The Complex Ginzburg-Landau Equation with Data in Sobolev Spaces of Negative Indices

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Abstract

The initial value problem for the complex Ginzburg-Landau equation of the form ($\sigma > 0, A \geq 0, a > 0, b > 0, \nu, \mu$ being real):

$$\partial_t u = Au + (a + i\nu)\Delta u - (b + i\mu)|u|^{2\sigma} u, \quad (x,t) \in \mathbb{R}^n \times (0, \infty)$$

with initial data $u(x, 0) = u_0(x) \in H_p^r$ is considered. The local well-posedness is established for $u_0 \in H_p^r$ if r and p satisfy

$$1 < p < \infty, \qquad \frac{1}{\sigma(2\sigma+1)} \leq \frac{n}{p} < \frac{1}{\sigma}$$

$$r_0 < r < -2\sigma r_0, \qquad r_0 = \frac{n}{p} - \frac{1}{\sigma}$$

This result reduces to H^r theory by setting $p = 2$ [13]. By taking $A = \nu = \mu = 0$, this reproduces a theorem for the nonlinear heat equation in [15].

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1 Introduction

In this paper we consider the initial value problem (IVP) for the complex Ginzburg-Landau (CGL) equation of the general form

$$\partial_t u = Au + (a + i\nu)\Delta u - (b + i\mu)|u|^{2\sigma} u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{1.2}$$

where u is a complex-valued function of space-time and $\sigma > 0, A \geq 0, a > 0, b > 0, \nu, \mu$ are real parameters. We are mainly interested in the well-posedness of the IVP (1.1), (1.2) with initial data $u_0 \in H_p^r$, the Sobolev spaces of negative indices (definition is given below).

The CGL equation has a long history in physics as a model equation describing the onset of instabilities in fluid mechanical systems as well as in the theory of pattern formation and superconductivity. The IVP (1.1), (1.2) has been studied recently ([11], [12], [13]) and the natural settings for the initial data are L^p, H^r and more generally H_p^r. These types of initial data spaces have been used in dealing with other nonlinear equations, for example, the Navier-Stokes equations ([8],[6], [9]), the quasi-geostrophic equation ([3]), the nonlinear heat (NLH) equation ([4],[14],[15]) and the nonlinear Schrödinger (NLS) equation ([2],[5]).

For $s \in \mathbb{R}$, $q \in [1,\infty)$, $H_q^s \equiv H_q^s(\mathbb{R}^n)$ denotes the Sobolev space consisting of all functions f such that

$$(1-\Delta)^{\frac{s}{2}} f \in L^q(\mathbb{R}^n), \quad \text{i.e.,} \quad \|f\|_{s,q} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^q} < \infty$$

Clearly, $H_2^s = H^s$. Some other properties concerning H_q^s can be found in [1]. Especially, we will need the imbedding theorem

$$H_{q_1}^{s_1} \subset H_{q_2}^{s_2}, \quad \text{if} \quad s_1 - \frac{n}{q_1} = s_2 - \frac{n}{q_2}, \quad 1 < q_1 \leq q_2 < \infty,$$

(see e.g. [1], p.153).

Our main result states that if $u_0 \in H_p^r$ with r, p satisfying

$$1 < p < \infty, \qquad \frac{1}{\sigma(2\sigma+1)} \leq \frac{n}{p} < \frac{1}{\sigma}$$

$$r_0 < r < -2\sigma r_0, \qquad r_0 = \frac{n}{p} - \frac{1}{\sigma}$$

then the IVP (1.1), (1.2) is locally well-posed. Precise statement is given in Theorem 3.1.

As we know, σ is said to be critical (resp. supercritical, subcritical) at the level of L^p if $\sigma = \frac{n}{p}$(resp. $>, <$) and at the level of H^r if $\sigma = \frac{2}{n-2r}$ (resp. $>, <$). We can generalize this notion by saying that $\sigma = \frac{p}{n-pr}$ (resp. $>, <$) is critical (resp. supercritical, subcritical) at the level of H_p^r. The local well-posedness result we establish is for the subcritical case. New devices may be necessary to extend the results to critical or supercritical case and the compactness method is a possible choice [5].

