J. Math. Pures Appl. 115 (2018) 187–217

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

The 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion

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ARTICLE INFO

Article history:
Received 3 January 2017
Available online 10 January 2018

MSC:
35Q35
35B65
76D03

Keywords:
2D Boussinesq equations
Fractional dissipation
Global regularity

ABSTRACT

This paper examines the global regularity problem on the two-dimensional (2D) incompressible Boussinesq equations with fractional horizontal dissipation and thermal diffusion. The goal is to establish the global existence and regularity for the Boussinesq equations with minimal dissipation and thermal diffusion. By working with this general 1D fractional Laplacian dissipation, we are no longer constrained to the standard partial dissipation and this study will help understand the issue on how much dissipation is necessary for the global regularity. Due to the nonlocality of these 1D fractional operators, some of the standard energy estimate techniques such as integration by parts no longer apply and new tools including several anisotropic embedding and interpolation inequalities involving fractional derivatives are derived. These tools allow us to obtain very sharp upper bounds for the nonlinearities.

RÉSUMÉ

Cet article examine le problème de la régularité globale sur les équations de Boussinesq bi-dimensionnels (2D) incompressibles avec dissipation horizontale fractionnaire et avec diffusion thermique. L'objectif est d'établir l'existence globale et la régularité pour les équations de Boussinesq avec dissipation minimale et diffusion thermique. En travaillant avec cette dissipation laplacienne fractionnaire unidimensionnelle assez générale, nous ne sommes plus limités à la dissipation partielle standard, et cette etude nous aidera à comprendre le problème sur combien de dissipation est nécessaire pour obtenir la régularité globale. A cause de la non-localité de ces opérateurs fractionnaires unidimensionnels, certaines des techniques d'estimation d'énergie standard, par exemple l'intégration par partie, ne s'applique plus, et des nouveaux outils comprenant l'injection anisotrope et les inégalités

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https://doi.org/10.1016/j.matpur.2018.01.006

d'interpolation concernant des dérivés fractionnaires sont dérivées. Ces outils nous permettent d'obtenir des bornes supérieures très sharp pour les non-linéarités.

1. Introduction

This paper concerns itself with the initial-value problem for the 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda_{x_1}^{2\alpha}u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\Lambda_{x_1}^{2\beta}\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $e_2 = (0,1)$, $u = (u_1(x, t), u_2(x, t))$ denotes the velocity field, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, and $\mu > 0$, $\nu > 0$, $\alpha \in [0,1]$ and $\beta \in [0,1]$ are real parameters. The horizontal fractional operator $\Lambda_{x_1} := \sqrt{-\partial_{x_1}^2}$ is defined through the Fourier transform, namely

$$\widehat{\Lambda_{x_1}^{\gamma} f}(\xi) = |\xi_1|^{\gamma}\hat{f}(\xi).$$

The goal here is to show the global regularity for (1.1) for smallest $\alpha, \beta \in [0,1]$.

We summarize some previous work closely related to our study here. To facilitate the description, we start with the general form of the 2D incompressible Boussinesq equations given by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\mathcal{L}_1 u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\mathcal{L}_2\theta = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.2}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Fourier multiplier operators, namely

$$\widehat{\mathcal{L}_1 u}(\xi) = m_1(\xi)\,\widehat{u}(\xi), \qquad \widehat{\mathcal{L}_2 u}(\xi) = m_2(\xi)\,\widehat{u}(\xi).$$

When $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, (1.2) becomes the standard model in geophysics as well as in the Rayleigh–Bérnard convection (see, e.g., [26,28,31]). When $\mu = \nu = 0$, (1.2) becomes completely inviscid. When $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by various special symbols, we recover various partial and fractional dissipation cases, which naturally bridges the fully dissipative Boussinesq and the complete inviscid Boussinesq equation.

The global regularity problem on (1.2) with partial or fractional dissipation has attracted considerable interests and there have been substantial developments. The global regularity for (1.2) with $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ can be obtained via similar approaches as those for the 2D Navier–Stokes equations. In fact, any L^2-initial data (u_0, θ_0) leads to a unique global solution to the fully dissipative 2D Boussinesq equation that becomes infinitely smooth for any time $t > 0$. In contrast, the completely inviscid Boussinesq equation is not well-understood and the global well-posedness remains an outstanding open problem. Due to the similarity between the 2D inviscid Boussinesq and the 3D axisymmetric Euler equations, the finite time singularity indicated by the numerical simulations on the 3D Euler in a bounded domain with special geometry and boundary data exhibits the complexity of this problem [25]. Sarria and Wu examined a special class of singular solutions [29]. Two 1D models as well as several 2D models of the inviscid Boussinesq equations have been proposed ([7,10,11,18,22]). These models all possess finite time blowup solutions and may shed light on the inviscid Boussinesq mystery.

The study on the global well-posedness of (1.2) with intermediate dissipation has gained substantial momentum recently. The work of Chae [6] and of Hou and Li [19] solved the global regularity problem for the case when either the full Laplacian dissipation or full Laplacian thermal diffusion is present. More recent pursues are on (1.2) with fractional dissipation, namely

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda^\alpha u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\Lambda^\beta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.3}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$. There have been very exciting developments. (1.3) with a broad range of α and β has been shown to be globally well-posed. More precisely, as proposed in [20], the indices α and β have been classified into three categories: the subcritical case when $\alpha+\beta>1$, the critical case when $\alpha+\beta=1$ and the supercritical case when $\alpha+\beta<1$. Plainly speaking, the critical case corresponds to exactly one-derivative dissipation in the whole system. Xu studied the subcritical case $\alpha+\beta\ge 2$ with $\alpha\ge 1$ and obtained the global regularity [33] while the well-posedness of (1.3) with $\mu=0$ and $1<\beta<2$ was established by Hmidi–Zerguine via the maximal regularity estimates for the semigroup $e^{-t\Lambda^\beta}$ [17]. The work of Miao and Xue [27] dealt with the subcritical case when α and β satisfy

$$\alpha\in((6-\sqrt{6})/4,1), \quad \beta\in(1-\alpha, \min((7+2\sqrt{6})\alpha/5-2, \alpha(1-\alpha)/(\sqrt{6}-2\alpha), 2-2\alpha)).$$

Constantin and Vicol [12] established the global regularity for the subcritical case with $\beta>\frac{2}{2+\alpha}$ by applying the pointwise inequalities for fractional Laplacian such as

$$\nabla f\cdot\Lambda^\alpha\nabla f(x) \ge \frac{1}{2}\Lambda^\alpha|\nabla f|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c\,\|f\|^\alpha_{L^\infty}}.$$

Later this global regularity result was improved by Yang, Jiu and Wu [34] and further by Ye and Xu [36]. Very recent work of Zhou, Li, Shang, Wu, Yuan and Zhao knocks out more subcritical cases [37]. We note that not all subcritical cases have been resolved. For instance, we do not know how to obtain the global regularity for the case when α and β are close to $\frac{1}{2}$ and $\alpha+\beta>1$. For the known results, we refer to the works [12,27,34–37] for more details.

The critical case is in general more difficult than the subcritical case. The two particular critical cases, $\alpha+\beta=1$ with $\alpha=1$ and $\beta=0$ or with $\alpha=0$ and $\beta=1$, were studied by Hmidi, Keraani and Rousset [15,16], who established the global regularity for both cases. By converting the Boussinesq equations with critical dissipation into the critical surface quasi-geostrophic equation, Jiu, Miao, Wu and Zhang [20] was able to establish the global regularity of the general critical case $\alpha+\beta=1$ with $1>\alpha>\frac{23-\sqrt{145}}{12}\approx 0.9132$. Stefanov and Wu made an improvement by allowing α in the larger range, $\alpha+\beta=1$ with $1>\alpha>\frac{\sqrt{1777}-23}{24}\approx 0.7981$ [30]. A recent work of Wu, Xu, Xue and Ye [32] further enlarged the range of α to $1>\alpha>10/13\approx 0.7692$.

The 2D Boussinesq equations with partial dissipation, another natural class of intermediate equations between the fully dissipative and the completely inviscid Boussinesq equations, have also received considerable attention. Danchin and Paicu [13] first examined the 2D Boussinesq equations with either horizontal dissipation ((1.2) with $\mu>0, \nu=0$, $\mathcal{L}_1=-\partial^2_{x_1}$) or horizontal thermal diffusion ((1.2) with $\mu=0, \nu>0$, $\mathcal{L}_2=-\partial^2_{x_1}$) and obtained the global regularity for both cases. Larios, Lunasin and Titi obtained a sharp uniqueness result and the global regularity for the horizontal dissipation case [23]. The global regularity problem for the 2D Boussinesq equations with vertical dissipation was difficult. Due to the mismatch between the partial derivatives of the vortex stretching term and the vertical partial dissipation, it was difficult to establish the global Sobolev bounds. The work of Cao and Wu solved this global regularity problem by proving a delicate global bound for the L^r-norm of the vertical component of the velocity and a double logarithmic interpolation inequality [5]. Adhikari, Cao and Wu obtained several interesting partial results before the work of Cao and Wu ([2,3]). Recently Li and Titi were able to weaken the assumption on the initial data from H^2 to H^1 [24]. Several other partial dissipation cases were dealt with in a very recent work [1].

This paper focuses on the 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion, namely (1.1). The aim is to prove the global existence and regularity for smallest possible indices $\alpha, \beta \in [0,1]$. By working with this general 1D fractional Laplacian dissipation, we are no longer constrained to the standard partial dissipation and the study of (1.1) will potentially lead to a full understanding of the issue on how much dissipation is necessary for the global regularity of (1.1). Due to the nonlocality of these 1D fractional operators, some of the standard energy estimate techniques such as integration by parts no longer apply and new tools have to be developed. Several anisotropic embedding and interpolation inequalities involving 1D fractional derivatives are derived to gain sharp bounds for the nonlinearities. Our main result can be stated as follows.

Theorem 1.1. *Assume that $\mu > 0$, $\nu > 0$, $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. If α and β are in any one of the two ranges:*

$$(1)\quad \alpha+\beta>1, \quad \frac{1}{2}<\alpha\le 1, \qquad \frac{1}{2}\ge\beta>\beta_0,$$
$$(2)\quad \alpha+\beta\ge 1, \qquad \beta>\frac{2+\sqrt{2}}{4},$$

where

$$\beta_0=\frac{2\alpha}{\sqrt{16\alpha^4-16\alpha^3+28\alpha^2-12\alpha+1}+4\alpha^2-2\alpha+1}, \tag{1.4}$$

then (1.1) admits a unique global solution that satisfies, for any given $T > 0$

$$u,\ \theta\in C([0,T];H^s(\mathbb{R}^2)), \qquad \Lambda_{x_1}^{\alpha}u,\ \Lambda_{x_1}^{\beta}\theta\in L^2([0,T];H^s(\mathbb{R}^2)).$$

To prove Theorem 1.1, our main efforts are devoted to establishing the global *a priori* bounds for (u, θ) in H^s. This global bound is obtained by successively improving the regularity of the solutions. The starting point is the natural energy bound for the velocity u and the global L^p-bound on θ for any $2 \le p \le \infty$. In addition to the global bound for $\|\theta\|_{L^p}$, the fractional dissipation also generates a lower bound (see Chamorro and Lemarié-Rieusset [8]), for any $\beta \in (0,1)$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Lambda_{x_1}^{2\beta}\theta(x_1,x_2)(|\theta(x_1,x_2)|^{p-2}\theta(x_1,x_2))\,dx_1dx_2 \ge C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2,$$

where $\dot{\mathcal{B}}^s_{p,r}$ denotes a homogeneous Besov space associated with the variable x_1. Invoking the vorticity equation

$$\partial_t\omega+u\cdot\nabla\omega+\Lambda_{x_1}^{2\alpha}\omega=\partial_{x_1}\theta,$$

we show that, for any $\alpha \in [0,1]$, $\beta \in [0,1]$ and $\alpha + \beta \ge 1$,

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha}\omega\|_{L^2}^2\, d\tau \le C(t, u_0, \theta_0).$$

We further prove the global L^p-bound for ω for any $\alpha+\beta>1$ and $2\le p<2(\alpha+\beta)$,

$$\|\omega(t)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}} \|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\, dx_2 d\tau \le C(t,\, u_0,\, \theta_0).$$

To prove this global bound, we make use of the upper bound of the form

$$\left\|\Lambda_{x_1}^{\kappa}(|\omega|^{p-2}\omega)(.,x_2)\right\|_{L_{x_1}^{\frac{p}{p-1}}} \le C\|\omega(.,x_2)\|_{\mathcal{B}^{\sigma}_{p,\frac{p}{p-1}}} \left\||\omega(.,x_2)|^{p-2}\right\|_{L_{x_1}^{\frac{p}{p-2}}}, \quad \sigma>\kappa,$$

which can be regarded as a generalization of the Kato–Ponce type inequality to the 1D fractional operators. Here $\mathcal{B}^s_{p,r}$ denotes an inhomogeneous Besov space associated with the variable x_1, whose norm is given by

$$\|f\|_{\mathcal{B}^s_{p,r}} \approx \|f\|_{\dot{\mathcal{B}}^s_{p,r}} + \|f\|_{L^p}.$$

By deriving several anisotropic embedding and interpolation inequalities involving 1D fractional derivatives, we are able to prove the global H^1-bound for θ for α and β in any one of the ranges:

$$\beta > \frac{1}{2}, \qquad \alpha+\beta \ge 1$$

and

$$\frac{1}{2} < \alpha \le 1, \qquad \frac{1}{2} \ge \beta > \beta_0, \qquad \alpha+\beta>1.$$

This is the reason why we need to consider α and β in two different ranges in Theorem 1.1. We then proceed to prove higher regularity global bounds. In the case when $\alpha > \frac{1}{2}$ and $\beta>\beta_0$, we are able to further prove the global bound for $\|\nabla\omega\|_{L^2}$,

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha}\nabla\omega(\tau)\|_{L^2}^2\, d\tau \le C(t,\, u_0,\, \theta_0),$$

which leads to the global bound

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau < \infty.$$

The global bound for $\|(u,\theta)\|_{H^s}$ follows as a consequence. In the case when $\beta>\frac{1}{2}$ and $\alpha+\beta\ge 1$, we need to further restrict β to the range $\beta > \frac{2+\sqrt{2}}{4}$. This restriction allows us to prove the global bound

$$\|\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta(s)\|_{L^2}^2\, ds \le C(t,\, u_0,\, \theta_0),$$

which leads to a global bound for $\int_0^t \|\nabla\theta\|_{L^\infty}\, d\tau$ and then for $\|(u,\theta)\|_{H^s}$.

