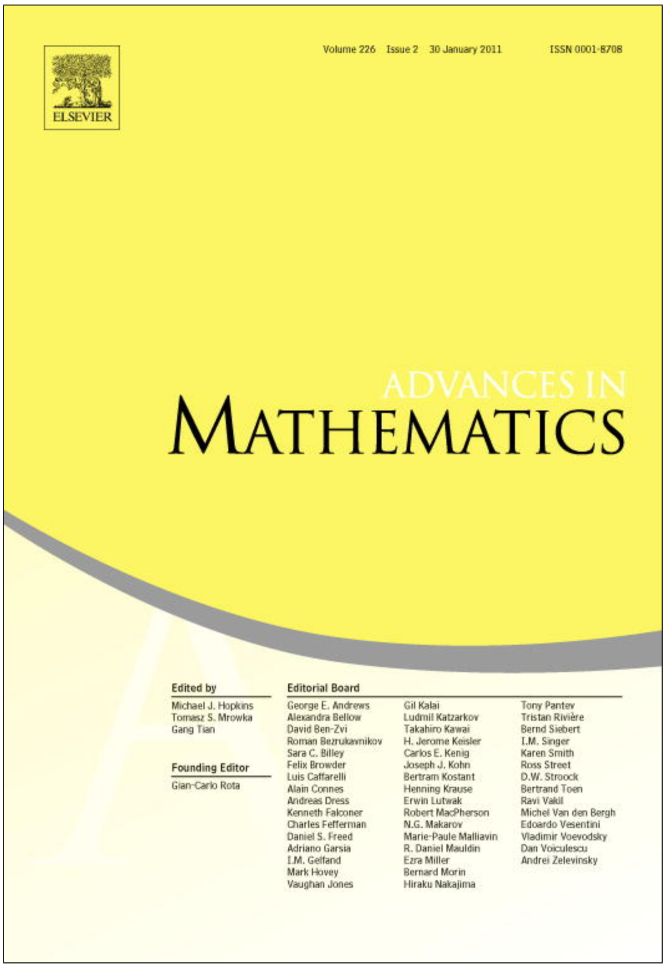
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Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion

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Abstract

Whether or not classical solutions of the 2D incompressible MHD equations without full dissipation and magnetic diffusion can develop finite-time singularities is a difficult issue. A major result of this paper establishes the global regularity of classical solutions for the MHD equations with mixed partial dissipation and magnetic diffusion. In addition, the global existence, conditional regularity and uniqueness of a weak solution is obtained for the 2D MHD equations with only magnetic diffusion.

MSC: 35B45; 35B65; 76W05

Keywords: Classical solutions; Global regularity; MHD equations; Partial dissipation and magnetic diffusion

1. Introduction

This paper concerns itself with the fundamental issue of whether classical solutions of the 2D incompressible MHD equations can develop finite-time singularities. The 2D MHD equations under consideration assume the form

$$u_t + u \cdot \nabla u = -\nabla p + \nu_1 u_{xx} + \nu_2 u_{yy} + b \cdot \nabla b, \tag{1}$$

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$$b_t + u \cdot \nabla b = \eta_1 b_{xx} + \eta_2 b_{yy} + b \cdot \nabla u, \tag{2}$$

$$\nabla \cdot u = 0, \tag{3}$$

$$\nabla \cdot b = 0, \tag{4}$$

where $(x, y) \in \mathbf{R}^2$, $t \geqslant 0$, $u = (u_1(x, y, t), u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ denotes the pressure, $b = (b_1(x, y, t), b_2(x, y, t))$ denotes the magnetic field, and ν_1, ν_2, η_1 and η_2 are nonnegative real parameters.

When $\nu_1 > 0$, $\nu_2 > 0$, $\eta_1 > 0$ and $\eta_2 > 0$, (1)–(4) has a unique global classical solution for every initial data $(u_0, b_0) \in H^m$ with $m \geqslant 2$ (see e.g. [4,9]). However, if any one of these parameters is zero, the global regularity issue has not been settled. This paper establishes the global regularity of classical solutions of (1)–(4) with either $\nu_1 = 0$, $\nu_2 = \nu > 0$, $\eta_1 = \eta > 0$ and $\eta_2 = 0$ or $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. More precisely, we have the following theorem.

Theorem 1. *Consider the 2D MHD equations* (1)–(4) *with* $\nu_1 = 0$*,* $\nu_2 = \nu > 0$*,* $\eta_1 = \eta > 0$ *and* $\eta_2 = 0$*. Assume* $u_0 \in H^2(\mathbf{R}^2)$ *and* $b_0 \in H^2(\mathbf{R}^2)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$*. Then* (1)–(4) *with the initial data* (u_0, b_0) *has a unique global classical solution* (u, b)*. In addition,* (u, b) *satisfies*

$$(u, b) \in L^\infty\big([0, \infty); H^2\big), \qquad \omega_y \in L^2\big([0, \infty); H^1\big), \qquad j_x \in L^2\big([0, \infty); H^1\big), \tag{5}$$

where $\omega = \nabla \times u$ *and* $j = \nabla \times b$ *represent the vorticity and the current density, respectively.*

A similar global regularity result can also be stated for (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$.

