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Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium

Jiahong Wu [a], Yi Zhu [b,*]

[a] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States

[b] Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China

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ABSTRACT

This paper focuses on the 3D incompressible magnetohydrodynamic (MHD) equations with mixed partial dissipation and magnetic diffusion. Our main result assesses the global stability of perturbations near the steady solution given by a background magnetic field. The stability problem on the MHD equations with partial or no dissipation has attracted considerable interests recently and there are substantial developments. The new stability result presented here is among the very few stability conclusions currently available for ideal or partially dissipated MHD equations. As a special consequence of the techniques introduced in this paper, we obtain the small data global well-posedness for the 3D incompressible Navier-Stokes equations without vertical dissipation.

* Corresponding author.
E-mail addresses: jiahong.wu@okstate.edu (J. Wu), zhuyim@ecust.edu.cn (Y. Zhu).

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1. Introduction

The magnetohydrodynamic (MHD) equations reflect the basic physics laws governing the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. The velocity field obeys the Navier-Stokes equations with Lorentz forcing generated by the magnetic field while the magnetic field satisfies the Maxwell's equations of electromagnetism. The MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [3,14,36]).

The MHD equations are also mathematically significant. The MHD equations share similarities with the Navier-Stokes equations, but they contain much richer structures than the Navier-Stokes equations. They are not merely a combination of two parallel Navier-Stokes type equations but an interactive and integrated system. Their distinctive features make analytic studies a great challenge but offer new opportunities.

Two fundamental problems on the MHD equations have recently attracted considerable interests. The first is the existence and uniqueness of solutions while the second concerns the stability of perturbations near physically relevant equilibrium. There have been substantial developments on these problems, especially on those MHD systems with only partial or fractional dissipation.

This paper focuses on a stability problem concerning the following 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u - \partial_1^2 u - \partial_2^2 u + \nabla P = B\cdot\nabla B, \\ \partial_t B + u\cdot\nabla B - \partial_3^2 B = B\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

where u represents the velocity field, P the total pressure and B the magnetic field. We provide some relevant physical backgrounds for the MHD system in (1.1). The Navier-Stokes equations with anisotropic viscous dissipation arise in several physical circumstances. It can model the turbulent diffusion of rotating fluids in Ekman layers. A standard reference is Chapter 4 of Pedlosky's book [34]. In addition, anisotropic viscous dissipation also arises in the modeling of reconnecting plasmas (see, e.g., [12,13]). When the resistivity of electrically conducting fluids such as certain plasmas and liquid metal is anisotropic and only in the vertical direction, the vertical magnetic diffusion may be relevant (see, e.g., [35]).

It is clear that a special solution of (1.1) is given by the zero velocity field and the background magnetic field $B^{(0)} = e_1$, where $e_1 = (1,0,0)$. The perturbation (u,b) around this equilibrium with $b = B - e_1$ obeys

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta_h u + \nabla P = b\cdot\nabla b + \partial_1 b, \\ \partial_t b + u\cdot\nabla b - \partial_3^2 b = b\cdot\nabla u + \partial_1 u, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1.2}$$

where, for notational convenience, we have written

$$\Delta_h = \partial_1^2 + \partial_2^2$$

and we shall also write $\nabla_h = (\partial_1, \partial_2)$.

This paper aims at the stability problem on the perturbation (u, b). Equivalently, we establish a small data global well-posedness result for (1.2) supplemented with the initial condition

$$u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x).$$

Our main result can be stated as follows. The notation $A \lesssim D$ means $A \le C\, D$ for a pure constant C.

Theorem 1.1. *Consider (1.2) with the initial data* $(u_0, b_0) \in H^3(\mathbb{R}^3)$ *and* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. *Then there exists a constant* $\epsilon > 0$ *such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \epsilon,$$

then (1.2) *has a unique global classical solution* (u, b) *satisfying, for any* $t > 0$,

$$\|u(t)\|_{H^3} + \|b(t)\|_{H^3} + \int_0^t \left(\|\nabla_h u\|_{H^3}^2 + \|\partial_3 b\|_{H^3}^2 + \|\partial_1 b\|_{H^2}^2\right) d\tau \lesssim \epsilon.$$

This new result constitutes an important contribution to the stability problem on the MHD equations. Prior to this stability result, we only know the stability of the background magnetic field for two cases, the ideal MHD equations and the MHD equations with kinematic dissipation and no magnetic diffusion. The nonlinear stability for the ideal MHD equations was established in several beautiful papers [2,6,22,33,41]. The stability problem for the MHD equations with no magnetic diffusion was first studied in [31], which inspired many further investigations. The stability has now been successfully established by several authors via different approaches (see, e.g., [1,15,23–25,31,33,37–39,43,44,51]). To give a more complete view of current studies on the stability and the global regularity problems, we also mention some of the other exciting results in [9,16,17,19–21,26,28,29,42,45–50,52] and the references therein.

A special consequence of Theorem 1.1 and its proof is the stability or small data global well-posedness of the 3D Navier-Stokes equations with only horizontal dissipation. It is not clear if the stability for the 3D Navier-Stokes still holds if there is only one directional dissipation (say, in x_1 or x_2 direction, but not both). The 3D Navier-Stokes equations with full dissipation have small data global well-posedness while the 3D incompressible Euler equations are ill-posed and have norm inflation in any Sobolev space H^k or C^k for any positive integer k [4,5,18].

The proof of Theorem 1.1 is not trivial. A natural starting point is to bound $\|u(t)\|_{H^3} + \|b(t)\|_{H^3}$ via the energy estimates. However, due to the lack of the vertical dissipation and the horizontal magnetic diffusion, some of the nonlinear terms can not be controlled in terms of $\|u(t)\|_{H^3} + \|b(t)\|_{H^3}$ or the dissipative parts $\|\nabla_h u\|_{H^3}$ and $\|\partial_3 b\|_{H^3}$. Consequently we are not able to obtain a closed differential inequality for

$$E_0(t) = \sup_{0\le\tau\le t} \left\{ \|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2 \right\} + 2\int_0^t \|\nabla_h u(\tau)\|_{H^3}^2 + \|\partial_3 b(\tau)\|_{H^3}^2 \, d\tau.$$

This forces us to include suitable extra terms in the energy estimates. We discover that the following term

$$E_1(t) = \int_0^t \|\partial_1 b(\tau)\|_{H^2}^2 \, d\tau$$

serves our purpose perfectly. All nonlinear terms involved in the estimates of $E_0(t)$ can be bounded in terms of $E_0(t)$ and $E_1(t)$. The selection of this term is based on the structure of (1.2) and through trial and error. We remark that the process of estimating $E_0(t)$ involves many terms and is very lengthy. Even with the combination of $E_0(t)$ and $E_1(t)$, it is still very difficulty to directly bound some of the nonlinear terms. Two of the most difficult ones are

$$\int_{\mathbb{R}^3} \partial_1 u_1 \, \partial_2^3 b_1 \, \partial_2^3 b_3 \, dx \quad \text{and} \quad \int_{\mathbb{R}^3} \partial_1 u_3 \, \partial_2^3 b_1 \, \partial_2^3 b_3 \, dx. \tag{1.3}$$

It does not appear possible to bound them directly in terms of $E_0(t)$ and $E_1(t)$. Our strategy is to make use of the special structure of the equation for b in (1.2) and replace $\partial_1 u_1$ and $\partial_1 u_3$ in (1.3) via the equation of b,

$$\partial_1 u = \partial_t b + u \cdot \nabla b - \partial_3^2 b - b \cdot \nabla u. \tag{1.4}$$

Substituting (1.4) in (1.3) generates more terms, but fortunately all the resulting terms can be bounded suitably by $E_0(t)$ and $E_1(t)$.

In addition, in order to make most efficient usage of the anisotropic dissipation, we employ extensively the following anisotropic bounds in the estimates of the nonlinear terms. These anisotropic bounds are extremely powerful in the study of global regularity and stability problems on partial differential equations with only partial dissipation. Similar inequalities have previously been used in the investigation of partially dissipated 2D MHD systems and related equations (see, e.g., [7,8]).