Even though there is a large literature on the CGL equation, not many papers deal with the IVP with weak initial data. In [13] Levermore and Oliver prove the local well-posedness with distributional data in $H^r(\mathbb{T}^n)$ in the subcritical case (i.e. $r > \frac{n}{2} - \frac{1}{\sigma}$). Our result reduces to H^r theory on $\mathbb{R}^n$ by setting $p = 2$. The proof in [13] is also based on contraction methods, but the technical details are quite different.

If we let $A = \nu = \mu = 0$, the local well-posedness result reduces to a parallel theorem obtained in [15] for the NLH equation. In [15] the initial data in taken in homogeneous Lebesgue space $\dot{H}_p^r$ (replacing $(1 - \Delta)$ by $-\Delta$ in the definition of H_p^r), rather than H_p^r.

The well-posedness result is proved by contraction mapping arguments. In Section 2 we establish estimates for the solution operator K to the linear equation and the operator G (defined in (2.5)) over spaces of weighted continuous functions in time (see Definition 2.2). These estimates are used in Section 3 to show the contraction.

2 Preliminary estimates

We now consider the linear equation

$$\partial_t u = Au + (a+i\nu)\Delta u, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{2.1}$$

and its solution operator $K(t) = e^{At+((a+i\nu)\Delta t}$. $K(t)$ is represented by the convolution in x, i.e.,

$$K(t)f = k_t * f \equiv \frac{1}{(4\pi(a+i\nu)t)^{n/2}} e^{At - \frac{|x|^2}{4(a+i\nu)t}} * f \tag{2.2}$$

and the kernel k_t satisfies

$$|k_t(x)| \leq \frac{1}{(4\pi|a+i\nu|t)^{n/2}} e^{At - \frac{a|x|^2}{4(a^2+\nu^2)t}}$$

Furthermore, we have

Lemma 2.1 *Let $A \geq 0$, $a > 0$ and ν be real parameters in (2.1).*

(a) *For $t > 0$, $\|k_t\|_{L^1} = \left(1 + \left[\frac{\nu}{a}\right]^2\right)^{\frac{n}{4}} e^{At}$ and therefore*

$$\|k_t * f\|_{L^p} \leq \left(1 + \left[\frac{\nu}{a}\right]^2\right)^{\frac{n}{4}} e^{At} \|f\|_{L^p}, \qquad 1 \leq p < \infty \tag{2.3}$$

(b) *For $s \geq 0$ and $q \in [1, \infty)$,*

$$\|k_t\|_{s,q} = \|(1-\Delta)^{\frac{s}{2}} k_t\|_{L^q} \leq C_k e^{At} t^{\frac{1}{2}\left[-s-n\left(1-\frac{1}{q}\right)\right]} \tag{2.4}$$

for some constant C_k depending on a, ν, s, q.

Proof (a) The value of $\|k_t\|_{L^1}$ is obtained by a simple calculation. The inequality (2.3) is a consequence of Young's inequality.

(b) If $\frac{s}{2}$ is an integer, (2.4) is obtained by a direct calculation. The general case can be dealt with by interpolation.

We shall solve the IVP (1.1), (1.2) in the space of weighted continuous functions in time, which we now introduce. Kato and his collaborators define this type of spaces in solving the IVP for the Navier-Stokes equations with various initial data ([6],[7],[8],[9],[10], [4]).

