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Generalized MHD equations

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Abstract

Solutions of the d-dimensional generalized MHD (GMHD) equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + b \cdot \nabla b - \nu(-\Delta)^{\alpha} u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - \eta(-\Delta)^{\beta} b \end{cases}$$

are studied in this paper. We pay special attention to the impact of the parameters ν, η, α and β on the regularity of solutions. Our investigation is divided into three major cases: (1) $\nu > 0$ and $\eta > 0$, (2) $\nu = 0$ and $\eta > 0$, and (3) $\nu = 0$ and $\eta = 0$. When $\nu > 0$ and $\eta > 0$, the GMHD equations with any $\alpha > 0$ and $\beta > 0$ possess a global weak solution corresponding to any L^2 initial data. Furthermore, weak solutions associated with $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$ are actually global classical solutions when their initial data are sufficiently smooth. As a special consequence, smooth solutions of the 3D GMHD equations with $\alpha \geqslant \frac{5}{4}$ and $\beta \geqslant \frac{5}{4}$ do not develop finite-time singularities. The study of the GMHD equations with $\nu = 0$ and $\eta > 0$ is motivated by their potential applications in magnetic reconnection. A local existence result of classical solutions and several global regularity conditions are established for this case. These conditions are imposed on either the vorticity $\omega = \nabla \times u$ or the current density $j = \nabla \times b$ (but not both) and are weaker than some of current existing ones. When $\nu = 0$ and $\eta = 0$, the GMHD equations reduce to the ideal MHD equations. It is shown here that the ideal MHD equations admit a unique local solution when the prescribed initial data is in a Hölder space C^r with $r > 1$.

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1. Introduction

We study in this paper a family of equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + b \cdot \nabla b - \nu(-\Delta)^\alpha u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - \eta(-\Delta)^\beta b, \end{cases} \tag{1.1}$$

where $\nu \geqslant 0$, $\eta \geqslant 0$, $\alpha > 0$ and $\beta > 0$ are real parameters, and $u = u(x,t) \in \mathbb{R}^d$, $b = b(x,t) \in \mathbb{R}^d$ and $P = P(x,t) \in \mathbb{R}$ are real-valued functions of $x \in \Omega$ and $t \geqslant 0$. The spatial domain Ω will be either the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$. A fractional power of the Laplace transform, $(-\Delta)^\alpha$, is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = 2\pi|\xi|^{2\alpha}\hat{f}(\xi).$$

More details on $(-\Delta)^\alpha$ can be found in Chapter 5 of Stein's book [15]. Sometimes we write $\Lambda = (-\Delta)^{\frac{1}{2}}$ for notational convenience. When $\alpha = \beta = 1$, (1.1) reduces to the usual MHD equations. In particular, if $\nu = \eta = 0$, (1.1) becomes the ideal MHD equations. It is therefore reasonable to call (1.1) a system of generalized MHD equations, or simply GMHD.

Dissipation corresponding to a fractional power of Laplacian can in principle arise from modeling real physical phenomena, but our motivation for studying (1.1) is mainly mathematical and the goal is to understand how the parameters affect the regularity of its solutions. The issue of whether singularities form in finite time in smooth solutions of the usual MHD equations is still open. For the 3D Euler equations, Beale, Kato and Majda (BKM) showed that no singularities can occur before the magnitude of the vorticity grows without a bound [1]. The work of Constantin [7] and Constantin et al. [8] generalizes the BKM result by linking the vorticity directions and the likelihood of blowup. Extending the BKM result, Caflisch, Klapper and Steele derived a necessary condition for singularity development in the ideal MHD equations [4]. Recently, Gibbon and Ohkitani studied the regularity of a class of stretched solutions to the 3D ideal MHD equations through analytical criteria and pseudo-spectral computations [10]. We hope that the study of the GMHD equations (1.1) will broaden our view on the issue of global regularity.

Our attention will be focused on existence, uniqueness and regularities of solutions in three major parameter domains: (1) $\nu > 0$ and $\eta > 0$, (2) $\nu = 0$ and $\eta > 0$, and (3) $\nu = \eta = 0$. A section will be devoted to each case. Section 2 is focused on (1.1) with $\nu > 0$ and $\eta > 0$. Our ultimate goal here is to determine whether (1.1) admits a global classical solution for any prescribed smooth initial data:

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \tag{1.2}$$

As our first step, we show that (1.1) with $\nu > 0$ and $\eta > 0$ does have a global weak solution if $(u_0, b_0) \in L^2$. The global existence of weak solutions is universal for $\alpha > 0$

and $\beta > 0$ and its proof takes advantage of the regularity available only when the dissipative terms are present. This point will be made precise in the first part of Section 2. The second part of Section 2 is devoted to a priori estimates. When combined with the already established global weak solutions, these bounds allow us to conclude that if

$$\alpha \geqslant \frac{1}{2} + \frac{d}{4} \quad \text{and} \quad \beta \geqslant \frac{1}{2} + \frac{d}{4},$$

then (1.1) with $\nu > 0$ and $\eta > 0$ possesses a global classical solution associated with each pair of smooth functions (u_0, b_0). As a consequence, we recover the classical result of global smooth solutions for the 2D MHD equations [9,14]. A more significant corollary is that smooth solutions of the 3D GMHD equations (1.1) with $\alpha \geqslant \frac{5}{4}$ and $\beta \geqslant \frac{5}{4}$ do not develop finite-time singularities. It is said that Ladyzhenskaya has shown a parallel result for the 3D equations

$$\partial_t u + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^\alpha u$$

with $\alpha \geqslant \frac{5}{4}$, but we were unable to find the reference. A simple proof of her result is provided in the appendix of this paper.

The study of the GMHD equations with $\nu = 0$ and $\eta > 0$ is motivated by their potential applications in magnetic reconnection. In a typical resistive process in MHD reconnection, the viscosity ν is often ignored because of its small effect. In Section 3, we examine in detail how the solutions of (1.1) with $\nu = 0$ behave in hope to achieve a better understanding of the physics models in the theory of MHD reconnection such as the Sweet–Parker model [13]. Our investigation starts with a local existence result for smooth solutions followed by an inquiry into their global (in time) extension. It is not clear whether or not these local solutions can be extended into global ones, but we show here that the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ are bounded for all time if we know before hand that ω or j is in a reasonably regular functional space. These regularity assumptions are imposed on either ω or j (but not both) and thus weaker than some of the existing ones [4,17]. For this purpose, we have also derived a closed form of the GMHD equations representing ω and j.

When $\nu = \eta = 0$, (1.1) reduces to the ideal MHD equations. Due to lack of smoothing mechanism and the strong coupling between the equations of u and b, theoretic issues concerning the ideal MHD equations such as existence and uniqueness are very challenging and few rigorous results are currently available in the literature. Partially motivated by the work of Chemin on the Euler equations [5], we study in Section 4 solutions of the ideal MHD equations with initial data in Hölder spaces. Our major conclusion is that for any initial data (u_0, b_0) in a Hölder space C^r with $r > 1$, the ideal MHD equations have a unique local classical solution. We remark that the regularity requirement on the initial data may be the minimal condition needed to establish any existence result concerning classical solutions. To make the presentation of this section self-contained, we will recall some basic facts concerning the characterization of C^r in terms of dyadic decompositions and the action of paraproduct on C^r.

At the end of this section we introduce several notations which will be used throughout the sequel. For a real number s, the space $H^s(\mathbb{R}^d)$ consists of functions u satisfying

$$||u||_{H^s}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty .$$

When the spatial domain is $\mathbb{T}^d$, H^s is similarly defined but in terms of Fourier series. Clearly, $H^0 = L^2$, but the norm in L^2 will be denoted $||\cdot||$ rather than $||\cdot||_{H^0}$. Finally, we will use $\mathscr{R}_i$ to denote the Riesz transform $\partial_i(-\Delta)^{\frac{1}{2}}$, i.e., $\mathscr{R}_i = \partial_i \Lambda$, $i = 1, \ldots, d$.

2. $\nu > 0$ and $\eta > 0$

In this section we study solutions of the GMHD equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + b \cdot \nabla b - \nu(-\Delta)^{\alpha} u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - \eta(-\Delta)^{\beta} b \end{cases} \tag{2.1}$$

with $\nu > 0$ and $\eta > 0$. Attention will be focused on existence and regularities of weak solutions of (2.1) with $\alpha > 0$ and $\beta > 0$. Naturally this section is divided into two subsections. The first subsection establishes the global existence of weak solutions corresponding to L^2 initial data. We remark that the result obtained here is in sharp contrast to the existing ones for (2.1) with $\nu = 0$ and $\eta = 0$, or the ideal MHD equations. So far only local weak solutions has been shown for the ideal MHD equations even in the 2D case. The proof of the global existence result in the $\nu > 0$ and $\eta > 0$ case takes the advantage of the regularity available only when the dissipative terms are present and its argument is no longer valid when $\nu = 0$ or $\eta = 0$.

One approach leading to global existence of classical solutions is to show that weak solutions corresponding to smooth initial data can be globally regularized. This amounts to proving certain a priori bounds. In the second subsection a priori bounds are obtained to establish that weak solutions of the GMHD equations with $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$ become classical solutions when their initial data are smooth. A special consequence is that smooth solutions of the 3D GMHD equations with $\alpha \geqslant \frac{5}{4}$ and $\beta \geqslant \frac{5}{4}$ do not develop finite-time singularities. It is said that Ladyzhenskaya has proven a parallel result for the 3D Navier–Stokes equations, but we are unable to locate her paper. We present a proof of her result in the appendix.

2.1. L^2 weak solutions

In this subsection we show that the GMHD equations (2.1) with $\nu>0$, $\eta>0$, $\alpha>0$ and $\beta>0$ have a global weak solution corresponding to any prescribed L^2 initial data. The proof presented here is for periodic boundary conditions, but minor modifications will make it work for the whole space case.

We start with a definition of weak solutions for (2.1) with L^2 initial data (u_0, b_0). Let $T>0$ be arbitrarily fixed.