Lemma 1.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\int_{\mathbb{R}^3} |fgh|dx \lesssim \|f\|_{L^2}^{\frac12}\|\partial_1 f\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}^{\frac12}\|\partial_3 h\|_{L^2}^{\frac12},$$

$$\int_{\mathbb{R}^3} |fghv|dx \lesssim \|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac14}\|\partial_1 g\|_{L^2}^{\frac14}\|\partial_2 g\|_{L^2}^{\frac14}\|\partial_1\partial_2 g\|_{L^2}^{\frac14}$$
$$\cdot \|h\|_{L^2}^{\frac12}\|\partial_3 h\|_{L^2}^{\frac12}\|v\|_{L^2}^{\frac12}\|\partial_3 v\|_{L^2}^{\frac12},$$

$$\left(\int_{\mathbb{R}^3} |fgh|^2 dx\right)^{\frac12} \lesssim \|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{H^2},$$

$$\int_{\mathbb{R}^3} |fgh|dx \lesssim \|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}.$$

Combining all aforementioned ingredients, we are able to drive the following energy inequalities

$$E_0(t) \lesssim E_0(0) + E_0(0)^{\frac32} + E_0(t)^{\frac32} + E_1(t)^{\frac32} + E_0(t)^2 + E_1(t)^2 \tag{1.5}$$

and

$$E_1(t) \lesssim E_0(0) + E_0(t) + E_0(t)^{\frac32} + E_1(t)^{\frac32}. \tag{1.6}$$

These inequalities, combined with the bootstrapping argument, allow us to prove Theorem 1.1.

We remark that many important results on the stability problem concerning the 3D anisotropic Navier-Stokes equations with horizontal dissipation have been obtained (see, e.g., [10,11,27,32]). Since the equation of the magnetic field involves only vertical dissipation, the situation here is different from two parallel Navier-Stokes equations with horizontal dissipation. In fact, the stability problem on the 3D Navier-Stokes equation with only vertical dissipation remains an outstanding open problem. The stability on the MHD system studied here is only made possible by fully exploiting the stabilizing effects of the background magnetic field and making use of the hidden structure in the MHD system.

The rest of this paper is divided into three sections. Section 2 provides the proofs of Theorem 1.1 and of Lemma 1.2. Section 3 derives the energy inequality (1.5) while Section 4 proves (1.6).

2. Proofs of Theorem 1.1 and Lemma 1.2

This section proves Theorem 1.1 and Lemma 1.2.

Proof of Theorem 1.1. We employ the bootstrapping argument (see, e.g., [40, p.20]). It follows from (1.5) and (1.6) that

$$E_0(t)+E_1(t) \lesssim E_0(0)+E_0(0)^{\frac{3}{2}}+E_0(t)^{\frac{3}{2}}+E_1(t)^{\frac{3}{2}}+E_0(t)^2+E_1(t)^2$$

or, for some pure constants C_0, C_1 and C_2,

$$\begin{aligned} E_0(t)+E_1(t) \le\, & C_0(E_0(0)+E_0(0)^{\frac{3}{2}})+C_1(E_0(t)^{\frac{3}{2}}+E_1(t)^{\frac{3}{2}}) \\ & +C_2(E_0(t)^2+E_1(t)^2). \end{aligned} \tag{2.1}$$

To initiate the bootstrapping argument, we make the ansatz

$$E_0(t)+E_1(t) \le \min\left\{\frac{1}{16C_1^2}, \frac{1}{4C_2}\right\}. \tag{2.2}$$

We then show that (2.1) allows us to conclude that $E_0(t)+E_1(t)$ actually admits an even smaller bound by taking the initial H^3-norm $E_0(0)$ sufficiently small. In fact, when (2.2) holds, (2.1) implies

$$\begin{aligned} E_0(t)+E_1(t) &\le C_0(E_0(0)+E_0(0)^{\frac{3}{2}})+C_1\sqrt{E_0+E_1}(E_0(t)+E_1(t)) \\ &\quad +C_2(E_0(t)+E_1(t))(E_0(t)+E_1(t)) \\ &\le C_0(E_0(0)+E_0(0)^{\frac{3}{2}})+\frac{1}{2}(E_0(t)+E_1(t)) \end{aligned}$$

or

$$E_0(t)+E_1(t) \le 2C_0(E_0(0)+E_0(0)^{\frac{3}{2}}). \tag{2.3}$$

Therefore, if we take $E_0(0)$ sufficiently small such that

$$2C_0(E_0(0)+E_0(0)^{\frac{3}{2}}) < \min\left\{\frac{1}{16C_1^2}, \frac{1}{4C_2}\right\}, \tag{2.4}$$

then $E_0(t)+E_1(t)$ actually admits an smaller bound in (2.3) than the one in the ansatz (2.2). The bootstrapping argument then assesses that (2.3) holds for all time when $E_0(0)$ obeys (2.4). This completes the proof. □

Next we prove Lemma 1.2. A simple fact to be used in the proof is the following version of Minkowski's inequality, for any $1 \le q \le p \le \infty$,

$$\|\|f\|_{L_y^q(\mathbb{R}^n)}\|_{L_x^p(\mathbb{R}^m)} \le \|\|f\|_{L_x^p(\mathbb{R}^m)}\|_{L_y^q(\mathbb{R}^n)},$$

where $f=f(x,y)$ with $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$ is a measurable function on $\mathbb{R}^m\times\mathbb{R}^n$. A more general version of Minkowski's inequality and its proof can be found in [30].

Proof of Lemma 1.2. The proof makes use of the following basic one-dimensional inequality, for $f \in H^1(\mathbb{R})$,

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\, \|f\|_{L^2(\mathbb{R})}^{\frac12} \, \|f'\|_{L^2(\mathbb{R})}^{\frac12}. \tag{2.5}$$

By Hölder's inequality and Minkowski's inequality,

$$\begin{aligned}
\int\limits_{\mathbb{R}^3} |fgh|\, dx &\le \|f\|_{L^2_{x_3} L^2_{x_2} L^\infty_{x_1}} \, \|g\|_{L^2_{x_3} L^\infty_{x_2} L^2_{x_1}} \, \|h\|_{L^\infty_{x_3} L^2_{x_2} L^2_{x_1}} \\
&\le \|f\|_{L^2_{x_3} L^2_{x_2} L^\infty_{x_1}} \, \|g\|_{L^2_{x_3} L^2_{x_1} L^\infty_{x_2}} \, \|h\|_{L^2_{x_2} L^2_{x_1} L^\infty_{x_3}} \\
&\le 2^{\frac32} \left\| \|f\|_{L^2_{x_1}}^{\frac12} \, \|\partial_1 f\|_{L^2_{x_1}}^{\frac12} \right\|_{L^2_{x_2 x_3}} \left\| \|g\|_{L^2_{x_2}}^{\frac12} \, \|\partial_2 g\|_{L^2_{x_2}}^{\frac12} \right\|_{L^2_{x_1 x_3}} \\
&\quad \times \left\| \|h\|_{L^2_{x_3}}^{\frac12} \, \|\partial_3 h\|_{L^2_{x_3}}^{\frac12} \right\|_{L^2_{x_1 x_2}} \\
&\le 2^{\frac32} \, \|f\|_{L^2}^{\frac12} \, \|\partial_1 f\|_{L^2}^{\frac12} \, \|g\|_{L^2}^{\frac12} \, \|\partial_2 g\|_{L^2}^{\frac12} \, \|h\|_{L^2}^{\frac12} \, \|\partial_3 h\|_{L^2}^{\frac12}.
\end{aligned}$$

Here $\|f\|_{L^2_{x_3} L^2_{x_2} L^\infty_{x_1}}$ represents the L^∞-norm in the x_1-variable, followed by the L^2-norm in x_2 and the L^2-norm in x_3. This finishes the proof of the first inequality. The proof of the second inequality is very similar. In fact, by Hölder's inequality and Minkowski's inequality,

$$\int |fghv|\, dx \le \|f\|_{L^2_{x_3} L^\infty_{x_1} L^\infty_{x_2}} \, \|g\|_{L^2_{x_3} L^\infty_{x_1} L^\infty_{x_2}} \, \|h\|_{L^2_{x_1 x_2} L^\infty_{x_3}} \, \|v\|_{L^2_{x_1 x_2} L^\infty_{x_3}}.$$

By (2.5) and Hölder's inequality,

$$\begin{aligned}
\|f\|_{L^2_{x_3} L^\infty_{x_1} L^\infty_{x_2}} &\le 2^{\frac12} \left\| \|f\|_{L^2_{x_2}}^{\frac12} \, \|\partial_1 f\|_{L^2_{x_2}}^{\frac12} \right\|_{L^2_{x_3} L^\infty_{x_1}} \\
&\le 2^{\frac12} \left\| \|f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2 x_3}}^{\frac12} \left\| \|\partial_2 f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2 x_3}}^{\frac12} \\
&\le 2^{\frac32} \, \|f\|_{L^2}^{\frac14} \, \|\partial_1 f\|_{L^2}^{\frac14} \, \|\partial_2 f\|_{L^2}^{\frac14} \, \|\partial_1 \partial_2 f\|_{L^2}^{\frac14}.
\end{aligned}$$

$\|g\|_{L^2_{x_3} L^\infty_{x_1} L^\infty_{x_2}}$ obeys a similar bound. $\|h\|_{L^2_{x_1 x_2} L^\infty_{x_3}}$ and $\|v\|_{L^2_{x_1 x_2} L^\infty_{x_3}}$ can be estimated as in the proof of the first inequality. Combining all these estimates leads to the desired second inequality in Lemma 1.2. The other two inequalities are obtained similarly. This completes the proof of Lemma 1.2. □

3. Proof of (1.5)

This section proves (1.5), namely

$$E_0(t) \lesssim E_0(0) + E_0(0)^{\frac32} + E_0(t)^{\frac32} + E_1(t)^{\frac32} + E_0(t)^2 + E_1(t)^2.$$

The proof of this inequality is very lengthy and involves the estimates of many terms.