Definition 2.2 *For $T > 0$ and $\alpha \geq 0$, we define the space*

$$C_{\alpha,s,q} \equiv \{f \in C((0,T); H_q^s), \quad \|f\|_{\alpha,s,q} < \infty\}$$

where the norm $\|f\|_{\alpha,s,q}$ is given by

$$\|f\|_{\alpha,s,q} = \sup\{t^\alpha \|f(\cdot,t)\|_{s,q} : \quad t \in (0,T)\}$$

$\dot{C}_{\alpha,s,q}$ is defined to be a subspace of $C_{\alpha,s,q}$

$$\dot{C}_{\alpha,s,q} = \{f \in C_{\alpha,s,q}((0,T); H_q^s) : \quad \lim_{t \to 0^+} t^\alpha \|f(\cdot,t)\|_{s,q} = 0\}$$

$\dot{C}_{\alpha,s,q}$ is a closed subspace of $C_{\alpha,s,q}$. $f \in C_{\alpha,s,q}$ (resp. $f \in \dot{C}_{\alpha,s,q}$) implies

$$\|f(\cdot,t)\|_{s,q} = O(t^{-\alpha}), \quad \text{(resp.} \quad \|f(\cdot,t)\|_{s,q} = o(t^{-\alpha})), \quad \text{as} \quad t \to 0^+$$

If $T < \infty$, $C_{\alpha,s,q} \subset \dot{C}_{\beta,s,q}$ when $\alpha < \beta$. The case $\alpha = 0$ is special. Clearly, $\dot{C}_{0,s,q} \subset BC([0,T); H_q^s) \subset C_{0,s,q}$, where $BC([0,T); H_q^s)$ denotes the space of H_q^s-valued bounded continuous functions on $[0,T)$. For simplicity, we'll write $\bar{C}_{s,q}$ for $BC([0,T); H_q^s)$.

To solve the IVP (1.1), (1.2), we need to establish estimates for the operators K and G (defined in (2.5)) over the type of spaces we just defined.

Proposition 2.3 *Let $A \geq 0, a > 0$ and ν be real parameters.*

(i) *For $q \in [1,\infty), s \in \mathbb{R}$,*

$$K(t)f \to f \quad \textit{in} \quad H_q^s \quad \textit{as } t \to 0^+$$

and K maps continuously from H_q^s to $\bar{C}_{s,q} \subset C_{0,s,q}$.

(ii) *If q_1, q_2, s_1, s_2 and α_2 satisfy*

$$q_1 \leq q_2, \qquad s_1 \leq s_2$$

$$\alpha_2 = \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$$

then K maps continuously from $H_{q_1}^{s_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$ (When $\alpha_2 = 0$, $\dot{C}_{\alpha_2,s_2,q_2}$ should be replaced by $\bar{C}_{s_2,q_2}$).

Proof. (i) follows from the semi-group properties of K on L^q.

We now show (ii). Let $s_0 = s_2 - s_1 \geq 0$ and $f \in H^{s_1}_{q_1}$. By Young's inequality,

$$\|K(t)f\|_{s_2,q_2} = \|(1-\Delta)^{\frac{s_0}{2}} k_t * \left((1-\Delta)^{\frac{s_1}{2}}\right) f\|_{L^{q_2}} \leq \|(1-\Delta)^{\frac{s_0}{2}} k_t\|_{L^{q_3}} \|f\|_{s_1,q_1}$$

where q_3 satisfies $1 - \frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{q_2}$. Using Lemma 2.1,

$$\|K(t)f\|_{s_2,q_2} \leq C_k e^{At} t^{\frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{n}{q_2}\right)} \|f\|_{s_1,q_1}$$

which implies that K maps continuously from $H^{s_1}_{q_1}$ to C_{α_2,s_2,q_2}. Furthermore, for $\alpha_2 > 0$, K maps into $\dot{C}_{\alpha_2,s_2,q_2}$, not just C_{α_2,s_2,q_2}. This is because K maps smooth functions into $C_{\alpha_2,s_2,q_2} \cap C_{0,s_2,q_2} \subset \dot{C}_{\alpha_2,s_2,q_2}$.

For $\alpha_2 = 0$, $\dot{C}$ is replaced by $\bar{C}$. Another easy proof can also be given in this case. Since now

$$s_1 - \frac{n}{q_1} = s_2 - \frac{n}{q_2}, \qquad q_1 \leq q_2$$

we can use the imbedding $H^{s_1}_{q_1} \subset H^{s_2}_{q_2}$ and (i) to show that K maps $H^{s_1}_{q_1}$ to $\bar{C}_{s_2,q_2}$.