Attention is also paid to the 2D MHD equations without dissipation but with magnetic diffusion, namely (1)–(4) with $\nu_1 = \nu_2 = 0$ but with $\eta_1 = \eta_2 = \eta > 0$. In this case, we obtain the following global *a priori* bound for $\omega = \nabla \times u$ and $j = \nabla \times b$,

$$\|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \eta \int_0^t \|\nabla j(\tau)\|_2^2 \, d\tau \leqslant C(\eta)\big(\|\omega(0)\|_2^2 + \|j(0)\|_2^2\big) \quad \text{for } t \geqslant 0,$$

where $C(\eta)$ is a constant depending on η only. One consequence of this global bound is the existence of a global H^1-weak solution. It is not clear if such weak solutions are unique or can be improved to global classical solutions. However, if we know the velocity field u of a solution obeys

$$\sup_{q \geqslant 2} \frac{1}{q} \int_0^T \|\nabla u(t)\|_q \, dt < \infty, \tag{6}$$

then this solution actually becomes a classical solution on $[0, T]$ and two weak solutions with one of their velocities satisfying this bound must coincide on $[0, T]$. We remark that (6) is weaker than the standard condition $\int_0^T \|\nabla u(t)\|_\infty \, dt < \infty$ and, as some preliminary evidence shows, is more likely to be proven true for (1)–(4) with $\eta_1 = \eta_2 = \eta > 0$.

This work is partially motivated by the recent progress made by Chae [2], Hou and Li [7] and Danchin and Paicu [3] on the 2D Boussinesq equations,

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + \theta e_2, \tag{7}$$

$$\nabla \cdot u = 0, \tag{8}$$

$$\theta_t + u \cdot \nabla \theta = \eta \Delta \theta, \tag{9}$$

where the 2D vector u represents the velocity field, the scalar θ the temperature, and $e_2 = (0, 1)$. Chae [2], and Hou and Li [7] established the global regularity of (7)–(8) with either dissipation or thermal diffusion. Further improvement was recently made by Hmidi, Keraani and Rousset, who reduced the Laplacian $-\Delta$ to $(-\Delta)^{\frac{1}{2}}$ [5,6]. Danchin and Paicu [3] constructed global solutions of (7)–(8) with either $\eta = 0$ and $\nu \Delta u$ replaced by νu_{xx} or $\nu = 0$ and $\eta \Delta \theta$ by $\eta \theta_{xx}$. We remark that the global regularity issue for the 2D MHD equations (1)–(4) is more sophisticated. The equations of u and b in (1)–(4) are both nonlinearly coupled vectors equations and the approaches in [2,3] and [7] do not appear to apply. In fact, it is not clear if (1)–(4) with $\eta_1 = \eta_2 = 0$ or (1)–(4) with $\nu_2 = \eta_2 = 0$ has global classical solutions.

The rest of this paper is divided into two sections. The second section is devoted to the global regularity of (1)–(4) with either $\nu_1 = 0$, $\nu_2 = \nu > 0$, $\eta_1 = \eta > 0$ and $\eta_2 = 0$ or $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. The third section handles (1)–(4) with $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 = \eta > 0$. Throughout these sections the L^p-norm of a function f is denoted by $\|f\|_p$, the H^s-norm by $\|f\|_{H^s}$ and the norm in the Sobolev space $W^{s,p}$ by $\|f\|_{W^{s,p}}$.

2. Mixed partial dissipation and magnetic diffusion

This section proves Theorem 1 as well as a parallel result for the case when $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. The proof of Theorem 1 is achieved through two stages. The first stage establishes a global bound for $\|\omega(t)\|_2$ and $\|j(t)\|_2$ while the second obtains a bound for $\|\nabla\omega(t)\|_2$ and $\|\nabla j(t)\|_2$. The following elementary lemma will play an important role.