Proof of (1.5). Due to the equivalence of $\|(u,b)\|_{H^3}$ with $\|(u,b)\|_{L^2}+\|(u,b)\|_{\dot{H}^3}$, it suffices to bound the L^2 and the homogeneous $\dot{H}^3$-norm of (u,b). By a simple energy estimate and $\nabla\cdot u=\nabla\cdot b=0$, we find that the L^2-norm of (u,b) obeys

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+2\int_0^t \|\nabla_h u(\tau)\|_{L^2}^2+\|\partial_3 b(\tau)\|_{L^2}^2\,d\tau=\|u(0)\|_{L^2}^2+\|b(0)\|_{L^2}^2.$$

The rest of the proof focuses on the $\dot{H}^3$ norm. Applying ∂_i^3 $(i=1,2,3)$ to (1.2) and then dotting by $(\partial_i^3 u,\partial_i^3 b)$, we find

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3(\|\partial_i^3 u\|_{L^2}^2+\|\partial_i^3 b\|_{L^2}^2)+\|\partial_i^3\nabla_h u\|_{L^2}^2+\|\partial_i^3\partial_3 b\|_{L^2}^2=I_1+I_2+I_3+I_4+I_5, \tag{3.1}$$

where

$$\begin{aligned}
I_1&=\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^3\partial_1 b\cdot\partial_i^3 u+\partial_i^3\partial_1 u\cdot\partial_i^3 b\,dx,\\
I_2&=-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx,\\
I_3&=\sum_{i=1}^3\int_{\mathbb{R}^3}[\partial_i^3(b\cdot\nabla b)-b\cdot\nabla\partial_i^3 b]\cdot\partial_i^3 u\,dx,\\
I_4&=-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\,dx,\\
I_5&=\sum_{i=1}^3\int_{\mathbb{R}^3}[\partial_i^3(b\cdot\nabla u)-b\cdot\nabla\partial_i^3 u]\cdot\partial_i^3 b\,dx.
\end{aligned}$$

By integration by parts, $I_1=0$. To bound I_2, we decompose it into two pieces,

$$\begin{aligned}
I_2&=-\sum_{i=1}^2\int_{\mathbb{R}^3}\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^3(u\cdot\nabla u)\cdot\partial_3^3 u\,dx\\
&= I_{2,1}+I_{2,2}.
\end{aligned}$$

By Hölder's inequality,

$$I_{2,1}\lesssim\|\nabla_h(u\cdot\nabla u)\|_{\dot{H}^2}\|\nabla_h u\|_{\dot{H}^3}\lesssim\|u\|_{H^3}\|\nabla_h u\|_{H^3}^2. \tag{3.2}$$

By Hölder's inequality and Lemma 1.2,

$$\begin{aligned}
I_{2,2} &= -\int_{\mathbb{R}^3} \partial_3^3(u_h \cdot \nabla_h u) \cdot \partial_3^3 u \, dx - \int_{\mathbb{R}^3} \partial_3^3(u_3 \partial_3 u) \cdot \partial_3^3 u \, dx \\
&= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k u_h \cdot \nabla_h \partial_3^{3-k} u \cdot \partial_3^3 u + \partial_3^k u_3 \partial_3^{4-k} u \cdot \partial_3^3 u \, dx \\
&\lesssim \sum_{k=1}^{3} \|\partial_3^k u_h\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_3^k u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \nabla_h \partial_3^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_3^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_3^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_3^{k-1} \nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_3^{k-1} \nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3^{4-k} u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_3^{4-k} u\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times \|\partial_3^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_3^3 u\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^3} \|\nabla_h u\|_{H^3}^2.
\end{aligned} \tag{3.3}$$

Now we turn to the next term I_3,

$$I_3 = \sum_{i=1}^{3} \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_i^k b \cdot \nabla \partial_i^{3-k} b \cdot \partial_i^3 u \, dx = I_{3,1} + I_{3,2} + I_{3,3}.$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}
I_{3,1} &\le \Big(\|\partial_1 b\|_{L^3} \|\nabla \partial_1^2 b\|_{L^2} + \|\partial_1^2 b\|_{L^3} \|\nabla \partial_1 b\|_{L^2} + \|\partial_1^3 b\|_{L^2} \|\nabla b\|_{L^3} \Big) \|\partial_1^3 u\|_{L^6} \\
&\lesssim \|b\|_{H^3} \|\partial_1 b\|_{H^2} \|\partial_1 u\|_{H^3}.
\end{aligned} \tag{3.4}$$

By Lemma 1.2,

$$\begin{aligned}
I_{3,2} &\lesssim \sum_{k=1}^{2} \|\partial_2^k b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_2^k b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_2^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \nabla \partial_2^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_2^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_2^3 u\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
I_{3,3} &\lesssim \sum_{k=1}^{2} \|\partial_3^k b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_3^k b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \nabla \partial_3^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\partial_3^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_3^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_3^3 b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_3^3 b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_3^3 u\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_3^3 u\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3}^{\frac{1}{2}} \|u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3} \|\partial_2 u\|_{H^3}^{\frac{1}{2}}.
\end{aligned} \tag{3.6}$$

The next term I_4 is naturally split into three parts,

$$I_4 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3(u \cdot \nabla b) \cdot \partial_i^3 b \, dx = I_{4,1} + I_{4,2} + I_{4,3}.$$

$I_{4,1}$ and $I_{4,3}$ involves the favorable partial derivatives in x_1 and x_3, respectively, and their handling is not very difficult. In contrast, $I_{4,2}$ has partials in terms of x_2 and the control of $I_{4,2}$ is extremely delicate. By Lemma 1.2,

$$\begin{aligned} I_{4,1} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1^k u \cdot \nabla \partial_1^{3-k} b \cdot \partial_1^3 b \, dx, \\ &\lesssim \|\partial_1 u\|_{L^\infty} \|\nabla \partial_1^2 b\|_{L^2} \|\partial_1^3 b\|_{L^2} + \sum_{k=2}^{3} \|\partial_1^k u\|_{L^6} \|\nabla \partial_1^{3-k} b\|_{L^3} \|\partial_1^3 b\|_{L^2} \\ &\lesssim \|b\|_{H^3} \|\partial_1 u\|_{H^3} \|\partial_1 b\|_{H^2} \end{aligned}$$

and

$$\begin{aligned} I_{4,3} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k u \cdot \nabla \partial_3^{3-k} b \cdot \partial_3^3 b \, dx, \\ &\lesssim \sum_{k=1}^{3} \|\partial_3^k u\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^k u\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla \partial_3^{3-k} b\|_{L^2}^{\frac{1}{2}} \|\partial_3^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{1}} \partial_3^3 b\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{3}{2}}. \end{aligned} \tag{3.7}$$

We now turn to $I_{4,2}$, one of the most difficult terms. We further decompose it into three terms,

$$\begin{aligned} I_{4,2} &= -\int_{\mathbb{R}^3} \big(\partial_2^3(u_1 \partial_1 b) + \partial_2^3(u_2 \partial_2 b) + \partial_2^3(u_3 \partial_3 b)\big) \cdot \partial_2^3 b \, dx \\ &= I_{4,2,1} + I_{4,2,2} + I_{4,2,3}. \end{aligned}$$