We now give estimate for G:

$$G(g)(t) = \int_0^t K(t-\tau) g(\tau) d\tau \tag{2.5}$$

Proposition 2.4 *If $q_1, q_2, s_1, s_2, \alpha_1$ and α_2 satisfy*

$$q_1 \leq q_2 \tag{2.6}$$

$$0 \leq s_2 - s_1 < 2 - \left[\frac{n}{q_1} - \frac{n}{q_2}\right] \tag{2.7}$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right] \tag{2.8}$$

Then G maps continuously from $\dot{C}_{\alpha_1,s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. Let $g \in \dot{C}_{\alpha_1,s_1,q_1}$ and $s_0 = s_2 - s_1$.

$$\|G(g)(t)\|_{s_2,q_2} = \int_0^t \|(1-\Delta)^{\frac{s_0}{2}} k_t * \left((1-\Delta)^{\frac{s_1}{2}} g(\tau)\right)\|_{L^{q_2}} d\tau$$

Using Young's inequality with $1 - \frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{q_2}$ and Lemma 2.1,

$$\|G(g)(t)\|_{s_2,q_2} \leq \int_0^t \|(1-\Delta)^{\frac{s_0}{2}} k_t\|_{L^{q_3}} \|g(\tau)\|_{s_1,q_1} d\tau$$

$$\leq C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} \int_0^t (t-\tau)^{\frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{1}{q_2}\right)} \tau^{-\alpha_1} d\tau$$

$$= C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} t^{-\alpha_1+1-\frac{1}{2}\left(-s_0-\frac{n}{q_1}+\frac{n}{q_2}\right)} \int_0^1 (1-\rho)^{\frac{1}{2}\left(-s_0-\frac{n}{q_1}+\frac{n}{q_2}\right)} \rho^{-\alpha_1} d\rho$$

Taking $\alpha_2 = \alpha_1 - 1 + \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$,

$$\|G(g)\|_{\alpha_2,s_2,q_2} \leq C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} B\left(1 - \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right], 1-\alpha_1\right)$$

We conclude the proof by noticing that the Beta function B is finite if

$$0 \leq s_2 - s_1 < 2 - \left[\frac{n}{q_1} - \frac{n}{q_2}\right], \quad \alpha_1 < 1$$

3 Well-posedness

We first state the main result.

Theorem 3.1 *Let p and r satisfy*

$$1 < p < \infty, \qquad \frac{1}{\sigma(2\sigma+1)} \leq \frac{n}{p} < \frac{1}{\sigma}, \tag{3.1}$$

$$r_0 < r < -2\sigma r_0, \qquad r_0 \equiv \frac{n}{p} - \frac{1}{\sigma} \tag{3.2}$$

Then for any $\delta > 0$, there is a $T = T(\delta) > 0$ such that for every $u_0 \in H_p^r$ with $\|u_0\|_{r,p} < \delta$ there is a unique solution u of the IVP (1.1), (1.2) on $(0, T)$ satisfying

$$u \in Z_T \equiv \bar{C}_{r,p} \cap \left(\cap_{p \leq q < \infty} \cap_{\lambda > r} \dot{C}_{\frac{\lambda - r}{2} + \frac{1}{2}\left(\frac{n}{p} - \frac{n}{q}\right), \lambda, q} \right) \tag{3.3}$$

Moreover, for any $T' \in (0, T)$, there is a neighborhood V of u_0 in H_p^r such that the solution map

$$\mathfrak{P}: \qquad V \longmapsto Z_{T'}, \quad u_0 \longmapsto u$$

is Lipschitz.

We make several remarks about this theorem.

