_{L^p}\,\|\Lambda^{(2-2\beta)\sigma}\nabla u\|_{L^q}\Big) \\
&\lesssim \langle t\rangle^{\sigma}\ \langle t\rangle^{-\frac{d+1}{2\beta}}\ Y(t)\,\|\tau\|_{L^\infty H^2} + \langle t\rangle^{\sigma}\ \|\tau\|_{L^\infty H^{r_1}} \sup_{t/2\le s\le t} \|\Lambda^{(2-2\beta)\sigma}\nabla u\|_{L^q},
\end{aligned}
$$

where $2 < p, q < \infty$ and r_1 satisfy

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad d\left(1 - \frac{2}{q}\right) - (8\beta-4)\sigma > 0, \qquad r_1 \ge d\left(\frac{1}{2} - \frac{1}{p}\right).
$$

By the Gagliardo-Nirenberg inequality,

$$
\|\Lambda^{(2-2\beta)\sigma}\nabla u\|_{L^q(\mathbb{R}^d)} \le C\,\|\nabla u\|_{L^\infty(\mathbb{R}^d)}^{n_1}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}^{n_2}\,\|\Lambda^{\sigma_1}\nabla u\|_{L^2(\mathbb{R}^d)}^{n_3},
$$

where

$$
(2-2\beta)\sigma < \sigma_1, \tag{5.14}
$$

$$
0 < n_1 < 1, \quad 0 < n_2 < 1, \quad 0 < n_3 < 1, \tag{5.15}
$$

$$
n_1 + n_2 + n_3 = 1, \qquad \frac{1}{q} - \frac{(2-2\beta)\sigma}{d} = \frac{1}{2}\,n_2 + \left(\frac{1}{2} - \frac{\sigma_1}{d}\right)\,n_3. \tag{5.16}
$$

Then J_{522} can be further bounded by

$$
\begin{aligned}
J_{522} &\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}+\sigma}\ Y(t)\,\|\tau\|_{L^\infty H^2} \\
&\qquad + \langle t\rangle^{\sigma}\ \|\tau\|_{L^\infty H^{r_1}}\ \langle t\rangle^{-n_1\frac{d+1}{2\beta}-n_2\frac{d+2}{4\beta}}\ Y^{n_1+n_2}(t)\,\|u\|_{L^\infty H^{1+\sigma_1}}^{n_3} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}+\sigma}\ Y(t)\,\|\tau\|_{L^\infty H^2} \\
&\qquad + \langle t\rangle^{-n_1\frac{d+1}{2\beta}-n_2\frac{d+2}{4\beta}+\sigma}\ \|\tau\|_{L^\infty H^{r_1}}\,(Y(t) + \|u\|_{L^\infty H^{1+\sigma_1}}).
\end{aligned}
$$

By further choosing $0 < \sigma < \sigma_1 \le \frac{d}{2}$, and $n_1 \in (0,1)$ and $n_2 \in (0,1)$ such that

$$
-\frac{d+1}{2\beta} + \sigma \le -\frac{d+2}{4\beta}, \qquad -n_1\frac{d+1}{2\beta} - n_2\frac{d+2}{4\beta} + \sigma \le -\frac{d+2}{4\beta}, \tag{5.17}
$$

we then obtain

$$J_{522} \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \|\tau\|_{L^\infty H^2} (Y(t) + \|u\|_{L^\infty H^{1+\sigma_1}}). \tag{5.18}$$

Since $r_1 < d/2 < 2$, we have replaced H^{r_1} by H^2 here. A simple calculation assures that we can indeed choose $\sigma \in (0,1)$, σ_1, n_1, n_2 and n_3 to satisfy (5.14), (5.15), (5.16) and (5.17). In fact, (5.16) reduces to

$$d\, n_1 + 2\sigma_1\, n_3 = d\left(1 - \frac{2}{q}\right) + 4(1-\beta)\sigma$$

and (5.17) to

$$4\beta\, \sigma \le d, \qquad (d+2)\, n_3 + 4\beta\, \sigma \le d n_1.$$

These conditions will be fulfilled if we choose

$$\begin{aligned}
&\sigma > 0 \quad \text{is sufficiently small}, \quad \sigma_1 = \frac{d}{2},\\
&d\left(1 - \frac{2}{q}\right) - (8\beta - 4)\sigma > 0,\\
&n_3 = \frac{d\left(1 - \frac{2}{q}\right) - (8\beta-4)\sigma}{d + 2 + 2\sigma_1},\\
&n_1 = \frac{4\beta\sigma}{d} + \frac{d+2}{d+2+2\sigma_1}\left(1 - \frac{2}{q} - \frac{(8\beta-4)\sigma}{d}\right),\\
&n_2 = 1 - (n_1 + n_3) = \frac{2}{q} - \frac{4(1-\beta)\sigma}{d}.
\end{aligned}$$

It is clear that, when $\sigma > 0$ is sufficiently small, n_1, n_2 and $n_3 \in (0,1)$. We summarize our estimates on J_5. In the case when $\beta = 1$, by (5.9) and (5.11),

$$|J_5| \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\, Y(t)\, \|\tau\|_{L^\infty L^2}.$$

In the case when $\frac{1}{2} \le \beta < 1$, by (5.9), (5.12), (5.13) and (5.18),

$$|J_5| \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \|\tau\|_{L^\infty H^2} \left(Y(t) + \|u\|_{L^\infty H^{1+\frac{d}{2}}}\right).$$

Combining the five estimates above can yield

$$\begin{aligned}
\|\nabla u\|_{L^2} \lesssim{}& \langle t\rangle^{-\frac{d+2}{4\beta}} \|(u_0,\tau_0)\|_{L^1\cap H^1} + \langle t\rangle^{-\frac{d+2}{4\beta}} \|u\|_{L^\infty H^{1+\frac{d}{2}}} \|\tau\|_{L^\infty H^2}\\
&+ \langle t\rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t) + Y(t))\|(u,\tau)\|_{L^\infty H^2}\right).
\end{aligned} \tag{5.19}$$

