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Quasi-geostrophic-type equations with initial data in Morrey spaces

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Abstract. This paper studies the well posedness of the initial value problem for the quasi-geostrophic type equations

$$\frac{\partial\theta}{\partial t} + u\nabla\theta + (-\Delta)^{\gamma}\theta = 0 \qquad \text{on } \mathbb{R}^n \times (0,\infty)$$
$$\theta(x,0) = \theta_0(x) \qquad x \in \mathbb{R}^n$$

where $\gamma (0 < \gamma \leqslant 1)$ is a fixed parameter and $u = (u_j)$ is divergence free and determined from θ through the Riesz transform $u_j = \pm\mathcal{R}_{\pi(j)}\theta$ ($\pi(j)$ being a permutation of j, $j = 1, 2, \cdots, n$). The initial data θ_0 is taken in certain Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (see text for the definition). The local well posedness is proved for

$$\tfrac{1}{2} < \gamma \leqslant 1 \qquad 1 < p < \infty \qquad \lambda = n - (2\gamma - 1)p \geqslant 0$$

and the solution is global for sufficiently small data. Furthermore, the solution is shown to be smooth.

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1. Introduction

We study the initial value problem (IVP) of the dissipative quasi-geostrophic-type (QGS) equations

$$\frac{\partial\theta}{\partial t} + u\nabla\theta + (-\Delta)^{\gamma}\theta = 0 \qquad \text{on } \mathbb{R}^n \times (0,\infty) \tag{1.1}$$
$$\theta(x,0) = \theta_0(x) \qquad x \in \mathbb{R}^n \tag{1.2}$$

where $\gamma (0 < \gamma \leqslant 1)$ is a fixed parameter and the velocity $u = (u_1, u_2, \ldots, u_n)$ is divergence free (i.e. $\nabla \cdot u = 0$) and determined from θ by

$$u_j = \pm\mathcal{R}_{\pi(j)}\theta \qquad \pi(j) \text{ is a permutation of } j,\ j = 1, 2, \ldots, n \tag{1.3}$$

where u_j may take either a plus or a minus sign and $\mathcal{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ are the Riesz transforms. Here the Riesz potential operator $(-\Delta)^{\alpha}$ is defined through the Fourier transform [18]:

$$\widehat{f}(\xi) = \int \mathrm{e}^{-\mathrm{i}x\xi} f(x)\,\mathrm{d}x$$
$$(\widehat{(-\Delta)^{\alpha} f})(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi).$$

If $n = 2$, equation (1.1) with u given in (1.3) becomes the two-dimensional QGS equation describing actual geophysical fluid flow on the two-dimensional boundary of a fast rotating three-dimensional half space with small Rossby and Ekman numbers [3, 14]. The scalar θ represents the potential temperature and u is the fluid velocity, which can be expressed by the stream function ψ:

$$u = (u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right) \qquad (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.4}$$

As developed by Constantin *et al* [3], the two-dimensional QGS equation is strikingly analogous to the three-dimensional Navier–Stokes equations both mathematically and physically. The Navier–Stokes equations with singular initial data have been extensively studied in coping with real physical phenomena. In this paper we study the existence, uniqueness and regularity of solutions to the QGS equation when the initial data is taken in Morrey spaces, $\theta_0 \in \mathcal{M}_{p,\lambda}$. We treat the general n−D equation for the completeness of mathematical theory. The usual Lebesgue spaces L^p and the space of finite measures $\mathcal{M}$ are special examples of Morrey spaces. Very singular data such as certain measures concentrated on smooth surfaces are also contained in Morrey spaces and these types of initial profiles are of real physical interest.

Although there is a large literature on quasi-geostrophic equations [14, 2, 7, 3], not many rigorous mathematical results concerning the solutions have been obtained. Constantin *et al* [3] proved finite-time existence results for smooth data and develop mathematical criteria characterizing blow-up for the two-dimensional non-dissipative QGS equation. Resnick [16] obtained solutions of two-dimensional QGS equations with L^2 data on the periodic domain by using the Galerkin approximation. In a previous paper [19] the vanishing dissipation limits and Gevrey class regularity [4] for the two-dimensional dissipative QGS equations are obtained.

In this paper we prove that if $\frac{1}{2} < \gamma \leqslant 1$ and $\theta_0 \in \mathcal{M}_{p,\lambda}$ with

$$1 < p < \infty \qquad \lambda = n - (2\gamma - 1)p \geqslant 0. \tag{1.5}$$

Then the IVP (1.1), (1.3), (1.2) is locally well posed and the solution is global for sufficiently small initial data. Furthermore, we prove that the solution is actually smooth. The precise statements are given in theorem 3.2 of section 3 and theorem 4.1 of section 4. The proof of the well posedness is achieved by using a key observation of Resnick [16] and the contraction-mapping principle. These results reduce to those in L^p theory by taking $\lambda = 0$. The contraction-mapping theorem is probably the most often used result in showing the existence of solutions and we give a brief introduction for the convenience of those who are not familiar with it. If $\mathbb{X}$ is a Banach space with norm $\|\cdot\|_{\mathbb{X}}$ and $\mathcal{P}$ maps from $\mathbb{X}$ to $\mathbb{X}$ and is a contraction, i.e. for some $b \in (0, 1)$

$$\|\mathcal{P}x - \mathcal{P}y\|_{\mathbb{X}} \leqslant b\|x - y\|_{\mathbb{X}} \qquad \text{for any } x, y \in \mathbb{X}.$$

Then there exists a unique $x_0 \in \mathbb{X}$ such that $\mathcal{P}x_0 = x_0$.