By Lemma 1.2,

$$\begin{aligned} I_{4,2,1} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_1 \partial_1 \partial_2^{3-k} b \cdot \partial_2^3 b \, dx \\ &\lesssim \sum_{k=2}^{3} \|\partial_2^k u_1\|_{L^6} \|\partial_1 \partial_2^{3-k} b\|_{L^3} \|\partial_2^3 b\|_{L^2} + \|\partial_2 u_1\|_{L^\infty} \|\partial_1 \partial_2^2 b\|_{L^2} \|\partial_2^3 b\|_{L^2} \\ &\lesssim \|b\|_{H^3} \|\partial_2 u_1\|_{H^3} \|\partial_1 b\|_{H^2} \end{aligned}$$

and

$$
\begin{aligned}
I_{4,2,3} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_3 \partial_3 \partial_2^{3-k} b \cdot \partial_2^3 b \, dx \\
&\lesssim \sum_{k=1}^{3} \|\partial_2^k u_3\|_{L^6} \|\partial_3 \partial_2^{3-k} b\|_{L^3} \|\partial_2^3 b\|_{L^2} \\
&\lesssim \|b\|_{H^3} \|\partial_2 u_3\|_{H^3} \|\partial_3 b\|_{H^3}.
\end{aligned}
$$

$I_{4,2,2}$ is much more challenging and we further break it down,

$$
\begin{aligned}
I_{4,2,2} &= -\sum_{k=2}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_2 \partial_2^{4-k} b \cdot \partial_2^3 b \, dx - 3 \int_{\mathbb{R}^3} \partial_2 u_2 \partial_2^3 b \cdot \partial_2^3 b \, dx \\
&= -\sum_{k=2}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_2 \partial_2^{4-k} b \cdot \partial_2^3 b \, dx + 3 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_2^3 b \cdot \partial_2^3 b \, dx \\
&\quad + 3 \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2^3 b \cdot \partial_2^3 b \, dx = I_{4,2,2,1} + I_{4,2,2,2} + I_{4,2,2,3}.
\end{aligned}
$$

By Lemma 1.2,

$$
\begin{aligned}
I_{4,2,2,1} &\lesssim \sum_{k=2}^{3} \|\partial_2^k u_2\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_2} \partial_2^k u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^{4-k} b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_2^{4-k} b\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2^3 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3} \|\partial_2 u_2\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}}.
\end{aligned}
$$

By integration by parts and Lemma 1.2,

$$
\begin{aligned}
I_{4,2,2,2} &\lesssim \|u_3\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_1} u_3\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_2} u_3\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_1} \boldsymbol{\partial_2} u_3\|_{L^2}^{\frac{1}{4}} \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^3 b\|_{L^2} \\
&\lesssim \|b\|_{H^3}^{\frac{1}{2}} \|u_3\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{3}{2}} \|\partial_2 u_3\|_{H^3}^{\frac{1}{2}}.
\end{aligned}
$$

$I_{4,2,2,3}$ can not be directly estimated to yield a suitable bound. If we attempt to directly apply Lemma 1.2 as follows,

$$
\int_{\mathbb{R}^3} \partial_1 u_1 \partial_2^3 b \partial_2^3 b \, dx \leq \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 b\|_{L^2}^{\frac{1}{2}},
$$

which involves $\|\partial_1 b\|_{H^3}$ and the differential inequality would not be closed. Fortunately the equations in (1.2) have a special structure. The equation of b allows us to replace $\partial_1 u_1$ via

$$\partial_1 u = \partial_t b + u \cdot \nabla b - \partial_3^2 b - b \cdot \nabla u$$

and bring us the hope of controlling $I_{4,2,2,3}$ suitably. We write $I_{4,2,2,3}$ as

$$\begin{aligned} I_{4,2,2,3} &= 3 \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2^3 b \cdot \partial_2^3 b \, dx \\ &= 3 \int_{\mathbb{R}^3} [\partial_t b_1 + u \cdot \nabla b_1 - \partial_3^2 b_1 - b \cdot \nabla u_1] |\partial_2^3 b|^2 \, dx \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

J_2, J_3 and J_4 are relatively easier to deal with. By Lemma 1.2,

$$\begin{aligned} J_2 &\lesssim \|u\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} u\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{2}} u\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \partial_{\mathbf{2}} u\|_{L^2}^{\frac{1}{4}} \|\nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \nabla b\|_{L^2}^{\frac{1}{4}} \\ &\quad \times \|\partial_{\mathbf{2}} \nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \partial_{\mathbf{2}} \nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_2^3 b\|_{L^2} \|\partial_{\mathbf{3}} \partial_2^3 b\|_{L^2} \\ &\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}. \end{aligned}$$

By Hölder's inequality,

$$J_3 \lesssim \|\partial_3 b_1\|_{L^\infty} \|\partial_3 \partial_2^3 b\|_{L^2} \|\partial_2^3 b\|_{L^2} \lesssim \|b\|_{H^3} \|\partial_3 b\|_{H^3}^2.$$

Again by Lemma 1.2,

$$\begin{aligned} J_4 &\lesssim \|\nabla u_1\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \nabla u_1\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{2}} \nabla u_1\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \partial_{\mathbf{2}} \nabla u_1\|_{L^2}^{\frac{1}{4}} \\ &\quad \times \|b\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} b\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{2}} b\|_{L^2}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \partial_{\mathbf{2}} b\|_{L^2}^{\frac{1}{4}} \|\partial_2^3 b\|_{L^2} \|\partial_{\mathbf{3}} \partial_2^3 b\|_{L^2} \\ &\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}. \end{aligned} \tag{3.8}$$

To deal with J_1, we rewrite it as

$$J_1 = 3 \frac{d}{dt} \int_{\mathbb{R}^3} b_1 \, |\partial_2^3 b|^2 \, dx - 3 \int_{\mathbb{R}^3} b_1 \partial_t |\partial_2^3 b|^2 \, dx = J_{1,1} + J_{1,2}. \tag{3.9}$$

To estimate $J_{1,2}$, we use the equation of b in (1.2) again to write is as

$$\begin{aligned} J_{1,2} &= -6 \int_{\mathbb{R}^3} b_1 \partial_2^3 b \cdot \big(- \partial_2^3 (u \cdot \nabla b) + \partial_2^3 \partial_3^2 b + \partial_2^3 (b \cdot \nabla u) + \partial_2^3 \partial_1 u \big) \, dx \\ &= J_{1,2,1} + J_{1,2,2} + J_{1,2,3} + J_{1,2,4}. \end{aligned}$$

By integrating by parts and applying Lemma 1.2, we have

$$
\begin{aligned}
J_{1,2,1} = & 6\sum_{k=1}^{3}\int_{\mathbb{R}^3} \mathcal{C}_3^k b_1\partial_2^3 b\cdot(\partial_2^k u\cdot\nabla\partial_2^{3-k} b)\,dx - 3\int_{\mathbb{R}^3} u\cdot\nabla b_1|\partial_2^3 b|^2\,dx\\
&\lesssim \|b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}} b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac14}\\
&\quad\times\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_2 u\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}}\partial_2 u\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}}\partial_2 u\|_{L^2}^{\frac14}\\
&\quad\times\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}^{\frac12}\,\|\nabla\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\nabla\partial_2^2 b\|_{L^2}^{\frac12}\\
&\quad+\sum_{k=2}^{3}\|b_1\|_{L^\infty}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_2^k u\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_2^k u\|_{L^2}^{\frac12}\\
&\qquad\times\|\nabla\partial_2^{3-k} b\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\nabla\partial_2^{3-k} b\|_{L^2}^{\frac12}+|J_2|\\
&\lesssim \|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}\|\nabla_h u\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^2}^{\frac12}\|\partial_3 b\|_{H^3}+\|b\|_{H^3}^2\|\nabla_h u\|_{H^3}\|\partial_1 b\|_{H^2}^{\frac12}\|\partial_3 b\|_{H^3}^{\frac12}.
\end{aligned}
$$

By integration by parts and Hölder's inequality,

$$
\begin{aligned}
J_{1,2,2} &= -3\int_{\mathbb{R}^3}\partial_3^2 b_1|\partial_2^3 b|^2\,dx+6\int_{\mathbb{R}^3} b_1|\partial_3\partial_2^3 b|^2\,dx\\
&\lesssim |J_3|+\|b_1\|_{L^\infty}\|\partial_3 b\|_{H^3}^2\\
&\lesssim \|b\|_{H^3}\|\partial_3 b\|_{H^3}^2.
\end{aligned}
$$