5.3. The estimate of $\|\mathcal{A}\|_{L^2}$

By (4.11),

$$\|\mathcal{A}\|_{L^2} = \|\widehat{\mathcal{A}}\|_{L^2} \lesssim \|\frac{|\xi|^2}{2}M_2\widehat{u_0}\|_{L^2} + \|M_3\widehat{\mathcal{A}_0}\|_{L^2} + \int_0^t \|\frac{|\xi|^2}{2}M_2(t-s)\widehat{G}(s)\|_{L^2}\,ds$$
$$+ \int_0^t \|M_3(t-s)\widehat{F}(s)\|_{L^2}\,ds + \int_0^t \|M_3(t-s)\widehat{H}(s)\|_{L^2}\,ds$$
$$\triangleq N_1 + N_2 + N_3 + N_4 + N_5.$$

According to (4.8), (4.9) and Lemma 2.2,

$$\begin{aligned} N_1 &= \frac{1}{2}\||\xi|^2 M_2\widehat{u_0}\|_{L^2(D_1)} + \frac{1}{2}\||\xi|^2 M_2\widehat{u_0}\|_{L^2(D_2)} \\ &\lesssim \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^2(D_1)} + \||\xi|^{1-2\beta}e^{-ct}|\xi|\widehat{u_0}\|_{L^2(D_2)} \\ &\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\|u_0\|_{L^1\cap L^2} + e^{-ct}\|\nabla u_0\|_{L^2} \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\|u_0\|_{L^1\cap H^1}, \end{aligned}$$

where we have used the simple fact that $|\xi|^{1-2\beta} \le C$ for any $\xi \in D_2$. Similarly,

$$N_2 \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\|\tau_0\|_{L^1\cap H^1}.$$

By (4.8), (4.9) and Lemma 2.2,

$$\begin{aligned} N_3 &= \int_0^t \|\frac{|\xi|^2}{2}M_2(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_1)}\,ds \\ &\quad + \int_0^t \|\frac{|\xi|^2}{2}M_2(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\lesssim \int_0^t \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\widehat{u\cdot\nabla u}\|_{L^2(D_1)}\,ds \\ &\quad + \int_0^t \||\xi|^{1-2\beta}e^{-c(t-s)}\||\xi|\widehat{u\cdot\nabla u}\|_{L^2(D_2)}\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}}\|u\cdot\nabla u\|_{L^1\cap L^2} + e^{-c(t-s)}\||\xi|\widehat{u\cdot\nabla u}\|_{L^2(D_2)}\,ds \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\nabla u\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty}) + e^{-c(t-s)} \\
&\quad\times \left(\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{u}\|_{L^2} + \|\widehat{u}\|_{L^1}\||\xi|^2\widehat{u}\|_{L^2}\right) ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}}\right) + e^{-c(t-s)} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{d}{2\beta}}\right) ds \\
&\quad\times (X(t)Y(t) + (X(t)+Y(t))\|u\|_{L^\infty H^2}) \\
&\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t)+Y(t))\|u\|_{L^\infty H^2}\right).
\end{aligned}
$$

N_4 can be estimated similarly as J_4,

$$
N_4 \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t)+Y(t))\|\tau\|_{L^\infty H^2}\right).
$$

The estimate of N_5 share some ideas with that for J_5. First, N_5 is naturally split into two parts,

$$
\begin{aligned}
N_5 &= \int_0^t \|M_3(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_1)}\, ds \\
&\quad + \int_0^t \|M_3(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\, ds \\
&\triangleq N_{51} + N_{52}.
\end{aligned}
$$

N_{51} can be bounded similarly as before,

$$
\begin{aligned}
N_{51} &\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_1)}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|Q(\tau,\nabla u)\|_{L^1\cap L^2}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\tau\|_{L^2}\left(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty}\right) ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+2}{4\beta}} \left(\langle s\rangle^{-\frac{d+2}{4\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right) dsY(t)\,\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\, Y(t)\,\|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

To bound N_{52}, we use the following upper bound for M_3 when $\xi \in D_2$ (see (4.9)),

$$|M_3(\xi,t)| \lesssim e^{-c\,|\xi|^{2\beta}t} + |\xi|^{2-4\beta}\, e^{-c|\xi|^{2-2\beta}t}.$$

Inserting this upper bound in N_{52} further divides N_{52} into two parts,

$$\begin{aligned} N_{52} &\lesssim \int_0^t \|e^{-c\,|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\quad + \int_0^t \||\xi|^{2-4\beta}\, e^{-c|\xi|^{2-2\beta}(t-s)}\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\triangleq \bar{N}_{52} + \widetilde{N}_{52}. \end{aligned}$$

For $\bar{N}_{52}$, we split the integral interval $[0,t]$ into $[0,t/2)$ and $[t/2,t]$,

$$\begin{aligned} \bar{N}_{52} &= \int_0^{t/2} \|e^{-c\,|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\quad + \int_{t/2}^t \|e^{-c\,|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\triangleq \bar{N}_{521} + \bar{N}_{522}. \end{aligned}$$

Using the same method as for the estimate of J_{521}, we infer

$$\bar{N}_{521} \lesssim \langle t\rangle^{-\frac{d+2}{4\beta}}\, \|u\|_{L^\infty H^2}\|\tau\|_{L^\infty H^2}.$$

The estimate of $\bar{N}_{522}$ is more complex. Our consideration is divided into two cases: $\beta = \frac{1}{2}$ and $\frac{1}{2} < \beta \le 1$. When $\frac{1}{2} < \beta \le 1$, the simple facts that $x^n e^{-cx} \le C$ for any $n \ge 0$ and $|\xi| \ge C$ for any $\xi \in D_2$ yield

$$\begin{aligned} \bar{N}_{522} &\lesssim \int_{t/2}^t \||\xi| e^{-\frac{c}{2}\,|\xi|^{2\beta}(t-s)} e^{-\frac{c}{2}\,|\xi|^{2\beta}(t-s)}\mathscr{F}\{Q(\tau,\nabla u)\}\|_{L^2(D_2)}\,ds \\ &\lesssim \int_{t/2}^t (t-s)^{-\frac{1}{2\beta}} e^{-\frac{c}{2}(t-s)}\, \|Q(\tau,\nabla u)\|_{L^2}\,ds \\ &\lesssim \int_{t/2}^t (t-s)^{-\frac{1}{2\beta}} e^{-\frac{c}{2}(t-s)}\, \|\tau\|_{L^\infty}\, \|\nabla u\|_{L^2}\,ds \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{t/2}^{t} (t-s)^{-\frac{1}{2\beta}} e^{-\frac{c}{2}(t-s)} \langle s \rangle^{-\frac{d+2}{4\beta}} \, ds \, Y(t) \|\tau\|_{L^\infty H^2} \\
&\lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \int_{t/2}^{t} (t-s)^{-\frac{1}{2\beta}} e^{-\frac{c}{2}(t-s)} \, ds \, Y(t) \|\tau\|_{L^\infty H^2} \\
&\lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \, Y(t) \|\tau\|_{L^\infty H^2},
\end{aligned}
$$

where the last step uses the simple fact that for any $\frac{1}{2} < \beta \leq 1$,

$$
\int_{t/2}^{t} (t-s)^{-\frac{1}{2\beta}} e^{-\frac{c}{2}(t-s)} \, ds \leq C(\beta).
$$