Definition 2.1. A weak solution of (2.1) is a pair of functions (u, b) satisfying

$$u\in L^\infty([0,T];L^2)\cap L^2([0,T];H^\alpha), \quad b\in L^\infty([0,T];L^2)\cap L^2([0,T];H^\beta)$$

and for any test function $v\in C^\infty(\mathbb{T}^d)$

$$\begin{aligned}\int u(t)v\,dx - \int u_0 v\,dx + \int_0^t \int [u\cdot(\nu\Lambda^{2\alpha} - u\cdot\nabla)v \\ + b\cdot\nabla v\cdot b - P(\nabla\cdot v)]\,dx\,d\tau = 0,\end{aligned} \tag{2.2}$$

$$\int b(t)v\,dx - \int b_0 v\,dx + \int_0^t \int [b\cdot(\eta\Lambda^{2\beta} - u\cdot\nabla)v + b\cdot\nabla v\cdot u]\,dx\,d\tau = 0, \tag{2.3}$$

where the spatial integrals are over $\mathbb{T}^d$.

The following theorem states that weak solutions of (2.1) are global.

Theorem 2.2. *Let $T>0$ be fixed. Let $\nu>0$, $\eta>0$, $\alpha>0$ and $\beta>0$. Assume that $u_0\in L^2$ and $b_0\in L^2$. Then the GMHD equations* (2.1) *possess a weak solution obeying Definition* 2.1 *over* $[0,T]$. *Furthermore,*

$$(\partial_t u, \partial_t b)\in L^{\frac{4\alpha}{d}}([0,T];H^{-1})\times L^{\frac{4\beta}{d}}([0,T];H^{-1}).$$

Proof. The tool is the Galerkin approximation. Denote by $\mathbb{P}_N$ the projection of L^2 onto the space spanned by $e^{ik\cdot x}$ with $0<|k|\leqslant N$. Consider a sequence of functions $\{(u^N, b^N)\}$ solving the equations

$$\frac{\partial}{\partial t}u^N + \mathbb{P}_N\{u^N\cdot\nabla u^N + \nabla P^N\} = \mathbb{P}_N\{b^N\cdot\nabla b^N\} + \nu\Lambda^{2\alpha}u^N \tag{2.4}$$

$$\frac{\partial}{\partial t}b^N + \mathbb{P}_N\{u^N\cdot\nabla b^N\} = \mathbb{P}_N\{b^N\cdot\nabla u^N\} + \eta\Lambda^{2\beta}b^N \tag{2.5}$$

with the initial condition

$$u^N(x,0) = u_0^N(x) \equiv \mathbb{P}_N\{u_0(x)\}, \quad b^N(x,0) = b_0^N(x) \equiv \mathbb{P}_N\{b_0(x)\}.$$

Clearly, for $t \in [0,T]$, (u^N, b^N) satisfies the energy inequality

$$\begin{aligned} &||u^N(\cdot,t)||^2 + ||b^N(\cdot,t)||^2 + 2\nu\int_0^t ||\Lambda^\alpha u^N(\cdot,\tau)||^2\, d\tau \\ &\quad + 2\eta\int_0^t ||\Lambda^\beta b^N(\cdot,\tau)||^2\, d\tau \leqslant ||u_0||^2 + ||b_0||^2. \end{aligned} \tag{2.6}$$

Therefore, $\{u^N\}$ is uniformly bounded in $L^\infty([0,T];L^2) \cap L^2([0,T];H^\alpha)$ and $\{b^N\}$ in $L^\infty([0,T];L^2) \cap L^2([0,T];H^\beta)$.

In addition, we show that $\{\partial_t u^N\}$ is bounded uniformly in $L^{\frac{4\alpha}{d}}([0,T];H^{-1})$ and $\{\partial_t b^N\}$ in $L^{\frac{4\beta}{d}}([0,T];H^{-1})$. For this purpose, we use (2.4) and (2.5) to write $\partial_t u^N$ and $\partial_t b^N$ equal to the remaining terms of these equations. It then suffices to show that the terms other than $\partial_t u^N$ and $\partial_t b^N$ in (2.4) and (2.5) are bounded uniformly. In fact, for any $v \in H^1$,

$$|((\mathbb{P}_N\{u^N \cdot \nabla u^N\}, v))| \leqslant ||v||_{H^1}||u^N||_{L^4}^2 \leqslant C||v||_{H^1}||u^N||^{2\gamma_1}||\Lambda^\alpha u^N||^{2-2\gamma_1},$$

where $\gamma_1 = 1 - \frac{d}{4\alpha}$. By the Gagliardo–Nirenberg inequality,

$$\begin{aligned} |(\mathbb{P}_N(b^N \cdot \nabla u^N), v)| &\leqslant C||v||_{H^1}||b^N||_{L^4}||u^N||_{L^4} \\ &\leqslant C||v||_{H^1}||b^N||^{\gamma_2}||\Lambda^\alpha b^N||^{1-\gamma_2}||u^N||^{\gamma_1}||\Lambda^\alpha u^N||^{1-\gamma_1} \\ &\leqslant C||v||_{H^1}||b^N||^{\gamma_2}||u^N||^{\gamma_1}(||\Lambda^\alpha b^N||^{2-2\gamma_2} + ||\Lambda^\alpha u^N||^{2-2\gamma_1}), \end{aligned} \tag{2.7}$$

where $\gamma_2 = 1 - \frac{d}{4\beta}$. To deal with the pressure term, we notice that

$$P^N = (-\Delta)^{-1}\nabla \cdot (u^N \cdot \nabla u^N - b^N \cdot \nabla b^N).$$

Thus

$$\begin{aligned} |(\mathbb{P}_N\{\nabla P^N\}, v)| &\leqslant C||v||_{H^1}(||u^N||_{L^4}^2 + ||b^N||_{L^4}^2) \\ &\leqslant C||v||_{H^1}(||u^N||^{2\gamma_1}||\Lambda^\alpha u^N||^{2-2\gamma_1} + ||b^N||^{2\gamma_2}||\Lambda^\alpha b^N||^{2-2\gamma_2}). \end{aligned}$$

Similar estimates can be obtained for the remaining terms. By (2.6),

$$||u^N||, ||b^N|| \in L^\infty([0,T]); \quad ||\Lambda^\alpha u^N||^2, ||\Lambda^\beta b^N||^2 \in L^1([0,T]).$$

Therefore

$$(\partial_t u^N, \partial_t b^N) \in L^{\frac{4\alpha}{d}}([0,T]; H^{-1}) \times L^{\frac{4\beta}{d}}([0,T]; H^{-1}). \tag{2.8}$$

Since $\{(u^N, b^N)\}$ is bounded uniformly in $L^\infty([0,T]; L^2)$, there exists $(\tilde{u}, \tilde{b}) \in L^\infty([0,T]; L^2)$ and a subsequence of (u^N, b^N) such that this subsequence converges weakly to $(\tilde{u}, \tilde{b})$. But this weak convergence does not allow one to pass to the limit in the nonlinear terms. Fortunately this subsequence actually converges in the strong norm of $L^2([0,T]; L^2)$. The strong convergence can be proven using the Lions–Aubin compactness theorem [6,16], which states in our situation that $L^2([0,T]; L^2)$ is compactly imbedded in the space

$$\left\{(u,b)\colon\ (u,b) \in L^2([0,T]; H^1), (\partial_t u, \partial_t b) \in L^{\frac{4\alpha}{d}}([0,T]; H^{-1}) \times L^{\frac{4\beta}{d}}([0,T]; H^{-1})\right\}.$$

According to (2.6) and (2.8), the sequence (u^N, b^N) belongs to the above space. Therefore, (u^N, b^N) has a subsequence that converges strongly to $(\tilde{u}, \tilde{b})$ in $L^2([0,T]; L^2)$.

The strong convergence of (u^N, b^N) to $(\tilde{u}, \tilde{b})$ in $L^2([0,T]; L^2)$ will allow us to show that $(\tilde{u}, \tilde{b})$ is indeed a weak solution of (2.1). Let $v \in C^\infty(\mathbb{T}^d)$. Dotting both sides of (2.4) and integrating over $\mathbb{T}^d \times [0,t]$, there obtains

$$\begin{aligned} 0 = &\int u^N(t) v\, dx - \int_0^N v\, dx \\ &+ \int_0^t \int [u^N \cdot (\nu \Lambda^{2\alpha} v^N - u^N \cdot \nabla v^N) + b^N \cdot (b^N \cdot \nabla v^N) - P^N (\nabla \cdot v^N)]\, dx\, d\tau. \end{aligned}$$

Let $N \to \infty$. Passing to the limits in the linear terms is trivial. A simple argument combined with the strong convergence (u^N, b^N) to $(\tilde{u}, \tilde{b})$ in $L^2([0,T]; L^2)$ will also allow one to pass to the limit in the nonlinear terms. For example, to show the convergence

$$\int u^N \cdot (u^N \cdot \nabla v^N)\, dx \to \int \tilde{u} \cdot (\tilde{u} \cdot \nabla v)\, dx,$$

we write

$$\begin{aligned} &\int [u^N \cdot (u^N \cdot \nabla v^N) - \tilde{u} \cdot (\tilde{u} \cdot \nabla v)]\, dx \\ &\quad = \int [(u^N - \tilde{u}) \cdot (u^N \cdot \nabla v^N) + \tilde{u} \cdot ((u^N - \tilde{u}) \cdot \nabla v^N) \\ &\qquad + \tilde{u} \cdot (\tilde{u} \cdot \nabla (v^N - v))]\, dx. \end{aligned} \tag{2.9}$$

The first term in (2.9) converges to zero because $u^N \to \tilde{u} \in L^2([0,T]; L^2)$. The weak convergence u^N to $\tilde{u}$ sends the middle term to zero and the last term is zero because of $v^N \to v$ in L^2. Thus, $(\tilde{u}, \tilde{b})$ satisfies (2.2). Similarly, invoking (2.5), one can show that $(\tilde{u}, \tilde{b})$ satisfies (2.3). This completes the proof of Theorem 2.2. $\square$

2.2. Global regularity of weak solutions

Weak solutions combined with appropriate a priori bounds leads to the global existence of classical solutions. In this subsection we establish a priori estimates for the GMHD equations (2.1) with $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$. This result and the previously established global weak solutions allow us to conclude that (2.1) has global classical solutions when $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$. In particular, we recover the global existence result for the 2D MHD equations [9,14]. Another special consequence is that smooth solutions of the 3D GMHD equations (2.1) with $\alpha \geqslant \frac{5}{4}$ and $\beta \geqslant \frac{5}{4}$ cannot develop finite-time singularities. The issue that remains open is whether or not the 3D GMHD equations with $\alpha = 1$ and $\beta = 1$, or the usual 3D MHD equations have a global smooth solution for any prescribed smooth initial data.