Remark 3.2 *Since we are mainly concerned about the case $r < 0$, the constraint $r < -2\sigma r_0$ for $r_0 < 0$ is virtually no condition.*

Remark 3.3 *By taking $p = 2$, Theorem 3.1 reduces to H^r theory for the CGL on $\mathbb{R}^n$, parallel to Theorem 4 in [13] where the IVP (1.1), (1.2) with data in $H^r(\mathbb{T}^n)$ is considered.*

Remark 3.4 *We observe that the time interval of existence depends only on $\|u_0\|_{r,p}$, the norm of the initial data. Therefore it is sufficient to obtain global control of H_p^r norm in order to show global existence.*

Theorem 3.1 is proved by contraction mapping arguments. To this end, we write Equation 1.1 in the integral form

$$u = Ku_0 + G(N(u)), \qquad N(u) = -(b + i\mu)|u|^{2\sigma}u \tag{3.4}$$

where K is the solution operator (2.2) to the linear equation (2.1) and G is defined in (2.5). Then we show there is a fixed point for the integral equation (3.4) in an appropriate space. As mentioned earlier, the space of weighted continuous functions in time is suitable.

Proof of Theorem 3.1. For simplicity of notations, we may include the factor e^{At} in the constants $C_1, C_2, C_3, \cdots$ that appear below. This practice causes no side effects because of the insignificant role played by e^{At} here.

Let the space X_T and the norm for $u \in X_T$ given by

$$X_T = \bar{C}_{r,p} \cap \dot{C}_{-\frac{r}{2},0,p}, \qquad \|u\|_{X_T} = \|u\|_{0,r,p} + \|u\|_{-\frac{r}{2},0,p}$$

Let $X_{T,R} \subset X_T$ be the closed ball of radius R centered at 0 and $X_{T,R}$ is a complete metric space. Consider the mapping $\mathcal{A}$ on $X_{T,R}$ with T, R yet to be determined:

$$\mathcal{A}(u) = Ku_0 + G(N(u)), \qquad N(u) = -(b + i\mu)|u|^{2\sigma}u$$

Applying (i) of Proposition 2.3 and (ii) with

$$q_1 = p = q_2, \quad s_1 = r, \quad s_2 = 0, \quad \alpha_2 = -\frac{r}{2}$$

we obtain for two constants C_1 and C_2

$$\|Ku_0\|_{0,r,p} \le C_1\|u_0\|_{r,p}, \qquad \|Ku_0\|_{-\frac{r}{2},0,p} \le C_2\|u_0\|_{r,p}$$

That is, $\|Ku_0\|_{X_T} \le C_3\|u_0\|_{r,p}$ with $C_3 = C_1 + C_2$.

We now estimate $G(N(u))$. Choose p_1 and r_1 such that

$$p_1 = \frac{p}{2\sigma + 1} < p, \qquad r_1 - \frac{n}{p_1} = r - \frac{n}{p}$$

and use the imbedding $H^{r_1}_{p_1} \subset H^r_p$ (see e.g. [1] , p.153) to obtain

$$\|G(N(u))\|_{0,r,p} \le C_4\|G(N(u))\|_{0,r_1,p_1}$$

We can now estimate $\|G(N(u))\|_{0,r_1,p_1}$ by applying Proposition 2.4 with

$$q_1 = q_2 = \frac{p}{2\sigma+1}$$

$$s_1 = 0, \qquad s_2 = r_1 \geq 0$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0, \qquad \alpha_2 = 0$$

because the conditions (2.6), (2.7) and (2.8) are now satisfied under the constraints (3.1), (3.2)

$$r_1 = r - \frac{n}{p} + \frac{n}{p_1} > (2\sigma+1)\left(\frac{n}{p} - \frac{1}{\sigma(2\sigma+1)}\right) \geq 0$$

$$0 < s_2 - s_1 = r_1 = r + 2\sigma\frac{n}{p} < -2\sigma r_0 + 2\sigma\frac{n}{p} = 2$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0 < -\frac{r_0}{2}(2\sigma+1) = -\left(\frac{n}{p} - \frac{1}{\sigma(2\sigma+1)}\right)\left(\sigma + \frac{1}{2}\right) + 1 < 1$$