When $\beta = \frac{1}{2}$, $\bar{N}_{522}$ can be bounded by the process in the estimate of J_{522},

$$
\bar{N}_{522} \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \, \|\tau\|_{L^\infty H^2} \left(Y(t) + \|u\|_{L^\infty H^{1+\frac{d}{2}}} \right).
$$

To estimate $\widetilde{N}_{52}$, we distinguish $\beta = 1$ from $\frac{1}{2} \leq \beta < 1$. When $\beta = 1$,

$$
\begin{aligned}
\widetilde{N}_{52} &\lesssim \int_0^t \| |\xi|^{-1} e^{-c(t-s)} \mathscr{F}\{Q(\tau, \nabla u)\} \|_{L^2(D_2)} \, ds \lesssim \int_0^t e^{-c(t-s)} \|\nabla u\|_{L^\infty} \|\tau\|_{L^2} \, ds \\
&\lesssim \int_0^t e^{-c(t-s)} \langle s \rangle^{-\frac{d+1}{2\beta}} \, ds \, Y(t) \, \|\tau\|_{L^\infty L^2} \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \, Y(t) \, \|\tau\|_{L^\infty L^2},
\end{aligned}
$$

where we have used the simple fact that $|\xi|^{-1} \leq C$ for any $\xi \in D_2$. When $\frac{1}{2} \leq \beta < 1$, $\widetilde{N}_{52}$ can be treated similarly as J_{52} in (5.12).

$$
\begin{aligned}
\widetilde{N}_{52} &= \int_0^{t/2} \| |\xi|^{2-4\beta} \, e^{-c|\xi|^{2-2\beta}(t-s)} \mathscr{F}\{\mathbb{P}\nabla \cdot Q(\tau, \nabla u)\} \|_{L^2(D_2)} \, ds \\
&\quad + \int_{t/2}^{t} \| |\xi|^{2-4\beta} \, e^{-c|\xi|^{2-2\beta}(t-s)} \mathscr{F}\{\mathbb{P}\nabla \cdot Q(\tau, \nabla u)\} \|_{L^2(D_2)} \, ds \\
&\triangleq \widetilde{N}_{521} + \widetilde{N}_{522}.
\end{aligned}
$$

By proceeding as in the estimates of J_{521} and J_{522} in the previous subsection, we obtain

$$
\widetilde{N}_{521} \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \, \|u\|_{L^\infty H^2} \|\tau\|_{L^\infty H^2}
$$

and

$$\widetilde{N}_{522} \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \|\tau\|_{L^\infty H^2} \left(Y(t) + \|u\|_{L^\infty H^{1+\frac{d}{2}}} \right).$$

Combining the estimates above, we find

$$N_5 \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \left(Y(t) + \|u\|_{L^\infty H^{1+\frac{d}{2}}} \right) \|\tau\|_{L^\infty H^2}.$$

Collecting the estimates of N_1 through N_5 yields

$$\begin{aligned} \|\mathcal{A}\|_{L^2} \lesssim \langle t \rangle^{-\frac{d+2}{4\beta}} \|(u_0, \tau_0)\|_{L^1 \cap H^1} + \langle t \rangle^{-\frac{d+2}{4\beta}} \|u\|_{L^\infty H^{1+\frac{d}{2}}} \|\tau\|_{L^\infty H^2} \\ + \langle t \rangle^{-\frac{d+2}{4\beta}} \left(X(t)Y(t) + (X(t) + Y(t)) \|(u, \tau)\|_{L^\infty H^2} \right). \end{aligned} \tag{5.20}$$

5.4. The estimate of $\|\widehat{u}\|_{L^1}$

By (4.11),

$$\begin{aligned} \|\widehat{u}\|_{L^1} &\lesssim \|M_1 \widehat{u_0}\|_{L^1} + \|M_2 \widehat{\mathcal{A}_0}\|_{L^1} + \int_0^t \|M_1(t-s) \widehat{G}(s)\|_{L^1} \, ds \\ &\quad + \int_0^t \|M_2(t-s) \widehat{F}(s)\|_{L^1} \, ds + \int_0^t \|M_2(t-s) \widehat{H}(s)\|_{L^1} \, ds \\ &\triangleq K_1 + K_2 + K_3 + K_4 + K_5. \end{aligned}$$

By (4.8) and (4.9),

$$\begin{aligned} K_1 &= \|M_1 \widehat{u_0}\|_{L^1(D_1)} + \|M_1 \widehat{u_0}\|_{L^1(D_2)} \\ &\lesssim \|e^{-\frac{\eta}{2}|\xi|^{2\beta} t} \widehat{u_0}\|_{L^1(D_1)} + \|e^{-ct} \widehat{u_0}\|_{L^1(D_2)} \\ &\lesssim \|e^{-\frac{\eta}{4}|\xi|^{2\beta} t}\|_{L^2(D_1)} \|e^{-\frac{\eta}{4}|\xi|^{2\beta} t} \widehat{u_0}\|_{L^2(D_1)} + e^{-ct} \|\widehat{u_0}\|_{L^1(D_2)}. \end{aligned}$$