As we shall find out in section 3, that the requirement $1 < p < \infty$ in (1.5) comes from estimating the singular integrals of the Calderon–Zygmund type when we bound u in terms of θ in Morrey spaces. $p = 1$ or $p = \infty$ is excluded since the Calderon–Zygmund-type singular operator is not bounded either in L^1 (but of weak type $(1, 1)$) or in L^∞ (but taking to BMO) [18]. For the Navier–Stokes equations, the singularity of the Biot–Savart kernel is milder and the borderline case $p = 1$ or $p = \infty$ is often included in the well posedness results [6, 9].

The index $\gamma = \frac{1}{2}$ seems special as we can see from this well posedness and other related results concerning the QGS equation [16, 19, 20]. For $\gamma > \frac{1}{2}$, the existence, uniqueness and regularity are obtained, but the same questions remain open for $\gamma \leqslant \frac{1}{2}$. Now the important question is if the index $\gamma = \frac{1}{2}$ is actually sharp and how this index is related to the singularity of the integral operator determining the velocity field. The answer to this question may provide a clue to the solution of the outstanding problem concerning the existence of smooth solutions for the Navier–Stokes equations.

This paper is organized as follows. In section 2 the definition and some properties of Morrey spaces are given as well as the solution operator of the linear equation and its properties. The well posedness results are stated and proved in section 3 while the smoothness of the solution is established in section 4.

2. Morrey spaces and the linear equation

In this section we shall list some basic properties of Morrey spaces and then study the the solution operator for the linear equation over Morrey spaces. First we recall the definition.

Definition 2.1. For $1 \leqslant p < \infty$, $0 \leqslant \lambda < n$, the Morrey space $\mathcal{M}_{p,\lambda}$ is defined as

$$\mathcal{M}_{p,\lambda} = \mathcal{M}_{p,\lambda}(\mathbb{R}^n) \equiv \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty\}$$

where the norm is given by

$$\|f\|_{p,\lambda} = \sup_{\{x\in\mathbb{R}^n, R>0\}} R^{-\frac{\lambda}{p}} \left(\int_{|y-x|\leqslant R} |f|^p(y)\, \mathrm{d}y \right)^{\frac{1}{p}}$$

$\ddot{\mathcal{M}}_{p,\lambda}$ is defined to be a subspace of $\mathcal{M}_{p,\lambda}$:

$$\ddot{\mathcal{M}}_{p,\lambda} = \{f \in \mathcal{M}_{p,\lambda} : \|f(\cdot - y) - f(\cdot)\|_{p,\lambda} \to 0, \text{as } y \to 0\}.$$

$\mathcal{M}_{p,\lambda}$ is a Banach space and $\ddot{\mathcal{M}}_{p,\lambda}$ is a closed subspace of $\mathcal{M}_{p,\lambda}$. For $p > 1$, $\mathcal{M}_{p,0} = L^p$ and $\mathcal{M}_{1,0} = \mathcal{M}$, where $\mathcal{M}$ is the space of finite measures. When not specified, the indices in the notation $\mathcal{M}_{p,\lambda}$ will be restricted in $1 \leqslant p < \infty$, $0 \leqslant \lambda < n$. Occasionally we consider $p = \infty$ and then $\mathcal{M}_{\infty,\lambda}$ simply means L^∞. The following properties of Morrey spaces (cf [1, 15, 9]) will be used.

Lemma 2.2. *For $1 \leqslant p, q, r \leqslant \infty$, we have*

(i) Inclusion relations:

$$\mathcal{M}_{p,\lambda} \subset \mathcal{M}_{q,\mu} \qquad \textit{if } \frac{n-\lambda}{p} = \frac{n-\mu}{q} \qquad q \leqslant p.$$

(ii) The Hölder inequality:

$$\|fg\|_{p,\lambda} \leqslant \|f\|_{q,\mu}\|g\|_{r,\nu} \qquad \textit{if } \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \qquad \frac{\lambda}{p} = \frac{\mu}{q} + \frac{\nu}{r}.$$

(iii) Continuous embedding in weighted Lebesgue space:

$$\mathcal{M}_{p,\lambda} \hookrightarrow L_{-\frac{\mu}{p},p} \qquad \textit{for } p > 1 \qquad \mu > \lambda$$

where $L_{s,p}$ is the weighted Lebesgue space consisting of functions f such that $(1+|x|^2)^{\frac{s}{2}} f \in L^p$ with the norm

$$\|f\|_{L_{s,p}} = \|(1+|x|^2)^{\frac{s}{2}} f\|_{L^p}.$$

We now consider the solution operator for the linear equation

$$\partial_t\theta + \Lambda^{2\gamma}\theta = 0 \qquad \Lambda = (-\Delta)^{\frac{1}{2}} \qquad \gamma \in (0, 1]$$

on the whole space $\mathbb{R}^n$. For a given initial data θ_0, the solution of this equation is given by $\theta = K(t)\theta_0 = \mathrm{e}^{-\Lambda^{2\gamma}t}\theta_0$, where $K(t)$ is a convolution operator with the kernel $k_t(x) = k_t^{(\gamma)}(x)$ being defined through its Fourier transform

$$\widehat{k}_t^{(\gamma)}(\xi) = \mathrm{e}^{-|\xi|^{2\gamma}t}. \tag{2.1}$$

In particular, k_t is the heat kernel for $\gamma = 1$ and the Poisson kernel for $\gamma = \frac{1}{2}$.