By Lemma 1.2,

$$
\begin{aligned}
J_{1,2,3} =& -6\sum_{k=0}^{3}\int_{\mathbb{R}^3}\mathcal{C}_3^k b_1\partial_2^3 b\cdot\partial_2^k b\cdot\nabla\partial_2^{3-k} u\,dx\\
=& -6\sum_{k=1}^{2}\int_{\mathbb{R}^3}\mathcal{C}_3^k b_1\partial_2^3 b\cdot(\partial_2^k b\cdot\nabla\partial_2^{3-k} u)\,dx-6\int_{\mathbb{R}^3} b_1\partial_2^3 b\cdot(\partial_2^3 b\cdot\nabla u)\,dx\\
&-6\int_{\mathbb{R}^3} b_1\partial_2^3 b\cdot(b\cdot\nabla\partial_2^3 u)\,dx\\
\lesssim& \sum_{k=1}^{2}\|b_1\|_{L^\infty}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_2^k b\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_2^k b\|_{L^2}^{\frac12}\|\nabla\partial_2^{3-k} u\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\nabla\partial_2^{3-k} u\|_{L^2}^{\frac12}\\
&+\|b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}} b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac14}\\
&\quad\times\|\nabla u\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\nabla u\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}}\nabla u\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}}\nabla u\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}\\
&+\|b_1\partial_2^3 b b\|_{L^2}\|\partial_2 u\|_{H^3}\\
\lesssim& \|b\|_{H^3}^2\|\partial_2 u\|_{H^3}\|\partial_1 b\|_{H^2}^{\frac12}\|\partial_3 b\|_{H^3}^{\frac12}+\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}\|\partial_h u\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^2}^{\frac12}\|\partial_3 b\|_{H^3}\\
&+\|b\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}} b\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}} b\|_{L^2}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}} b\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}^{\frac12}\|b\|_{H^2}\|\partial_2 u\|_{H^3}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|b\|_{H^3}^2\|\partial_2 u\|_{H^3}\|\partial_1 b\|_{H^2}^{\frac{1}{2}}\|\partial_3 b\|_{H^3}^{\frac{1}{2}} + \|u\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{3}{2}}\|\partial_h u\|_{H^3}^{\frac{1}{2}}\|\partial_1 b\|_{H^2}^{\frac{1}{2}}\|\partial_3 b\|_{H^3} \\
&\quad + \|b\|_{H^3}^2\|\partial_2 u\|_{H^3}\|\partial_1 b\|_{H^2}^{\frac{1}{2}}\|\partial_3 b\|_{H^3}^{\frac{1}{2}}.
\end{aligned}
$$

The last term $J_{1,2,4}$ can also be bounded via Lemma 1.2,

$$
\begin{aligned}
J_{1,2,4} &\lesssim \|b_1\|_{L^2}^{\frac{1}{4}}\|\partial_{\mathbf{1}} b_1\|_{L^2}^{\frac{1}{4}}\|\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac{1}{4}}\|\partial_{\mathbf{1}}\partial_{\mathbf{2}} b_1\|_{L^2}^{\frac{1}{4}}\|\partial_2^3 b\|_{L^2}^{\frac{1}{2}}\|\partial_{\mathbf{3}}\partial_2^3 b\|_{L^2}^{\frac{1}{2}}\|\partial_2^3\partial_1 u\|_{L^2} \\
&\lesssim \|b\|_{H^3}\|\partial_1 u\|_{H^3}\|\partial_1 b\|_{H^2}^{\frac{1}{2}}\|\partial_3 b\|_{H^3}^{\frac{1}{2}}.
\end{aligned}
$$

It remains to estimate I_5,

$$
I_5 = \sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big(\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u\Big)\cdot\partial_i^3 b\,dx = I_{5,1}+I_{5,2}+I_{5,3}.
$$

By Hölder's and Sobolev's inequalities,

$$
\begin{aligned}
I_{5,1} &= \sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_1^k b\cdot\nabla\partial_1^{3-k}u\cdot\partial_1^3 b\,dx \\
&\lesssim \sum_{k=1}^{2}\|\partial_1^k b\|_{L^3}\|\nabla\partial_1^{3-k}u\|_{L^6}\|\partial_1^3 b\|_{L^2} + \|\partial_1^3 b\|_{L^2}^2\|\nabla u\|_{L^\infty} \\
&\lesssim \|b\|_{H^3}\|\partial_1 u\|_{H^3}\|\partial_1 b\|_{H^2} + \|u\|_{H^3}\|\partial_1 b\|_{H^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_{5,3} &= \sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k b\cdot\nabla\partial_3^{3-k}u\cdot\partial_3^3 b\,dx \\
&\lesssim \sum_{k=1}^{3}\|\partial_3^k b\|_{L^3}\|\nabla\partial_3^{3-k}u\|_{L^2}\|\partial_3^3 b\|_{L^6} \\
&\lesssim \|u\|_{H^3}\|\partial_3 b\|_{H^3}^2.
\end{aligned}
$$

The difficult term is $I_{5,2}$, which is further decomposed into

$$
\begin{aligned}
I_{5,2} &= \int_{\mathbb{R}^3}\Big[\partial_2^3(b_1\partial_1 u) + \partial_2^3(b_2\partial_2 u) + \partial_2^3(b_3\partial_3 u) - b\cdot\nabla\partial_2^3 u\Big]\cdot\partial_2^3 b\,dx \\
&= I_{5,2,1}+I_{5,2,2}+I_{5,2,3}.
\end{aligned}
$$

The last two terms $I_{5,2,2}$ and $I_{5,2,3}$ can be directly bounded. By $\nabla\cdot b = 0$,

$$
\begin{aligned}
I_{5,2,2} &= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_2 \partial_2 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx \\
&\lesssim \sum_{k=1}^{2} \|\partial_2^{k-1}(\partial_1 b_1 + \partial_3 b_3)\|_{L^3} \|\partial_2 \partial_2^{3-k} u\|_{L^6} \|\partial_2^3 b\|_{L^2} \\
&\quad + \|\partial_2^2(\partial_1 b_1 + \partial_3 b_3)\|_{L^2} \|\partial_2 u\|_{L^\infty} \|\partial_2^3 b\|_{L^2} \\
&\lesssim \|b\|_{H^3} \|\partial_2 u\|_{H^3} (\|\partial_1 b\|_{H^2} + \|\partial_3 b\|_{H^3}).
\end{aligned}
$$

By integration by parts and Lemma 1.2,

$$
\begin{aligned}
I_{5,2,3} &= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \partial_3 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx \\
&= - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k \partial_3 b_3 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \partial_2^{3-k} u \partial_2^3 \cdot \partial_3 b \, dx \\
&\lesssim \sum_{k=1}^{3} \|\partial_2^k \partial_3 b_3\|_{L^2} \|\partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_1} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_2} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \\
&\qquad \times \|\boldsymbol{\partial_1} \boldsymbol{\partial_2} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\partial_2^3 b\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2^3 b\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_1} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_2} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \\
&\qquad \times \|\boldsymbol{\partial_1} \boldsymbol{\partial_2} \partial_2^{3-k} u\|_{L^2}^{\frac{1}{4}} \|\partial_2^3 \partial_3 b\|_{L^2} \\
&\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{3}{2}}.
\end{aligned}
$$

The estimate for $I_{5,2,1}$ is much more complex. We further break it down,

$$
\begin{aligned}
I_{5,2,1} &= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \partial_1 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx \\
&= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_1 \partial_2^3 b_1 + \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_2 \partial_2^3 b_2 + \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_3 \partial_2^3 b_3 \, dx \\
&= I_{5,2,1,1} + I_{5,2,1,2} + I_{5,2,1,3}.
\end{aligned}
$$

We estimate $I_{5,2,1,1}$ and $I_{5,2,1,2}$ directly. By Lemma 1.2,

$$
I_{5,2,1,1} = \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_1 \partial_2^3 b_1 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_1 \partial_1 u_1 \partial_2^3 b_1 \, dx
$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{2} \|\partial_2^k b_1\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{1}} \partial_2^k b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^{3-k} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_1 \partial_2^{3-k} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \partial_2^3 b_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + |I_{4,2,2,3}| \\
&\lesssim \|b\|_{H^3} \|\nabla_h u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}} + |I_{4,2,2,3}|.
\end{aligned}
$$