Theorem 2.3. *Let $T > 0$. Assume that $v > 0$, $\eta > 0$, α and β satisfy*

$$\alpha \geqslant \frac{1}{2} + \frac{d}{4} \quad \text{and} \quad \beta \geqslant \frac{1}{2} + \frac{d}{4}. \tag{2.10}$$

Let $u_0 \in H^s$ and $b_0 \in H^s$ with $s \geqslant \max\{2\alpha, 2\beta\}$. Then the corresponding weak solution (u, b) of the GMHD equations (2.1) *established in the previous subsection is actually a classical solution satisfying*

$$u \in L^\infty([0,T]; H^s) \cap L^2([0,T]; H^{s+\alpha}), \quad b \in L^\infty([0,T]; H^s) \cap L^2([0,T]; H^{s+\beta}).$$

Remark. When $d = 3$, (2.10) reduces to $\alpha \geqslant \frac{5}{4}$ and $\beta \geqslant \frac{5}{4}$. Ladyzhenskaya has obtained a parallel result for the equation

$$\partial_t u + u \cdot \nabla u = -\nabla p - v(-\Delta)^\alpha u$$

with $\alpha \geqslant \frac{5}{4}$, but we were unable to find the reference. We will provide a proof of her result in the appendix. In a recent work Mattingly and Sinai [12] gave a proof of Ladyzhenskaya's result using the methods of dynamical systems, but their proof does not include the case $\alpha = \frac{5}{4}$.

Proof of Theorem 2.3. First recall that u and b satisfy

$$||u||^2 + ||b||^2 + 2v \int_0^t ||\Lambda^\alpha u||^2 \, d\tau + 2\eta \int_0^t ||\Lambda^\beta b||^2 \, d\tau = ||u_0||^2 + ||b_0||^2. \tag{2.11}$$

Multiplying the first equation of (2.1) by Δu and the second by Δb, integrating by parts and adding the results, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int [|\Lambda u|^2+|\Lambda b|^2]\,dx + \int [\nu|\Lambda^{\alpha+1}u|^2+\eta|\Lambda^{\beta+1}b|^2]\,dx\\ = \int \partial_k u_i\cdot\partial_i u_j\cdot\partial_k u_j\,dx + \int \partial_k u_i\cdot\partial_i b_j\cdot\partial_k b_j\,dx.\\ -\int \partial_k b_i\cdot\partial_i b_j\cdot\partial_k u_j\,dx - \int \partial_k b_i\cdot\partial_i u_j\cdot\partial_k b_j\,dx.\end{aligned}$$

The first term on the right is bounded by $||\nabla u||_{L^3}^3$ and the other three terms are bounded by $||\nabla u||_{L^3}||\nabla b||_{L^3}^2$. For α and β satisfying (2.10), we set

$$a_1 = 1-\frac{1}{3\alpha}\left(1+\frac{d}{2}\right),\quad a_2=\frac{1}{3},\quad a_3=\frac{1}{3\alpha}\left(1-\alpha+\frac{d}{2}\right)$$

and

$$e_1 = 1-\frac{1}{3\beta}\left(1+\frac{d}{2}\right),\quad e_2=\frac{1}{3},\quad e_3=\frac{1}{3\beta}\left(1-\beta+\frac{d}{2}\right).$$

Obviously, these indices satisfy

$$a_1+a_2+a_3=1,\quad a_1+\alpha a_2+(1+\alpha)a_3 = 1+\frac{d}{6}$$

and

$$e_1+e_2+e_3=1,\quad e_1+\beta e_2+(1+\beta)e_3 = 1+\frac{d}{6}.$$

Therefore, by the Gagliardo–Nirenberg inequality,

$$||\nabla u||_{L^3}\leqslant C||\nabla u||^{a_1}||\Lambda^\alpha u||^{a_2}\,||\Lambda^{\alpha+1}u||^{a_3}$$

and

$$||\nabla b||_{L^3}\leqslant C||\nabla b||^{e_1}||\Lambda^\beta b||^{e_2}||\Lambda^{\beta+1}b||^{e_3}.$$

Since $\frac{3a_1}{2}+\frac{3a_3}{2}=1$ and $\frac{3e_1}{2}+\frac{3e_3}{2}=1$, we apply Young's inequality to obtain

$$||\nabla u||_{L^3}^3\leqslant\frac{\nu}{8}||\Lambda^{\alpha+1}u||^2 + C_\nu||\Lambda^\alpha u||^{2a_2/a_1}||\nabla u||^2$$

$$\begin{aligned}||\nabla u||_{L^3}||\nabla b||_{L^3}^2 &\leqslant C||\nabla u||^{a_1}||\Lambda^\alpha u||^{a_2}||\Lambda^{\alpha+1}u||^{a_3}\cdot||\nabla b||^{2e_1}||\Lambda^\beta b||^{2e_2}||\Lambda^{\beta+1}b||^{2e_3}\\ &\leqslant \frac{\nu}{8}||\Lambda^{\alpha+1}u||^2+\frac{\eta}{8}||\Lambda^{\beta+1}b||^2 + C_\nu||\Lambda^\alpha u||^{2a_2/a_1}||\nabla u||^2\\ &\quad + C_\eta||\Lambda^\beta b||^{2e_2/e_1}||\nabla b||^2.\end{aligned}$$

Combining these estimates, we reach the closed inequality

$$\frac{d}{dt}\int [|\Lambda u|^2 + |\Lambda b|^2]\,dx + \int [\nu|\Lambda^{\alpha+1}u|^2 + \eta|\Lambda^{\beta+1}b|^2]\,dx$$
$$\leqslant C_\nu ||\Lambda^\alpha u||^{2a_2/a_1}||\nabla u||^2 + C_\eta ||\Lambda^\beta b||^{2e_2/e_1}||\nabla b||^2. \tag{2.12}$$

For $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$, $\frac{2a_2}{a_1} \leqslant 2$ and $\frac{2e_2}{e_1} \leqslant 2$. It then follows from (2.11) that for any $T > 0$ and $t \in [0, T]$

$$\int_0^t ||\Lambda^\alpha u||^{2a_2/a_1}\,d\tau < \infty, \quad \int_0^t ||\Lambda^\beta u||^{2e_2/e_1}\,d\tau < \infty.$$

We then conclude from (2.12) that

$$u \in L^\infty([0,T]; H^1) \cap L^2([0,T]; H^{1+\alpha}), \quad b \in L^\infty([0,T]; H^1) \cap L^2([0,T]; H^{1+\beta}).$$

We remark that higher regularities can be established inductively. For example, to bound the second derivatives, one starts with the basic inequality

$$\frac{1}{2}\frac{d}{dt}\int [|\Lambda^2 u|^2 + |\Lambda^2 b|^2]\,dx + \int [\nu|\Lambda^{\alpha+2}u|^2 + \eta|\Lambda^{\beta+2}b|^2]\,dx$$
$$\leqslant C||\nabla u||\left(||\Lambda^2 u||_{L^4}^2 + ||\Lambda^2 b||_{L^4}^2\right) + C||\nabla b||||\Lambda^2 u||_{L^4}||\Lambda^2 b||_{L^4}.$$

A closed inequality is then obtained after inserting the estimates

$$||\Lambda^2 u||_{L^4}^2 \leqslant C||\Lambda^2 u||^{2-\frac{d}{2\alpha}}||\Lambda^{2+\alpha}u||^{\frac{d}{2\alpha}} \leqslant \frac{\nu}{2}||\Lambda^{2+\alpha}u||^2 + C_\nu||\Lambda^2 u||^2,$$

$$||\Lambda^2 b||_{L^4}^2 \leqslant C||\Lambda^2 b||^{2-\frac{d}{2\beta}}||\Lambda^{2+\beta}u||^{\frac{d}{2\beta}} \leqslant \frac{\eta}{2}||\Lambda^{2+\beta}b||^2 + C_\eta||\Lambda^2 b||^2$$

and using the bounds related to the first-order derivatives

$$||\nabla u(\cdot,t)||, ||\nabla b|| \in L^\infty([0,T]); \quad ||\Lambda^\alpha u||^2, ||\Lambda^\beta b||^2 \in L^1([0,T]).$$

We omit further details. This completes the proof of Theorem 2.3. □

3. $\nu = 0$ and $\eta > 0$

In this section, attention will be directed to (1.1) with $\nu = 0$, namely

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla P + b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b = b\cdot\nabla u - \eta(-\Delta)^\beta b, \end{cases} \tag{3.1}$$

where $\eta > 0$ and $\beta > 0$. The study of (3.1) is partially motivated by its potential applications in magnetic reconnection. In a typical resistive process of MHD reconnection, the viscosity ν has a small effect but the magnetic diffusion plays a crucial role. Our ultimate goal is to have a complete understanding of several well-known models in MHD reconnection such as the Sweet–Parker model [13].

The spatial domain is assumed to be either the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$. We are primarily concerned with the existence of global classical solutions of (3.1) associated with the initial condition

$$u(x,0) = u_0(x) \quad \text{and} \quad b(x,0) = b_0(x). \tag{3.2}$$

For clarity, this section is divided into two subsections. In the first subsection we prove the existence of a local classical solution. In the second subsection we derive several conditions under which the local solution can be extended into a global one. We emphasize that these conditions are valid for a whole range of magnetic diffusion and are weaker than some of the known ones [4,17]. For this purpose, we derive the MHD equations representing the vorticity ω and the current density j as well as the vorticity and the magnetic flux function ψ.

3.1. Local classical solutions

We show in this subsection that the initial-value problem (3.1)–(3.2) possesses a unique local classical solution when u_0 and b_0 are sufficiently smooth. The arguments in the proof apply to both $\eta > 0$ and $\eta = 0$ cases.