We remark that the regularity assumption on the initial data may not be optimal. The optimal regularity assumption depends on α and β, and it is a complex process to make the assumption sharp. However, when α is very small, it may be necessary to assume that $u_0 \in H^s$ with $s > 2$. This assumption would lead to the necessary regularity on the vorticity $\omega = \nabla \times u$.

The rest of this paper is divided into four sections. Section 2 establishes the global H^1-bound and global $W^{1,p}$ bound for the velocity. Section 3 proves the global H^1-bound for θ. The proof is divided into two cases dealing two different parameter ranges. Anisotropic embedding and interpolation inequalities are derived in this section. Higher regularity bounds are provided in Sections 4 and 5. Section 4 deals with the case $\alpha > \frac{1}{2}$ and $\beta > \beta_0$ while Section 5 handles the case $\beta > \frac{2+\sqrt{2}}{4}$ and $\alpha + \beta \geq 1$. For the rest of the paper, we automatically assume (without stating) that $\mu = 1$ and $\nu = 1$.

2. Global H^1-bound and $W^{1,p}$ bound for the velocity

As we have remarked in the introduction, the proof of Theorem 1.1 boils down to establishing global *a priori* bounds on the solutions. This section proves the global H^1-bound and $W^{1,p}$ bound for the velocity. More precisely, we prove the following propositions. The global H^1-bound holds for any $\alpha + \beta \geq 1$.

Proposition 2.1. *Let $\alpha + \beta \geq 1$. Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 1.1 and let (u, θ) be the corresponding solution. Then, for any $t > 0$,*

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha} \nabla u(\tau)\|_{L^2}^2 \, d\tau \leq C(t, u_0, \theta_0), \tag{2.1}$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

When $\alpha + \beta > 1$, we also obtain a global L^p bound for ω, or $W^{1,p}$ bound for u. This global bound plays an important role in the proof of a global H^1 bound for θ in the case when $\beta \leq \frac{1}{2}$.

Proposition 2.2. *Let $\alpha + \beta > 1$. Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 1.1 and let (u, θ) be the corresponding solution. Then, for any $2 \leq p < 2(\alpha + \beta)$ and for any $t > 0$,*

$$\|\omega(t)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}} \|\omega(., x_2)\|_{\dot{\mathcal{B}}_{p,p}^{\frac{2\alpha}{p}}}^p \, dx_2 d\tau \leq C(t, u_0, \theta_0), \tag{2.2}$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

The rest of this section is devoted to the proofs of these two propositions. To do so, we first state the L^2-level energy estimates.

Lemma 2.3. *Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 1.1. Then the corresponding solution (u, θ) of (1.1) admits the following bounds, for any $t > 0$,*

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta} \theta(\tau)\|_{L^2}^2 \, d\tau \leq \|\theta_0\|_{L^2}^2, \tag{2.3}$$

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}, \tag{2.4}$$

$$\|\theta(t)\|_{L^p}^p + C\int_0^t \int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2 d\tau \le \|\theta_0\|^p_{L^p}, \quad \forall p\in[2,\infty), \tag{2.5}$$

$$\|u(t)\|^2_{L^2} + \int_0^t \|\Lambda^{\alpha}_{x_1} u(\tau)\|^2_{L^2}\,d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2, \tag{2.6}$$

where $\dot{\mathcal{B}}^s_{p,r}$ denotes the homogeneous Besov space associated with the variable x_1.

Proof of Lemma 2.3. Taking the L^2 inner product of the equation $(1.1)_2$ with θ and using the divergence-free condition, we find

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2_{L^2} + \|\Lambda^{\beta}_{x_1}\theta\|^2_{L^2} = 0.$$

Integrating in time yields the desired estimate (2.3). Taking the L^2 inner product of $(1.1)_1$ with u and using the divergence-free condition, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\Lambda^{\alpha}_{x_1} u\|^2_{L^2} = \int_{\mathbb{R}^2} \theta e_2\cdot u\,dx \le \|u\|_{L^2}\|\theta\|_{L^2},$$

which together with (2.3) gives

$$\|u(t)\|^2_{L^2} + \int_0^t \|\Lambda^{\alpha}_{x_1} u(\tau)\|^2_{L^2}\,d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Multiplying $(1.1)_2$ by $|\theta|^{p-2}\theta$ and integrating by parts yield

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|^p_{L^p} + \int_{\mathbb{R}^2} \Lambda^{2\beta}_{x_1}\theta(|\theta|^{p-2}\theta)\,dx = 0.$$

Invoking the lower bound

$$\begin{aligned}\int_{\mathbb{R}^2} \Lambda^{2\beta}_{x_1}\theta(|\theta|^{p-2}\theta)\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Lambda^{2\beta}_{x_1}\theta(x_1,x_2)(|\theta(x_1,x_2)|^{p-2}\theta(x_1,x_2))\,dx_1dx_2 \\ &\ge C\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\Lambda^{\beta}_{x_1}|\theta(x_1,x_2)|^{\frac{p}{2}}\right)^2 dx_1dx_2 \end{aligned} \tag{2.7}$$

yields

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \qquad 2\le p\le\infty.$$

Especially, (2.4) holds. Alternatively, if we employ the lower bound of [8, Theorem 2],

$$\int_{\mathbb{R}} \Lambda^{2s}_{x_1} f(x_1,x_2)(|f(x_1,x_2)|^{p-2}f(x_1,x_2))\,dx_1 \ge C(s,p)\|f(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2s}{p}}_{p,p}}, \tag{2.8}$$

we obtain

$$\int_{\mathbb{R}^2} \Lambda_{x_1}^{2\beta}\theta(|\theta|^{p-2}\theta)\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \Lambda_{x_1}^{2\beta}\theta(x_1,x_2)(|\theta(x_1,x_2)|^{p-2}\theta(x_1,x_2))\,dx_1dx_2$$
$$\geq C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2$$

and consequently

$$\frac{d}{dt}\|\theta(t)\|^p_{L^p} + C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2 \leq 0,$$

which yields (2.5). This finishes the proof. □

We are ready to prove Propositions 2.1 and 2.2.

Proof of Proposition 2.1. We make use of the vorticity equation

$$\partial_t\omega + u\cdot\nabla\omega + \Lambda_{x_1}^{2\alpha}\omega = \partial_{x_1}\theta. \tag{2.9}$$

Taking the inner product of (2.9) with ω yields

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2_{L^2} + \|\Lambda^\alpha_{x_1}\omega\|^2_{L^2} = \int \omega\,\partial_{x_1}\theta\,dx. \tag{2.10}$$

For $\alpha+\beta\geq 1$,

$$\int \omega\,\partial_{x_1}\theta\,dx \leq \|\Lambda^\alpha_{x_1}\omega\|_{L^2}\|\Lambda^{1-\alpha}_{x_1}\theta\|_{L^2} \leq \frac{1}{2}\|\Lambda^\alpha_{x_1}\nabla u\|^2_{L^2} + C(\|\theta\|^2_{L^2} + \|\Lambda^\beta_{x_1}\theta\|^2_{L^2}).$$

The desired bound in (2.1) then follows after integrating in time and invoking the fact that $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$. □

Proof of Proposition 2.2. Multiplying (2.9) by $|\omega|^{p-2}\omega$ and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{p}\frac{d}{dt}\|\omega(t)\|^p_{L^p} + \int_{\mathbb{R}^2}\Lambda^{2\alpha}_{x_1}\omega(|\omega|^{p-2}\omega)\,dx = \int_{\mathbb{R}^2}\partial_{x_1}\theta(|\omega|^{p-2}\omega)\,dx. \tag{2.11}$$

Applying (2.8), we have

$$\int_{\mathbb{R}^2}\Lambda^{2\alpha}_{x_1}\omega(|\omega|^{p-2}\omega)\,dx \geq \widetilde{C}\int_{\mathbb{R}}\|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\,dx_2. \tag{2.12}$$

By the interpolation inequality and the Young inequality, it concludes

$$\Big|\int_{\mathbb{R}^2}\partial_{x_1}\theta(|\omega|^{p-2}\omega)\,dx\Big|$$
$$= \Big|\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_{x_1}\theta(|\omega|^{p-2}\omega)\,dx_1dx_2\Big|$$
$$\leq C\int_{\mathbb{R}}\|\Lambda^{1-\kappa}_{x_1}\theta(.,x_2)\|_{L^p_{x_1}}\big\|\Lambda^\kappa_{x_1}(|\omega|^{p-2}\omega)(.,x_2)\big\|_{L^{\frac{p}{p-1}}_{x_1}}\,dx_2$$

$$
\begin{aligned}
&\le C\int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\kappa}\theta(.,x_2)\|_{\mathcal{B}^0_{p,2}}\|\omega(.,x_2)\|_{\mathcal{B}^\sigma_{p,\frac{p}{p-1}}}\||\omega(.,x_2)|^{p-2}\|_{L^{\frac{p}{p-2}}_{x_1}}\,dx_2\\
&\le C\int_{\mathbb{R}} \|\theta(.,x_2)\|_{\mathcal{B}^{1-\kappa}_{p,2}}\|\omega(.,x_2)\|_{\mathcal{B}^\sigma_{p,\frac{p}{p-1}}}\|\omega(.,x_2)\|^{p-2}_{L^p_{x_1}}\,dx_2\\
&\le C\int_{\mathbb{R}} \|\theta(.,x_2)\|_{\mathcal{B}^{\frac{2\beta}{p}}_{p,p}}\|\omega(.,x_2)\|_{\mathcal{B}^{\frac{2\alpha}{p}}_{p,p}}\|\omega(.,x_2)\|^{p-2}_{L^p_{x_1}}\,dx_2\\
&\le C\int_{\mathbb{R}} (\|\theta(.,x_2)\|_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}+\|\theta(.,x_2)\|_{L^p_{x_1}})(\|\omega(.,x_2)\|_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}+\|\omega(.,x_2)\|_{L^p_{x_1}})\|\omega(.,x_2)\|^{p-2}_{L^p_{x_1}}\,dx_2\\
&\le \frac{\widetilde{C}}{2}\int_{\mathbb{R}} \|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\,dx_2 + C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2\\
&\quad +C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{L^p_{x_1}}\,dx_2 + C\int_{\mathbb{R}} \|\omega(.,x_2)\|^p_{L^p_{x_1}}\,dx_2\\
&\le \frac{\widetilde{C}}{2}\int_{\mathbb{R}} \|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\,dx_2 + C\int_{\mathbb{R}} \|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2 + C\|\theta\|^p_{L^p}+C\|\omega\|^p_{L^p}, \qquad (2.13)
\end{aligned}
$$

where in the fourth line we have used the following inequality (see Lemma A.4)

$$
\big\|\Lambda^\kappa_{x_1}(|\omega|^{p-2}\omega)(.,x_2)\big\|_{L^{\frac{p}{p-1}}_{x_1}} \le C\|\omega(.,x_2)\|_{\mathcal{B}^\sigma_{p,\frac{p}{p-1}}}\big\||\omega(.,x_2)|^{p-2}\big\|_{L^{\frac{p}{p-2}}_{x_1}}, \quad \sigma>\kappa, \qquad (2.14)
$$

and κ in the sixth line satisfies

$$
1-\frac{2\beta}{p}<\kappa<\sigma<\frac{2\alpha}{p} \quad \text{or} \quad p<2(\alpha+\beta).
$$

Combining (2.11), (2.12) and (2.13) yields

$$
\frac{d}{dt}\|\omega(t)\|^p_{L^p}+\int_{\mathbb{R}}\|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\,dx_2 \le C\int_{\mathbb{R}}\|\theta(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}}\,dx_2 + C\|\theta(t)\|^p_{L^p}+C\|\omega(t)\|^p_{L^p},
$$

which, together with (2.5), imply

$$
\|\omega(t)\|^p_{L^p}+\int_0^t\int_{\mathbb{R}}\|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}}\,dx_2d\tau \le C(t,\,u_0,\,\theta_0).
$$

The proof of Proposition 2.2 is then completed. □

3. Global H^1-bound for θ

This section provides a global bound for $\|\nabla\theta\|_{L^2}$. We distinguish between two cases: $\beta>\frac{1}{2}$ and $\beta\le\frac{1}{2}$. In the case when $\beta>\frac{1}{2}$, the global bound holds for any $\alpha+\beta\ge 1$. In the case when $\beta\le\frac{1}{2}$, we need $\beta>\beta_0$ and $\alpha>\frac{1}{2}$. The precise results are stated in the following propositions.