By Hölder's inequality and $\partial_2 b_2 = -\partial_1 b_1 - \partial_3 b_3$,

$$
\begin{aligned}
I_{5,2,1,2} &= \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_2 \partial_2^3 b_2 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_1 \partial_1 u_2 \partial_2^3 b_2 \, dx \\
&\lesssim \sum_{k=1}^{2} \|\partial_2^k b_1\|_{L^3} \|\partial_1 \partial_2^{3-k} u_2\|_{L^6} \|\partial_2^3 b_2\|_{L^2} + \|\partial_2^3 b_1\|_{L^2} \|\partial_1 u_2\|_{L^\infty} \|\partial_2^3 b_2\|_{L^2} \\
&\lesssim \|b\|_{H^3} \|\partial_1 u\|_{H^3} (\|\partial_1 b\|_{H^2} + \|\partial_3 b\|_{H^3}).
\end{aligned}
$$

The last term $I_{5,2,1,3}$ contains a part that can not be directly handled,

$$
\begin{aligned}
I_{5,2,1,3} &= \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \partial_1 \partial_2^{3-k} u_3 \partial_2^3 b_3 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_1 \partial_1 u_3 \partial_2^3 b_3 \, dx \\
&\lesssim \sum_{k=1}^{2} \|\partial_2^k b_1\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{1}} \partial_2^k b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^{3-k} u_3\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_1 \partial_2^{3-k} u_3\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times \|\partial_2^3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \partial_2^3 b_3\|_{L^2}^{\frac{1}{2}} + K_1 \\
&\lesssim \|b\|_{H^3} \|\partial_h u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}} + K_1
\end{aligned}
$$

where

$$
K_1 = \int_{\mathbb{R}^3} \partial_2^3 b_1 \partial_1 u_3 \partial_2^3 b_3 \, dx.
$$

It does not appear to be possible to give a direct estimate on K_1. As in the estimate of $I_{4,2,2,3}$, we use the special structure of the equation for b in (1.2) and make the substitution

$$
\partial_1 u_3 = \partial_t b_3 + u \cdot \nabla b_3 - \partial_3^2 b_3 - b \cdot \nabla u_3.
$$

Then K_1 can be rewritten as

$$
\begin{aligned}
K_1 &= \int_{\mathbb{R}^3} \Big(\partial_t b_3 + u \cdot \nabla b_3 - \partial_3^2 b_3 - b \cdot \nabla u_3\Big) \partial_2^3 b_1 \partial_2^3 b_3 \, dx \\
&= K_{1,1} + K_{1,2} + K_{1,3} + K_{1,4}.
\end{aligned}
$$

We estimate $K_{1,2}$, $K_{1,3}$ and $K_{1,4}$ similarly as J_2, J_3, J_4 to obtain

$$\begin{aligned}
|K_{1,2}| &\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}, \\
|K_{1,3}| &\lesssim \|b\|_{H^3} \|\partial_3 b\|_{H^3}^2, \\
|K_{1,4}| &\lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}.
\end{aligned}$$

By integration by parts,

$$K_{1,1} = \frac{d}{dt} \int_{\mathbb{R}^3} b_3 \partial_2^3 b_1 \partial_2^3 b_3 \, dx - \int_{\mathbb{R}^3} b_3 \partial_t (\partial_2^3 b_1 \partial_2^3 b_3) \, dx = K_{1,1,1} + K_{1,1,2}.$$

According to the equation of b in (1.2),

$$\begin{cases}
\partial_2^3 \partial_t b_1 + \partial_2^3 (u \cdot \nabla b_1) - \partial_2^3 \partial_3^2 b_1 = \partial_2^3 (b \cdot \nabla u_1) + \partial_2^3 \partial_1 u_1, \\
\partial_2^3 \partial_t b_3 + \partial_2^3 (u \cdot \nabla b_3) - \partial_2^3 \partial_3^2 b_3 = \partial_2^3 (b \cdot \nabla u_3) + \partial_2^3 \partial_1 u_3.
\end{cases}$$

Hence,

$$\begin{aligned}
K_{1,1,2} &= \int_{\mathbb{R}^3} b_3 \partial_2^3 [u \cdot \nabla b_1 - \partial_3^2 b_1 - b \cdot \nabla u_1 - \partial_1 u_1] \partial_2^3 b_3 \\
&\quad + b_3 \partial_2^3 [u \cdot \nabla b_3 - \partial_3^2 b_3 - b \cdot \nabla u_3 - \partial_1 u_3] \partial_2^3 b_1 \, dx \\
&= \int_{\mathbb{R}^3} b_3 [\partial_2^3 (u \cdot \nabla b_1) \partial_2^3 b_3 + \partial_2^3 (u \cdot \nabla b_3) \partial_2^3 b_1] - b_3 [\partial_2^3 \partial_3^2 b_1 \partial_2^3 b_3 + \partial_2^3 \partial_3^2 b_3 \partial_2^3 b_1] \\
&\quad - b_3 [\partial_2^3 (b \cdot \nabla u_1) \partial_2^3 b_3 + \partial_2^3 (b \cdot \nabla u_3) \partial_2^3 b_1] - b_3 [\partial_2^3 \partial_1 u_1 \partial_2^3 b_3 + \partial_2^3 \partial_1 u_3 \partial_2^3 b_1] \\
&= K_{1,1,2,1} + K_{1,1,2,2} + K_{1,1,2,3} + K_{1,1,2,4}.
\end{aligned}$$

As in the estimate of the term $J_{1,2,1}$, we have

$$K_{1,1,2,1} \lesssim \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3} + \|b\|_{H^3}^2 \|\nabla_h u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}}.$$

Similar to the terms $J_{1,2,2}$, $J_{1,2,3}$ and $J_{1,2,4}$, we have

$$\begin{aligned}
K_{1,1,2,2} &= \int_{\mathbb{R}^3} 2 b_3 \partial_2^3 \partial_3 b_1 \partial_2^3 \partial_3 b_3 + \partial_3 b_3 [\partial_2^3 \partial_3 b_3 \partial_2^3 b_1 + \partial_2^3 \partial_3 b_1 \partial_2^3 b_3] \, dx \\
&\lesssim \|b\|_{L^\infty} \|\partial_3 b\|_{H^3}^2 + \|\partial_3 b_3\|_{L^6} \|\partial_3 b\|_{H^3} \|b\|_{H^3} \\
&\lesssim \|b\|_{H^3} \|\partial_3 b\|_{H^3}^2,
\end{aligned}$$

$$
\begin{aligned}
K_{1,1,2,3} \lesssim\ & \|b\|_{H^3}^2 \|\partial_2 u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}} \\
& + \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\partial_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3} \\
& + \|b\|_{H^3}^2 \|\partial_2 u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}}
\end{aligned}
$$

and

$$
K_{1,1,2,4} \lesssim\ \|b\|_{H^3} \|\partial_1 u\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}}.
$$

Integrating (3.1) in time, namely

$$
E_0(t) \lesssim\ E_0(0) + \int_0^t (I_2(\tau) + I_3(\tau) + I_4(\tau) + I_5(\tau))\, d\tau
$$

and inserting all the bounds obtained above for I_2 through I_5, we obtain (1.5) after applying Hölder's inequality. To be clear, we provide some details. The bounds for I_2 in (3.2) and (3.3) yield

$$
\begin{aligned}
\int_0^t I_2(\tau)\, d\tau & \lesssim \int_0^t \|u\|_{H^3}\, \|\nabla_h u\|_{H^3}^2\, d\tau \\
& \leq \sup_{0\leq\tau\leq t} \|u(\tau)\|_{H^3} \int_0^t \|\nabla_h u\|_{H^3}^2\, d\tau \leq E_0^{\frac{3}{2}}(t).
\end{aligned}
$$

The bounds for I_3 in (3.4), (3.5) and (3.6) lead to, by Hölder's inequality,

$$
\begin{aligned}
\int_0^t I_3(\tau)\, d\tau \lesssim & \int_0^t \|b\|_{H^3} \|\partial_1 b\|_{H^2} \|\partial_1 u\|_{H^3}\, d\tau \\
& + \int_0^t \|b\|_{H^3} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}\, d\tau \\
& + \int_0^t \|b\|_{H^3}^{\frac{1}{2}}\, \|u\|_{H^3}^{\frac{1}{2}}\, \|\partial_1 b\|_{H^2}^{\frac{1}{2}}\, \|\partial_3 b\|_{H^3} \|\partial_2 u\|_{H^3}^{\frac{1}{2}}\, d\tau \\
\lesssim\ & E_0(t)^{\frac{1}{2}}\, E_1(t)^{\frac{1}{2}}\, E_0(t)^{\frac{1}{2}} + E_0(t)^{\frac{1}{2}} E_1(t)^{\frac{1}{4}}\, E_0(t)^{\frac{1}{4}}\, E_0(t)^{\frac{1}{2}} \\
\lesssim\ & E_0^{\frac{3}{2}}(t) + E_1^{\frac{3}{2}}(t).
\end{aligned}
$$