Theorem 3.1. *Let $\eta \geqslant 0$ and $\beta > 0$. Assume that $u_0 \in H^m$* and *$b_0 \in H^m$ with $m > \max\{2, \beta\} + d/2$. Then there exists a T depending only on u_0 and b_0 such that* (3.1)–(3.2) *possesses a unique classical solution (u, b), which remains in $\in L^\infty([0, T]; H^m)$.*

Proof. Let $\varepsilon > 0$ be a small parameter and $0 < \alpha \leqslant \beta$ be fixed. Consider the regularized equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + b \cdot \nabla b - \varepsilon(-\Delta)^\alpha u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - \eta(-\Delta)^\beta b \end{cases} \tag{3.3}$$

with initial data $u(x,0) = u_0(x)$ and $b(x,0) = b_0(x)$. Theorem 2.2 of the previous section implies that (3.3) possesses a global weak solution $(u_\varepsilon, b_\varepsilon)$. We now show that there exists some finite-time interval $[0, T]$ over which $(u_\varepsilon, b_\varepsilon)$ is regular and converges to a classical solution (u, b) of (3.1). This mounts to establishing certain a priori bounds for $(u_\varepsilon, b_\varepsilon)$ thanks to the standard approximating procedures [3,6]. More specifically, we need to show that $(u_\varepsilon, b_\varepsilon)$ is uniformly bounded in H^m over $[0, T]$ and $(\partial_t u_\varepsilon, \partial_t b_\varepsilon)$ in H^s for some $s < m$. This is accomplished in the following lemma.

Lemma 3.2. *There exist a $T>0$ and a constant c independent of* ε *such that*

$$\sup_{t\in[0,T]} (||u_\varepsilon(\cdot,t)||_{H^m} + ||b_\varepsilon(\cdot,t)||_{H^m}) \leqslant c, \tag{3.4}$$

$$\sup_{t\in[0,T]} (||\partial_t u_\varepsilon(\cdot,t)||_{H^s} + ||\partial_t b_\varepsilon(\cdot,t)||_{H^s}) \leqslant c, \tag{3.5}$$

where $s = m - \beta$.

Proof. Let σ be a multi-index with $|\sigma| = m$. One easily verifies that

$$\frac{1}{2}\frac{d}{dt}||D^\sigma u_\varepsilon||^2 + \varepsilon \int \Lambda^{2\alpha}(D^\sigma u_\varepsilon)\cdot(D^\sigma u_\varepsilon)\,dx$$
$$= -\int (D^\sigma u_\varepsilon)\cdot\nabla u_\varepsilon\cdot(D^\sigma u_\varepsilon)\,dx + \int (D^\sigma b_\varepsilon)\cdot\nabla b_\varepsilon\cdot(D^\sigma u_\varepsilon)\,dx$$

and

$$\frac{1}{2}\frac{d}{dt}||D^\sigma b_\varepsilon||^2 + \eta \int \Lambda^{2\beta}(D^\sigma b_\varepsilon)\cdot(D^\sigma b_\varepsilon)\,dx$$
$$= -\int (D^\sigma u_\varepsilon)\cdot\nabla b_\varepsilon\cdot(D^\sigma b_\varepsilon)\,dx + \int (D^\sigma b_\varepsilon)\cdot\nabla u_\varepsilon\cdot(D^\sigma b_\varepsilon)\,dx.$$

Therefore, for some constant c,

$$\frac{d}{dt}(||D^\sigma u_\varepsilon||^2 + ||D^\sigma b_\varepsilon||^2)\,dx \leqslant c(||\nabla u_\varepsilon||_{L^\infty} + ||\nabla b_\varepsilon||_{L^\infty})(||D^\sigma u_\varepsilon||^2 + ||D^\sigma b_\varepsilon||^2).$$

Applying Sobolev's imbedding theorem, we have for $m>1+d/2$

$$||\nabla u_\varepsilon||_{L^\infty} \leqslant C||u_\varepsilon||_{H^m} \quad \text{and} \quad ||\nabla b_\varepsilon||_{L^\infty} \leqslant C||b_\varepsilon||_{H^m}.$$

Thus we have deduced that for some constant c

$$\frac{d}{dt}(||u_\varepsilon||_{H^m} + ||b_\varepsilon||_{H^m}) \leqslant c(||u_\varepsilon||^2_{H^m} + ||b_\varepsilon||^2_{H^m}).$$

This inequality implies immediately that there exists a $T = T(||u_0||_m, ||b_0||_m)$ such that $(u_\varepsilon, b_\varepsilon)$ is bounded uniformly with respect to both $\varepsilon \geqslant 0$ and $t\in[0,T]$. This proves (3.4).

To show (3.5), we recall the basic inequality that for $s>d/2$

$$||f\cdot g||_{H^s} \leqslant C||f||_{H^s}||g||_{H^s}. \tag{3.6}$$

Using (3.3), we can rewrite $\partial_t u_\varepsilon$ and $\partial_t b_\varepsilon$ in terms of the other terms in (3.3). (3.5) is then obtained by taking H^s norm and applying (3.6). $\square$

Proof of Theorem 3.1 (Conclusion). Lemma 3.2 will allow us to use the Aubin–Nitche compactness theorem [11]. According to their compactness theorem, $\{(u_\varepsilon, b_\varepsilon)\}$ is compact in $C([0,T]; H^s)$. By Sobolev's imbedding theorem, it is also compact in $C([0,T]; C^2)$. Therefore, one can pass to the limit in the nonlinear terms of (3.3). This completes the proof of Theorem 3.1. $\square$

3.2. *Conditions for global solutions*

Local smooth solutions obtained in the previous subsection are not known to be global, but they can be extended into global solutions if they further satisfy appropriate regularity conditions. Several such conditions are derived in this subsection for the two-dimensional case. These conditions are expressed in terms of the vorticity and current density as well as the magnetic field. This subsection consists of three parts labeled explicitly for the clarity of presentation.

3.2.1. *Equations of ω and j*

In this portion we derive a special form of the MHD equations representing ω and j. The v-term is kept in the derivation for the purpose of future references. As we have mentioned before, we restrict our consideration to the two-dimensional case.

Taking the curl of Eqs. (1.1), we find that the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy the equations

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j - v(-\Delta)^\alpha \omega, \\ \partial_t j + u \cdot \nabla j = b \cdot \nabla \omega - 2\{u_n, b_n\} - \eta(-\Delta)^\beta j. \end{cases} \tag{3.7}$$

Here the repeated index n is summed and the notation $\{f, g\}$ stands for the Poisson bracket of two scalar functions f and g, i.e.,

$$\{f, g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}.$$

The Poisson bracket can be verified to possess some fine properties. If $\nabla \cdot u = 0$ and $D_t = \partial_t + u \cdot \nabla$ denotes the material derivative, then the following product rule holds:

$$D_t\{f, g\} = \{D_t f, g\} + \{f, D_t g\}.$$

Furthermore, for any frozen-in vector field b, i.e., $[D_t, b \cdot \nabla] = 0$,

$$b \cdot \nabla\{f, g\} = \{b \cdot \nabla f, g\} + \{f, b \cdot \nabla g\},$$

where $[\ \ ]$ denotes the commutator operator.

The term $\{u_n, b_n\}$ in (3.7) can be further expressed in terms of ω and j. In fact, if $\mathscr{P}$ and $\mathscr{Q}$ denote the deformation tensors for u and b, respectively, namely

$$\mathscr{P}_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad \mathscr{Q}_{ij} = \frac{1}{2}(\partial_i b_j + \partial_j b_i),$$

then $\{u_n, b_n\}$ can be written as

$$\{u_n, b_n\} = \varepsilon_{il}\mathscr{P}_{ik}\mathscr{Q}_{lk},$$

where ε_{il} is the standard permutation symbol and the repeated indices are summed. In addition, the deformation tensors $\mathscr{P}$ and $\mathscr{Q}$ can be recovered from w and j through a singular integral operator with the kernel K. More precisely,

$$\mathscr{P} = K * \omega \quad \text{and} \quad \mathscr{Q} = K * j,$$

where K assumes the explicit form

$$K(x) = K((x_1, x_2)) = \frac{1}{|x|^4}\begin{pmatrix} 2x_1x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1x_2 \end{pmatrix}. \tag{3.8}$$

Thus, we have obtained a "closed form" of the MHD equations of ω and j

$$\begin{cases} D_t\omega = b \cdot \nabla j - \nu(-\Delta)^{\alpha}\omega, \\ D_t j = b \cdot \nabla\omega - 2\varepsilon_{il}(K_{ik} * \omega)(K_{lk} * j) - \eta(-\Delta)^{\beta} j. \end{cases} \tag{3.9}$$

When the spatial domain is either the whole space $\mathbb{R}^2$ or the torus $\mathbb{T}^2$, the above equations are supplemented with initial conditions

$$\omega(x, 0) = \omega_0(x) \quad \text{and} \quad j(x, 0) = j_0(x).$$

3.2.2. Conditions in terms of ω and j

In this portion we set $\nu = 0$ and derive regularity conditions based on the equations of ω and j, namely (3.9). Our intention is to establish maximal regularities for solutions of (3.9) by imposing minimal conditions on either ω or j (but not both). A natural type of conditions is regularity in $L^q([0, T]; L^p)$ for $1 \leqslant p \leqslant \infty$. We also remark that some of the results here are valid not only for $\eta > 0$ but also for $\eta = 0$.

Theorem 3.3. *Assume that $\eta \geqslant 0$. Let ω_0 and j_0 be smooth, say, $(\omega_0, j_0) \in H^m$ with $m > \max\{2, \beta\} + d/2$ and (ω, j) be the corresponding solution of the 2D MHD equation* (3.9). *If, for some $T > 0$,*

$$\int_0^T ||j(\cdot, t)||_{L^\infty}\, dt < \infty, \tag{3.10}$$

then $\omega, j \in L^\infty([0, \infty]; L^2)$, and $j \in L^2([0, T]; H^\beta)$ when $\eta > 0$.

Remark. Theorem 3.3 is valid for both $\eta > 0$ and $\eta = 0$.