If $t \geq 1$,

$$\begin{aligned} \|e^{-c|\xi|^{2\beta} t}\|_{L^2(D_1)} &= \left(\int_{|\xi| < \eta^{-\frac{1}{2\beta - 1}}} e^{-2c|\xi|^{2\beta} t} \, d\xi \right)^{\frac{1}{2}} \\ &= t^{-\frac{d}{4\beta}} \left(\int_{|v| < t^{\frac{1}{2\beta}} \eta^{-\frac{1}{2\beta - 1}}} e^{-2c|v|^{2\beta}} \, dv \right)^{\frac{1}{2}} \lesssim t^{-\frac{d}{4\beta}}. \end{aligned}$$

If $0 \le t < 1$,

$$\|e^{-c|\xi|^{2\beta}t}\|_{L^2(D_1)} \le C(d)\,\eta^{-\frac{d}{2\beta-1}}.$$

Putting the two inequalities above together, we have, for any $t \ge 0$,

$$\|e^{-c|\xi|^{2\beta}t}\|_{L^2(D_1)} \lesssim \langle t\rangle^{-\frac{d}{4\beta}}. \tag{5.21}$$

By Lemma 2.2,

$$\|e^{-\frac{\eta}{4}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^2(D_1)} \lesssim \langle t\rangle^{-\frac{d}{4\beta}}\ \|u_0\|_{L^1\cap L^2}.$$

Therefore,

$$K_1 \lesssim \langle t\rangle^{-\frac{d}{2\beta}}\ \|u_0\|_{L^1\cap L^2} + e^{-ct}\|u_0\|_{H^{r_2}} \lesssim \langle t\rangle^{-\frac{d}{2\beta}}\ \|u_0\|_{L^1\cap H^{r_2}},$$

for any $r_2 > \frac{d}{2}$. Similarly,

$$K_2 \lesssim \langle t\rangle^{-\frac{d}{2\beta}}\ \|\tau_0\|_{L^1\cap H^{r_2}}.$$

By (4.8), (4.9), (5.21) and Lemma 2.2,

$$\begin{aligned}
K_3 &= \int_0^t \|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_1)}\,ds \\
&\quad + \int_0^t \|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_2)}\,ds \\
&\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_1)}\,ds \\
&\quad + \int_0^t \|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_2)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{2\beta}}\ \|u\cdot\nabla u\|_{L^1\cap L^2} + e^{-c(t-s)}\|\widehat{u\cdot\nabla u}\|_{L^1}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{2\beta}}\ \|\nabla u\|_{L^2}(\|u\|_{L^2} + \|u\|_{L^\infty}) + e^{-c(t-s)}\|\widehat{u}\|_{L^1}\||\xi|\widehat{u}\|_{L^1}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d}{2\beta}} \left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}}\right) + e^{-c(t-s)} \langle s\rangle^{-\frac{2d+1}{2\beta}} \, ds \, X(t)\, Y(t) \\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}} \, X(t)\, Y(t).
\end{aligned}
$$

To bound K_4, we first split it as

$$
\begin{aligned}
K_4 &= \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_1)}\, ds \\
&\quad + \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_2)}\, ds \\
&\triangleq K_{41} + K_{42}.
\end{aligned}
$$

We first compute K_{42},

$$
\begin{aligned}
K_{42} &\lesssim \int_0^t \||\xi|^{1-2\beta} e^{-c(t-s)} \widehat{u\cdot\nabla\tau}\|_{L^1(D_2)}\, ds \lesssim \int_0^t e^{-c(t-s)} \|\widehat{u}\|_{L^1} \||\xi|\widehat{\tau}\|_{L^1}\, ds \\
&\lesssim \int_0^t e^{-c(t-s)} \langle s\rangle^{-\frac{d}{2\beta}} \, ds \, X(t)\, \|\tau\|_{L^\infty H^{r_3}} \lesssim \langle t\rangle^{-\frac{d}{2\beta}} \, X(t)\, \|\tau\|_{L^\infty H^{r_3}},
\end{aligned}
$$

where $r_3 > 1 + \frac{d}{2}$. To estimate K_{41}, we divide the time integral interval $[0, t]$ into $[0, \frac{t}{2})$ and $[\frac{t}{2}, t]$,

$$
\begin{aligned}
K_{41} &= \int_0^{\frac{t}{2}} \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_1)}\, ds \\
&\quad + \int_{\frac{t}{2}}^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_1)}\, ds \\
&\triangleq K_{411} + K_{412}.
\end{aligned}
$$

Making use of (4.8), (5.4), Hölder inequality, (5.21) and Lemma 2.2, K_{411} is bounded by

$$
K_{411} \lesssim \int_0^{\frac{t}{2}} \||\xi| e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)} \widehat{u\otimes\tau}\|_{L^1(D_1)}\, ds
$$

$$
\begin{aligned}
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{1}{2\beta}} \, \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)} \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\widehat{u\otimes\tau}\|_{L^2(D_1)}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \, \|u\otimes\tau\|_{L^1}\, ds \lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \, \|\tau\|_{L^2}\|u\|_{L^2}\, ds \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{d}{4\beta}} \, ds\, X(t)\, \|\tau\|_{L^\infty L^2} \lesssim \langle t\rangle^{-\frac{d}{2\beta}} \, X(t)\, \|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

Again, by (4.8), (5.4), Hölder inequality and (5.21), we obtain

$$
\begin{aligned}
K_{412} &\lesssim \int_{\frac{t}{2}}^{t} \||\xi| e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\widehat{u\otimes\tau}\|_{L^1(D_1)}\, ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{1}{2\beta}} \, \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)} \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\widehat{u\otimes\tau}\|_{L^2(D_1)}\, ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \, \|\widehat{u\otimes\tau}\|_{L^2}\, ds \lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \, \|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^2}\, ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \, \langle s\rangle^{-\frac{d}{2\beta}} \, ds\, X(t)\, \|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}} \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \, ds\, X(t)\, \|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}} \, X(t)\, \|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

Therefore,

$$
K_4 \lesssim \langle t\rangle^{-\frac{d}{2\beta}} \, X(t)\, \|\tau\|_{L^\infty H^{r_3}}, \quad \text{with} \quad r_3 > 1 + \frac{d}{2}.
$$

To estimate K_5, we first write it as

$$\begin{aligned}K_5 &= \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\,ds \\ &\quad + \int_0^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_2)}\,ds \\ &\triangleq K_{51}+K_{52}.\end{aligned}$$