Proposition 3.1. *Let $\beta>\frac{1}{2}$ and $\alpha+\beta\ge 1$. Assume $(u_0,\,\theta_0)$ satisfies the assumptions stated in Theorem 1.1 and let (u,θ) be the corresponding solution. Then, for any $t>0$,*

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta}\nabla\theta(\tau)\|_{L^2}^2\, d\tau \le C(t,\, u_0,\, \theta_0), \tag{3.1}$$

where $C(t,\, u_0,\, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

In the complement case when $\beta \le \frac{1}{2}$, the global bound is given by the following proposition.

Proposition 3.2. *Assume α and β satisfy*

$$\frac{1}{2} < \alpha \le 1, \qquad \frac{1}{2} \ge \beta > \beta_0, \qquad \alpha + \beta > 1. \tag{3.2}$$

Assume $(u_0,\, \theta_0)$ satisfies the assumptions stated in Theorem 1.1 and let (u, θ) be the corresponding solution. Then, for any $t > 0$,

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta}\nabla\theta(\tau)\|_{L^2}^2\, d\tau \le C(t,\, u_0,\, \theta_0), \tag{3.3}$$

where $C(t,\, u_0,\, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

Remark 3.3. In the proofs of the two propositions above, we shall restrict to the case $\alpha,\, \beta \in (0,\, 1)$. Then remaining cases either have been dealt with or are simpler than the case handled here. As stated in the introduction, Danchin and Paicu [13] established the global regularity for the system (1.1) with $\nu = 0,\ \alpha = 1$ or $\mu = 0,\ \beta = 1$. When $\alpha > 1$ or $\beta > 1$, the dissipative terms provide more smoothing and are mathematically easier than the case $\alpha,\, \beta \in (0,\, 1)$.

In order to prove the propositions above, we need anisotropic Sobolev inequalities.

Lemma 3.4. *Assume $f \in L^q_{x_2}L^p_{x_1}(\mathbb{R}^2)$ with $p,\, q \in [2,\, \infty]$, and $g,\ h,\ \Lambda_{x_1}^{\gamma_1} g,\ \Lambda_{x_2}^{\gamma_2} h$ are all in $L^2(\mathbb{R}^2)$ with $\gamma_1 > \frac{1}{p}$ and $\gamma_2 > \frac{1}{q}$. Then, for a constant $C = C(p, q, \gamma_1, \gamma_2) > 0$,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f\, g\, h|\; dx_1 dx_2 \le C\, \|f\|_{L^q_{x_2}L^p_{x_1}}\, \|g\|_{L^2}^{1-\frac{1}{\gamma_1 p}}\, \|\Lambda_{x_1}^{\gamma_1} g\|_{L^2}^{\frac{1}{\gamma_1 p}}\; \|h\|_{L^2}^{1-\frac{1}{\gamma_2 q}}\, \|\Lambda_{x_2}^{\gamma_2} h\|_{L^2}^{\frac{1}{\gamma_2 q}}, \tag{3.4}$$

where we have used the notation

$$\|h\|_{L^q_{x_2}L^p_{x_1}} := \Big(\int_{\mathbb{R}} \|h(.,x_2)\|_{L^p_{x_1}}^q\; dx_2\Big)^{\frac{1}{q}}.$$

In particular, if $f,\ g,\ h \in L^2(\mathbb{R}^2)$ and $\Lambda_{x_1}^{\gamma_1} g,\ \Lambda_{x_2}^{\gamma_2} h \in L^2(\mathbb{R}^2)$ with $\gamma_1,\ \gamma_2 > \frac{1}{2}$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f\, g\, h|\; dx_1 dx_2 \le C\, \|f\|_{L^2}\, \|g\|_{L^2}^{1-\frac{1}{2\gamma_1}}\, \|\Lambda_{x_1}^{\gamma_1} g\|_{L^2}^{\frac{1}{2\gamma_1}}\; \|h\|_{L^2}^{1-\frac{1}{2\gamma_2}}\, \|\Lambda_{x_2}^{\gamma_2} h\|_{L^2}^{\frac{1}{2\gamma_2}}, \tag{3.5}$$

where C is a constant depending on γ_1 and γ_2 only.

Lemma 3.5. *Let $p, q \in [2, \infty]$. Then,*

$$\|f\|_{L^q_{x_2}L^p_{x_1}} \le C\, \|f\|_{L^2_{x_1x_2}}^{\gamma_1\,\gamma_2}\, \|\Lambda_{x_2}^{\sigma_2} f\|_{L^2_{x_1x_2}}^{\gamma_1\,(1-\gamma_2)}\, \|\Lambda_{x_1}^{\sigma_1} f\|_{L^2_{x_1x_2}}^{(1-\gamma_1)\,\gamma_2}\, \|\Lambda_{x_2}^{\sigma_2}\Lambda_{x_1}^{\sigma_1} f\|_{L^2_{x_1x_2}}^{(1-\gamma_1)\,(1-\gamma_2)}, \tag{3.6}$$

where C is a constant depending on the parameters, and $\gamma_1, \gamma_2 \in [0,1]$, $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ satisfy

$$(1-\gamma_1)\,\sigma_1 = \frac{1}{2} - \frac{1}{p}, \qquad (1-\gamma_2)\,\sigma_2 = \frac{1}{2} - \frac{1}{q}.$$

Furthermore,

$$\begin{aligned}\|f\|_{L^q_{x_2} L^p_{x_1}} \leq C\, \|f\|^{\rho}_{L^2_{x_1x_2}}\, \|\Lambda^{\sigma_1+\sigma_2}_{x_2} f\|_{L^2_{x_1x_2}}^{\frac{\sigma_2}{\sigma_1+\sigma_2}\gamma_1\,(1-\gamma_2)}\, \|\Lambda^{\sigma_1+\sigma_2}_{x_1} f\|_{L^2_{x_1x_2}}^{\frac{\sigma_1}{\sigma_1+\sigma_2}(1-\gamma_1)\,\gamma_2} \\ \times \|\Lambda^{\sigma_2}_{x_2}\Lambda^{\sigma_1}_{x_1} f\|_{L^2_{x_1x_2}}^{(1-\gamma_1)\,(1-\gamma_2)},\end{aligned} \tag{3.7}$$

where ρ is given by

$$\rho = \gamma_1\gamma_2 + \frac{\sigma_1}{\sigma_1+\sigma_2}\gamma_1(1-\gamma_2) + \frac{\sigma_2}{\sigma_1+\sigma_2}(1-\gamma_1)\gamma_2.$$

In the special case when $p = q = \infty$, (3.6) becomes

$$\|f\|_{L^\infty} \leq C\, \|f\|_{L^2}^{(1-\frac{1}{2\sigma_1})(1-\frac{1}{2\sigma_2})}\, \|\Lambda^{\sigma_2}_{x_2} f\|_{L^2}^{(1-\frac{1}{2\sigma_1})\,\frac{1}{2\sigma_2}}\, \|\Lambda^{\sigma_1}_{x_1} f\|_{L^2}^{\frac{1}{2\sigma_1}(1-\frac{1}{2\sigma_2})}\, \|\Lambda^{\sigma_1}_{x_1}\Lambda^{\sigma_2}_{x_2} f\|_{L^2}^{\frac{1}{4\sigma_1\sigma_2}},$$

where $\sigma_1, \sigma_2 > \frac{1}{2}$.

We provide the proofs of the lemmas presented above.

Proof of Lemma 3.4. We start with the one-dimensional Sobolev inequality

$$\|g\|_{L^{\frac{2p}{p-2}}_{x_1}(\mathbb{R})} \leq C\|g\|_{L^2_{x_1}(\mathbb{R})}^{1-\frac{1}{\gamma_1 p}} \|\Lambda^{\gamma_1}_{x_1} g\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{\gamma_1 p}}, \qquad p \in [2,\infty], \quad \gamma_1 > \frac{1}{p}, \tag{3.8}$$

where we have used the sub-index x_1 with the Lebesgue spaces to emphasize that the norms are taken in one-dimensional Lebesgue spaces with respect to x_1. Here and in what follows, we adopt the convention $\frac{2p}{p-2} = \infty$ for $p = 2$. By Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |f\,g\,h|\, dx_1 dx_2 &\leq C \int_{\mathbb{R}} \|f\|_{L^p_{x_1}}\, \|g\|_{L^{\frac{2p}{p-2}}_{x_1}}\, \|h\|_{L^2_{x_1}}\, dx_2 \\ &\leq C \int_{\mathbb{R}} \|f\|_{L^p_{x_1}}\, \|g\|_{L^2_{x_1}}^{1-\frac{1}{\gamma_1 p}}\, \|\Lambda^{\gamma_1}_{x_1} g\|_{L^2_{x_1}}^{\frac{1}{\gamma_1 p}}\, \|h\|_{L^2_{x_1}}\, dx_2 \\ &\leq C\Big(\int_{\mathbb{R}} \|f\|^q_{L^p_{x_1}}\, dx_2\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}} \|g\|^2_{L^2_{x_1}}\, dx_2\Big)^{\frac{\gamma_1 p-1}{2\gamma_1 p}} \\ &\quad \times \Big(\int_{\mathbb{R}} \|\Lambda^{\gamma_1}_{x_1} g\|^2_{L^2_{x_1}}\, dx_2\Big)^{\frac{1}{2\gamma_1 p}}\, \|h\|_{L^{\frac{2q}{q-2}}_{x_2} L^2_{x_1}} \\ &= C\|f\|_{L^q_{x_2} L^p_{x_1}}\, \|g\|_{L^2}^{1-\frac{1}{\gamma_1 p}}\, \|\Lambda^{\gamma_1}_{x_1} g\|_{L^2}^{\frac{1}{\gamma_1 p}}\, \|h\|_{L^{\frac{2q}{q-2}}_{x_2} L^2_{x_1}}.\end{aligned} \tag{3.9}$$

By Minkowski's inequality and (3.8),

$$\begin{aligned}
\|h\|_{L^{\frac{2q}{q-2}}_{x_2} L^2_{x_1}} &\le C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|^2_{L^{\frac{2q}{q-2}}_{x_2}}\, dx_1\Big)^{\frac12} \\
&\le C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|^{2-\frac{2}{\gamma_2 q}}_{L^2_{x_2}} \|\Lambda^{\gamma_2}_{x_2} h(x_1,.)\|^{\frac{2}{\gamma_2 q}}_{L^2_{x_2}}\, dx_1\Big)^{\frac12} \\
&\le C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|^2_{L^2_{x_2}}\, dx_1\Big)^{\frac{\gamma_2 q-1}{2\gamma_2 q}} \Big(\int_{\mathbb{R}} \|\Lambda^{\gamma_2}_{x_2} h(x_1,.)\|^2_{L^2_{x_2}}\, dx_1\Big)^{\frac{1}{2\gamma_2 q}} \\
&= C\|h\|^{1-\frac{1}{\gamma_2 q}}_{L^2} \|\Lambda^{\gamma_2}_{x_2} h\|^{\frac{1}{\gamma_2 q}}_{L^2}.
\end{aligned} \tag{3.10}$$

Inserting (3.10) in (3.9) gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f\, g\, h|\, dx_1 dx_2 \le C\, \|f\|_{L^q_{x_2} L^p_{x_1}}\, \|g\|^{1-\frac{1}{\gamma_1 p}}_{L^2} \|\Lambda^{\gamma_1}_{x_1} g\|^{\frac{1}{\gamma_1 p}}_{L^2}\, \|h\|^{1-\frac{1}{\gamma_2 q}}_{L^2} \|\Lambda^{\gamma_2}_{x_2} h\|^{\frac{1}{\gamma_2 q}}_{L^2}.$$

This completes the proof. □

Proof of Lemma 3.5. By Sobolev's inequality for 1D functions,

$$\|f\|_{L^p_{x_1}} \le C\, \|f\|^{\gamma_1}_{L^2_{x_1}}\, \|\Lambda^{\sigma_1}_{x_1} f\|^{1-\gamma_1}_{L^2_{x_1}},$$

where $\gamma_1 \in [0,1]$ and $\sigma_1 \ge 0$ satisfy

$$\frac1p = \frac12\,\gamma_1 + (1-\gamma_1)\left(\frac12 - \sigma_1\right) \quad \text{or} \quad (1-\gamma_1)\,\sigma_1 = \frac12 - \frac1p.$$