Remark. We have made no efforts here to optimize the regularity requirement for the initial condition. Our main concern is the global existence associated with sufficiently smooth data.

Remark. Caflisch, Klapper and Steele have shown in [4] that a smooth solution of the ideal MHD equations is global if the corresponding vorticity ω and current density j obeys

$$\int_0^\infty (||\omega(\cdot,t)||_{L^\infty} + ||j(\cdot,t)||_{L^\infty})\, dt < \infty. \tag{3.11}$$

Theorem 3.3 above indicates that it suffices to impose a condition solely on j, i.e., (3.10) in order for ω and j to remain in $L^\infty([0,T];L^2)$. However, it seems that (3.10) alone is not sufficient to show that $\nabla\omega$ and ∇j are in $L^\infty([0,T];L^2)$. We will explain a little bit more at the end of this portion.

Proof of Theorem 3.3. One easily deduces from (3.9) that

$$\frac{d}{dt}(||\omega||^2 + ||j||^2) + 2\eta \int |\Lambda^\beta j|^2\, dx = -2\varepsilon_{il} \int (K_{ik} * \omega)(K_{lk} * j) \cdot j\, dx. \tag{3.12}$$

Since K is the kernel of a standard singular integral operator, we have for some pure constant C that

$$||K_{ik} * \omega||_{L^2} \leqslant C||\omega||_{L^2} \quad \text{and} \quad ||K_{jk} * j||_{L^2} \leqslant C||j||_{L^2}.$$

Therefore the right-hand side of (3.12) is bounded by

$$\begin{aligned}\left|2\varepsilon_{ij} \int (K_{ik} * \omega)(K_{jk} * j) \cdot j\right| &\leqslant C||j||_{L^\infty}||||\omega||_{L^2}\, ||j||_{L^2}\\ &\leqslant C||j||_{L^\infty}||(||\omega||_{L^2}^2 + ||j||_{L^2}^2).\end{aligned}$$

Inserting this bound in (3.12) and applying Gronwall's inequality, one obtains

$$\omega, j \in L^\infty([0,T];L^2) \quad \text{and} \quad j \in L^2([0,T];H^\beta).$$

This concludes the proof of Theorem 3.3. $\square$

The following theorem exploits the regularity associated with the η-term.

Theorem 3.4. *Let* $(\omega_0, j_0) \in H^m$ *with* $m > \max\{2,\beta\} + d/2$ *and* (ω, j) *be the corresponding solution of the MHD equations* (3.9) *with* $\eta > 0$. *Let* p *and* q *be two* *indices satisfying*

$$p > \frac{1}{\beta}, \quad q > 1, \quad \frac{1}{\beta p} + \frac{1}{q} = 1.$$

If, for some $T > 0$,

$$\int_0^T \|\omega(\cdot,t)\|_{L^p}^q \, dt < \infty,$$

then $(\omega, j) \in L^\infty([0,T]; L^2)$ *and* $j \in L^2([0,T]; H^\beta)$.

Proof. For $q_1 > 1$ and $r_1 > 1$ satisfying $1/p + 1/q_1 + 1/r_1 = 1$,

$$\begin{aligned} \left| \int (K_{ik} * \omega)(K_{lk} * j) j \, dx \right| &\leqslant C \|K_{ik} * \omega\|_{L^p} \|K_{lk} * j\|_{L^{q_1}} \|j\|_{L^{r_1}} \\ &\leqslant C \|\omega\|_{L^p} \|j\|_{L^{q_1}} \|j\|_{L^{r_1}}, \end{aligned}$$

where C is pure constant. Using the Gagliardo–Nirenberg inequalities

$$\|j\|_{L^{q_1}} \leqslant C \|j\|_{L^2}^{1-\frac{1}{\beta}+\frac{2}{\beta q_1}} \|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}-\frac{2}{\beta q_1}} \quad \text{and} \quad \|j\|_{L^{r_1}} \leqslant C \|j\|_{L^2}^{1-\frac{1}{\beta}+\frac{2}{\beta r_1}} \|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}-\frac{2}{\beta r_1}},$$

we obtain

$$\left| \int (K_{ik} * \omega)(K_{lk} * j) j \, dx \right| \leqslant C \|\omega\|_{L^p} \|j\|_{L^2}^{2-\frac{2}{\beta p}} \|\Lambda^\beta j\|_{L^2}^{\frac{2}{\beta p}}.$$

By Young's inequality,

$$\left| \int (K_{ik} * \omega)(K_{lk} * j) j \, dx \right| \leqslant \frac{\eta}{2} \|\Lambda^\beta j\|_{L^2}^2 + C_\eta \|\omega\|_{L^p}^q \|j\|_{L^2}^2,$$

where C_η is a constant depending on η and $1/(\beta p) + 1/q = 1$. Inserting this bound in (3.12), we obtain

$$\frac{d}{dt} (\|j\|^2 + \|\omega\|^2) + \eta \|\Lambda^\beta j\|^2 \leqslant C_\eta \|\omega\|_{L^p}^q \|j\|_{L^2}^2.$$

The proof is then completed after applying Gronwall's inequality. $\square$

In the next theorem we require the finiteness of a quantity involving L^1-norm of ω, the lowest norm of vorticity magnitude. We will need a lemma which provides a L^∞ bound for the operator $K*$ (K is defined in (3.8)). $K*$ is a bounded operator from L^p to L^p for $1 < p < \infty$, but there are explicit examples showing that $K*$ does not always map L^∞ into L^∞. The following lemma states that $K * V$ is in L^∞ if V is locally Hölder continuous.

Lemma 3.5. *Let $V \in L^1(\mathbb{R}^2)$ be locally Hölder continuous in the sense that there exists some $\rho \in (0,1)$ and constant C*

$$|V(x) - V(y)| \leqslant C|x-y|^\rho \tag{3.13}$$

for any $x, y \in \mathbb{R}^2$ with $|x-y| < ||V||_{L^1}$. Then $K * V$ is in L^∞ with

$$||K * V||_{L^\infty} \leqslant C||V||_{L^1}^{\frac{\rho}{2+\rho}}.$$

Proof. Recall that for any $r > 0$

$$\int_{|x|<r} K(x)\,dx = 0.$$

Using this fact, we have

$$\begin{aligned} K * V &= \int_{\mathbb{R}^2} (V(x-y) - V(x))K(y)\,dx \\ &= \int_{S_1}\int_0^\infty \frac{V(x-re^{i\theta}) - V(x)}{r} K_1(\theta)\,dr\,d\theta, \end{aligned}$$

where S_1 denotes the unit circle and

$$K_1(\theta) = \begin{pmatrix} 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \\ \sin^2\theta - \cos^2\theta & -2\sin\theta\cos\theta \end{pmatrix}.$$

Now set $\delta = ||V||_{L^1}^{\frac{1}{2+\rho}}$ and split the integral with respect to r into two parts: from 0 to δ and from δ to ∞. We handle the two parts accordingly. Using the local Hölder continuity (3.13), we obtain

$$\left|\int_{S_1}\int_0^\delta \frac{V(x-re^{i\theta}) - V(x)}{r} K_1(\theta)\,d\theta\,dr\right| \leqslant C\delta^\rho = C||V||_{L^1}^{\frac{\rho}{2+\rho}}.$$

Since V is also in L^1,

$$\int_{S_1}\int_\delta^\infty \frac{V(x-re^{i\theta}) - V(x)}{r} K_1(\theta)\,d\theta\,dr \leqslant C\frac{||V||_{L^1}}{\delta^2} = C||V||_{L^1}^{\frac{\rho}{2+\rho}}.$$

Therefore $||K * V||_{L^\infty} \leqslant C||V||_{L^1}^{\frac{\rho}{2+\rho}}$. $\square$

Theorem 3.6. *Let* $(\omega_0, j_0) \in H^m$ *with* $m > \max\{2, \beta\} + d/2$ *and* (ω, j) *be the corresponding solution of the MHD equations* (3.9) *with* $\eta \geqslant 0$. *Let* $T > 0$. *If* $\omega \in L^1$ *is a locally Hölder continuous function with index* $\rho \in (0,1)$ *and satisfies*

$$\int_0^T ||\omega(\cdot,t)||_{L^1}^{\frac{\rho}{2+\rho}}\, dt < \infty, \tag{3.14}$$

then $\omega, j \in L^\infty([0,T]; L^2)$, *and* $j \in L^2([0,T]; H^\beta)$ *if* $\eta > 0$.

Proof. According to Lemma 3.5,

$$\begin{aligned}\left|\int (K_{ik} * \omega)(K_{lk} * j) j\right| &\leqslant C||K_{ik} * \omega||_{L^\infty} ||K_{lk} * j||_{L^2} ||j||_{L^2} \\ &\leqslant C||\omega(\cdot,t)||_{L^1}^{\frac{\rho}{2+\rho}} ||j||_{L^2}^2.\end{aligned}$$

Inserting the above estimate in (3.12) and applying Gronwall's inequality result in $\omega, j \in L^\infty([0,T]; L^2)$, and $j \in L^2([0,T]; H^\beta)$ for $\eta > 0$. $\square$

As we have remarked before, the assumptions of the theorems in this portion do not seem to yield a uniform bound for $||\nabla\omega||$ or $||\nabla j||$. We now briefly explore why (3.14) fails to control certain terms and where the proof for uniform bounds on $||\nabla\omega||$ and $||\nabla j||$ breaks down. For smooth ω and j, $\nabla\omega$ and ∇j satisfy

$$\frac{1}{2}\frac{d}{dt}\int [|\nabla\omega|^2 + |\nabla j|^2]\, dx + \eta \int |\Lambda^\beta j|^2\, dx = R_1 + R_2 + R_3,$$

where R_1, R_2 and R_3 are given by

$$R_1 = -\int \partial_m u_i \cdot (\partial_i \omega \cdot \partial_m \omega + \partial_i j \cdot \partial_m j)\, dx,$$

$$R_2 = \int \partial_m b_i \cdot (\partial_i \omega \cdot \partial_m j + \partial_i j \cdot \partial_m \omega)\, dx,$$

$$R_3 = -2\varepsilon_{il} \int [(K_{ik} * \partial_m \omega)(K_{lk} * \omega) + (K_{ik} * \omega)(K_{lk} * \partial_m \omega)]\partial_m j\, dx.$$