We obtain

$$\|G(N(u))\|_{0,r_1,p_1} \leq C_5\|N(u)\|_{-\frac{r}{2}-\sigma r_0,0,\frac{p}{2\sigma+1}} \leq C_5 T^{\sigma(r-r_0)}\|N(u)\|_{-\frac{(2\sigma+1)r}{2},0,\frac{p}{2\sigma+1}}$$

Here we've used $r > r_0$ to pick up the factor $T^{\sigma(r-r_0)}$. Thus,

$$\|G(N(u))\|_{0,r,p} \leq C_6 T^{\sigma(r-r_0)}\|u\|^{2\sigma+1}_{-\frac{r}{2},0,p} \leq C_6 T^{\sigma(r-r_0)}\|u\|^{2\sigma+1}_{X_T} \tag{3.5}$$

where $C_6 = C_4 C_5\sqrt{b^2+\mu^2}$.

To estimate $\|G(N(u))\|_{-\frac{r}{2},0,p}$, we apply Proposition 2.4 with

$$q_1 = \frac{p}{2\sigma+1}, \qquad q_2 = p,$$

$$s_1 = 0, \qquad s_2 = 0,$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0, \quad \alpha_2 = -\frac{r}{2}$$

and it is easy to check that the conditions (2.6), (2.7) and (2.8) are satisfied because of the constraints (3.1) , (3.2) and $r_0 < 0$. We obtain for some constant C_7

$$\|G(N(u))\|_{-\frac{r}{2},0,p} \leq C_7\|N(u)\|_{-\frac{r}{2}-\sigma r_0,0,\frac{p}{2\sigma+1}}$$

$$\leq C_8 T^{\sigma(r-r_0)} \|u\|^{2\sigma+1}_{-\frac{r}{2},0,p} \leq C_8 T^{\sigma(r-r_0)} \|u\|^{2\sigma+1}_{X_T} \tag{3.6}$$

where $C_8 = C_7\sqrt{b^2+\mu^2}$.

Now we show that $\mathcal{A}$ is a contraction. Using the inequality

$$||u|^{2\sigma}u - |\tilde{u}|^{2\sigma}\tilde{u}| \leq C_9 \left(|u|^{2\sigma} + |\tilde{u}|^{2\sigma}\right) |u-\tilde{u}|$$

for $\sigma > 0$ and some constant C_9, we have for any $u, \tilde{u} \in X_T$

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} = \|G(N(u)) - G(N(\tilde{u}))\|_{X_T}$$

$$\leq C_{10}\|G((|u|^{2\sigma} + |\tilde{u}|^{2\sigma})|u-\tilde{u}|)\|_{X_T}$$

where $C_{10} = \sqrt{b^2+\mu^2}C_9$. Estimating G in X_T similarly as in (3.5) and (3.6),

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} \leq C_{11} T^{\sigma(r-r_0)} \|(|u|^{2\sigma} + |\tilde{u}|^{2\sigma})|u-\tilde{u}|\|_{-(2\sigma+1)\frac{r}{2},0,\frac{p}{2\sigma+1}}$$

$$\leq C_{11} T^{\sigma(r-r_0)} \left(\|u\|^{2\sigma}_{-\frac{r}{2},0,p} + \|\tilde{u}\|^{2\sigma}_{-\frac{r}{2},0,p}\right) \|u-\tilde{u}\|_{-\frac{r}{2},0,p}$$

$$= C_{11} T^{\sigma(r-r_0)} \left(\|u\|^{2\sigma}_{X_T} + \|\tilde{u}\|^{2\sigma}_{X_T}\right) \|u-\tilde{u}\|_{X_T}$$

Summing up, we've found that for some constants C_H, C_I, C_J

$$\|\mathcal{A}(u)\|_{X_T} \leq C_H \|u_0\|_{r,p} + C_I T^{\sigma(r-r_0)} \|u\|^{2\sigma+1}_{X_T}$$

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} \leq C_J T^{\sigma(r-r_0)} \left(\|u\|^{2\sigma}_{X_T} + \|\tilde{u}\|^{2\sigma}_{X_T}\right) \|u-\tilde{u}\|_{X_T}$$