We first compute K_{52},

$$\begin{aligned}K_{52} &\lesssim \int_0^t \||\xi|^{1-2\beta}e^{-c(t-s)}\widehat{Q(\tau,\nabla u)}\|_{L^1(D_2)}\,ds \lesssim \int_0^t e^{-c(t-s)}\||\xi|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^1}\,ds \\ &\lesssim \int_0^t e^{-c(t-s)}\,\langle s\rangle^{-\frac{d+1}{2\beta}}\,ds\,Y(t)\,\|\tau\|_{L^\infty H^{r_4}} \lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\,Y(t)\,\|\tau\|_{L^\infty H^{r_4}}\end{aligned}$$

for any $r_4 > \frac{d}{2}$. We divide K_{51} into two parts,

$$\begin{aligned}K_{51} &= \int_0^{\frac{t}{2}} \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\,ds \\ &\quad + \int_{\frac{t}{2}}^t \|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\,ds \\ &\triangleq K_{511}+K_{512}.\end{aligned}$$

By (4.8), Hölder inequality, (5.21) and Lemma 2.2, K_{511} is bounded by

$$\begin{aligned}K_{511} &\lesssim \int_0^{\frac{t}{2}} \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\widehat{Q(\tau,\nabla u)}\|_{L^1(D_1)}\,ds \\ &\lesssim \int_0^{\frac{t}{2}} \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)}\|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\,ds \\ &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d}{2\beta}}\,\|Q(\tau,\nabla u)\|_{L^1}\,ds \\ &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d}{2\beta}}\,\|\nabla u\|_{L^2}\|\tau\|_{L^2}\,ds\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d}{2\beta}} \langle s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2}\\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}} \int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2}\\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}}\, Y(t)\, \|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

For K_{512}, invoking (4.8), Hölder inequality, (5.21) and Young's inequality, we have

$$
\begin{aligned}
K_{512} &\lesssim \int_{\frac{t}{2}}^{t} \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)} \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds\\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d}{4\beta}}\, \|\widehat{Q(\tau,\nabla u)}\|_{L^2}\, ds\\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d}{4\beta}}\, \||\xi|\widehat{u}\|_{L^1} \|\tau\|_{L^2}\, ds\\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d}{4\beta}} \langle s\rangle^{-\frac{d+1}{2\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2}\\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d}{4\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2}\\
&\lesssim \langle t\rangle^{-\frac{d}{2\beta}}\, Y(t)\, \|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

Therefore

$$
K_5 \lesssim \langle t\rangle^{-\frac{d}{2\beta}}\, Y(t)\, \|\tau\|_{L^\infty H^{r_4}}, \quad \text{with} \quad r_4 > \frac{d}{2}.
$$

Combining the estimates above, we find, for $r_3 > 1 + \frac{d}{2}$,

$$
\begin{aligned}
\|\widehat{u}\|_{L^1} &\lesssim \langle t\rangle^{-\frac{d}{2\beta}}\, \|(u_0,\tau_0)\|_{L^1\cap H^{r_3}} + \langle t\rangle^{-\frac{d}{2\beta}}\, X(t)Y(t)\\
&\quad + \langle t\rangle^{-\frac{d}{2\beta}}\, (X(t)+Y(t))\|\tau\|_{L^\infty H^{r_3}}.
\end{aligned} \tag{5.22}
$$

5.5. The estimate of $\||\xi|\widehat{u}\|_{L^1}$

By (4.11),

$$\begin{aligned}\||\xi|\widehat{u}\|_{L^1} &\lesssim \||\xi|M_1\widehat{u_0}\|_{L^1} + \||\xi|M_2\widehat{\mathcal{A}_0}\|_{L^1} + \int_0^t \||\xi|M_1(t-s)\widehat{G}\|_{L^1}\,ds \\ &\quad + \int_0^t \||\xi|M_2(t-s)\widehat{F}\|_{L^1}\,ds + \int_0^t \||\xi|M_2(t-s)\widehat{H}\|_{L^1}\,ds \\ &\triangleq L_1 + L_2 + L_3 + L_4 + L_5.\end{aligned}$$

By (4.8), (4.9), (5.4), (5.21), Hölder inequality and Lemma 2.2,

$$\begin{aligned}L_1 &= \||\xi|M_1\widehat{u_0}\|_{L^1(D_1)} + \||\xi|M_1\widehat{u_0}\|_{L^1(D_2)} \\ &\lesssim \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^1(D_1)} + \||\xi|e^{-ct}\widehat{u_0}\|_{L^1(D_2)} \\ &\lesssim \langle t\rangle^{-\frac{1}{2\beta}}\|e^{-\frac{\eta}{8}|\xi|^{2\beta}t}\|_{L^2(D_1)}\|e^{-\frac{\eta}{8}|\xi|^{2\beta}t}\widehat{u_0}\|_{L^2(D_1)} + e^{-ct}\||\xi|\widehat{u_0}\|_{L^1(D_2)} \\ &\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\|u_0\|_{L^1\cap L^2} + e^{-ct}\|u_0\|_{H^{r_5}} \\ &\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\|u_0\|_{L^1\cap H^{r_5}},\end{aligned}$$

for $r_5 > 1 + \frac{d}{2}$. Similarly,

$$L_2 \lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\|\tau_0\|_{L^1\cap H^{r_5}}.$$