To estimate R_1, we recall that $\mathscr{P}_{ij} = (\partial_i u_j + \partial_j u_i)/2$. Because of symmetry,

$$\begin{aligned}R_1 &= -2\int \mathscr{P}_{mi}(\partial_i \omega \cdot \partial_m \omega + \partial_i j \cdot \partial_m j)\, dx \\ &= -2\int (K_{mi} * \omega) \cdot (\partial_i \omega \cdot \partial_m \omega + \partial_i j \cdot \partial_m j)\, dx.\end{aligned}$$

It then follows from Lemma 3.5 that

$$|R_1| \leqslant C\|K_{mi} * \omega\|_{L^\infty}[\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2] \leqslant C\|\omega\|_{L^1}^{\frac{\rho}{2+\rho}}[\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2].$$

R_3 can be estimated in a similar fashion. By Lemma 3.5,

$$\begin{aligned} R_3 &\leqslant 2(\|K_{ik} * \partial_m\omega\|_{L^2}\|K_{lk} * \omega\|_{L^\infty} + \|K_{ik} * \omega\|_{L^\infty}\|K_{lk} * \partial_m\omega)\|\partial_m j\|_{L^2} \\ &\leqslant C\|\omega\|_{L^1}^{\frac{\rho}{2+\rho}}\|\nabla\omega\|_{L^2}\|\nabla j\|_{L^2} \leqslant C\|\omega\|_{L^1}^{\frac{\rho}{2+\rho}}(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2). \end{aligned}$$

So a condition like (3.14) would take care of R_1 and R_3. The term that is really troublesome is R_2. Recalling that $\mathcal{Q}_{ij} = (\partial_i b_j + \partial_j b_i)/2$, we can rewrite R_2 as

$$R_2 = 2\int \mathcal{Q}_{mi} \cdot \partial_m\omega \cdot \partial_i j\, dx.$$

It seems that (3.14) alone is insufficient for the control of R_2 and this is where the proof for uniform bounds on $\|\nabla\omega\|$ and $\|\nabla j\|$ breaks down. But a similar conditions on j as in (3.14) would allow one to show that $\|\Lambda^m\omega\|$ and $\|\Lambda^m j\|$ are bounded uniformly in $t\in[0,\infty)$ for any fixed $m\geqslant 1$. We state this conclusion in the following theorem.

Theorem 3.7. *Let $T>0$ and (ω, j) be a local smooth solution of the MHD equations* (3.9) *with $\eta\geqslant 0$. If ω and j are both local Hölder continuous with the index ρ and satisfy*

$$\omega, j \in L^{\frac{\rho}{2+\rho}}([0,T]; L^1),$$

then ω and j are both in $L^\infty([0,T]; H^m)$ for any $m\geqslant 0$. In particular, if $T=\infty$, then (ω, j) is a global classical solution of (3.9).

3.2.3. Conditions in terms of b

Without loss of generality, we shall assume in this part that $\beta = 1$ in (3.7). If ψ denotes the magnetic flux function, i.e., $b = \nabla^\perp\psi$, then (3.7) is equivalent to the following equations of ψ and ω:

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = b\cdot\nabla j \\ \partial_t\psi + u\cdot\nabla\psi = \eta\Delta\psi \end{cases} \tag{3.15}$$

as far as smooth solutions are concerned. Smooth solutions of (3.15) are currently not known to be global. Numerical simulations seem to indicate exponential growths of ω rather than finite-time singularities [2]. In this section we explore conditions on b under which ω grows at most exponentially. Since the condition involves the time-derivative of b, the estimate may not be helpful in practice.

Theorem 3.8. *Let* $(\psi_0, \omega_0) \in H^s$ *with* $s > \max\{2, \beta\} + d/2$ *and* (ψ, ω) *be the solution of* (3.15) *with initial data* (ψ_0, ω_0). *If the magnetic field* $b = \nabla^\perp \times \psi$ *satisfies*

$$\int_0^\infty (||\partial_t b(\cdot, t)||_{L^4}^4 + ||\nabla b(\cdot, t)||_{L^4}^4)\, dt < \infty,$$

then ω *does not develop any finite-time singularities and* $||\omega||$ *grows at most exponentially.*

Proof. Noticing $\Delta\psi = j$, we can combine the equations in (3.15) to eliminate j,

$$\eta(\partial_t \omega + u \cdot \nabla\omega) = b \cdot \nabla(\partial_t \psi + u \cdot \nabla\psi).$$

Multiplying both sides by ω and integrating over the spatial domain, we have

$$\frac{\eta}{2}\frac{d}{dt}\int |\omega|^2\, dx = \int (b \cdot \nabla \partial_t \psi)\omega\, dx + \int (b \cdot \nabla(u \cdot \nabla\psi))\omega\, dx. \tag{3.16}$$

Applying basic inequalities to the right-hand side of (3.16) eventually leads to

$$\begin{aligned}\frac{\eta}{2}\frac{d}{dt}||\omega(\cdot, t)||^2 \leqslant\ & C(1 + ||b|| + ||b||^{\frac{1}{2}}\, ||u||^{\frac{3}{4}} + ||\nabla b||_{L^4}^4)||\omega||^2 \\ & + C(||b||^2 ||\nabla b||^2 + ||\partial_t b||_{L^4}^4 + ||\nabla b||_{L^4}^4)\end{aligned}$$

Finally we use Gronwall's inequality. $\square$

4. $\boldsymbol{v} = 0$ and $\boldsymbol{\eta} = 0$

We now turn our attention to (1.1) with both $v = 0$ and $\eta = 0$. When $v = \eta = 0$, (1.1) reduces to the ideal MHD equations, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u. \end{cases} \tag{4.1}$$

In this section the spatial domain is taken to be $\mathbb{R}^d$ and we are primarily concerned with solutions of (4.1) supplemented with the initial condition

$$u(x, 0) = u_0(x) \quad \text{and} \quad b(x, 0) = b_0(x), \quad x \in \mathbb{R}^d. \tag{4.2}$$

We shall show that for each $(u_0, b_0) \in C^r$ with $r > 1$, the initial-value problem (4.1)–(4.2) has a unique classical solution (u, b), which remains in C^r for some finite time. Here C^r denotes the Hölder space characterized by the usual dyadic decomposition. Some basic facts concerning C^r will be reviewed in the first subsection. The precise theorem and its proof are given in the second subsection. The proof is a modification of Chemin's approach [5].

Using Elsässer's variables $z^+ = u + b$ and $z^- = u - b$, the ideal MHD equations can be written in the following symmetric form:

$$\begin{cases} \partial_t z^+ + z^- \cdot \nabla z^+ = -\nabla P, \\ \partial_t z^- + z^+ \cdot \nabla z^- = -\nabla P, \end{cases} \tag{4.3}$$

which we find more convenient for our purpose here. Correspondingly, the initial data for z^+ and z^- are $z_0^+ = u_0 + b_0$ and $z_0^- = u_0 - b_0$, respectively. The pressure P in (4.3) is determined by z^+ and z^- through the relation

$$P = P(z^+, z^-) = (-\Delta)^{-1}\nabla \cdot (z^- \cdot \nabla z^+) \quad \text{or} \quad P(z^+, z^-) = \mathscr{R}_i \mathscr{R}_j (z_i^+ z_j^-).$$

4.1. C^r and paraproduct

We review here the characterizations of the Hölder space C^r and the action of paraproduct on C^r. We start with a dyadic decomposition of $\mathbb{R}^d$. It can be verified that there exist two radial functions $\chi \in C_0^\infty$ and $\phi \in C_0^\infty$ satisfying

$$supp\, \chi \subset \{\xi\colon\ |\xi| \leqslant 4/3\}, \quad supp\, \phi \subset \{\xi\colon\ 3/4 < |\xi| < 8/3\},$$

$$\chi(\xi) + \sum_{j \geqslant 0} \phi(2^{-j}\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^d.$$

For the purpose of isolating different Fourier frequencies, we define the operators Δ_i for $i \in \mathbb{Z}$ as follows:

$$\Delta_i u = \begin{cases} 0 & \text{if } i \leqslant -2, \\ \chi(D)u = \int h(y)u(x-y)\, dy & \text{if } i = -1, \\ \phi(2^{-i}D)u = 2^{id} \int g(2^i y)u(x-y)\, dy & \text{if } i \geqslant 0, \end{cases} \tag{4.4}$$

where $h = \chi^\vee$ and $g = \phi^\vee$ are the inverse Fourier transforms of χ and ϕ, respectively.

For $i \in \mathbb{Z}$, S_i is the sum of Δ_j with $j \leqslant i - 1$, i.e.,

$$S_i u = \Delta_{-1} u + \Delta_0 u + \Delta_1 u + \cdots + \Delta_{i-1} u = \int_{\mathbb{R}^d} h(2^i y) u(x-y)\, dy.$$

It can be shown for any tempered distribution f that $S_i f \to f$ in the distributional sense, as $i \to \infty$.

For any real number r (not necessarily positive), the Hölder space C^r contains tempered distributions u that satisfy

$$\sup_i 2^{ir} \|\Delta_i u\|_{L^\infty} < \infty.$$

Note that the space C^0 defined here is not equivalent to L^∞. In fact, L^∞ is a subspace of C^0. On the other hand, if a function is in C^ε for any $\varepsilon > 0$, then it is in L^∞.

The usual product uv of two functions u and v can be decomposed into three parts. More precisely, using the notion of paraproduct, we can write

$$uv = T_u v + T_v u + R(u, v), \tag{4.5}$$

where

$$T_u v = \sum_j S_{j-1} u \cdot \Delta_j v, \quad R(u, v) = \sum_{|i-j| \leqslant 1} \Delta_i u \cdot \Delta_j v.$$

We remark that the decomposition in (4.5) allows one to distinguish different types of terms in the product of uv. The Fourier frequencies of u and v in $T_u v$ and $T_v u$ are separated from each other while those of the terms in $R(u, v)$ are close to each other. Using the decomposition in (4.5), one can show that for $s > 0$

$$||uv||_{C^s} \leqslant C(||u||_{C^s} ||v||_{L^\infty} + ||u||_{L^\infty} ||v||_{C^s}). \tag{4.6}$$

4.2. Local C^r solutions

We now state our main theorem of this section.