So the conditions that $\mathcal{A}$ map $X_{T,R}$ into itself and be a contraction are

$$C_H\delta + C_I T^{\sigma(r-r_0)} R^{2\sigma+1} \leq R, \qquad 2C_J T^{\sigma(r-r_0)} R^{2\sigma} < 1 \tag{3.7}$$

For $\delta > 0$, we choose $R > 0$ and $T = T(\delta) > 0$ such that (3.7) is satisfied. Hence it follows from the contraction mapping principle that there is a fixed point $u \in X_{T,R}$ for $\mathcal{A}$, i.e., $u = \mathcal{A}(u)$. That is, u solves (3.4).

We now show that u is actually in Z_T (defined in (3.3)). First we notice that

$$u = \mathcal{A}(u) = Ku_0 + G(N(u))$$

For $p \leq q < \infty$, $\lambda > r$, $Ku_0 \in \dot{C}_{\frac{\lambda-r}{2}+\frac{1}{2}\left(\frac{n}{p}-\frac{n}{q}\right),\lambda,q}$ is an easy consequence of Proposition 2.3. Thus $Ku_0 \in Z_T$. To show $G(N(u)) \in Z_T$, we follow similar procedures of establishing (3.5) and obtain for any $p \leq q < \infty$ and $\lambda > r$

$$\|G(N(u))\|_{\frac{\lambda-r}{2}+\frac{1}{2}\left(\frac{n}{p}-\frac{n}{q}\right),\lambda,q}$$

$$\leq C_{12}T^{\sigma(r-r_0)}\|N(u)\|_{-\frac{(2\sigma+1)r}{2},0,\frac{p}{2\sigma+1}} \leq C_{13}T^{\sigma(r-r_0)}\|u\|_{X_T}^{2\sigma+1}$$

Therefore $u \in Z_T$.

The proof of the uniqueness and the local Lipschitz continuity is routine and thus omitted.

References

[1] , J. Bergh, J. Löfström, Interpolation spaces, Springer-Verlag Berlin Heidelberg New York, 1976.

[2] J. Bourgain, *Nonlinear Schrödinger equations*, Lecture notes.

[3] P. Constantin, A. Majda, E. Tabak, *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar*, Nonlinearity, **7**(1994), 1495-1533

[4] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Diff. Eq., **62**(1986), 186-212.

[5] J. Ginibre, G. Velo, *The Cauchy problem in local spaces for the complex Ginzburg-Landau equation I. compactness methods*, Physica D, **95**(1996), 191-228.

[6] T. Kato, *Strong L^p- solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z., **187**(1984), 471-480.

[7] T. Kato, *The Navier-Stokes solutions for an incompressiblefluid in $\mathbb{R}^2$ with measure as the initial vorticity*, Diff. Integral Equ., **7**(1994), 949-966.

[8] T. Kato, H. Fujita, *On the nonstationary Navier-Stokes equations*, Rend. Sem. mat. Univ. Padova, **32**(1962), 243-260.

[9] T. Kato, G. Ponce, *The Navier-Stokes equations with weak initial data*, Int. Math. Research Notices, **10**(1994), 435-444.

[10] T. Kato, G. Ponce, *Well-posedness of the Euler and the Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbb{R}^2)$*, Rev. Mat. Iber. **2**(1986), 73-88.

[11] C.R. Doering, J.D. Gibbon, C.D. Levermore, *Weak and strong solutions of the complex Ginzburg-Landau equation*, Phsica D, **71**(1994), 285-318.

[12] C.D. Levermore and M. Oliver, *The complex Ginzburg-Landau equation as a model problem*, Lectures in Appl. Math., Vol. **31** (1996), 141-189.

[13] C.D. Levermore and M. Oliver, *Distribution-valued initial data for the complex Ginzburg-Landau equation*, Comm. PDE, 1997, **22**(1997), 39-48.

[14] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math., **38**(1981), 29-40.

[15] J. Wu, *Well-posedness of a semilinear heat equation with weak initial data*, J. Fourier Anal. Appl., **4** (1998), 629-642.