Therefore, by Hölder's inequality with $q_1 = \frac{q}{\gamma_1}$ and $q_2 = \frac{q}{1-\gamma_1}$,

$$\begin{aligned}
\|f\|_{L^q_{x_2} L^p_{x_1}} &\le C\, \Big\| \|f\|^{\gamma_1}_{L^2_{x_1}}\, \|\Lambda^{\sigma_1}_{x_1} f\|^{1-\gamma_1}_{L^2_{x_1}} \Big\|_{L^q_{x_2}} \\
&\le C\, \Big\| \|f\|^{\gamma_1}_{L^2_{x_1}} \Big\|_{L^{q_1}_{x_2}}\, \Big\| \|\Lambda^{\sigma_1}_{x_1} f\|^{1-\gamma_1}_{L^2_{x_1}} \Big\|_{L^{q_2}_{x_2}} \\
&\le C\, \| \|f\|_{L^2_{x_1}} \|^{\gamma_1}_{L^q_{x_2}}\, \| \|\Lambda^{\sigma_1}_{x_1} f\|_{L^2_{x_1}} \|^{1-\gamma_1}_{L^q_{x_2}}.
\end{aligned} \tag{3.11}$$

By Minkowski's inequality, for $q \ge 2$,

$$\| \|f\|_{L^2_{x_1}} \|_{L^q_{x_2}} \le \| \|f\|_{L^q_{x_2}} \|_{L^2_{x_1}}, \qquad \| \|\Lambda^{\sigma_1}_{x_1} f\|_{L^2_{x_1}} \|_{L^q_{x_2}} \le \| \|\Lambda^{\sigma_1}_{x_1} f\|_{L^q_{x_2}} \|_{L^2_{x_1}}. \tag{3.12}$$

By Sobolev's inequality for 1D functions,

$$\|f\|_{L^q_{x_2}} \le C\, \|f\|^{\gamma_2}_{L^2_{x_2}}\, \|\Lambda^{\sigma_2}_{x_2} f\|^{1-\gamma_2}_{L^2_{x_2}}, \tag{3.13}$$

$$\|\Lambda^{\sigma_1}_{x_1} f\|_{L^q_{x_2}} \le C\, \|\Lambda^{\sigma_1}_{x_1} f\|^{\gamma_2}_{L^2_{x_2}}\, \|\Lambda^{\sigma_2}_{x_2} \Lambda^{\sigma_1}_{x_1} f\|^{1-\gamma_2}_{L^2_{x_2}}, \tag{3.14}$$

where $\gamma_2 \in [0,1]$ and $\sigma_2 \ge 0$ satisfy

$$\frac1q = \frac12\,\gamma_2 + (1-\gamma_2)\left(\frac12 - \sigma_2\right) \quad \text{or} \quad (1-\gamma_2)\,\sigma_2 = \frac12 - \frac1q.$$

Inserting (3.12), (3.13) and (3.14) in (3.11), we obtain

$$\|f\|_{L^q_{x_2}L^p_{x_1}} \le C \left\| \|f\|^{\gamma_2}_{L^2_{x_2}} \|\Lambda^{\sigma_2}_{x_2} f\|^{1-\gamma_2}_{L^2_{x_2}} \right\|^{\gamma_1}_{L^2_{x_1}} \left\| \|\Lambda^{\sigma_1}_{x_1} f\|^{\gamma_2}_{L^2_{x_2}} \|\Lambda^{\sigma_2}_{x_2}\Lambda^{\sigma_1}_{x_1} f\|^{1-\gamma_2}_{L^2_{x_2}} \right\|^{1-\gamma_1}_{L^2_{x_1}}$$
$$\le C \|f\|^{\gamma_1\gamma_2}_{L^2_{x_1x_2}} \|\Lambda^{\sigma_2}_{x_2} f\|^{\gamma_1(1-\gamma_2)}_{L^2_{x_1x_2}} \|\Lambda^{\sigma_1}_{x_1} f\|^{(1-\gamma_1)\gamma_2}_{L^2_{x_1x_2}} \|\Lambda^{\sigma_2}_{x_2}\Lambda^{\sigma_1}_{x_1} f\|^{(1-\gamma_1)(1-\gamma_2)}_{L^2_{x_1x_2}},$$

which is exactly (3.6). This completes the proof of Lemma 3.5. (3.7) follows from (3.6) and the embedding inequalities

$$\|\Lambda^{\sigma_2}_{x_2} f\|_{L^2_{x_1x_2}} \le C \|f\|\|^{\frac{\sigma_1}{\sigma_1+\sigma_2}}_{L^2_{x_1x_2}} \|\Lambda^{\sigma_1+\sigma_2}_{x_2} f\|^{\frac{\sigma_2}{\sigma_1+\sigma_2}}_{L^2_{x_1x_2}}.$$

This completes the proof of Lemma 3.5. □

We also need the following simple interpolation inequality, which will be used frequently without mentioning.

$$\|\Lambda^{\gamma}_{x_i} f\|_{L^2} \le C\|f\|^{1-\frac{\gamma}{\varrho}}_{L^2} \|\Lambda^{\varrho}_{x_i} f\|^{\frac{\gamma}{\varrho}}_{L^2}, \quad 0 \le \gamma \le \varrho, \quad i = 1, 2. \tag{3.15}$$

The proof of (3.15) is quite simple and we give the proof in the Appendix for reader's convenience.

The following standard commutator estimate will also be used as well, which can be found in [21, p. 614].

Lemma 3.6. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then*

$$\|\Lambda^s(f\,g) - g\,\Lambda^s f - f\,\Lambda^s g\|_{L^p(\mathbb{R}^d)} \le C\,\|g\|_{L^\infty(\mathbb{R}^d)}\,\|\Lambda^s f\|_{L^p(\mathbb{R}^d)},$$

where $d \ge 1$ denotes the spatial dimension and $C = C(d,s,p)$ is a constant.

We are now ready to prove the propositions.

Proof of Proposition 3.1. Applying ∇ to $(1.1)_2$ and then dotting it with by $\nabla\theta$ yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2_{L^2} + \|\Lambda^{\beta}_{x_1}\nabla\theta\|^2_{L^2} = -\int_{\mathbb{R}^2} \sum_{i=1}^{2}\sum_{j=1}^{2} \partial_{x_i}\theta\partial_{x_i}u_j\partial_{x_j}\theta\,dx. \tag{3.16}$$

The term on the righthand side of (3.16) is a quadratic form and can be written as

$$\begin{aligned} -\int_{\mathbb{R}^2} \sum_{i=1}^{2}\sum_{j=1}^{2} \partial_{x_i}\theta\partial_{x_i}u_j\partial_{x_j}\theta\,dx &= -\int_{\mathbb{R}^2} \partial_{x_1}u_1\partial_{x_1}\theta\partial_{x_1}\theta\,dx - \int_{\mathbb{R}^2} \partial_{x_1}u_2\partial_{x_2}\theta\partial_{x_1}\theta\,dx \\ &\quad - \int_{\mathbb{R}^2} \partial_{x_2}u_1\partial_{x_1}\theta\partial_{x_2}\theta\,dx - \int_{\mathbb{R}^2} \partial_{x_2}u_2\partial_{x_2}\theta\partial_{x_2}\theta\,dx \\ &:= K_1 + K_2 + K_3 + K_4. \end{aligned} \tag{3.17}$$

These terms can be bounded as follows. For any $0 < \alpha, \beta < 1$ and $\alpha + \beta \ge 1$, K_1 and K_2 can be bounded suitably. For $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, we set

$$\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{2}, \qquad \frac{1}{q} = \frac{1}{2} - \frac{\beta}{2}.$$

By Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
|K_1| &\le \|\partial_{x_1}\theta\|_{L^2}\,\|\partial_{x_1}u_1\|_{L^p}\,\|\partial_{x_1}\theta\|_{L^q}\\
&\le C\,\|\partial_{x_1}\theta\|_{L^2}\,\|\Lambda^{\alpha}\partial_{x_1}u_1\|_{L^2}\,\|\Lambda^{\beta}\partial_{x_1}\theta\|_{L^2}\\
&\le \frac{1}{16}\,\|\Lambda^{\beta}\partial_{x_1}\theta\|_{L^2}^2 + C\,\|\Lambda^{\alpha}\partial_{x_1}u_1\|_{L^2}^2\,\|\partial_{x_1}\theta\|_{L^2}^2.
\end{aligned}
$$

Due to Plancherel's Theorem and the simple inequalities

$$
\xi_1^2\,(\xi_1^2+\xi_2^2)^{\alpha} \le \xi_1^{2\alpha}\,(\xi_1^2+\xi_2^2), \qquad \xi_1^2\,(\xi_1^2+\xi_2^2)^{\beta} \le \xi_1^{2\beta}\,(\xi_1^2+\xi_2^2),
$$

we have

$$
\|\Lambda^{\alpha}\partial_{x_1}u_1\|_{L^2} \le \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}, \qquad \|\Lambda^{\beta}\partial_{x_1}\theta\|_{L^2} \le \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}.
$$

Therefore,

$$
|K_1| \le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2\,\|\partial_{x_1}\theta\|_{L^2}^2.
$$

Very similarly,

$$
|K_2| \le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2\,\|\partial_{x_2}\theta\|_{L^2}^2.
$$

When $\alpha+\beta>1$, one can perform the estimates above for $\widetilde{\alpha}=1-\beta$ and β to obtain

$$
\begin{aligned}
|K_1| &\le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\Lambda_{x_1}^{\widetilde{\alpha}}\nabla u\|_{L^2}^2\,\|\partial_{x_1}\theta\|_{L^2}^2,\\
|K_2| &\le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\Lambda_{x_1}^{\widetilde{\alpha}}\nabla u\|_{L^2}^2\,\|\partial_{x_2}\theta\|_{L^2}^2
\end{aligned}
$$

and invoke the interpolation inequality

$$
\|\Lambda_{x_1}^{\widetilde{\alpha}}\nabla u\|_{L^2} \le \|\nabla u\|_{L^2}^{1-\frac{\widetilde{\alpha}}{\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{\widetilde{\alpha}}{\alpha}}
$$

to get to the bounds

$$
\begin{aligned}
|K_1| &\le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,(\|\nabla u\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2)\,\|\partial_{x_2}\theta\|_{L^2}^2,\\
|K_2| &\le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,(\|\nabla u\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2)\,\|\partial_{x_2}\theta\|_{L^2}^2.
\end{aligned}
$$

To bound K_3, we need the assumption that $\beta>\frac{1}{2}$. For $\beta>\frac{1}{2}$, we apply (3.5) with $\gamma_1=\gamma_2=\beta>\frac{1}{2}$ in Lemma 3.4,

$$
\begin{aligned}
K_3 &\le C\,\|\nabla u\|_{L^2}\,\|\nabla\theta\|_{L^2}^{1-\frac{1}{2\beta}}\,\|\Lambda_{x_2}^{\beta}\partial_{x_1}\theta\|_{L^2}^{\frac{1}{2\beta}}\,\|\nabla\theta\|_{L^2}^{1-\frac{1}{2\beta}}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{2\beta}}\\
&\le C\,\|\nabla u\|_{L^2}\,\|\nabla\theta\|_{L^2}^{1-\frac{1}{2\beta}}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{2\beta}}\,\|\nabla\theta\|_{L^2}^{1-\frac{1}{2\beta}}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{2\beta}}\\
&\le \frac{1}{16}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\nabla u\|_{L^2}^{\frac{2\beta}{2\beta-1}}\,\|\nabla\theta\|_{L^2}^2.
\end{aligned}
$$

K_4 is handled differently. By (3.8) and Hölder's inequality,

$$
\begin{aligned}
K_4 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_{x_1} u_1 \partial_{x_2}\theta \partial_{x_2}\theta \, dx_1 dx_2 \\
&\le \Big| \int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(.,x_2)\|_{L^2_{x_1}} \|\Lambda_{x_1}^{\beta}(\partial_{x_2}\theta\partial_{x_2}\theta)(.,x_2)\|_{L^2_{x_1}} \, dx_2 \Big| \\
&\le \Big| \int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(.,x_2)\|_{L^2_{x_1}} \|\partial_{x_2}\theta(.,x_2)\|_{L^\infty_{x_1}} \|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta(.,x_2)\|_{L^2_{x_1}} \, dx_2 \Big| \\
&\le \Big| \int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(.,x_2)\|_{L^2_{x_1}} \|\partial_{x_2}\theta(.,x_2)\|_{L^2_{x_1}}^{1-\frac{1}{2\beta}} \|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta(.,x_2)\|_{L^2_{x_1}}^{\frac{1}{2\beta}} \\
&\qquad \times \|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta(.,x_2)\|_{L^2_{x_1}} \, dx_2 \Big| \\
&\le C \|\Lambda_{x_1}^{1-\beta} u_1\|_{L^\infty_{x_2} L^2_{x_1}} \Big(\int_{\mathbb{R}} \|\nabla\theta(.,x_2)\|_{L^2_{x_1}}^2 \, dx_2 \Big)^{\frac{2\beta-1}{4\beta}} \\
&\qquad \times \Big(\int_{\mathbb{R}} \|\Lambda_{x_1}^{\beta}\nabla\theta(.,x_2)\|_{L^2_{x_1}}^2 \, dx_2 \Big)^{\frac{2\beta+1}{4\beta}}.
\end{aligned} \tag{3.18}
$$