Theorem 4.1. *Let $u_0 \in C^r$ and $b_0 \in C^r$ with $r > 1$. Consider solutions of* (4.3) *with initial data*

$$z^+(x, 0) = z_0^+(x) \equiv u_0(x) + b_0(x), \quad z^-(x, 0) = z_0^-(x) \equiv u_0(x) - b_0(x).$$

Then there exists a T depending on u_0 and b_0 only such that (4.3) *has a unique solution (z^+, z^-) over the interval $[0, T]$. Furthermore, $z^+, z^- \in L^\infty([0, T]; C^r)$ and $P \in L^\infty([0, T]; C^{r+1})$.*

Remark. It is not known if the local solution established in this theorem can be extended into a global one. But if we know that the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy

$$\int_0^\infty (||\omega(\cdot, t)||_{L^\infty} + ||j(\cdot, t)||_{L^\infty})\, dt < \infty, \tag{4.7}$$

then (u, b) can be extended into a global solution. The verification of this assertion involves bounding $||u||_{C^r}$ and $||b||_{C^r}$ in terms of the quantity on the left of (4.7). We omit more details.

Remark. In Theorem 4.1 the initial data are assumed to be in C^r with $r > 1$. It is not clear whether similar results hold if u_0 and b_0 are merely in C^1. Our guess is that conditions involving initial vorticity and current density may be needed.

Proof of Theorem 4.1. Consider two approximating sequences of $\{z_n^+\}$ and $\{z_n^-\}$ satisfying

$$z_1^+ = S_2(z_0^+), \quad z_1^- = S_2(z_0^-),$$

and

$$\begin{cases} \partial_t z_{n+1}^+ + z_n^- \cdot \nabla z_{n+1}^+ = -\nabla P(z_n^+, z_n^-), \\ \partial_t z_{n+1}^- + z_n^+ \cdot \nabla z_{n+1}^- = -\nabla P(z_n^+, z_n^-), \\ z_{n+1}^+(0) = S_{n+2} z_0^+, \quad z_{n+1}^-(0) = S_{n+2} z_0^-. \end{cases} \tag{4.8}$$

We shall show that there exists a time interval $[0, T]$ over which $\{z_n^+\}$ and $\{z_n^-\}$ are uniformly bounded in $L^\infty([0, T]; C^r)$ and Cauchy in $L^\infty([0, T]; C^{r-1})$.

Let $j \geqslant -1$. Applying the operator Δ_j to both sides of the first equation in (4.8), there obtains

$$\partial_t(\Delta_j z_{n+1}^+) + z_n^- \cdot \nabla(\Delta_j z_{n+1}^+) = -\nabla P(z_n^+, z_n^-) + [z_n^- \cdot \nabla, \Delta_j] z_{n+1}^+,$$

where $[\,,\,]$ denotes the commutator. This equation can be written in the following integral form:

$$\begin{aligned} \Delta_j z_{n+1}^+(x,t) &= (S_{n+2} z_0^+)((\psi^-)^{-1}(x,t)) \\ &\quad + \int_0^t (-\Delta_j \nabla P(z_n^+, z_n^-) + [z_n^- \cdot \nabla, \Delta_j] z_{n+1}^+)(y_n^-, \tau)\, d\tau, \end{aligned}$$

where $y_n^- = y_n^-(\tau) = \psi_n^-((\psi_n^-)^{-1}(x,t), \tau)$ and ψ_n^- is the stream function corresponding to the field z_n^-. Taking L^∞-norm and then multiplying by 2^{jr}, we obtain from the above equation that

$$\begin{aligned} ||z_{n+1}^+||_{C^r} \leqslant ||z_0^+||_{C^r} &+ \int_0^t ||\nabla P(z_n^+, z_n^-)||_{C^r}\, d\tau \\ &+ \int_0^t \sup_j 2^{jr} ||[z_n^- \cdot \nabla, \Delta_j] z_{n+1}^+||_{L^\infty}\, d\tau \end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned} ||z_{n+1}^-||_{C^r} \leqslant ||z_0^-||_{C^r} &+ \int_0^t ||\nabla P(z_n^+, z_n^-)||_{C^r}\, d\tau \\ &+ \int_0^t \sup_j 2^{jr} ||[z_n^+ \cdot \nabla, \Delta_j] z_{n+1}^-||_{L^\infty}\, d\tau. \end{aligned} \tag{4.10}$$

We need to bound $||\nabla P(z_n^+, z_n^-)||_{C^r}$ and $||[z_n^+ \cdot \nabla, \Delta_j] z_{n+1}^-||_{L^\infty}$. The following lemmas provide the estimates. Notice that these estimates hold for any $r > -1$.

Lemma 4.2. *Let $r > -1$ and $P(v,w) = \mathcal{R}_i \mathcal{R}_j (v_i w_j)$. Then for some constant C depending on r*

$$||\nabla P(v,w)||_{C^r} \leqslant \begin{cases} C(||v||_{C^r} ||\nabla w||_{L^\infty} + ||w||_{C^r} ||\nabla v||_{L^\infty}) & \text{for } r > 1, \\ C \min\{||\nabla v||_{L^\infty} ||w||_{C^r}, ||\nabla w||_{L^\infty} ||v||_{C^r}\} & \text{for } r \in (-1,1). \end{cases}$$

Proof. The bound follows from Proposition 2.5.1 of [5, p. 40]. □

Lemma 4.3. *Let $r > -1$ and $j \geqslant -1$. Assume that v and w are both divergence-free. Then for some constant C depending on r only*

$$||[w \cdot \nabla, \Delta_j] v||_{L^\infty} \leqslant C 2^{-jr} (||\nabla w||_{L^\infty} + ||\nabla w||_{C^{r-1}}) ||v||_{C^r}.$$

Proof. We sketch the proof and details can be found in [5, p. 67]. Using the decomposition (4.5), we have

$$[w \cdot \nabla, \Delta_j] v = ([\Delta_j, T_{w_k} \partial_k] + [\Delta_j, T_{\partial_k} w_k] + [\Delta_j, \partial_k R(w_k, \cdot)]) v.$$

We estimate the three terms on the right-hand side. Using the definition of Δ_j, we find for the first term that

$$||[\Delta_j, T_{w_k} \partial_k] v||_{L^\infty} \leqslant C 2^{-jr} ||\nabla w||_{L^\infty} ||v||_{C^r}.$$

For the second term, we have

$$\begin{aligned} ||[\Delta_j, T_{\partial_k} w_k] v||_{L^\infty} &= ||\Delta_j T_{\partial_k v} w_k - T_{\partial_k (\Delta_j v)} w_k||_{L^\infty} \\ &\leqslant ||\Delta_j T_{\partial_k v} w_k||_{L^\infty} + ||T_{\partial_k (\Delta_j v)} w_k||_{L^\infty} \leqslant C 2^{-jr} ||\nabla w||_{C^{r-1}} ||v||_{C^r}. \end{aligned}$$

For the third part, it can be shown that

$$\begin{aligned} ||[\Delta_j, \partial_k R(w_k, \cdot)]||_{L^\infty} &\leqslant ||\Delta_j \partial_k R(w_k, v)||_{L^\infty} + ||\partial_k R(w_k, \Delta_j v)||_{L^\infty} \\ &\leqslant C 2^{-jr} ||w||_{C^1} ||v||_{C^r} \leqslant C 2^{-jr} ||\nabla w||_{L^\infty} ||v||_{C^r}. \qquad \square \end{aligned}$$

Proof of Theorem 4.1 (Conclusion). Applying Lemmas 4.2 and 4.3 to the terms in (4.9) and (4.10), we find that

$$Y_{n+1}(t) \leqslant (\|z_0^+\|_{C^r} + \|z_0^-\|_{C^r}) + C\int_0^t (\|\nabla z_n^+\|_{L^\infty} + \|\nabla z_n^-\|_{L^\infty})\, Y_n(\tau)\, d\tau$$
$$+\ C\int_0^t (\|\nabla z_n^+\|_{L^\infty} + \|\nabla z_n^-\|_{L^\infty} + \|z_n^+\|_{C^{r-1}} + \|z_n^-\|_{C^{r-1}})\, Y_{n+1}(\tau)\, d\tau, \quad (4.11)$$

where $Y_n(t) = \|z_n^+\|_{C^r} + \|z_n^-\|_{C^r}$. We now show inductively the uniform boundedness of the sequence $\{Y_n\}$ in $L^\infty([0, T_1]; C^r)$ for some $T_1 > 0$. It is obvious that

$$Y_1 \leqslant CY_0 \equiv C(\|z_0^+\|_{C^r} + \|z_0^-\|_{C^r})$$

for some constant C. More generally, there exists a $T_1 > 0$ such that for $t \leqslant T_1$ and any n,

$$Y_n(t) \leqslant C_0 \equiv 4CY_0. \quad (4.12)$$

To show (4.12), one first realize that $\|\nabla z_n^+\|_{L^\infty}$ and $\|\nabla z_n^+\|_{L^\infty}$ can be bounded in terms of Y_n and then apply an inductive argument to (4.11).