The estimate of L_3 is more complex,

$$\begin{aligned}L_3 &= \int_0^t \||\xi|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_1)}\,ds \\ &\quad + \int_0^t \||\xi|M_1(t-s)\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_2)}\,ds \\ &\lesssim \int_0^t \||\xi|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_1)}\,ds \\ &\quad + \int_0^t \||\xi|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla u)\}\|_{L^1(D_2)}\,ds\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|u\cdot\nabla u\|_{L^1\cap L^2} + e^{-c(t-s)}\||\xi|\widehat{u\cdot\nabla u}\|_{L^1}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|\nabla u\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty})\,ds \\
&\quad + \int_0^t e^{-c(t-s)}(\||\xi|\widehat{u}\|_{L^1}^2 + \|\widehat{u}\|_{L^1}\||\xi|^2\widehat{u}\|_{L^1})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|\nabla u\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty})\,ds \\
&\quad + \int_0^t e^{-c(t-s)}\left\{\||\xi|\widehat{u}\|_{L^1}^2 + \|\widehat{u}\|_{L^1}\left(\||\xi|^2\widehat{u}\|_{L^1(|\xi|\le\langle s\rangle^{\frac{1}{2\beta}})} + \||\xi|^2\widehat{u}\|_{L^1(|\xi|>\langle s\rangle^{\frac{1}{2\beta}})}\right)\right\} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}} \||\xi|\widehat{u}\|_{L^2}(\|\widehat{u}\|_{L^2}+\|\widehat{u}\|_{L^1})\,ds \\
&\quad + \int_0^t e^{-c(t-s)}\left\{\||\xi|\widehat{u}\|_{L^1}^2 + \|\widehat{u}\|_{L^1}\left(\langle s\rangle^{\frac{1}{2\beta}}\||\xi|\widehat{u}\|_{L^1} + \langle s\rangle^{-\frac{1}{2\beta}}\||\xi|^{3+\frac{d}{2}}\widehat{u}\|_{L^2}\right)\right\} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}}\left(\langle s\rangle^{-\frac{d+1}{2\beta}} + \langle s\rangle^{-\frac{3d+2}{4\beta}}\right) + e^{-c(t-s)}\left(\langle s\rangle^{-\frac{d+1}{\beta}} + \langle s\rangle^{-\frac{d}{\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right) ds \\
&\quad \times \left(X(t)Y(t) + Y(t)^2 + X(t)\|u\|_{L^\infty H^{3+\frac{d}{2}}}\right) \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\left(X(t)Y(t) + Y(t)^2 + X(t)\|u\|_{L^\infty H^{3+\frac{d}{2}}}\right).
\end{aligned}
$$

To estimate L_4, we first write it as

$$
\begin{aligned}
L_4 = &\int_0^t \||\xi|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_1)}\,ds \\
&+ \int_0^t \||\xi|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\}\|_{L^1(D_2)}\,ds \\
&\triangleq L_{41} + L_{42}.
\end{aligned}
$$

In order to generate enough decay in L_{41}, we invoke (5.7) to write

$$
\begin{aligned}
L_{41} &\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\widehat{u\otimes\mathcal{A}}\|_{L^1(D_1)}\,ds \\
&\quad + \sum_{i=1}^d \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{[\mathbb{P}\nabla\cdot,u_i]\tau\}\|_{L^1(D_1)}\,ds \\
&\triangleq L_{411}+L_{412}.
\end{aligned}
$$

By (5.4), Hölder inequality (5.21) and Lemma 2.2,

$$
\begin{aligned}
L_{411} &\lesssim \int_0^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\widehat{u\otimes\mathcal{A}}\|_{L^1(D_1)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{2\beta}}\,\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)}\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\widehat{u\otimes\mathcal{A}}\|_{L^2(D_1)}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}}\left(\|u\otimes\mathcal{A}\|_{L^1}+\|\widehat{u\otimes\mathcal{A}}\|_{L^2}\right)ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}}\left(\|u\|_{L^2}\|\mathcal{A}\|_{L^2}+\|\widehat{u}\|_{L^1}\|\widehat{\mathcal{A}}\|_{L^2}\right)ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{d+1}{2\beta}}\left(\langle s\rangle^{-\frac{d+1}{2\beta}}+\langle s\rangle^{-\frac{3d+2}{4\beta}}\right)ds\,X(t)\,Y(t)\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\;X(t)\,Y(t).
\end{aligned}
$$

To estimate L_{412}, we split the interval $[0,t]$ into $[0,\frac{t}{2})$ and $[\frac{t}{2},t]$ and invoke (4.8), (5.4), Hölder inequality, (5.21), Lemma 2.2, (2.5) and (5.8) to obtain

$$
\begin{aligned}
L_{412} &= \sum_{i=1}^d \int_0^{\frac{t}{2}} \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{[\mathbb{P}\nabla\cdot,u_i]\tau\}\|_{L^1(D_1)}\,ds \\
&\quad + \sum_{i=1}^d \int_{\frac{t}{2}}^t \|e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)}|\xi|\mathscr{F}\{[\mathbb{P}\nabla\cdot,u_i]\tau\}\|_{L^1(D_1)}\,ds \\
&\lesssim \sum_{i=1}^d \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{1}{2\beta}}\,\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)}\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\mathscr{F}\{[\mathbb{P}\nabla\cdot,u_i]\tau\}\|_{L^2(D_1)}\,ds \\
&\quad + \sum_{i=1}^d \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{1}{2\beta}}\,\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)}\|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\mathscr{F}\{[\mathbb{P}\nabla\cdot,u_i]\tau\}\|_{L^2(D_1)}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{i=1}^{d} \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|[\mathbb{P}\nabla\cdot, u_i]\tau\|_{L^1}\, ds \\
&\quad + \sum_{i=1}^{d} \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\mathscr{F}\{[\mathbb{P}\nabla\cdot, u_i]\tau\}\|_{L^2}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \left(\|\nabla u\|_{L^2}\|\tau\|_{L^2} + \|u\|_{L^2}\|\mathcal{A}\|_{L^2}\right) ds \\
&\quad + \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \||\xi|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^2}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \left(\langle s\rangle^{-\frac{d+2}{4\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right) ds\, Y(t)(X(t) + \|\tau\|_{L^\infty L^2}) \\
&\quad + \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \langle s\rangle^{-\frac{d+1}{2\beta}}\, ds\, Y(t)\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \int_0^{\frac{t}{2}} \left(\langle s\rangle^{-\frac{d+2}{4\beta}} + \langle s\rangle^{-\frac{d+1}{2\beta}}\right) ds\, Y(t)(X(t) + \|\tau\|_{L^\infty L^2}) \\
&\quad + \langle t\rangle^{-\frac{d+1}{2\beta}} \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\, Y(t)(X(t) + \|\tau\|_{L^\infty L^2}).
\end{aligned}
$$

To estimate L_{42}, we make use of (4.9), for $\xi \in D_2$,

$$
|M_2(t)| \lesssim |\xi|^{-2\beta} e^{-c|\xi|^{2-2\beta}t} \lesssim |\xi|^{-2\beta} e^{-c\,t}. \tag{5.23}
$$

In addition, in order to generate enough decay, we invoke (2.4) to write the upper bound into three parts.