Furthermore, by (3.8) and Hölder's inequality,

$$
\begin{aligned}
\|\Lambda_{x_1}^{1-\beta} u_1\|_{L^\infty_{x_2} L^2_{x_1}} &\le \Big(\int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(x_1,.)\|_{L^\infty_{x_2}}^2 \, dx_1 \Big)^{\frac12} \\
&\le C \Big(\int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(x_1,.)\|_{L^2_{x_2}}^{\frac{2\beta-1}{\beta}} \|\Lambda_{x_2}^{\beta}\Lambda_{x_1}^{1-\beta} u_1(x_1,.)\|_{L^2_{x_2}}^{\frac{1}{\beta}} \, dx_1 \Big)^{\frac12} \\
&\le C \Big(\int_{\mathbb{R}} \|\Lambda_{x_1}^{1-\beta} u_1(x_1,.)\|_{L^2_{x_2}}^2 \, dx_1 \Big)^{\frac{2\beta-1}{4\beta}} \Big(\int_{\mathbb{R}} \|\Lambda_{x_2}^{\beta}\Lambda_{x_1}^{1-\beta} u_1(x_1,.)\|_{L^2_{x_2}}^2 \, dx_1 \Big)^{\frac{1}{4\beta}} \\
&\le C \|\Lambda_{x_1}^{1-\beta} u_1\|_{L^2}^{\frac{2\beta-1}{2\beta}} \|\nabla u_1\|_{L^2}^{\frac{1}{2\beta}} \\
&\le C (\|u\|_{L^2}^{\beta} \|\nabla u\|_{L^2}^{1-\beta})^{\frac{2\beta-1}{2\beta}} \|\nabla u\|_{L^2}^{\frac{1}{2\beta}},
\end{aligned} \tag{3.19}
$$

where we have used

$$
\|\Lambda_{x_2}^{\beta}\Lambda_{x_1}^{1-\beta} u\|_{L^2} = \big\| |\xi_2|^{\beta} |\xi_1|^{1-\beta} \widehat{u}(\xi) \big\|_{L^2} \le C \| |\xi| \widehat{u}(\xi)\|_{L^2} \le C \|\nabla u\|_{L^2}.
$$

Inserting (3.19) in (3.18) yields

$$
\begin{aligned}
K_4 &\le C (\|u\|_{L^2}^{\beta} \|\nabla u\|_{L^2}^{1-\beta})^{\frac{2\beta-1}{2\beta}} \|\nabla u\|_{L^2}^{\frac{1}{2\beta}} \|\nabla\theta\|_{L^2}^{\frac{2\beta-1}{2\beta}} \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{2\beta+1}{2\beta}} \\
&\le \frac{1}{16} \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\, \|u\|_{L^2}^{2\beta} \|\nabla u\|_{L^2}^{\frac{2\beta(3-2\beta)}{2\beta-1}} \|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{3.20}
$$

Combining the estimates above leads to

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 &\le C(\|\nabla u\|_{L^2}^{\frac{2\beta}{2\beta-1}} + \|u\|_{L^2}^{2\beta} \|\nabla u\|_{L^2}^{\frac{2\beta(3-2\beta)}{2\beta-1}}) \|\nabla\theta\|_{L^2}^2 \\
&\quad + C\, (\|\nabla u\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2)\, \|\partial_{x_2}\theta\|_{L^2}^2,
\end{aligned}
$$

which yields the desired global bound. This completes the proof of Proposition 3.1. □

We now turn to the proof of Proposition 3.2.

Proof of Proposition 3.2. As in (3.16) and (3.17), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 = K_1 + K_2 + K_3 + K_4. \tag{3.21}$$

Since the bounds for K_1 and K_2 in the proof of Proposition 3.1 hold for any $0 \le \alpha, \beta \le 1$ with $\alpha + \beta \ge 1$, K_1 and K_2 can be estimated exactly the same as before. We now bound K_4. The estimates for K_3 are complex and will be presented later. To estimate K_4, we invoke the assumption that

$$\frac{1}{2} < \alpha < 1, \quad 0 < \beta \le \frac{1}{2}, \quad \alpha + \beta \ge 1.$$

We choose

$$2 < q, \widetilde{q} < \infty, \qquad \frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{1}{2}$$

and apply Hölder's inequality to obtain

$$\begin{aligned} K_4 &= \int \partial_{x_1} u_1\, \partial_{x_2}\theta \partial_{x_2}\theta\, dx \\ &\le \|\partial_{x_1} u_1\|_{L^\infty_{x_2} L^q_{x_1}} \|\partial_{x_2}\theta\|_{L^2} \|\partial_{x_2}\theta\|_{L^2_{x_2} L^{\widetilde{q}}_{x_1}}. \end{aligned}$$

By Lemma 3.5, for $(1-\beta)\gamma_0 = \frac{1}{2} - \frac{1}{\widetilde{q}}$ and $\gamma_0 \in [0,1]$,

$$\|\partial_{x_2}\theta\|_{L^2_{x_2} L^{\widetilde{q}}_{x_1}} \le C\, \|\partial_{x_2}\theta\|_{L^2}^{\gamma_0} \|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta\|_{L^2}^{1-\gamma_0}.$$

For $\alpha > \frac{1}{2}$, we choose $\frac{1}{2} < \sigma_2 < \alpha$, $\sigma_1 + \sigma_2 = \alpha$ and

$$(1-\gamma_1)\sigma_1 = \frac{1}{2} - \frac{1}{q}, \qquad (1-\gamma_2)\sigma_2 = \frac{1}{2}$$

and apply Lemma 3.5,

$$\begin{aligned} \|\partial_{x_1} u_1\|_{L^\infty_{x_2} L^q_{x_1}} &\le C\, \|\partial_{x_1} u_1\|_{L^2}^{\gamma_1\gamma_2} \|\Lambda_{x_2}^{\sigma_2}\partial_{x_1} u_1\|_{L^2}^{\gamma_1(1-\gamma_2)} \\ &\quad \times \|\Lambda_{x_1}^{\sigma_1}\partial_{x_1} u_1\|_{L^2}^{(1-\gamma_1)\gamma_2} \|\Lambda_{x_1}^{\sigma_1}\Lambda_{x_2}^{\sigma_2}\partial_{x_1} u_1\|_{L^2}^{(1-\gamma_1)(1-\gamma_2)} \\ &\le C\, \|\partial_{x_1} u_1\|_{L^2}^{\rho} \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{1-\rho}, \end{aligned}$$

where

$$\rho = \gamma_1\gamma_2 + \frac{\sigma_1}{\sigma_1+\sigma_2}\gamma_1(1-\gamma_2) + \frac{\sigma_2}{\sigma_1+\sigma_2}(1-\gamma_1)\gamma_2.$$

Inserting the bounds above in K_4 and applying Young's inequality, we have

$$\begin{aligned} |K_4| &\le \frac{1}{16}\|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta\|_{L^2}^2 + C\, \|\partial_{x_2}\theta\|_{L^2}^2 \|\partial_{x_1} u_1\|_{L^2}^{\frac{2\rho}{1+\gamma_0}} \|\Lambda^{\alpha}\nabla u\|_{L^2}^{\frac{2(1-\rho)}{1+\gamma_0}} \\ &\le \frac{1}{16}\|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta\|_{L^2}^2 + C\, \|\partial_{x_2}\theta\|_{L^2}^2 \|\partial_{x_1} u_1\|_{L^2}^{\frac{2\rho}{1+\gamma_0}} (1 + \|\Lambda^{\alpha}\nabla u\|_{L^2}^2). \end{aligned}$$

A particular set of values of q, $\widetilde{q}$, γ_0, σ_1, σ_2, γ_1, γ_2 and ρ are

$$q = \frac{4}{3-2\alpha}, \quad \widetilde{q} = \frac{4}{2\alpha-1}, \quad \gamma_0 = \frac{3-2\alpha}{4(1-\beta)}$$

and

$$\sigma_1 = \frac{2\alpha-1}{4}, \quad \sigma_2 = \frac{2\alpha+1}{4}, \quad \gamma_1 = 0, \quad \gamma_2 = \frac{2\alpha-1}{2\alpha+1}, \quad \rho = \frac{1}{2} - \frac{1}{4\alpha}.$$

We now turn to K_3. Our efforts are devoted to establishing sharpest possible bounds. One way to bound it is to apply Lemma 3.4. In fact, by (3.4) with

$$p = q > \frac{1}{\beta}, \qquad \gamma_1 = \gamma_2 = \beta, \tag{3.22}$$

we have

$$\begin{aligned}
|K_3| &= \left| \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_1}\theta \partial_{x_2}\theta \, dx \right| \\
&\le C \, \|\partial_{x_2} u_1\|_{L^p} \, \|\partial_{x_1}\theta\|_{L^2}^{1-\frac{1}{\beta p}} \|\Lambda_{x_2}^{\beta}\partial_{x_1}\theta\|_{L^2}^{\frac{1}{\beta p}} \, \|\partial_{x_2}\theta\|_{L^2}^{1-\frac{1}{\beta p}} \|\Lambda_{x_1}^{\beta}\partial_{x_2}\theta\|_{L^2}^{\frac{1}{\beta p}} \\
&\le C \|\omega\|_{L^p} \|\nabla\theta\|_{L^2}^{1-\frac{1}{\beta p}} \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{\beta p}} \|\nabla\theta\|_{L^2}^{1-\frac{1}{\beta p}} \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{\beta p}} \\
&\le \frac{1}{16} \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C \, \|\omega\|_{L^p}^{\frac{\beta p}{\beta p-1}} \|\nabla\theta\|_{L^2}^2.
\end{aligned}$$

According to (2.2), if $\alpha+\beta > 1$ and $2 \le p < 2(\alpha+\beta)$, then $\|\omega\|_{L^p}$ is globally bounded. Therefore, we need

$$\frac{1}{\beta} < p < 2(\alpha+\beta) \quad \text{or} \quad \beta > \frac{\sqrt{\alpha^2+2}-\alpha}{2}. \tag{3.23}$$

We now provide an alternative estimate for K_3, which holds for a larger range of β. By integration by parts,

$$K_3 = \int \theta \, \partial_{x_1}\partial_{x_2} u_1 \partial_{x_2}\theta + \int \theta \, \partial_{x_2} u_1 \, \partial_{x_1}\partial_{x_2}\theta := K_{31} + K_{32}.$$

By Hölder's inequality and the boundedness of Riesz transform on L^2,

$$K_{31} = \int \Lambda_{x_1}^{\alpha}\Lambda_{x_1}^{-1}\partial_{x_1}\partial_{x_2}u_1 \, \Lambda_{x_1}^{1-\alpha}(\theta\,\partial_{x_2}\theta) \le \|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2} \, \|\Lambda_{x_1}^{1-\alpha}(\theta\,\partial_{x_2}\theta)\|_{L^2}.$$

By Lemma 3.6,

$$\begin{aligned}
\|\Lambda_{x_1}^{1-\alpha}(\theta\,\partial_{x_2}\theta)\|_{L^2} &\le C \, \|\theta\|_{L^\infty} \, \|\Lambda_{x_1}^{1-\alpha}\partial_{x_2}\theta\|_{L^2} + \|\Lambda_{x_1}^{1-\alpha}\theta\,\partial_{x_2}\theta\|_{L^2} \\
&\le C \, \|\theta\|_{L^\infty} \, \|\Lambda_{x_1}^{1-\alpha}\partial_{x_2}\theta\|_{L^2} + \, \|\partial_{x_2}\theta\|_{L^2} \, \|\Lambda_{x_1}^{1-\alpha}\theta\|_{L^\infty}.
\end{aligned}$$