We now show that there exists a $T \in [0, T_1]$ such that z_n^+ and z_n^- are both Cauchy sequences in $L^\infty([0, T]; C^{r-1})$. For $m, n \in \mathbb{Z}^+$, consider the differences

$$w_{m,n}^+ = z_m^+ - z_n^+, \quad w_{m,n}^- = z_m^- - z_n^-$$

which satisfy the equations

$$\partial_t w_{m+1,n+1}^+ + z_m^- \cdot \nabla w_{m+1,n+1}^+ + w_{m,n}^- \cdot \nabla z_{n+1}^+$$
$$= -\nabla P(w_{m,n}^+, z_m^-) - \nabla P(z_n^+, w_{m,n}^-), \quad (4.13)$$

$$\partial_t w_{m+1,n+1}^- + z_m^+ \cdot \nabla w_{m+1,n+1}^- + w_{m,n}^+ \cdot \nabla z_{n+1}^-$$
$$= -\nabla P(w_{m,n}^+, z_m^-) - \nabla P(z_n^+, w_{m,n}^-). \quad (4.14)$$

Taking C^{r-1} norm in (4.13) and applying Lemma 4.3, we obtain

$$\|w_{m+1,n+1}^+\|_{C^{r-1}} \leqslant C\int_0^t V_m^-(\tau)\|w_{m+1,n+1}^+(\cdot,\tau)\|_{C^{r-1}}\, d\tau$$
$$+ \int_0^t \|w_{m,n}^- \cdot \nabla z_{n+1}^+\|_{C^{r-1}}\, d\tau$$
$$+ \int_0^t \|\nabla P(w_{m,n}^+, z_m^-) + \nabla P(z_n^+, w_{m,n}^-)\|_{C^{r-1}}\, d\tau,$$

where $V_m^- = ||\nabla z_m^-||_{L^\infty} + ||z_m^-||_{C^{r-1}}$. Applying (4.6) and Lemma 4.2, we have

$$\begin{aligned} ||w_{m+1,n+1}^+||_{C^{r-1}} \leqslant C \int_0^t V_m^-(\tau)||w_{m+1,n+1}^+(\cdot,\tau)||_{C^{r-1}}\, d\tau \\ + \int_0^t [||w_{m,n}^-||_{C^{r-1}}||\nabla z_{n+1}^+||_{L^\infty} + ||w_{m,n}^-||_{L^\infty}||\nabla z_{n+1}^+||_{C^{r-1}}]\, d\tau \\ + \int_0^t [||w_{m,n}^+||_{C^{r-1}}||\nabla z_m^-||_{L^\infty} + ||w_{m,n}^-||_{C^{r-1}}||\nabla z_n^-||_{L^\infty}]\, d\tau. \end{aligned}$$

A similar estimate can be deduced for $||w_{m+1,n+1}^-||_{C^{r-1}}$ from (4.14).

$$\begin{aligned} ||w_{m+1,n+1}^-||_{C^{r-1}} \leqslant C \int_0^t V_m^+(\tau)||w_{m+1,n+1}^-(\cdot,\tau)||_{C^{r-1}}\, d\tau \\ + \int_0^t [||w_{m,n}^+||_{C^{r-1}}||\nabla z_{n+1}^-||_{L^\infty} + ||w_{m,n}^+||_{L^\infty}||\nabla z_{n+1}^-||_{C^{r-1}}]\, d\tau \\ + \int_0^t [||w_{m,n}^+||_{C^{r-1}}||\nabla z_m^-||_{L^\infty} + ||w_{m,n}^-||_{C^{r-1}}||\nabla z_n^-||_{L^\infty}]\, d\tau. \end{aligned}$$

Now we use the uniform boundedness of $\{z_k^+\}$ and $\{z_k^-\}$ in C^r to conclude that for some constant C

$$Z_{m+1,n+1}(t) \leqslant C \int_0^t Z_{m,n}(\tau)\, d\tau + C \int_0^t V_{m,n}(\tau) Z_{m+1,n+1}(\tau)\, d\tau,$$

where $Z_{m+1,n+1} = ||w_{m+1,n+1}^+||_{C^{r-1}} + ||w_{m+1,n+1}^-||_{C^{r-1}}$ and $V_{m,n} = V_m^- + V_m^+$. By Gronwall's inequality,

$$Z_{m+1,n+1}(t) \leqslant C_T Z_{m+1,n+1}(0) + TC_T Z_{m,n}(t) \tag{4.15}$$

for some constant C_T depending on $T \leqslant T_1$. According to the definition of $Z_{m,n}$,

$$Z_{m+1,n+1}(0) = ||S_{m+2}z_0^+ - S_{n+2}z_0^+||_{C^{r-1}} + ||S_{m+2}z_0^- - S_{n+2}z_0^-||_{C^{r-1}},$$

which approaches zero as m and n tend to ∞. Now choose T such that $TC_T \leqslant \frac{1}{2}$. We can then conclude from (4.15) that $\{Z_{m,n}\}$ approaches zero as m, n approaches infinity. That is, $\{z_n^+\}$ and $\{z_n^-\}$ are Cauchy sequences in $L^\infty([0,T]; C^{r-1})$. Therefore, there exist two functions z^+ and z^- such that $z_n^+ \to z^+$ and $z_n^- \to z^-$ in $L^\infty([0,T]; C^{r-1})$ as $n \to \infty$. To show that (z^+, z^-) solves (4.3), we realize that the nonlinear terms are continuous on $C^{r-1} \times C^{r-1}$ functions and thus the nonlinear terms in (4.8) converge to the corresponding ones in (4.3).

The uniqueness of solutions can be established in a similar fashion as in the proof showing that $\{z_n^+\}$ and $\{z_n^-\}$ are Cauchy. We omit more details. This concludes the proof of Theorem 4.1. $\square$

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Appendix

In this appendix we prove that the 3D equations

$$\partial_t u + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^{\alpha} u$$

with $\alpha \geqslant 5/4$ has a unique global smooth solution for any prescribed smooth initial data. The spatial domain is assumed to be either the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$. It is known to the community of mathematical fluids that Ladyzhenskaya has previously obtained such a result, but we were unable to locate her paper.

Theorem A.1. *Let $\nu > 0$, $\alpha \geqslant 5/4$ and $u_0 \in H^s$ with $s > 2\alpha$. Then the 3D Navier–Stokes type equations*

$$\partial_t u + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^{\alpha} u \tag{A.1}$$

with initial condition

$$u(x, 0) = u_0(x)$$

has a unique global classical solution.

Proof. The proof of this theorem is parallel to that of Theorem 2.3. Arguing in a similar fashion as in Section 2, one can show that (A.1) has a global weak solution u satisfying

$$||u(\cdot, t)||^2 + 2\nu \int_0^t ||\Lambda^{\alpha} u(\cdot, \tau)||^2 \, d\tau \leqslant ||u_0||^2. \tag{A.2}$$

Let $T > 0$. It then suffices to establish a priori bounds for $||u||_{H^m}$ on the time interval $[0, T]$, where $m \leqslant s$. We start with $m = 1$. Consider the equation for ∇u,

$$\frac{d}{dt} ||\nabla u||^2 + 2\nu \int_0^t ||\Lambda^{\alpha+1} u(\cdot, \tau)||^2 \, d\tau = -2 \int \partial_j u_k \cdot \partial_j u_i \cdot \partial_i u_k \, dx. \tag{A.3}$$

The term on the right-hand side can be bounded as follows:

$$\left|\int \partial_j u_k \cdot \partial_j u_i \cdot \partial_i u_k \, dx\right| \leqslant C\|\nabla u\|_{L^3}^3 \leqslant C\|\nabla u\|^{a_1}\|\Lambda^\alpha u\|^{a_2}\|\Lambda^{\alpha+1}u\|^{a_3}$$
$$\leqslant \frac{\nu}{4}\|\Lambda^{\alpha+1}u\|^2 + C_\nu\|\Lambda^\alpha u\|^{\frac{2a_2}{a_1}}\|\nabla u\|^2,$$

where $a_1 = 1 - \frac{5}{6\alpha}$, $a_2 = \frac{1}{3}$ and $a_3 = \frac{5}{6\alpha} - \frac{1}{3}$. Inserting this estimate in (A.3), we obtain

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\int_0^t \|\Lambda^{\alpha+1}u(\cdot,\tau)\|^2 \, d\tau \leqslant C_\nu\|\Lambda^\alpha u\|^{\frac{2a_2}{a_1}}\|\nabla u\|^2. \tag{A.4}$$

When $\alpha \geqslant \frac{5}{4}$, $\frac{2a_2}{a_1} \leqslant 2$. According to (A.2),

$$\int_0^t \|\Lambda^\alpha u\|^{\frac{2a_2}{a_1}} \, d\tau < C(T)$$

for any $t \leqslant T$. We can then infer from (A.4) that $u \in L^\infty([0,T]; H^1)$. Higher-order regularities can be established inductively, but we omit more details. This completes the proof of Theorem A.1. $\square$

References

[1] J.T. Beale, T. Kato, A. Majda, Remarks on breakdown of smooth solutions for the 3D Euler equations, Commun. Math. Phys. 94 (1984) 61–66.

[2] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, Cambridge, 1993.

[3] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, Commun. Pure Appl. Math. 35 (1982) 771–831.

[4] R.E. Caflisch, I. Klapper, G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Commun. Math. Phys. 184 (1997) 443–455.

[5] J.-Y. Chemin, Perfect Incompressible Fluids, Clarendon Press, Oxford, 1998.

[6] P. Constantin, Navier–Stokes equations and area of interfaces, Commun. Math. Phys. 129 (1990) 241–266.

[7] P. Constantin, Geometric statistics in turbulence, SIAM Rev. 36 (1994) 73–98.

[8] P. Constantin, C. Fefferman, A. Majda, Geometric constraints on potentially singular solutions of the 3-D Euler equations, Commun. Partial Differential Equations 21 (1996) 559–571.

[9] G. Duvaut, J.L. Lions, Inequations en thermoelasticite et magnetohydrodynamique, Arch. Rational Mech. Anal. 46 (1972) 241–279.

[10] J.D. Gibbon, K. Ohkitani, Singularity formation in a class of stretched solutions of the equations for ideal megneto-hydrodynamics, Nonlinearity 14 (2001) 1239–1264.

[11] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2001.

[12] J. Mattingly, Y. Sinai, An elementary proof of the existence and uniqueness theorem for the Navier–Stokes equation, Commun. Contemp. Math. 1 (1999) 497–516.

[13] E. Priest, T. Forbes, Magnetic Reconnection, MHD Theory and Applications, Cambridge University Press, Cambridge, 2000.

[14] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, Commun. Pure Appl. Math. 36 (1983) 635–664.
[15] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, USA, 1970.
[16] R. Temam, Navier–Stokes Equations, North-Holland Publishing Company, Amsterdam, 1979.
[17] J. Wu, Viscous and inviscid magneto-hydrodynamics equations, J. Anal. Math. 73 (1997) 251–265.