As observed by Resnick [16], $k_t^{(\gamma)}$ can be expressed as an average of the heat kernel,

$$k_1^{(\gamma)}(x) = \int_0^\infty k_s^{(1)}(x)\,\mathrm{d}P_\gamma(s) \qquad k_t^{(\gamma)} = t^{-\frac{n}{2\gamma}} k_1^{(\gamma)}\left(\frac{x}{t^{\frac{1}{2\gamma}}}\right) \tag{2.2}$$

where P_γ is some probability distribution. Furthermore,

Lemma 2.3. *For* $0 < \gamma \leqslant 1$,

(a) $k_t^{(\gamma)} \geqslant 0$, *and is a non-increasing radial function on* $\mathbb{R}^n$ *with*

$$|x|^l k_t^{(\gamma)}(|x|) \longmapsto 0 \qquad \textit{as } |x| \to \infty$$

for any real power l.

(b) For $t > 0$, $\|k_t^{(\gamma)}\|_{L^1} = 1$ *and thus for* $f_t = k_t^{(\gamma)} * f$,

$$|f_t|^p \leqslant k_t^{(\gamma)} * |f|^p \qquad 1 \leqslant p < \infty.$$

Proof. The proof of (a) is an easy consequence of equation (2.2) and the corresponding properties of the heat kernel. We now show (b). $\|k_t^{(\gamma)}\|_{L^1} = 1$ is obvious. For any fixed x,

$$\begin{aligned} |f_t(x)|^p &= \left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\,\widehat{f_t}(\xi)\,\mathrm{d}\xi\right|^p \\ &= \left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\mathrm{e}^{-|\xi|^{2\gamma}t}\,\widehat{f}(\xi)\,\mathrm{d}\xi\right|^p \\ &= \left|\int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} \mathrm{e}^{-|\xi|^{2\gamma}t}\mathrm{e}^{\mathrm{i}(x-y)\xi}\,\mathrm{d}\xi\,\mathrm{d}y\right|^p. \end{aligned}$$

By Hölder's inequality, with $1/p + 1/q = 1$,

$$|f_t(x)|^p \leqslant \left|\int_{\mathbb{R}^n} |f(y)|^p \int_{\mathbb{R}^n} \mathrm{e}^{-|\xi|^{2\gamma}t}\mathrm{e}^{\mathrm{i}(x-y)\xi}\,\mathrm{d}\xi\,\mathrm{d}y\right| II$$

where II is given by

$$II = \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-|\xi|^{2\gamma}t}\mathrm{e}^{\mathrm{i}(x-y)\xi}\,\mathrm{d}\xi\,\mathrm{d}y\right|^{p/q} = \left|\int_{\mathbb{R}^n} k_t^{(\gamma)}(x-y)\,\mathrm{d}y\right|^{p/q} = 1.$$

Therefore,

$$|f_t(x)|^p \leqslant \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\mathrm{e}^{-|\xi|^{2\gamma}t}\,\widehat{|f|^p}(\xi)\,\mathrm{d}\xi = k_t^{(\gamma)} * |f|^p(x).$$

An easier proof can be given by a direct use of Hölder's inequality.

We now establish estimates for the operator $K(t)$ between Morrey spaces.

Proposition 2.4. *Let $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ and $0 \leqslant \lambda_1 = \lambda_2 < n$. For any $t > 0$, the operators $K(t)$, $W(t) = \nabla K(t)$ and $\partial_t K(t)$ are bounded operators from $\mathcal{M}_{q_1,\lambda_1}$ to $\ddot{\mathcal{M}}_{q_2,\lambda_2}$ and depend on t continuously, where ∇ denotes the space derivative. Furthermore, we have for $f \in \mathcal{M}_{q_1,\lambda_1}$,*

$$t^{\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|K(t)f\|_{q_2,\lambda_2} \leqslant C\|f\|_{q_1,\lambda_1} \tag{2.3}$$

$$t^{\frac{1}{2\gamma}+\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|W(t)f\|_{q_2,\lambda_2} \leqslant C\|f\|_{q_1,\lambda_1} \tag{2.4}$$

$$t^{1+\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|\partial_t K(t)f\|_{q_2,\lambda_2} \leqslant C\|f\|_{q_1,\lambda_1} \tag{2.5}$$

where $\alpha_i = \frac{n-\lambda_i}{q_i}(i = 1, 2)$ and constants C depend on $\gamma, q_1, q_2, \lambda_1, \lambda_2$.

The proposition is a modification of lemma 2.1 in [9] and its proof will be given in the appendix.

3. Well posedness in Morrey spaces

In this section we deal with the IVP (1.1), (1.3), (1.2) for the QGS equations with initial data in Morrey spaces. The solutions are found to be in the spaces of weighted continuous functions in time, which we now introduce. Kato and collaborators first define such spaces in solving the IVP for the Navier–Stokes equations [9, 10, 13, 12].