Applying Lemma 3.5 with σ_1 and σ_2 satisfying

$$\frac{1}{2} < \sigma_1 < \alpha, \qquad \frac{1}{2} < \sigma_2 < \alpha, \qquad \sigma_1 + \sigma_2 < \alpha + \beta,$$

we obtain

$$\begin{aligned}\|\Lambda_{x_1}^{1-\alpha}\theta\|_{L^\infty} &\le C\,\|\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2(\mathbb{R}^2)}^{(1-\frac{1}{2\sigma_1})(1-\frac{1}{2\sigma_2})}\,\|\Lambda_{x_2}^{\sigma_1}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2\sigma_1}(1-\frac{1}{2\sigma_2})}\\ &\quad\times\|\Lambda_{x_1}^{\sigma_2}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2\sigma_2}(1-\frac{1}{2\sigma_1})}\,\|\Lambda_{x_1}^{\sigma_2}\Lambda_{x_2}^{\sigma_1}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4\sigma_2\sigma_1}}\\ &\le C\,(\|\theta\|_{L^2}+\|\nabla\theta\|_{L^2})^{1-\frac{1}{4\sigma_2\sigma_1}}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{4\sigma_2\sigma_1}},\end{aligned}$$

where we have used the simple facts

$$\begin{aligned}&\|\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2}\le\|\theta\|_{L^2}+\|\nabla\theta\|_{L^2},\qquad \|\Lambda_{x_2}^{\sigma_1}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2}\le\|\theta\|_{L^2}+\|\nabla\theta\|_{L^2},\\ &\|\Lambda_{x_1}^{\sigma_2}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2}\le\|\theta\|_{L^2}+\|\nabla\theta\|_{L^2},\qquad \|\Lambda_{x_1}^{\sigma_2}\Lambda_{x_2}^{\sigma_1}\Lambda_{x_1}^{1-\alpha}\theta\|_{L^2}\le\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2},\\ &\left(1-\frac{1}{2\sigma_1}\right)\left(1-\frac{1}{2\sigma_2}\right)+\frac{1}{2\sigma_1}\left(1-\frac{1}{2\sigma_2}\right)+\frac{1}{2\sigma_2}\left(1-\frac{1}{2\sigma_1}\right)=1-\frac{1}{4\sigma_2\sigma_1}.\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}|K_{31}|&\le C\,\|\theta_0\|_{L^\infty}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}\\ &\quad+C\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}\,\left(\|\theta_0\|_{L^2}+\|\nabla\theta\|_{L^2}\right)^{2-\frac{1}{4\sigma\gamma_1}}\,\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1}{4\sigma\gamma_1}}\\ &\le\frac{1}{16}\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2+C\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}^2\,\left(\|\theta_0\|_{L^2}+\|\nabla\theta\|_{L^2}\right)^2.\end{aligned}$$

To bound K_{32}, we first apply Hölder's inequality and Lemma 3.6 to obtain

$$\begin{aligned}K_{32}&=\int\Lambda_{x_1}^{\beta}\Lambda_{x_1}^{-1}\partial_{x_1}\partial_{x_2}\theta\,\Lambda_{x_1}^{1-\beta}(\theta\partial_{x_2}u_1)\\ &\le C\,\|\Lambda_{x_1}^{\beta}\,\partial_{x_2}\theta\|_{L^2}\,\|\Lambda_{x_1}^{1-\beta}(\theta\,\partial_{x_2}u_1)\|_{L^2}\\ &\le C\,\|\theta_0\|_{L^\infty}\,\|\Lambda_{x_1}^{\beta}\,\partial_{x_2}\theta\|_{L^2}\,\|\Lambda_{x_1}^{1-\beta}\partial_{x_2}u_1\|_{L^2}\\ &\quad+C\,\|\Lambda_{x_1}^{\beta}\,\partial_{x_2}\theta\|_{L^2}\,\|\Lambda_{x_1}^{1-\beta}\theta\,\partial_{x_2}u_1\|_{L^2}.\end{aligned}$$

To estimate $\|\Lambda_{x_1}^{1-\beta}\theta\,\partial_{x_2}u_1\|_{L^2}$, we invoke the bound derived in Proposition 2.2,

$$\|\omega(t)\|_{L^p}\le C(t,\,u_0,\,\theta_0)\quad\text{for}\quad 2\le p<2(\alpha+\beta).\tag{3.24}$$

Since $\alpha>\frac{1}{2}$, we set

$$q=\frac{(2\alpha-1)p+2}{2\alpha},\qquad q<\frac{1}{\beta},\qquad \sigma_2>\frac{1}{2}$$

and apply the interpolation inequalities to obtain

$$\begin{aligned}\|\Lambda_{x_1}^{1-\beta}\theta\,\partial_{x_2}u_1\|_{L^2}&\le\|\partial_{x_2}u_1\|_{L^q_{x_2}L^\infty_{x_1}}\,\|\Lambda_{x_1}^{1-\beta}\theta\|_{L^{\frac{2q}{q-2}}_{x_2}L^2_{x_1}}\\ &\le C\,\|\partial_{x_2}u_1\|_{L^p_{x_2}L^p_{x_1}}^{1-\frac{2}{(2\alpha-1)p+2}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2_{x_2}L^2_{x_1}}^{\frac{2}{(2\alpha-1)p+2}}\,\|\Lambda_{x_2}^{\beta}\Lambda_{x_1}^{1-\beta}\theta\|_{L^2}^{1-\frac{1-q\beta}{q(\sigma_2-\beta)}}\\ &\quad\times\|\Lambda_{x_2}^{\sigma_2}\Lambda_{x_1}^{1-\beta}\theta\|_{L^2}^{\frac{1-q\beta}{q(\sigma_2-\beta)}}\\ &\le C\,\|\partial_{x_2}u_1\|_{L^p}^{1-\frac{2}{(2\alpha-1)p+2}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}^{\frac{2}{(2\alpha-1)p+2}}\,\|\nabla\theta\|_{L^2}^{1-\frac{1-q\beta}{q(\sigma_2-\beta)}}\\ &\quad\times\|\Lambda_{x_2}^{\sigma_2}\Lambda_{x_1}^{1-\beta}\theta\|_{L^2}^{\frac{1-q\beta}{q(\sigma_2-\beta)}}\end{aligned}$$

$$\leq C\,\|\omega\|_{L^p}^{1-\frac{2}{(2\alpha-1)p+2}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}^{\frac{2}{(2\alpha-1)p+2}}\,\|\nabla\theta\|_{L^2}^{1-\frac{1-q\beta}{q(\sigma_2-\beta)}}$$
$$\times\,\|\Lambda_{x_2}^{\sigma_2}\Lambda_{x_1}^{1-\beta}\theta\|_{L^2}^{\frac{1-q\beta}{q(\sigma_2-\beta)}}.$$

We further restrict our consideration to $\beta > \frac{1}{4}$ and set

$$\sigma_2 = 2\beta.$$

It then follows from Young's inequality that

$$K_{32} \leq \frac{1}{16}\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + C\,\|\theta_0\|_{L^\infty}^2\left(\|\partial_{x_2}u_1\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}^2\right)$$
$$+\,C\,\|\omega\|_{L^p}^{\frac{2q\beta}{2q\beta-1}\cdot\frac{(2\alpha-1)p}{(2\alpha-1)p+2}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\|_{L^2}^{\frac{2q\beta}{2q\beta-1}\cdot\frac{2}{(2\alpha-1)p+2}}\,\|\nabla\theta\|_{L^2}^2.$$

In order to close the energy estimate, we need

$$\frac{2q\beta}{2q\beta-1}\cdot\frac{2}{(2\alpha-1)p+2} \leq 2,$$

which, due to $q = \frac{(2\alpha-1)p+2}{2\alpha}$, is reduced to

$$q \geq \frac{\alpha+\beta}{2\alpha\beta} \qquad \text{or} \qquad p \geq \frac{\alpha-\beta}{(2\alpha-1)\beta}.$$

In addition, the condition that $q < \frac{1}{\beta}$ in terms of p is

$$p < \frac{2(\alpha-\beta)}{(2\alpha-1)\beta}.$$

Consequently, p should satisfy

$$\max\left\{2,\ \frac{\alpha-\beta}{(2\alpha-1)\beta}\right\} \leq p < \min\left\{2(\alpha+\beta),\ \frac{2(\alpha-\beta)}{(2\alpha-1)\beta}\right\}. \tag{3.25}$$

Since $\alpha+\beta > 1$, this condition is the same as

$$\frac{2(\alpha-\beta)}{(2\alpha-1)\beta} < 2(\alpha+\beta),$$

which is equivalent to

$$\beta > \frac{\sqrt{16\alpha^4-16\alpha^3+28\alpha^2-12\alpha+1}-4\alpha^2+2\alpha-1}{4(2\alpha-1)}$$

or

$$\beta > \beta_0 \equiv \frac{2\alpha}{\sqrt{16\alpha^4-16\alpha^3+28\alpha^2-12\alpha+1}+4\alpha^2-2\alpha+1}. \tag{3.26}$$

It is clear that, for $\alpha \geq \frac{1}{2}$,

$$\frac{\sqrt{\alpha^2+2}-\alpha}{2} > \frac{2\alpha}{\sqrt{16\alpha^4-16\alpha^3+28\alpha^2-12\alpha+1}+4\alpha^2-2\alpha+1}.$$

That is, (3.26) is less restrictive than (3.23). Inserting the estimates above in (3.21) yields

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 \le C(1 + \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \times \left(1 + \|\omega\|_{L^p}^{\frac{\alpha\beta(2\alpha-1)p}{\beta(2+(2\alpha-1)p)-\alpha}}\right)\|\nabla\theta\|_{L^2}^2,$$

where p satisfies (3.25). Gronwall's inequality then implies the desired global bound. This concludes the proof of Proposition 3.2. □

4. Higher regularity for the case $\alpha > \frac{1}{2}$ and $\beta > \beta_0$

This section proves Theorem 1.1 for the first case when

$$\frac{1}{2} < \alpha \le 1, \qquad \frac{1}{2} \ge \beta > \beta_0, \qquad \alpha + \beta > 1.$$

As we remarked before, it suffices to establish the global *a priori* bound for $\|(u,\theta)\|_{H^s}$. This is achieved in the following proposition.

Proposition 4.1. *Assume that $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. Assume α and β satisfy*

$$\frac{1}{2} < \alpha \le 1, \qquad \frac{1}{2} \ge \beta > \beta_0, \qquad \alpha + \beta > 1,$$

then the corresponding solution of (1.1) obeys the following global bound

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t (\|\Lambda_{x_1}^{\alpha} u(\tau)\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta}\theta(\tau)\|_{H^s}^2)\, d\tau < \infty.$$

To prove (4.1), we first derive a global H^2-bound for u.

Proposition 4.2. *Assume the conditions in Proposition 4.1. Then $\omega = \nabla \times u$ satisfies*

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha}\nabla\omega(\tau)\|_{L^2}^2\, d\tau \le C(t, u_0, \theta_0), \tag{4.1}$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

Proof of Proposition 4.2. Taking the gradient of the vorticity equation

$$\partial_t\omega + (u\cdot\nabla)\omega + \Lambda_{x_1}^{2\alpha}\omega = \partial_{x_1}\theta, \tag{4.2}$$

and dotting it with $\nabla\omega$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(\nabla u\cdot\nabla\omega)\cdot\nabla\omega\, dx - \int_{\mathbb{R}^2}\partial_{x_1}\theta\Delta\omega\, dx \\ &:= N_1 + N_2.\end{aligned} \tag{4.3}$$

N_1 is a quadratic form and can be explicitly written as

$$N_1 = -\int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1}\omega \partial_{x_1}\omega\, dx - \int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_2}\omega \partial_{x_1}\omega\, dx - \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_1}\omega \partial_{x_2}\omega\, dx$$
$$- \int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_2}\omega \partial_{x_2}\omega\, dx \ := N_{11} + N_{12} + N_{13} + N_{14}.$$

Applying (3.5) with $\gamma_1 = \gamma_2 = \alpha > \frac{1}{2}$ and Hölder's inequality, we obtain

$$\begin{aligned} N_{11},\ N_{12} &\le C\,\|\nabla\omega\|_{L^2}\,\|\partial_{x_1} u\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\partial_{x_1}\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_1}\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le C\,\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\,\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le C\,\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\,\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le \frac{1}{16}\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 + C\,\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{2}{4\alpha-1}}\,\|\nabla\omega\|_{L^2}^2, \end{aligned}$$

where in the third line we have applied the following estimate

$$\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\|_{L^2} = \big\||\xi_2|^{\alpha}|\xi_1|\widehat{u}(\xi)\big\|_{L^2} \le \big\||\xi_1|^{\alpha}|\xi|\widehat{u}(\xi)\big\|_{L^2} = \|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}. \tag{4.4}$$

Once again, the inequalities (3.5) and (4.4) entail

$$\begin{aligned} N_{13} &\le C\,\|\partial_{x_2}\omega\|_{L^2}\,\|\partial_{x_2} u_1\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2} u_1\|_{L^2}^{\frac{1}{2\alpha}}\,\|\partial_{x_1}\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_2}^{\alpha}\partial_{x_1}\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le C\,\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\,\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2} u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\Lambda_{x_2}^{\alpha}\partial_{x_1}\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le C\,\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\,\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le \frac{1}{16}\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 + C\,\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{2}{4\alpha-1}}\,\|\nabla\omega\|_{L^2}^2, \\ N_{14} &= \int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_2}\omega\partial_{x_2}\omega\, dx \\ &\le C\,\|\partial_{x_2}\omega\|_{L^2}\,\|\partial_{x_1} u_1\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u_1\|_{L^2}^{\frac{1}{2\alpha}}\,\|\partial_{x_2}\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\partial_{x_2}\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le C\,\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\,\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{1}{2\alpha}}\,\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^{\frac{1}{2\alpha}} \\ &\le \frac{1}{16}\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 + C\,\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}\,\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{2}{4\alpha-1}}\,\|\nabla\omega\|_{L^2}^2. \end{aligned}$$

By Hölder's inequality and the interpolation inequality (3.15),

$$\begin{aligned} |N_2| &\le C\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}\|\Lambda_{x_1}^{1-\alpha}\nabla\theta\|_{L^2} \\ &\le C\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^{\frac{1-\alpha}{\beta}}\,\|\nabla\theta\|_{L^2}^{\frac{\alpha+\beta-1}{\beta}} \\ &\le \frac{1}{16}\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \end{aligned} \tag{4.5}$$

Combining the estimates above and invoking the fact $\frac{2}{4\alpha-1} \le 2$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 \\
&\le C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{2}{4\alpha-1}}\|\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\
&\le C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}(1+\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2)\|\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Recalling the global bounds in (2.1), (3.1) and (3.3) and applying the Gronwall inequality lead to the global bound in (4.1). This concludes the proof of Lemma 4.2. □

Now we are ready to establish the global H^s-estimate.