The bounds for I_4 involve a lot of terms and we shall just choose some typical ones to bound the time integral of I_4. For example, the time integrals of the bounds for $I_{4,3}$ in (3.7), J_4 in (3.8) and $J_{1,1}$ in (3.9) obey, by Hölder's inequality,

$$\int_0^t \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^3}^{\frac{3}{2}} \, d\tau \leq E_0(t)^{\frac{1}{2}} \, E_0(t)^{\frac{1}{4}} \, E_0(t)^{\frac{3}{4}} \; = E_0^{\frac{3}{2}}(t),$$

$$\begin{aligned}\int_0^t \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} \|\nabla_h u\|_{H^3}^{\frac{1}{2}} \|\partial_1 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 b\|_{H^3} \, d\tau &\leq E_0(t) \, E_0(t)^{\frac{1}{4}} \, E_1(t)^{\frac{1}{4}} \, E_0(t)^{\frac{1}{2}} \\ &\lesssim E_0^2(t) + E_1^2(t)\end{aligned}$$

and

$$\begin{aligned}\int_0^t J_{1,1} \, d\tau &= 3 \int_{\mathbb{R}^3} b_1 (\partial_2^3 b)^2 \, dx - 3 \int_{\mathbb{R}^3} b_1(x,0) (\partial_2^3 b)^2(x,0) \, dx \\ &\lesssim \|b_1(0)\|_{L^\infty} \, \|b(0)\|_{H^3}^2 + \|b_1(t)\|_{L^\infty} \, \|b(t)\|_{H^3}^2 \\ &\lesssim E_0(0)^{\frac{3}{2}} + E_0^{\frac{3}{2}}(t).\end{aligned}$$

The time integral of I_5 is similarly bounded. This completes the proof of (1.5). $\square$

4. Proof of (1.6)

This section proves (1.6), namely

$$E_1(t) \lesssim E_1(0) + E_0(t) + E_0(t)^{\frac{3}{2}} + E_1(t)^{\frac{3}{2}}.$$

Proof of (1.6). Due to the equivalence of the norm $\|\partial_1 b\|_{H^2}$ and the norm $\|\partial_1 b\|_{L^2} + \|\partial_1 b\|_{\dot{H}^2}$, it suffices to estimate the L^2-norm and the homogeneous $\dot{H}^2$-norm of $\partial_1 b$. We make use of the velocity equation in (1.2) to write

$$\partial_1 b = \partial_t u + u \cdot \nabla u - \Delta_h u + \nabla P - b \cdot \nabla b.$$

Therefore,

$$\begin{aligned}\|\partial_1 b\|_{L^2}^2 &= \int_{\mathbb{R}^3} \partial_t u \cdot \partial_1 b \, dx + \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \partial_1 b \, dx \\ &\quad - \int_{\mathbb{R}^3} \Delta_h u \cdot \partial_1 b \, dx - \int_{\mathbb{R}^3} b \cdot \nabla b \cdot \partial_1 b \, dx \\ &= N_1 + N_2 + N_3 + N_4,\end{aligned} \tag{4.1}$$

where we have eliminated the pressure term due to $\nabla \cdot b = 0$. We integrate by parts and use the equation of b in (1.2) to obtain

$$
\begin{aligned}
N_1 =& \frac{d}{dt}\int_{\mathbb{R}^3} u\cdot\partial_1 b\,dx - \int_{\mathbb{R}^3} u\cdot\partial_1(-u\cdot\nabla b+\partial_3^2 b+b\cdot\nabla u+\partial_1 u)dx \\
=& N_{1,0}+N_{1,1}+N_{1,2}+N_{13}+N_{1,4}.
\end{aligned}
$$

By Lemma 1.2 and Hölder's inequality,

$$
\begin{aligned}
N_{1,1} =& -\int_{\mathbb{R}^3} u\cdot(\partial_1 u\cdot\nabla b+u\cdot\nabla\partial_1 b)dx \\
\lesssim& \|\partial_1 u\|_{L^2}\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1\partial_2 u\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_3 b\|_{L^2}^{\frac12} \\
&+\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla\partial_1 b\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_3 b\|_{L^2}^{\frac12} \\
\le& \|\nabla_h u\|_{H^3}^{\frac32}\|\partial_3 b\|_{H^3}^{\frac12}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}+\|\nabla_h u\|_{H^3}\|\partial_1 b\|_{H^2}\|u\|_{H^3},
\end{aligned}
$$

$$
N_{1,2} = -\int_{\mathbb{R}^3} u\cdot\partial_1\partial_3^2 b\,dx \le \|\nabla_h u\|_{H^3}\|\partial_3 b\|_{H^3},
$$

$$
\begin{aligned}
N_{1,3} =& \int_{\mathbb{R}^3}(u\cdot(\partial_1 b\cdot\nabla u)+u\cdot(b\cdot\nabla\partial_1 u))\,dx \\
\lesssim& \|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12} \\
&+\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|b\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\nabla\partial_1 u\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2 u\|_{L^2}^{\frac12} \\
\le& \|\nabla_h u\|_{H^3}\|\partial_1 b\|_{H^2}\|u\|_{H^3}+\|\nabla_h u\|_{H^3}^{\frac32}\|\partial_3 b\|_{H^3}^{\frac12}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}
\end{aligned}
$$

and

$$
N_{1,4} = -\int_{\mathbb{R}^3} u\cdot\partial_1^2 u\,dx \le \|\nabla_h u\|_{H^3}^2.
$$

Similarly,

$$
\begin{aligned}
N_2 =& \int_{\mathbb{R}^3} u\cdot\nabla u\cdot\partial_1 b\,dx \\
\lesssim& \|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla\partial_2 u\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12} \\
\le& \|\nabla_h u\|_{H^3}\|\partial_1 b\|_{H^2}\|u\|_{H^3},
\end{aligned}
$$

$$
N_3 = -\int_{\mathbb{R}^3}\Delta_h u\cdot\partial_1 b\,dx \le \|\nabla_h u\|_{H^3}\|\partial_1 b\|_{H^2}
$$

and

$$
\begin{aligned}
N_4 =& -\int_{\mathbb{R}^3} b\cdot\nabla b\cdot\partial_1 b\,dx\\
\lesssim& \|b\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_3 b\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1\partial_2 b\|_{L^2}^{\frac12}\\
\leq& \|\partial_1 b\|_{H^2}^{\frac32}\|\partial_3 b\|_{H^3}^{\frac12}\|b\|_{H^3}.
\end{aligned}
$$

This finishes the L^2 estimate of $\partial_1 b$. Now we turn to the $\dot{H}^2$ estimate. According to the equation of u in (1.2), we have

$$
\begin{aligned}
\sum_{i=1}^3\|\partial_i^2\partial_1 b\|_{L^2}^2 =& \sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2\partial_t u\cdot\partial_i^2\partial_1 b\,dx\\
&+\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2(u\cdot\nabla u)\cdot\partial_i^2\partial_1 b\,dx\\
&-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2\Delta_h u\cdot\partial_i^2\partial_1 b\,dx\\
&-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2(b\cdot\nabla b)\cdot\partial_i^2\partial_1 b\,dx\\
=& M_1+M_2+M_3+M_4.
\end{aligned}
\tag{4.2}
$$

To bound M_1, we integrate by parts and use the equation of b in (1.2) to obtain

$$
\begin{aligned}
M_1 =& \frac{d}{dt}\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2 u\cdot\partial_i^2\partial_1 b\,dx\\
&-\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2 u\cdot\partial_i^2\partial_1(-u\cdot\nabla b+\partial_3^2 b+b\cdot\nabla u+\partial_1 u)\,dx\\
=& M_{1,0}+M_{1,1}+M_{1,2}+M_{1,3}+M_{1,4}.
\end{aligned}
$$