Definition 3.1. Let $0 < T < \infty$. For a given Banach space X and a real number $\alpha \geqslant 0$, we denote by $C_\alpha((0, T); X)$ the space of X−valued continuous functions f on $(0, T)$ with the norm

$$\|f\|_{\alpha,X} = \sup_{0<t<T} t^\alpha \|f(\cdot, t)\|_X < \infty.$$

In particular, $C_0((0, T); X) = BC((0, T); X)$ is the space of bounded continuous functions (note here that $C_0((0, T); X) \neq C((0, T); X)$, the space of continuous functions).

$\dot{C}_\alpha = \dot{C}_\alpha((0, T); X)$ denotes a subspace of $C_\alpha((0, T); X)$ consisting of all functions f with

$$\lim_{t\to 0}\sup t^\alpha \|f(\cdot, t)\|_X = 0.$$

In the following Morrey spaces will play the role of X and the norm in $C_\alpha((0, T); \mathcal{M}_{p,\lambda})$ will be abbreviated as $\|\|_{\alpha,p,\lambda}$.

Now we state the main theorem of this section and then prove it.

Theorem 3.2. *Let $\frac{1}{2} < \gamma \leqslant 1$ and $\theta_0 \in \mathcal{M}_{p,\lambda}$ with*

$$1 < p < \infty \qquad \text{and} \qquad \lambda = n - (2\gamma - 1)p \geqslant 0.$$

Then there is a $\delta > 0$ such that if $\|\theta_0\|_{p,\lambda} < \delta$, the IVP (1.1), (1.3), (1.2) has a solution θ on $(0, T)$ for some $T > 0$ satisfying

$$\theta \in BC((0, T); \ddot{\mathcal{M}}_{p,\lambda}) \cap (\cap_{1<q\leqslant p} \cap_{k>n-(2\gamma-1)q} BC([0, T); L_{-k/q,q})). \tag{3.1}$$

Furthermore, for any $p < q \leqslant \infty$

$$\theta \in \mathbb{Y} \equiv C_{m'}((0, T); \ddot{\mathcal{M}}_{q,\lambda}) \qquad m' = \left(1 - \frac{1}{2\gamma}\right)\left(1 - \frac{p}{q}\right) \tag{3.2}$$

and θ is the unique solution in the class of functions satisfying (3.2) with small norm $\|u\|_{q,\lambda}$ for $p < q \leqslant \infty$. Moreover, the mapping

$$\mathcal{Q} : \theta_0 \longmapsto \theta \qquad V \longmapsto \mathbb{Y}$$

is Lipschitz for some neighbourhood V of θ_0.

Remark 3.3. It is easy to see from the proof of this theorem that the solution is actually global (i.e. $T = \infty$) if the norm $\|\theta_0\|_{p,\lambda}$ is sufficiently small.

Remark 3.4. Note that (3.1), in particular, implies that for $1 < q \leqslant p$

$$\theta \to \theta_0 \qquad \text{in } L_{-\frac{k}{q},q} \text{ as } t \to 0.$$

In general we do not anticipate $\theta \to \theta_0$ in $\mathcal{M}_{p,\lambda}$ for any $\theta_0 \in \mathcal{M}_{p,\lambda}$ since $K(t)$ is not a C_0 semigroup on $\mathcal{M}_{p,\lambda}$. Furthermore, we have

Proposition 3.5. *Let $f \in \mathcal{M}_{q,\lambda}$. Then $f \in \ddot{\mathcal{M}}_{q,\lambda}$ if and only if*

$$K(t)f - f \to 0 \qquad \textit{as } t \to 0 \textit{ in } \mathcal{M}_{q,\lambda}.$$

This proposition shows that $\ddot{\mathcal{M}}_{p,\lambda}$ is the maximal closed subspace of $\mathcal{M}_{p,\lambda}$ on which $K(t)$ is a C_0 semigroup. As a consequence of this proposition, (3.1) can be improved to

$$\theta \in BC([0, T); \ddot{\mathcal{M}}_{p,\lambda})$$

if the initial data $\theta_0 \in \ddot{M}_{p,\lambda}$.

Remark 3.6. For $\|\theta_0\|_{p,\lambda} < \delta$, (3.2) also gives us the decay rate of θ for large t, especially,

$$\|\theta\|_{L^\infty} = \mathrm{O}(t^{\frac{1}{2\gamma}-1}).$$

We prove this theorem by the method of integral equations and contraction mapping arguments. Following standard practice [5, 6, 9, 8, 11, 13], we write the QGS equation (1.1) into the integral form:

$$\theta(t) = K(t)\theta_0 - G(u,\theta)(t) \equiv K(t)\theta_0 - \int_0^t K(t-\tau)(u\nabla\theta)(\tau)\,\mathrm{d}\tau \tag{3.3}$$

where $K(t)$ is the solution operator of the linear equation

$$\partial_t\theta + \Lambda^{2\gamma}\theta = 0.$$

We observe that $u \cdot \nabla\theta = \sum_j u_j\partial_j\theta = \nabla \cdot (u\theta)$ provided that $\nabla \cdot u = 0$. This provides an alternative expression for G:

$$G(u,\theta)(t) = G(u\theta)(t) = \int_0^t \nabla \cdot K(t-\tau)(u\theta)(\tau)\,\mathrm{d}\tau.$$

We shall solve (3.3) in the spaces of weighted continuous functions in time introduced at the beginning of this section. To this end we need estimates for the operator G acting between these spaces.