Proof of Proposition 4.1. It follows from energy estimates that

$$\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \|\Lambda_{x_1}^{\alpha}u\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^2 \\
&\le C(1+\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2).
\end{aligned}$$

To bound $\|\nabla u\|_{L^\infty}$, we recall the interpolation inequality (see [13, Lemma A.2])

$$\|h\|_{L^\infty} \le C(\|h\|_{L^2} + \|\Lambda_{x_1}^{\delta_1}h\|_{L^2} + \|\Lambda_{x_2}^{\delta_2}h\|_{L^2}), \qquad \frac{1}{\delta_1}+\frac{1}{\delta_2} < 2, \tag{4.6}$$

which implies that

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C(\|\nabla u\|_{L^2} + \|\Lambda_{x_1}^{1+\alpha}\nabla u\|_{L^2} + \|\Lambda_{x_2}\nabla u\|_{L^2}) \\
&\le C(\|\omega\|_{L^2} + \|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2} + \|\nabla\omega\|_{L^2}).
\end{aligned}$$

The global H^2-bound in (4.1) then yields

$$\int_0^t \|\nabla u(s)\|_{L^\infty}^2\, ds < \infty.$$

This bound leads to a global bound for $\|\nabla\theta\|_{L^\infty}$. In fact, for any $q\in[1,\infty]$,

$$\|\nabla\theta\|_{L^q} \le \|\nabla\theta_0\|_{L^q}\exp\Big[\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big].$$

It is then clear that

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t (\|\Lambda_{x_1}^{\alpha}u(\tau)\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta}\theta(\tau)\|_{H^s}^2)\, d\tau < \infty.$$

This completes the proof of Proposition 4.1. □

5. Higher regularity for the case $\alpha+\beta\ge 1$ with $\beta > \frac{2+\sqrt{2}}{4}$

This section deals with the second case of the indices in Theorem 1.1. The goal is still to establish a global bound for $\|(u,\theta)\|_{H^s}$. This is done in the following proposition.

Proposition 5.1. *Assume that* $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$. *Assume* α *and* β *satisfy*

$$\alpha + \beta \geq 1, \qquad \beta > \frac{2+\sqrt{2}}{4}.$$

Then the corresponding solution of (1.1) obeys the following global bound

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t (\|\Lambda_{x_1}^{\alpha} u(\tau)\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta} \theta(\tau)\|_{H^s}^2)\, d\tau < \infty.$$

In order to prove this global bound, we first bound $\int_0^t \|\Lambda_{x_1}^{\delta}\theta(s)\|_{L^2}\, ds$ for δ as large as possible. As a matter of fact, we are able to prove the following result.

Proposition 5.2. *Assume* (u_0, θ_0) *satisfies the assumptions stated in Theorem 1.1. If* $\alpha + \beta \geq 1$ *and* $\beta > \frac{1}{2}$, *then there holds*

$$\|\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta(s)\|_{L^2}^2\, ds \leq C(t, u_0, \theta_0), \tag{5.1}$$

where $C(t, u_0, \theta_0)$ *is a constant depending on* t *and the initial data.*

Proof of Proposition 5.2. We apply the operator $\Lambda_{x_1}^{\beta+\frac{1}{2}}$ to the equation $(1.1)_2$, multiply the resultant by $\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta$ and integrate over $\mathbb{R}^2$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} \Lambda_{x_1}^{\beta+\frac{1}{2}}(u\cdot\nabla\theta)\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta\, dx \\
&\leq C\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2}\|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2} \\
&\leq \frac{1}{2}\|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2}^2 + C\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2}^2 + C\|u\|_{H^1}^2(\|\nabla\theta\|_{L^2}^2 + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2),
\end{aligned} \tag{5.2}$$

where we have applied the following estimate

$$\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2} \leq C\|u\|_{H^1}(\|\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}).$$

This inequality will be proven in the following Lemma 5.3. By (2.1) and (3.1), integrating (5.2) with respect to time t leads to the desired estimate

$$\|\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta(s)\|_{L^2}^2\, ds < \infty. \tag{5.3}$$

This concludes the proof of Proposition 5.2. $\square$

The bound in the following lemma has been used in the proof above. We provide a proof for this bound.

Lemma 5.3. *For any* $\beta > \frac{1}{2}$,

$$\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2} \le C\|u\|_{H^1}(\|\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}). \tag{5.4}$$

Proof of Lemma 5.3. For any fixed x_2, one has

$$\|(u\cdot\nabla\theta)(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}} \le C\|u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}}, \quad 0<\epsilon\le\beta-\frac{1}{2}.$$

The above inequality is an easy consequence of Littlewood–Paley technique. By the Bony decomposition, one gets

$$(u\cdot\nabla\theta)(.,x_2) = T_u\nabla\theta(.,x_2) + T_{\nabla\theta}u(.,x_2) + R(u,\nabla\theta)(.,x_2),$$

which together with the properties of the paraproduct operators T and R yields

$$\begin{aligned}
\|T_u\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}} &\le C\|u(.,x_2)\|_{\mathcal{B}^{-\epsilon}_{\infty,\infty}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}}\\
&\le C\|u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}},\\
\|T_{\nabla\theta}u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}} &\le C\|\nabla\theta(.,x_2)\|_{L^\infty}\|u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}}\\
&\le C\|u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}},\\
\|R(u,\nabla\theta)(.,x_2)(.,x_2)\|_{H_{x_1}^{\beta+\frac{1}{2}}} &\le C\|u(.,x_2)\|_{\mathcal{B}^{-\epsilon}_{\infty,\infty}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}}\\
&\le C\|u(.,x_2)\|_{H_{x_1}^{\frac{1}{2}}}\|\nabla\theta(.,x_2)\|_{H_{x_1}^{\frac{1}{2}+\epsilon}}.
\end{aligned}$$

Now we can show

$$\begin{aligned}
\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2} &\le C\|(u\cdot\nabla\theta)(x_1,x_2)\|_{L^2_{x_2}H_{x_1}^{\frac{1}{2}}}\\
&\le C\|u(.,x_2)\|_{L^\infty_{x_2}H_{x_1}^{\frac{1}{2}}}\|\nabla\theta(.,x_2)\|_{L^2_{x_2}H_{x_1}^{\frac{1}{2}+\epsilon}}\\
&\le C\|u\|_{H^1}\|\nabla\theta(.,x_2)\|_{L^2_{x_2}H_{x_1}^{\beta}}\\
&\le C\|u\|_{H^1}(\|\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}),
\end{aligned}$$

where in the third line we have used the fact $\epsilon \le \beta - \frac{1}{2}$, and

$$\|u(.,x_2)\|_{L^\infty_{x_2}H_{x_1}^{\frac{1}{2}}} \le C\|u\|_{H^1}. \tag{5.5}$$

(5.5) is a direct consequence of the following trace theorem

$$\|u(.,x_n)\|_{L^\infty_{x_n}H_{x'}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \le C\|u\|_{H^s(\mathbb{R}^n)}, \quad x' = (x_1,x_2,\cdots,x_{n-1}), \tag{5.6}$$

which can be proven as follows. We write

$$\begin{aligned}
u(x) &= u(x',x_n)\\
&= \int_{\mathbb{R}^n} e^{ix_n\xi_n}e^{ix'\cdot\xi'}\widehat{u}(\xi',\xi_n)\,d\xi' d\xi_n
\end{aligned}$$

$$= \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \Big(\int_{\mathbb{R}} e^{ix_n\xi_n} \widehat{u}(\xi', \xi_n) d\xi_n \Big) d\xi'.$$

One thus deduces

$$\widehat{u_h}(\xi', x_n) = \int_{\mathbb{R}} e^{ix_n\xi_n} \widehat{u}(\xi', \xi_n) d\xi_n,$$

where $\widehat{u_h}(\xi_h, x_n)$ denotes the Fourier transform of u in terms of the variable x'. Now it is easy to see that for any $s > \frac{1}{2}$

$$\begin{aligned}
|\widehat{u_h}(\xi', x_n)|^2 &\le C \int_{\mathbb{R}} |\widehat{u}(\xi', \xi_n)|^2 (1 + |\xi|^2)^s d\xi_n \int_{\mathbb{R}} (1 + |\xi|^2)^{-s} d\xi_n \\
&\le C \int_{\mathbb{R}} |\widehat{u}(\xi', \xi_n)|^2 (1 + |\xi|^2)^s d\xi_n \int_{\mathbb{R}} (1 + |\xi'|^2 + |\xi_n|^2)^{-s} d\xi_n \\
&\le C(1 + |\xi'|^2)^{-s+\frac{1}{2}} \int_{\mathbb{R}} |\widehat{u}(\xi', \xi_n)|^2 (1 + |\xi|^2)^s d\xi_n \int_0^\infty (1 + \tau^2)^{-s} d\tau \\
&\le C(1 + |\xi'|^2)^{-s+\frac{1}{2}} \int_{\mathbb{R}} |\widehat{u}(\xi', \xi_n)|^2 (1 + |\xi|^2)^s d\xi_n,
\end{aligned}$$

where in the third we have used the change of variable $|\xi_n| = \sqrt{1 + |\xi'|^2}\tau$. Multiplying the above estimate by $(1 + |\xi'|^2)^{s-\frac{1}{2}}$ and integrating the resultant over $\mathbb{R}^{n-1}$, we immediately obtain

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{s-\frac{1}{2}} |\widehat{u_h}(\xi', x_n)|^2 d\xi' &\le C \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |\widehat{u}(\xi', \xi_n)|^2 (1 + |\xi|^2)^s d\xi_n d\xi' \\
&= C \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi,
\end{aligned}$$

which implies

$$\|u(., x_n)\|_{L^\infty_{x_n} H^{s-\frac{1}{2}}_{x'}(\mathbb{R}^{n-1})} \le C\|u\|_{H^s(\mathbb{R}^n)}.$$

This completes the proof of Lemma 5.3. □

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. It follows from standard energy estimates that, for all $s > 0$,

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{H^s}^2 + \|\Lambda_{x_1}^\beta \theta\|_{H^s}^2 \le C(\|\nabla u\|_{L^\infty}\|\theta\|_{H^s}^2 + \|\theta\|_{L^\infty}\|\omega\|_{H^s}\|\theta\|_{H^s}). \tag{5.7}$$

According to (4.6), if β satisfies

$$\frac{1}{\beta} + \frac{1}{2\beta - \frac{1}{2}} < 2 \quad \text{or} \quad \beta > \frac{2 + \sqrt{2}}{4},$$

then

$$\begin{aligned}\|\partial_{x_1}\theta\|_{L^\infty} &\le C(\|\partial_{x_1}\theta\|_{L^2} + \|\Lambda_{x_2}^{\beta}\partial_{x_1}\theta\|_{L^2} + \|\Lambda_{x_1}^{2\beta-\frac{1}{2}}\partial_{x_1}\theta\|_{L^2})\\ &\le C(\|\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2}).\end{aligned}$$

Consequently, due to the global bounds in (3.1) and (5.1),

$$\int_0^t \|\partial_{x_1}\theta(s)\|_{L^\infty}^2\, ds < \infty.$$

It then follows from the vorticity equation

$$\partial_t\omega + (u\cdot\nabla)\omega + \Lambda_{x_1}^{2\alpha}\omega = \partial_{x_1}\theta$$

that, for any $2 \le q \le \infty$,

$$\|\omega(t)\|_{L^q} < \infty.$$

According to the classical estimate on Calderón–Zygmund operators (see, e.g., [9, Theorem 3.1.1]),

$$\|\nabla u(t)\|_{L^q} \le Cq\|\omega(t)\|_{L^q} \le Cq\|\omega(t)\|_{L^2\cap L^\infty}, \quad \forall q\in[2,\infty),$$

and thus

$$\sup_{q\ge 2}\frac{\|\nabla u(t)\|_{L^q}}{q} \le C\|\omega(t)\|_{L^2\cap L^\infty} < \infty. \tag{5.8}$$