$$
\begin{aligned}
L_{42} &\lesssim \int_0^{t} \|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(u\cdot\nabla\mathcal{A})\}\|_{L^1(D_2)} ds \\
&\quad + \int_0^{t} \|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(\nabla u\cdot\nabla\tau)\}\|_{L^1(D_2)} ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \|e^{-c(t-s)}\mathscr{F}\{\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau)\}\|_{L^1(D_2)}ds\\
\lesssim&\int_0^t e^{-c(t-s)}\left(\|\widehat{u}\|_{L^1}\||\xi|\widehat{\mathcal{A}}\|_{L^1}+\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{\tau}\|_{L^1}\right)ds\\
\lesssim&\int_0^t e^{-c(t-s)}\|\widehat{u}\|_{L^1}\left(\||\xi|\widehat{\mathcal{A}}\|_{L^1(|\xi|\le\langle s\rangle^{\frac{d}{2\beta(2+d)}})}+\||\xi|\widehat{\mathcal{A}}\|_{L^1(|\xi|>\langle s\rangle^{\frac{d}{2\beta(2+d)}})}\right)ds\\
&+\int_0^t e^{-c(t-s)}\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{\tau}\|_{L^1(D_2)}ds\\
\lesssim&\int_0^t e^{-c(t-s)}\Big\{\|\widehat{u}\|_{L^1}\Big(\langle s\rangle^{\frac{d}{4\beta}}\|\widehat{\mathcal{A}}\|_{L^2}+\langle s\rangle^{-\frac{1}{2\beta}}\||\xi|^{2+\frac{2}{d}}\widehat{\mathcal{A}}\|_{L^1(|\xi|>\langle s\rangle^{\frac{d}{2\beta(2+d)}})}\Big)\\
&\qquad+\||\xi|\widehat{u}\|_{L^1}\||\xi|\widehat{\tau}\|_{L^1(D_2)}\Big\}ds\\
\lesssim&\int_0^t e^{-c(t-s)}\Big\{\|\widehat{u}\|_{L^1}\Big(\langle s\rangle^{\frac{d}{4\beta}}\|\widehat{\mathcal{A}}\|_{L^2}+\langle s\rangle^{-\frac{1}{2\beta}}\|\tau\|_{\dot{H}^{4+\frac{d}{2}+\frac{2}{d}}}\Big)\\
&\qquad+\||\xi|\widehat{u}\|_{L^1}\|\tau\|_{\dot{H}^{2+\frac{d}{2}}}\Big\}ds\\
\lesssim&\int_0^t e^{-c(t-s)}\langle s\rangle^{-\frac{d+1}{2\beta}}ds\left(X(t)Y(t)+(X(t)+Y(t))\|\tau\|_{L^\infty H^{4+\frac{d}{2}+\frac{2}{d}}}\right)\\
\lesssim&\langle t\rangle^{-\frac{d+1}{2\beta}}\left(X(t)Y(t)+(X(t)+Y(t))\|\tau\|_{L^\infty H^{4+\frac{d}{2}+\frac{2}{d}}}\right).
\end{aligned}
$$

Therefore,

$$
L_4\lesssim\langle t\rangle^{-\frac{d+1}{2\beta}}\left(X(t)Y(t)+(X(t)+Y(t))\|\tau\|_{L^\infty H^{4+\frac{d}{2}+\frac{2}{d}}}\right).
$$

To bound L_5, we first split it into two parts,

$$
\begin{aligned}
L_5=&\int_0^t \||\xi|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\,ds\\
&+\int_0^t \||\xi|M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_2)}\,ds\\
\triangleq&\, L_{51}+L_{52}.
\end{aligned}
$$

We further divide L_{51} into two parts,

$$
\begin{aligned}
L_{51} &= \int_0^{\frac{t}{2}} \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\, ds \\
&\quad + \int_{\frac{t}{2}}^{t} \||\xi| M_2(t-s)\mathscr{F}\{\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\}\|_{L^1(D_1)}\, ds \\
&\triangleq L_{511} + L_{512}.
\end{aligned}
$$

By (4.8), (5.4), (5.21), Hölder inequality and Lemma 2.2,

$$
\begin{aligned}
L_{511} &\lesssim \int_0^{\frac{t}{2}} \||\xi| e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^1(D_1)}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{1}{2\beta}} \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)}\|_{L^2(D_1)} \|e^{-\frac{\eta}{8}|\xi|^{2\beta}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|Q(\tau,\nabla u)\|_{L^1}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \|\nabla u\|_{L^2}\|\tau\|_{L^2}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{d+1}{2\beta}} \langle s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\|\tau\|_{L^\infty L^2} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\, Y(t)\, \|\tau\|_{L^\infty L^2}.
\end{aligned}
$$

L_{512} is bounded by

$$
\begin{aligned}
L_{512} &\lesssim \int_{\frac{t}{2}}^{t} \||\xi| e^{-\frac{\eta}{2}|\xi|^{2\beta}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^1(D_1)}\, ds \lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \|\widehat{Q(\tau,\nabla u)}\|_{L^2}\, ds \\
&\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \||\xi|\widehat{u}\|_{L^1}\|\widehat{\tau}\|_{L^2}\, ds \lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}} \langle s\rangle^{-\frac{d+1}{2\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2}
\end{aligned}
$$

$$\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{d+2}{4\beta}}\, ds\, Y(t)\, \|\tau\|_{L^\infty L^2} \lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\, Y(t)\, \|\tau\|_{L^\infty L^2}.$$

To bound L_{52}, we use the upper bound (4.9) or (5.23).