Proposition 3.7. *Let $f \in C_h((0, T); \mathcal{M}_{q_1,\lambda_1})$ and $g \in C_l((0, T); \mathcal{M}_{q_2,\lambda_2})$ with*

$$1 \leqslant q_1 \leqslant \infty \qquad 1 \leqslant q_2 \leqslant \infty \qquad \frac{1}{q_1} + \frac{1}{q_2} \leqslant 1 \qquad h + l < 1.$$

Assume that q and λ satisfy: $1 \leqslant q \leqslant \infty$

$$\frac{1}{q} \leqslant \frac{1}{q_1} + \frac{1}{q_2} \qquad 0 \leqslant \epsilon \equiv \frac{n-\lambda_1}{q_1} + \frac{n-\lambda_2}{q_2} - \frac{n-\lambda}{q} < 2\gamma - 1.$$

Then

$$G(f, g) \in C_m((0, T); \mathcal{M}_{q,\lambda})$$

and

$$\|G(f, g)\|_{C_m((0,T);\mathcal{M}_{q,\lambda})} \leqslant C\|f\|_{C_h((0,T);\mathcal{M}_{q_1,\lambda_1})}\|g\|_{C_l((0,T);\mathcal{M}_{q_2,\lambda_2})} \tag{3.4}$$

where $m = h + l + \frac{1}{2\gamma}(1+\epsilon) - 1$ and C is a constant.

Proof. To prove this proposition, we use proposition 2.4 and the Hölder inequality in lemma 2.2

$$\|G(f,g)(t)\|_{q,\lambda} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2\gamma}(1+\epsilon)}\|fg\|_{\frac{q_1q_2}{q_1+q_2},\frac{\lambda_1q_2}{q_1+q_2}+\frac{\lambda_2q_1}{q_1+q_2}}(\tau)\,\mathrm{d}\tau$$
$$\leqslant C\int_0^t (t-\tau)^{-\frac{1}{2\gamma}(1+\epsilon)}\|f(\cdot,\tau)\|_{\lambda_1,q_1}\|g(\cdot,\tau)\|_{\lambda_2,q_2}\,\mathrm{d}\tau.$$

Therefore for $m = h + l + \frac{1}{2\gamma}(1+\epsilon) - 1$,

$$\|G(f,g)(t)\|_{q,\lambda} \leqslant C\|f\|_{h,q_1,\lambda_1}\|g\|_{l,q_2,\lambda_2}t^{-m}\int_0^1 (1-\sigma)^{-\frac{1}{2\gamma}(1+\epsilon)}\sigma^{-(h+l)}\,\mathrm{d}\sigma.$$

The integral in the last inequality can be written as the Beta function

$$B\left(1-\frac{1}{2\gamma}[1+\epsilon], 1-h-l\right).$$

Using the fact that the Beta function $B(a,b)$ is finite if $a > 0$ and $b > 0$, we obtain that

$$\|G(f,g)\|_{m,q,\lambda} \leqslant C\|f\|_{h,q_1,\lambda_1}\|g\|_{l,q_2,\lambda_2}$$

under the assumptions of this proposition. The estimate (3.4) is thus proved. The continuity of $G(f,g)$ in t comes from applying dominated convergence theorem to the quantity

$$t^m G(f,g)(t) = t^m\int_0^t W(t-\tau)(fg)(\tau)\,\mathrm{d}\tau.$$

We will need a result concerning the Calderon–Zygmund-type singular integral operators on Morrey spaces. The Riesz transform is a particular example of these types of singular integral operators.

Lemma 3.8. *Let $\mathcal{Z}$ be a Calderon–Zygmund-type singular integral operator, i.e. $\mathcal{Z}$: $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a homogeneous continuous function of degree $-n$ and the integral on the unit sphere vanishes. Let $\mathcal{M}_{q,\mu}$ with $1 < q < \infty, 0 \leqslant \mu < n$ be a Morrey space. Then $\mathcal{Z}$ is bounded on $\mathcal{M}_{q,\mu}$ to itself.*

This result can be found in [17] but see [9] for an elementary proof.

Proof of theorem 3.2. Let X denote the Banach space

$$X = C_{\left(\frac{1}{2}-\frac{1}{4\gamma}\right)}((0,T);\mathcal{M}_{2p,\lambda}) \tag{3.5}$$

and X_R denote the complete metric space of the closed ball in X centred at 0 and of radius R, where T and R are yet to be determined. Consider the nonlinear map $\mathcal{A}$ on X_R given by

$$\mathcal{A}(\theta)(t) = K(t)\theta_0 - G(u\theta)(t) \qquad t \in (0,T).$$

We now start proving that $\mathcal{A}$ maps X_R to itself and is a contraction. Applying proposition 2.4 with

$$q_1 = p \qquad q_2 = 2p \qquad \lambda_1 = \lambda_2 = \lambda = n - (2\gamma - 1)p$$

we obtain

$$\|K\theta_0\|_X = \|K\theta_0\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \leqslant C_0\|\theta_0\|_{p,\lambda}$$

where constant C_0 may depend on n, p, γ.