Applying standard energy estimates on the vorticity equation yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{H^s}^2 + \|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}^2\\ &\le C\|\nabla u\|_{L^\infty}\|\omega\|_{H^s}^2 + \|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}\|\Lambda_{x_1}^{1-\alpha}\theta\|_{H^s}\\ &\le C\|\nabla u\|_{L^\infty}\|\omega\|_{H^s}^2 + \|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}\|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^{\frac{1-\alpha}{\beta}}\|\theta\|_{H^s}^{\frac{\alpha+\beta-1}{\beta}}\\ &\le \frac{1}{2}\|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}^2 + \frac{1}{2}\|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^2 + C(1+\|\nabla u\|_{L^\infty})(\|\omega\|_{H^s}^2 + \|\theta\|_{H^s}^2),\end{aligned} \tag{5.9}$$

Adding (5.7) and (5.9) leads to

$$\begin{aligned}&\frac{d}{dt}(\|\theta(t)\|_{H^s}^2 + \|\omega(t)\|_{H^s}^2) + \|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^2 + \|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}^2\\ &\le C(1+\|\theta\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty})(\|\omega\|_{H^s}^2 + \|\theta\|_{H^s}^2).\end{aligned} \tag{5.10}$$

To control $\|\nabla u\|_{L^\infty}$, we use the logarithmic Sobolev interpolation inequality

$$\|\nabla u\|_{L^\infty} \le C + C\Big(\sup_{q\ge 2}\frac{\|\nabla u\|_{L^q}}{q}\Big)\ln\Big(e + \|\omega\|_{H^\sigma}\Big), \qquad \forall\sigma > 1. \tag{5.11}$$

Inserting (5.11) in (5.10) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\theta(t)\|_{H^s}^2+\|\omega(t)\|_{H^s}^2)+\|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^2+\|\Lambda_{x_1}^{\alpha}\omega\|_{H^s}^2\\
&\le C\left(1+\|\theta_0\|_{L^\infty}^2+\Big(\sup_{q\ge 2}\frac{\|\nabla u\|_{L^q}}{q}\Big)\ln\Big(e+\|\omega\|_{H^s}\Big)\right)(\|\omega\|_{H^s}^2+\|\theta\|_{H^s}^2)\\
&\le C\left(1+\Big(\sup_{q\ge 2}\frac{\|\nabla u\|_{L^q}}{q}\Big)\right)\ln\Big(e+\|\omega\|_{H^s}^2+\|\theta\|_{H^s}^2\Big)(\|\omega\|_{H^s}^2+\|\theta\|_{H^s}^2).
\end{aligned}$$

Thanks to (5.8) and the Osgood inequality, we obtain the desired global H^s-bound. (5.11) is consequence of the high-low frequency techniques. By the Littlewood–Paley decomposition,

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le \|S_N\nabla u\|_{L^\infty}+\sum_{j\ge N}\|\Delta_j\nabla u\|_{L^\infty}\\
&\le C2^{\frac{2N}{N}}\|S_N\nabla u\|_{L^N}+C\sum_{j\ge N}2^j\|\Delta_j\nabla u\|_{L^2}\\
&\le CN\frac{\|S_N\nabla u\|_{L^N}}{N}+C\sum_{j\ge N}2^{j(1-\sigma)}2^{j\sigma}\|\Delta_j\omega\|_{L^2}\\
&\le CN\sup_{q\ge 2}\frac{\|\nabla u\|_{L^q}}{q}+C2^{N(1-\sigma)}\|\omega\|_{H^\sigma}\qquad(\sigma>1)\\
&\le C+C\Big(\sup_{q\ge 2}\frac{\|\nabla u\|_{L^q}}{q}\Big)\ln\Big(e+\|\omega\|_{H^\sigma}\Big),
\end{aligned}\tag{5.12}$$

where in the last inequality we have selected

$$N=\left[\frac{\ln\Big(e+\|\omega\|_{H^\sigma}\Big)}{(\sigma-1)\ln 2}\right]+1.$$

This concludes the proof of Proposition 5.1. □

Acknowledgements

The authors sincerely wish to express their thanks to the anonymous referees and the associated editor for their careful reading and also their useful and constructive comments, which improve the presentation of this paper. J. Wu was supported by NSF grant DMS 1614246, and by the AT&T Foundation at Oklahoma State University and by NNSFC (No. 11471103, a grant awarded to Professor B. Yuan). X. Xu was partially supported by the National Natural Science Foundation of China (No. 11371059; No. 11471220; No. 11771045). Z. Ye was supported by the Foundation of Jiangsu Normal University (No. 16XLR029), the Natural Science Foundation of Jiangsu Province (No. BK20170224), the National Natural Science Foundation of China (No. 11701232).

Appendix A. Besov spaces and several inequalities

In this section, we show some common notations about the Besov spaces and several inequalities. Now let us begin with the Littlewood–Paley theory (see for instance [4]). We choose some smooth radial non increasing function χ with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B}:=\{\xi\in\mathbb{R}^n,|\xi|\le\frac{4}{3}\}$ and with value 1 on $\{\xi\in\mathbb{R}^n,|\xi|\le\frac{3}{4}\}$, then we set $\varphi(\xi)=\chi\big(\frac{\xi}{2}\big)-\chi(\xi)$. One easily verifies that $\varphi\in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^n,\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and satisfy

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u = 0, \quad j \leq -2; \qquad \Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^n} \widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u = \chi(2^{-j}D)u = \sum_{-1\leq k\leq j-1} \Delta_k u = 2^{jn} \int_{\mathbb{R}^n} \widetilde{h}(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

Meanwhile, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{Z}.$$

We denote the function spaces of rapidly decreasing functions by $S(\mathbb{R}^n)$, tempered distributions by $S'(\mathbb{R}^n)$, and polynomials by $\mathcal{P}(\mathbb{R}^n)$. Let us now recall the definition of homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

Definition A.1. Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\dot{B}^s_{p,r} = \{f \in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n); \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\|\dot{\Delta}_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & \forall\, r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}, & \forall\, r = \infty. \end{cases}$$

Definition A.2. Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r} = \{f \in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \Big(\sum_{j\geq -1} 2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & \forall\, r < \infty, \\ \sup_{j\geq -1} 2^{js}\|\Delta_j f\|_{L^p}, & \forall\, r = \infty. \end{cases}$$

For $s > 0, (p, r) \in [1, +\infty]^2$, the inhomogeneous Besov space norm $B^s_{p,r}$ is equal to

$$\|f\|_{B^s_{p,r}} \approx \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,r}}. \tag{A.1}$$

An another equivalent norm of the homogeneous Besov space $\dot{B}^s_{p,q}$ with $s \in (0,1)$ is given by

$$\|f\|_{\dot{B}^s_{p,q}} \approx \left[\int_{\mathbb{R}^d} \frac{\|f(x+\cdot) - f(\cdot)\|^q_{L^p(\mathbb{R}^d)}}{|x|^{d+sq}} \, dx \right]^{\frac{1}{q}} \tag{A.2}$$

for any $f \in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

Next, we introduce the Bernstein lemma which is fundamental in the analysis involving Besov spaces.

Lemma A.3 *(see [4]). Let $\alpha \geq 0$, $j \geq 0$ and $1 \leq a \leq b \leq \infty$, then there exists a constant $C = C(\alpha, a, b)$ such that the following inequalities hold*

$$\|\Lambda^\alpha S_j f\|_{L^b} \leq C\, 2^{j\alpha + jn(\frac{1}{a} - \frac{1}{b})} \|S_j f\|_{L^a},$$
$$C^{-1}\, 2^{j\alpha} \|\Delta_j f\|_{L^b} \leq \|\Lambda^\alpha \Delta_j f\|_{L^b} \leq C\, 2^{j\alpha + jn(\frac{1}{a} - \frac{1}{b})} \|\Delta_j f\|_{L^a}.$$

Now let us state the following generalized Kato–Ponce inequality, which was used in the proof of (2.13).

Lemma A.4. *Let $0 < \kappa < \sigma < 1$, $2 \leq m < \infty$, and $p, q, r \in (1, \infty)^3$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, there exists $C = C(\kappa, \sigma, m, p, q, r)$ such that*

$$\big\||f|^{m-2} f\big\|_{L^p} + \|\Lambda^\kappa(|f|^{m-2} f)\|_{L^p} \leq C \|f\|_{B^\sigma_{q,\,p}} \|f\|^{m-2}_{L^{r(m-2)}}. \tag{A.3}$$

Proof of Lemma A.4. Lemma A.4 was established in [14, Lemma 2.5]. For the convenience of readers, we restate it here. For $0 < \kappa < \sigma$ and $p, \tilde{p} \in [1, \infty]^2$, one has

$$\|g\|_{L^p} + \|\Lambda^\kappa g\|_{L^p} \leq C\, \|g\|_{B^\sigma_{p,p}} \equiv C\, (\|g\|_{L^p} + \|g\|_{\dot{B}^\sigma_{p,p}}). \tag{A.4}$$

Actually, Lemma A.3 ensures

$$\begin{aligned} \|\Lambda^\kappa g\|_{L^p} &\leq \sum_{k \geq -1} \|\Lambda^\kappa \Delta_k g\|_{L^p} = \|\Lambda^\kappa \Delta_{-1} g\|_{L^p} + \sum_{k \geq 0} \|\Lambda^\kappa \Delta_k g\|_{L^p} \\ &\leq C\, \|g\|_{L^p} + \sum_{k \geq 0} 2^{-(\sigma - \kappa)k} \|\Lambda^\sigma \Delta_k g\|_{L^p} \\ &\leq C\, \|g\|_{L^p} + C\, \|g\|_{\dot{B}^\sigma_{p,p}}. \end{aligned}$$

By the equivalence definition of $\dot{B}^\sigma_{p,p}$ in (A.2), it gives

$$\|\Lambda^\kappa(|f|^{m-2} f)\|^p_{L^p} \leq C\, \big\||f|^{m-2} f\big\|^p_{L^p} + C \int_{\mathbb{R}^n} \frac{\||f|^{m-2} f(x+\cdot) - |f|^{m-2} f(\cdot)\|^p_{L^p}}{|x|^{n+\sigma p}} \, dx.$$

According to the Hölder inequality, we have

$$\big\||f|^{m-2} f\big\|^p_{L^p} \leq \|f\|^p_{L^q} \||f|^{m-2}\|^p_{L^r} = \|f\|^p_{L^q} \|f\|^{p(m-2)}_{L^{r(m-2)}}.$$

Due to the simple inequality

$$\Big||a|^{m-2}a-|b|^{m-2}b\Big|\le C(m)|a-b|(|a|^{m-2}+|b|^{m-2})$$

and Hölder's inequality,

$$\begin{aligned}\||f|^{m-2}f(x+\cdot)-|f|^{m-2}f(\cdot)\|_{L^p}&\le C\|f(x+\cdot)-f(\cdot)\|_{L^q}\||f|^{m-2}\|_{L^r}\\&\le C\|f(x+\cdot)-f(\cdot)\|_{L^q}\|f\|^{m-2}_{L^{r(m-2)}}.\end{aligned}$$

We therefore obtain

$$\begin{aligned}\|\Lambda^\kappa(|f|^{m-2}f)\|^p_{L^p}&\le C\,\|f\|^p_{L^q}\|f\|^{p(m-2)}_{L^{r(m-2)}}+C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\int\limits_{\mathbb{R}^n}\frac{\|f(x+\cdot)-f(\cdot)\|^p_{L^q}}{|x|^{n+\sigma p}}\,dx\\&\le C\,\|f\|^p_{L^q}\|f\|^{p(m-2)}_{L^{r(m-2)}}+C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\|f\|^p_{\dot B^\sigma_{q,\,p}}\\&\le C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\|f\|^p_{B^\sigma_{q,\,p}}.\end{aligned}$$

This ends the proof of (A.3). □

Proof of (3.15). For example, we just consider $i=1$. Simple computations yield

$$\begin{aligned}\|\Lambda^\gamma_{x_1}f\|_{L^2}&=\left(\int\limits_{\mathbb{R}}\|\Lambda^\gamma_{x_1}f(.,x_2)\|^2_{L^2_{x_1}}\,dx_2\right)^{\frac12}\\&\le C\left(\int\limits_{\mathbb{R}}\|f(.,x_2)\|^{2(1-\frac{\gamma}{\varrho})}_{L^2_{x_1}}\|\Lambda^\varrho_{x_1}f(.,x_2)\|^{\frac{2\gamma}{\varrho}}_{L^2_{x_1}}\,dx_2\right)^{\frac12}\\&\le C\left(\int\limits_{\mathbb{R}}\|f(.,x_2)\|^2_{L^2_{x_1}}\,dx_2\right)^{\frac{\varrho-\gamma}{2\varrho}}\left(\int\limits_{\mathbb{R}}\|\Lambda^\varrho_{x_1}f(.,x_2)\|^2_{L^2_{x_1}}\,dx_2\right)^{\frac{\gamma}{2\varrho}}\\&=C\|f\|^{1-\frac{\gamma}{\varrho}}_{L^2}\|\Lambda^\varrho_{x_1}f\|^{\frac{\gamma}{\varrho}}_{L^2},\end{aligned}$$

which is (3.15) with the case $i=1$.

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