$$\begin{aligned}
L_{52} &\lesssim \int_0^t \|e^{-c(t-s)}|\xi|\widehat{Q(\tau,\nabla u)}\|_{L^1(D_2)} ds \\
&\lesssim \int_0^t e^{-c(t-s)} \left(\||\xi|\widehat{u}\|_{L^1(D_2)}\||\xi|\widehat{\tau}\|_{L^1(D_2)} + \|\widehat{\tau}\|_{L^1(D_2)}\||\xi|^2\widehat{u}\|_{L^1(D_2)}\right) ds \\
&\lesssim \int_0^t e^{-c(t-s)}\Big\{\||\xi|\widehat{u}\|_{L^1}\|\tau\|_{\dot{H}^{2+\frac{d}{2}}} + \|\tau\|_{\dot{H}^{1+\frac{d}{2}}} \\
&\qquad \times \left(\||\xi|^2\widehat{u}\|_{L^1(|\xi|\le\langle s\rangle^{\frac14})} + \||\xi|^2\widehat{u}\|_{L^1(|\xi|>\langle s\rangle^{\frac14})}\right)\Big\} ds \\
&\lesssim \int_0^t e^{-c(t-s)}\Big\{\||\xi|\widehat{u}\|_{L^1}\|\tau\|_{\dot{H}^{2+\frac{d}{2}}} + \|\tau\|_{\dot{H}^{1+\frac{d}{2}}} \\
&\qquad \times \left(\langle s\rangle^{\frac14}\||\xi|\widehat{u}\|_{L^1} + \langle s\rangle^{-\frac{d+1}{2\beta}}\||\xi|^{2+\frac{2d+2}{\beta}}\widehat{u}\|_{L^1(|\xi|\ge\langle s\rangle^{\frac14})}\right)\Big\} ds \\
&\lesssim \int_0^t e^{-c(t-s)}\||\xi|\widehat{u}\|_{L^1}\|\tau\|_{\dot{H}^{2+\frac{d}{2}}} ds \\
&\quad + \int_0^t e^{-c(t-s)}\|\tau\|^{\frac12}_{\dot{H}^{1+\frac{d}{2}}}\langle s\rangle^{\frac14}\||\xi|\widehat{u}\|_{L^1} ds \|\tau\|^{\frac12}_{L^\infty\dot{H}^{1+\frac{d}{2}}} \\
&\quad + \int_0^t e^{-c(t-s)}\langle s\rangle^{-\frac{d+1}{2\beta}}\|u\|_{\dot{H}^{3+\frac{d}{2}+\frac{2d+2}{\beta}}}\|\tau\|_{\dot{H}^{1+\frac{d}{2}}} ds \\
&\lesssim \int_0^t e^{-c(t-s)}\langle s\rangle^{-\frac{d+1}{2\beta}} ds \Big\{Y(t)\left(\|\tau\|_{L^\infty\dot{H}^{2+\frac{d}{2}}} + \|\tau\|^{\frac12}_{L^\infty\dot{H}^{1+\frac{d}{2}}}\right) \\
&\qquad + \|u\|_{L^\infty\dot{H}^{3+\frac{d}{2}+\frac{2d+2}{\beta}}}\|\tau\|_{L^\infty\dot{H}^{1+\frac{d}{2}}}\Big\} \\
&\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}}\Big\{Y(t)\left(\|\tau\|_{L^\infty H^{2+\frac{d}{2}}} + \|\tau\|^{\frac12}_{L^\infty H^{1+\frac{d}{2}}}\right) + \|u\|_{L^\infty H^{3+\frac{d}{2}+\frac{2d+2}{\beta}}}\|\tau\|_{L^\infty H^{1+\frac{d}{2}}}\Big\},
\end{aligned}$$

where we used the fact in Theorem 1.2, which is

$$\|\tau(s)\|_{\dot{H}^{1+\frac{d}{2}}} \le C\langle s\rangle^{-\frac12}.$$

Thus, we have

$$L_5 \lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \left\{ Y(t) \left(\|\tau\|_{L^\infty H^{2+\frac{d}{2}}} + \|\tau\|^{\frac{1}{2}}_{L^\infty H^{1+\frac{d}{2}}} \right) + \|u\|_{L^\infty H^{3+\frac{d}{2}+\frac{2d+2}{\beta}}} \|\tau\|_{L^\infty H^{1+\frac{d}{2}}} \right\}.$$

Collecting the estimates of L_1 through L_5 leads to

$$\begin{aligned} \||\xi|\widehat{u}\|_{L^1} &\lesssim \langle t\rangle^{-\frac{d+1}{2\beta}} \|(u_0,\tau_0)\|_{L^1\cap H^{r_5}} + \langle t\rangle^{-\frac{d+1}{2\beta}} \left(X(t)Y(t) + Y(t)^2 + X(t)\|u\|_{L^\infty H^{3+\frac{d}{2}}} \right) \\ &\quad + \langle t\rangle^{-\frac{d+1}{2\beta}} \left(X(t) + Y(t) \right) \left(\|\tau\|_{L^\infty H^{4+\frac{d}{2}+\frac{2}{d}}} + \|\tau\|^{\frac{1}{2}}_{L^\infty H^{1+\frac{d}{2}}} \right) \\ &\quad + \langle t\rangle^{-\frac{d+1}{2\beta}} \|u\|_{L^\infty H^{3+\frac{d}{2}+\frac{2d+2}{\beta}}} \|\tau\|_{L^\infty H^{1+\frac{d}{2}}}, \end{aligned} \tag{5.24}$$

where $r_5 > 1 + \frac{d}{2}$.

5.6. *Verification of (5.1) and (5.2)*

Finally we combine the estimates in the previous subsections to verify (5.1) and (5.2). We remark that (5.6) involves only H^1-norm of τ, (5.19) and (5.20) involve the H^2-norm of τ and $H^{1+\frac{d}{2}}$ of u, (5.22) involves the H^{r_3}-norm of τ with $r_3 > 1 + \frac{d}{2}$, and (5.24) involves the $H^{3+\frac{d}{2}+\frac{2d+2}{\beta}}$-norm of u and $H^{4+\frac{d}{2}+\frac{2}{d}}$-norm of τ. In order to accommodate all these requirements, we choose the functional setting to be H^r with $r = 3 + \frac{d}{2} + \frac{2d+2}{\beta}$.

According to (5.6) with (5.22), for $r = 3 + \frac{d}{2} + \frac{2d+2}{\beta}$,

$$X(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^r} + X(t)Y(t) + (X(t)+Y(t))\|\tau\|_{L^\infty H^r}.$$

By (5.19), (5.20) and (5.24),

$$Y(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^r} + \|(u,\tau)\|^2_{L^\infty H^r} + (X(t)+Y(t)) \left(Y(t) + \|(u,\tau)\|_{L^\infty H^r} \right).$$

This completes the proof of Theorem 1.3. □

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