The application of proposition 3.7 with

$$q_1 = q_2 = q = 2p \qquad \lambda_1 = \lambda_2 = \lambda = n - (2\gamma - 1)p \qquad m = h = l = \frac{1}{2} - \frac{1}{4\gamma}$$

shows that

$$\|G(u\theta)\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \leqslant C\|u\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda}\|\theta\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda}.$$

Using lemma 3.8, we obtain for $\theta \in X_R$

$$\|G(u\theta)\|_X = \|G(u\theta)\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \leqslant C_1 R^2.$$

where C and C_1 are constants.

Furthermore, for any θ and $\tilde{\theta} \in X_R$,

$$\|\mathcal{A}(\theta) - \mathcal{A}(\tilde{\theta})\|_X = \|G(u\theta) - G(\tilde{u}\tilde{\theta})\|_X$$

where $\tilde{u} = (\tilde{u}_j)$ with $\tilde{u}_j = \pm\mathcal{R}_{\pi(j)}\tilde{\theta}(j = 1, 2, \ldots, n)$. Using proposition 3.7 to estimate G again,

$$\begin{aligned}\|\mathcal{A}(\theta) - \mathcal{A}(\tilde{\theta})\|_X &\leqslant \|G((\tilde{u} - u)\tilde{\theta})\|_X + \|G(u(\theta - \tilde{\theta}))\|_X \\ &\leqslant C(\|\tilde{u} - u\|_X\|\tilde{\theta}\|_X + \|\theta - \tilde{\theta}\|_X\|u\|_X).\end{aligned}$$

Since $(\tilde{u} - u)_j = \pm\mathcal{R}_{\pi(j)}(\tilde{\theta} - \theta)$, lemma 3.8 implies that for some constant $C_{p,\lambda}$

$$\|u\|_X \leqslant C_{p,\lambda}\|\theta\|_X \qquad \|\tilde{u} - u\|_X \leqslant C_{p,\lambda}\|\tilde{\theta} - \theta\|_X.$$

Therefore for constant $C_2 = 2CC_{p,\lambda}$,

$$\|\mathcal{A}(\theta, \theta_0) - \mathcal{A}(\tilde{\theta}, \theta_0)\|_X \leqslant \frac{C_2}{2}(\|\tilde{\theta}\|_X + \|\theta\|_X)\|\tilde{\theta} - \theta\|_X \leqslant C_2 R\|\tilde{\theta} - \theta\|_X.$$

It then follows that the conditions that $\mathcal{A}$ map X_R into itself and be a contraction are

$$C_0\|\theta_0\|_{p,\lambda} + C_1 R^2 \leqslant R \qquad C_2 R < 1.$$

These conditions are met if

$$\|\theta_0\|_{p,\lambda} < \delta = \frac{1}{4C_0C_1} \qquad R < \min\left\{\frac{1}{2C_1}, \frac{1}{C_2}\right\}.$$

Hence for the above chosen δ and R, $\mathcal{A}$ has a unique fixed point $\theta \in X_R$ satisfying $\theta(t) = \mathcal{A}(\theta)(t)$ for $t \in (0, T)$ ($T > 0$ and $T = \infty$ if $\|\theta_0\|_{p,\lambda}$ is small enough).

To show that θ satisfies (3.1), we first note that

$$\theta = \mathcal{A}(\theta) \equiv K\theta_0 - G(u\theta).$$

For $\theta_0 \in \mathcal{M}_{p,\lambda}$ with $\lambda = n - (2\gamma - 1)p$,

$$K\theta_0 \in BC((0, T); \dot{\mathcal{M}}_{p,\lambda})$$

as implied by proposition 2.4. Furthermore, because of the continuous embedding for $1 < q \leqslant p$ and $k > n - (2\gamma - 1)q$

$$\mathcal{M}_{p,\lambda} \subset \mathcal{M}_{q,n-(2\gamma-1)q} \subset L_{-\frac{k}{q},q}$$

and the fact that $K(t)$ forms a C_0 semigroup on $L_{-k/q,q}$ for $1 < q$, we have $K\theta_0 \in BC([0, T); L_{-k/q,q})$ for $q > 1$ and $k > n - (2\gamma - 1)q$.

For the term $G(u\theta)$, we first apply proposition 3.7 with

$$\lambda_1 = \lambda_2 = \lambda \qquad q_1 = q_2 = 2p \qquad q = p \qquad h = l = \frac{1}{2} - \frac{1}{4\gamma} \qquad m = 0$$

to show that

$$G(u\theta) \in BC((0,T); \ddot{\mathcal{M}}_{p,\lambda}). \tag{3.6}$$

Now we use (3.6) and apply proposition 3.7 with

$$\lambda_1 = \lambda_2 = \lambda \qquad q_1 = q_2 = p \qquad q = \frac{p}{1+\eta}$$
$$h = l = 0 \qquad m = -\eta\left(1 - \frac{1}{2\gamma}\right) \qquad \eta > 0 \text{ is small}$$

to obtain that

$$G(u\theta) \in C_{-\eta\left(1-\frac{1}{2\gamma}\right)}((0,T); \mathcal{M}_{q,\lambda}) \qquad \text{for } q = \frac{p}{1+\eta} < p$$

which implies that

$$G(u\theta) \to 0 \qquad \text{as } t \to 0 \text{ in } \mathcal{M}_{q,\lambda}$$

because of the embedding $\mathcal{M}_{q,\lambda} \subset L_{-k/q,q}$ for $k > n - (2\gamma - 1)q \geqslant \lambda$,

$$G(u\theta) \to 0 \qquad \text{in } L_{-k/q,q}, \text{ as } t \to 0.$$

Therefore $G(u\theta) \in BC([0,T); L_{-\frac{k}{q},q})$ for $1 < q \leqslant p$ and $k > n - (2\gamma - 1)q$. Summing up, we have shown that $\theta = K\theta_0 - G(u\theta)$ is exactly in the class defined by (3.1).