By Lemma 1.2,

$$
\begin{aligned}
M_{1,1} =& \int_{\mathbb{R}^3}\partial_1^2 u\cdot\partial_1^3(u\cdot\nabla b)+\partial_2^2 u\cdot\partial_2^2\partial_1(u\cdot\nabla b)+\partial_3^3\partial_1 u\cdot\partial_3(u\cdot\nabla b)\,dx\\
\leq& \|\nabla_h u\|_{H^3}\Big(\|\nabla_h u\|_{H^3}\|b\|_{H^3}+\|\partial_1 b\|_{H^2}\|u\|_{H^3}\Big)\\
&+\|\nabla_h u\|_{H^3}\Big(\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_3 b\|_{L^2}^{\frac12}\\
&\quad+\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1\partial_2 u\|_{L^2}^{\frac14}\|\partial_3\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_3^2 b\|_{L^2}^{\frac12}\Big)
\end{aligned}
$$

$$\le \|\nabla_h u\|_{H^3}\Big(\|\nabla_h u\|_{H^3}\|b\|_{H^3} + \|\partial_1 b\|_{H^2}\|u\|_{H^3} + \|\nabla_h u\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^3}^{\frac12}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\Big).$$

Clearly,

$$M_{1,2} = -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_1\partial_i^3 u\cdot\partial_i\partial_3^2 b\,dx \le \|\nabla_h u\|_{H^3}\|\partial_3 b\|_{H^3}.$$

By Lemma 1.2,

$$\begin{aligned}
M_{1,3} = &-\int_{\mathbb{R}^3} \partial_1^2 u\cdot\partial_1^3(b\cdot\nabla u) + \partial_2^2 u\cdot\partial_2^2\partial_1(b\cdot\nabla u) + \partial_3^3\partial_1 u\cdot\partial_3(b\cdot\nabla u)\,dx\\
\le&\|\nabla_h u\|_{H^3}\Big(\|\nabla_h u\|_{H^3}\|b\|_{H^3} + \|\partial_1 b\|_{H^2}\|u\|_{H^3}\Big)\\
&+ \|\nabla_h u\|_{H^3}\Big(\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_3^2 b\|_{L^2}^{\frac12}\\
&\quad + \|\nabla\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_3\nabla u\|_{L^2}^{\frac14}\|\partial_2\partial_3\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3\nabla u\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\Big)\\
\le&\|\nabla_h u\|_{H^3}\Big(\|\nabla_h u\|_{H^3}\|b\|_{H^3} + \|\partial_1 b\|_{H^2}\|u\|_{H^3} + \|\nabla_h u\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^3}^{\frac12}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\Big).
\end{aligned}$$

Obviously,

$$M_{1,4} \le \sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big|\partial_i^2\partial_1 u\Big|^2\,dx \le \|\nabla_h u\|_{H^3}^2.$$

By Lemma 1.2, M_2 is bounded by

$$\begin{aligned}
M_2 = &\int_{\mathbb{R}^3} \partial_1^2(u\cdot\nabla u)\cdot\partial_1^3 b + \partial_2^2(u\cdot\nabla u)\partial_2^2\cdot\partial_1 b + \partial_1(u\cdot\nabla u)\cdot\partial_3^4 b\,dx\\
\le&\|\partial_1 b\|_{H^2}\|\nabla_h u\|_{H^3}\|u\|_{H^3}\\
&+ \|\partial_3 b\|_{H^3}\Big(\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\\
&\quad + \|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1\partial_2 u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|\partial_1\partial_3\nabla u\|_{L^2}^{\frac12}\Big)\\
\le&\|\partial_1 b\|_{H^2}\|\nabla_h u\|_{H^3}\|u\|_{H^3} + \|\partial_3 b\|_{H^3}\|\nabla_h u\|_{H^3}\|u\|_{H^3}.
\end{aligned}$$

The bound for M_3 is straightforward,

$$M_3 = \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2\Delta_h u\partial_i^2\partial_1 b\,dx \le \|\partial_h u\|_{H^3}\|\partial_1 b\|_{H^2}.$$

The last term M_4 can be bounded via Lemma 1.2,

$$
\begin{aligned}
M_4 = &\int_{\mathbb{R}^3} \partial_1^2(b\cdot\nabla b)\cdot\partial_1^3 b + \partial_3^2(b\cdot\nabla b)\cdot\partial_3^2\partial_1 b + \partial_2^2(b\cdot\nabla b)\cdot\partial_2^2\partial_1 b\,dx\\
&\le \|\partial_1 b\|_{H^2}^2\|b\|_{H^3} + \|\partial_1 b\|_{H^2}\|\partial_3 b\|_{H^3}\|b\|_{H^3}\\
&\quad + \|\partial_2^2\partial_1 b\|_{L^2}\Big(\|\nabla b\|_{L^2}^{\frac14}\|\partial_1\nabla b\|_{L^2}^{\frac14}\|\partial_2\nabla b\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla b\|_{L^2}^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_2^2\partial_3 b\|_{L^2}^{\frac12}\\
&\qquad + \|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1\partial_2 b\|_{L^2}^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac14}\|\partial_1\partial_2^2 b\|_{L^2}^{\frac14}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\partial_2\partial_3\nabla b\|_{L^2}^{\frac12}\\
&\qquad + \|b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1\partial_2 b\|_{L^2}^{\frac14}\|\partial_2^2\nabla b\|_{L^2}^{\frac12}\|\partial_2^2\partial_3\nabla b\|_{L^2}^{\frac12}\Big)\\
&\le \|\partial_1 b\|_{H^2}^2\|b\|_{H^3} + \|\partial_1 b\|_{H^2}\|\partial_3 b\|_{H^3}\|b\|_{H^3} + \|\partial_1 b\|_{H^2}^{\frac32}\|\partial_3 b\|_{H^3}^{\frac12}\|b\|_{H^3}.
\end{aligned}
$$

Adding (4.1) and (4.2), integrating in time, invoking the bound for N_1 through N_4 and M_1 through M_4, and applying Hölder's inequality to the time integrals, we obtain (1.6). For the sake of clarity, we provide the details. The time integrals of $N_{1,0}$ and $M_{1,0}$ are bounded by

$$
\begin{aligned}
\int_0^t N_{1,0}\,d\tau &= \int_{\mathbb{R}^3} u(x,t)\cdot\partial_1 b(x,t)\,dx - \int_{\mathbb{R}^3} u(x,0)\cdot\partial_1 b(x,0)\,dx\\
&\le E_0(t)+E_0(0),\\
\int_0^t M_{1,0}\,d\tau &= \sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2 u(x,t)\cdot\partial_i^2\partial_1 b(x,t)\,dx\\
&\quad - \sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^2 u(x,0)\cdot\partial_i^2\partial_1 b(x,0)\,dx \le E_0(t)+E_0(0).
\end{aligned}
$$

By Hölder's inequality,

$$
\int_0^t N_{1,1}d\tau \lesssim E_0^{\frac32}(t)+E_1^{\frac32}(t), \qquad \int_0^t N_{1,3}\,d\tau \lesssim E_0^{\frac32}(t)+E_1^{\frac32}(t).
$$

Clearly,

$$
\int_0^t N_{1,2}\,d\tau \le E_0(t), \qquad \int_0^t N_{1,4}\,d\tau \le E_0(t)
$$

and

$$
\int_0^t N_2\,d\tau \le E_0^{\frac32}(t)+E_1^{\frac32}(t), \qquad \int_0^t N_4\,d\tau \le E_0^{\frac32}(t)+E_1^{\frac32}(t).
$$

The integral of N_3 is slightly different. By Hölder's inequality,

$$\int_0^t N_3\,d\tau \le E_0(t)^{\frac12}\,E_1(t)^{\frac12} \le \frac14 E_1(t) + C\,E_0(t).$$

Furthermore,

$$\int_0^t M_{1,1}\,d\tau,\ \int_0^t M_{1,2}\,d\tau \lesssim E_0^{\frac32}(t) + E_1^{\frac32}(t), \quad \int_0^t M_{1,2}\,d\tau,\ \int_0^t M_{1,4}\,d\tau \lesssim E_0(t),$$
$$\int_0^t M_2\,d\tau \lesssim E_0^{\frac32}(t), \quad \int_0^t M_4\,d\tau \lesssim E_0^{\frac32}(t) + E_1^{\frac32}(t)$$

and

$$\int_0^t M_3\,d\tau \le E_0(t)^{\frac12}\,E_1(t)^{\frac12} \le \frac14 E_1(t) + C\,E_0(t).$$

Combining all the bounds above yields

$$E_1(t) \le E_0(0) + \frac12 E_1(t) + C\,E_0(t) + C\,E_0(t)^{\frac32} + C\,E_1(t)^{\frac32},$$

which gives (1.6). This completes the proof of (1.6). □

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