We now prove that θ satisfies (3.2). $K\theta_0$ satisfying (3.2) is an easy consequence of proposition 2.4. We apply proposition 3.7 to $G(u\theta)$ with

$$h = l = \left(\frac{1}{2} - \frac{1}{4\gamma}\right) \qquad m = m'$$
$$q = q \qquad q_1 = q_2 = 2p \qquad \lambda = \lambda_1 = \lambda_2$$

and use the fact that $\theta \in X$ (3.5) to show that $G(u\theta)$ is in the class defined by (3.2). This argument, combined with the uniqueness of θ in (3.5), indicates the uniqueness of θ in (3.2).

The proof of the Lipschitz property is routine (see i.e. [20]) and is therefore omitted.

4. Further regularity

In this section we prove that the solution θ obtained in theorem 3.2 is actually smooth. More precisely, we have

Theorem 4.1. *Let θ be the solution obtained in theorem 3.2, Then for any $p \leqslant q < \infty$ and $k, j = 0, 1, 2, \cdots$*

$$\partial_t^k \nabla^j \theta \in C((0,T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.1}$$

where $\lambda = n - (2\gamma - 1)p$, ∇ denotes the space derivative and $C((0,T); X)$ is the space of X-valued continuous functions on $(0,T)$.

Proof. The smoothness of θ is proved by standard schemes. First we consider the case when $k = 0$. For $j = 0$, (4.1) can be seen from (3.1), (3.2) in theorem 3.2. We now prove that (4.1) is true for $j = 1$. We take any $t_1 > 0$ and prove the results for $t > t_1$.

We take ∇ of G and apply the Leibnitz rule to obtain

$$\nabla G(u, \theta) = G_1(\nabla u, \theta) + G_2(u, \nabla\theta)$$

where G_1 and G_2 are integral operators on (t_1, T) with the same properties as G. First let $q > p$ and X be the space consisting of functions θ such that

$$\theta \in C([t_1, T); \ddot{\mathcal{M}}_{q,\lambda}) \qquad \nabla\theta \in C_{1-\frac{1}{2\gamma}}((t_1, T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.2}$$

and X_R be the closed ball of radius R in X. The idea is to apply contraction mapping arguments to $\mathcal{A}$ on X_R with T and R to be determined. First we choose R appropriately such that $K(t)\theta_1 \in X_R$, where $\theta_1 = \theta(t_1)$ is the value of θ at t_1. As in the proof of theorem 3.2, we apply proposition 3.7 to G, G_1 and G_2 to show that for $\theta \in X_R$

$$G(u, \theta) \in C_{\left(-\left(1-\frac{1}{2\gamma}\right)\left(1-\frac{p}{q}\right)\right)}((0, T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.3}$$

$$\nabla G(u, \theta) \in C_{\left(1-\frac{1}{2\gamma}\right)\left(1-\frac{p}{q}\right)}((0, T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.4}$$

and lemma 3.8 is used in estimating u and ∇u as usual. Equations (4.3) and (4.4) imply not only $G \in X_R$, but also that the norm of $G(u, \theta)$ in X_R has a small factor $(T - t_1)^{\varrho}$ if $T - t_1$ is small, where

$$\varrho = \min\left\{\left(1 - \frac{1}{2\gamma}\right)\left(1 - \frac{p}{q}\right), \left(1 - \frac{1}{2\gamma}\right)\frac{p}{q}\right\}.$$

If $T - t_1$ is chosen small and R taken as above, then $\mathcal{A}$ maps X_R to itself. Furthermore , it can be shown in the same spirit as in the proof of theorem 3.2 that $\mathcal{A}$ is a contraction map on X_R. Therefore $\mathcal{A}$ has a fixed point θ in X_R, which solves (3.3). The uniqueness result in theorem 3.2 indicates that this θ is just the original θ obtained in theorem 3.2. Thus we have shown that $\theta \in C((t_1, T); \ddot{\mathcal{M}}_{q,\lambda})$, which implies that $\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda})$ because of the arbitrariness of t_1.

For the case $q = p$, the result $\nabla\theta \in C((t_1, T); \ddot{\mathcal{M}}_{p,\lambda})$ can be established by applying proposition 3.7 to ∇G again and using the relation in (4.2).

Repeating the same argument for higher spatial derivatives of θ, we obtain the result $\nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda})$. This finishes the proof for the case $k = 0$.

We now prove (4.1) for $k = 1$. It is easy to see from the regularity result we have just obtained that

$$\nabla^j\theta \qquad \nabla^j u \qquad (-\Delta)^{\lambda}\nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda}) \qquad j = 0, 1, 2, \ldots$$

for any $p \leqslant q < \infty$. Using equation (1.1)

$$\partial_t\theta = -u\nabla\theta - (-\Delta)^{\gamma}\theta$$

and the Hölder inequality for the Morrey spaces (i.e. (ii) of lemma 2.2), we obtain for $j = 0, 1, 2, \ldots$

$$\partial_t\nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda}).$$

The result for general k can be established by induction. This concludes the proof of theorem 4.1.

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Appendix

We prove proposition 2.4 in this appendix. For simplicity of notation, we will drop γ when writing the kernel $k_t^{(\gamma)}$ and use f_t for $k_t * f$.

We first prove estimate (2.3). For $q_1 = q_2 = \infty$, (2.3) is obvious. For $q_1 = q_2 < \infty$, we use (b) of lemma 2.3, i.e.

$$\|k_t\|_{L^1} = 1 \qquad |f_t|^p \leqslant k_t * |f|^p \tag{A.1}$$

to obtain for $q_2 = q_1$, $\lambda_2 = \lambda_1$

$$\|f_t\|_{q_2,\lambda_2} \leqslant \|f\|_{q_1,\lambda_1}. \tag{A.2}$$

To prove (2.3) in the general case, we first estimate $\|f_t\|_{L^\infty}$. By (A.1),

$$\begin{aligned} |f_t(0)|^{q_1} &\leqslant \int_{\mathbb{R}^n} k_t(|x|)|f(x)|^{q_1}\,\mathrm{d}x = \int_0^\infty k_t(\rho)\,\mathrm{d}r(\rho) \\ &\leqslant \int_0^\infty |k_t'(\rho)|r(\rho)\,\mathrm{d}\rho \\ &\leqslant \|f\|_{q_1,\lambda_1}^{q_1} \int_0^\infty |k_t'(\rho)|\rho^{\lambda_1}\,\mathrm{d}\rho \end{aligned}$$

where $r(\rho) = \int_{|y|<\rho} |f(y)|^{q_1}\,\mathrm{d}y$ and we have used the fact that $r(\rho) \leqslant \|f\|_{q_1,\lambda_1}^{q_1}\rho^{\lambda_1}$. The decay properties of k_t (see lemma 2.3) are also used here. Since $x = 0$ is not special, we obtain

$$\|f_t\|_{L^\infty}^{q_1} \leqslant c\|f\|_{q_1,\lambda_1}^{q_1} t^{-\frac{n+1}{2\gamma}} \int_0^\infty |k_1'(\rho t^{-\frac{1}{2\gamma}})|\rho^{\lambda_1}\,\mathrm{d}\rho \leqslant c\|f\|_{q_1,\lambda_1}^{q_1} t^{\frac{\lambda_1-n}{2\gamma}} \tag{A.3}$$

which also proves (2.3) for $q_1 < q_2 = \infty$.

We can now estimate $\|f_t\|_{q_2,\lambda_2}$ in terms of $\|f\|_{q_1,\lambda_1}$ for $q_1 \leqslant q_2 < \infty$. For any real number $R > 0$ and $x \in \mathbb{R}^n$,

$$R^{-\lambda_2}\int_{|x-y|<R} |f_t(y)|^{q_2}\,\mathrm{d}y \leqslant R^{-\lambda_2}(\|f_t\|_{L^\infty})^{(q_2-q_1)} \int_{|x-y|<R} |f_t(y)|^{q_1}\,\mathrm{d}y.$$

Using estimate (A.3) and noting that $\lambda_1 = \lambda_2$,

$$R^{-\lambda_2}\int_{|x-y|<R} |f_t(y)|^{q_2}\,\mathrm{d}y \leqslant c(\|f\|_{q_1,\lambda_1})^{(q_2-q_1)} t^{\frac{1}{2\gamma}\left(\frac{(\lambda_1-n)}{q_1}\right)(q_2-q_1)} \|f_t\|_{q_1,\lambda_1}^{q_1}.$$

Because of $\|f_t\|_{q_1,\lambda_1} \leqslant \|f\|_{q_1,\lambda_1}$ as in (A.2),

$$R^{-\lambda_2}\int_{|x-y|<R} |f_t(y)|^{q_2}\,\mathrm{d}y \leqslant c\|f\|_{q_1,\lambda_1}^{q_2} t^{\frac{q_2}{2\gamma}(\alpha_2-\alpha_1)}$$

where $\alpha_i = \frac{n-\lambda_i}{q_i} (i = 1, 2)$. By definition,

$$\|f_t\|_{q_2,\lambda_2} \leqslant \mathrm{ct}^{\frac{1}{2\gamma}(\alpha_2-\alpha_1)}\|f\|_{q_1,\lambda_1}$$

where constants c depend on the indices $\gamma, q_1, q_2, \lambda_1, \lambda_2$.

Estimates (2.3), (2.4) can be proved similarly by using the identities

$$(\partial_x k_t)(x) = \mathrm{ct}^{-\frac{1}{2\gamma}} g_t(x) \qquad (\partial_t k_t)(x) = \mathrm{ct}^{-1} g_t(x)$$

where g_t is another radial function enjoying the same properties as $k_t(x)$ does (the properties of k_t are listed in lemma 2.3).

$f_t \in \ddot{\mathcal{M}}_{q_2,\lambda_2}$ follows directly from (2.3). and the continuity of $K(t)$ in t is a consequence of (2.4). The continuity of W follows from a similar estimate for $\partial_t W$ as in (2.4) and that of $\partial_t K(t)$ from the identity $\partial_t K = -\Lambda^{2\gamma} K$.

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