2.1. An elementary lemma

Lemma 1. *Assume that f, g, g_y, h and h_x are all in $L^2(\mathbf{R}^2)$. Then,*

$$\iint |fgh|\,dx\,dy \leqslant C\|f\|_2 \|g\|_2^{1/2} \|g_y\|_2^{1/2} \|h\|_2^{1/2} \|h_x\|_2^{1/2}. \tag{10}$$

Proof. Applying Hölder's inequality and the elementary inequality

$$\sup_{x\in\mathbf{R}} \bigl|F(x)\bigr| \leqslant \sqrt{2}\biggl(\int \bigl|F(x)\bigr|^2\,dx\biggr)^{\frac{1}{4}} \biggl(\int \bigl|F_x(x)\bigr|^2\,dx\biggr)^{\frac{1}{4}}, \tag{11}$$

we have

$$\iint |fgh|\,dx\,dy$$
$$\leqslant C\int \biggl[\biggl(\int |f|^2\,dx\biggr)^{1/2} \biggl(\int |g|^2\,dx\biggr)^{1/2} \Bigl(\sup_{-\infty<x<\infty} h\Bigr)\biggr]\,dy$$

$$\leqslant C\int\biggl[\biggl(\int|f|^2\,dx\biggr)^{1/2}\biggl(\int|g|^2\,dx\biggr)^{1/2}\biggl(\int|h|^2\,dx\biggr)^{1/4}\biggl(\int|h_x|^2\,dx\biggr)^{1/4}\biggr]\,dy$$

$$\leqslant C\|f\|_2\biggl(\sup_{-\infty<y<\infty}\biggl(\int|g|^2\,dx\biggr)^{1/2}\biggr)\|h\|_2^{\frac{1}{2}}\|h_x\|_2^{\frac{1}{2}}. \tag{12}$$

In addition, by (11) again,

$$\begin{aligned}
\sup_{-\infty<y<\infty}\biggl(\int|g|^2\,dx\biggr)^4 &\leqslant C\biggl[\int\biggl(\int|g|^2\,dx\biggr)^2\,dy\biggr]\biggl[\int\biggl(\int|g||g_y|\,dx\biggr)^2\,dy\biggr]\\
&\leqslant C\biggl(\int\biggl(\int|g|^4\,dy\biggr)^{1/2}\,dx\biggr)^2\int\biggl[\biggl(\int|g|^2\,dx\biggr)\biggl(\int|g_y|^2\,dx\biggr)\biggr]\,dy\\
&\leqslant C\biggl(\int\biggl[\biggl(\int|g|^2\,dy\biggr)^{3/4}\biggl(\int|g_y|^2\,dy\biggr)^{1/4}\biggr]\,dx\biggr)^2\\
&\quad\times\biggl(\sup_{-\infty<y<\infty}\int|g|^2\,dx\biggr)\biggl(\iint|g_y|^2\,dx\,dy\biggr)\\
&\leqslant C\|g\|_2^3\|g_y\|_2\biggl(\sup_{-\infty<y<\infty}\int|g|^2\,dx\biggr)\|g_y\|_2^2.
\end{aligned}$$

That is,

$$\sup_{-\infty<y<\infty}\int|g|^2\,dx\leqslant C\|g\|_2\|g_y\|_2. \tag{13}$$

Combining (12) and (13) yields (10). This completes the proof of Lemma 1. □

2.2. A priori *bounds for* $\|\omega\|_2$ *and* $\|j\|_2$

This subsection establishes *a priori* bounds for $\|\omega\|_2$ and $\|j\|_2$ as stated in the following proposition.

Proposition 2. *If* (u,b) *solves* (1)–(4) *with* $\nu_1=0$, $\nu_2=\nu>0$, $\eta_1=\eta>0$ *and* $\eta_2=0$, *then the vorticity* $\omega=\nabla\times u$ *and the current density* $j=\nabla\times b$ *satisfy*

$$\|\omega(t)\|_2^2+\|j(t)\|_2^2+\nu\int_0^t\|\omega_y(\tau)\|_2^2\,d\tau+\eta\int_0^t\|j_x(\tau)\|_2^2\,d\tau\leqslant C\bigl(\|\omega_0\|_2^2+\|j_0\|_2^2\bigr) \tag{14}$$

where C *is a constant depending on* ν, η, $\|u_0\|_2$ *and* $\|b_0\|_2$ *only, and* $\omega_0=\nabla\times u_0$ *and* $j_0=\nabla\times b_0$.

Proof. Taking the inner products of (1) with u and (2) with b, adding the results and integrating by parts, we obtain

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu\int_0^t \|u_y(\tau)\|_2^2\,d\tau + 2\eta\int_0^t \|b_x(\tau)\|_2^2\,d\tau \leqslant \|u_0\|_2^2 + \|b_0\|_2^2. \tag{15}$$

Since ω and j satisfy

$$\omega_t + u\cdot\nabla\omega = \nu\omega_{yy} + b\cdot\nabla j, \tag{16}$$

$$j_t + u\cdot\nabla j = \eta j_{xx} + b\cdot\nabla\omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \tag{17}$$

we find that $X(t) = \|\omega(t)\|_2^2 + \|j(t)\|_2^2$ obeys

$$\frac{1}{2}\frac{dX(t)}{dt} + \nu\|\omega_y\|_2^2 + \eta\|j_x\|_2^2 \leqslant 2\left|\int \big[\partial_x b_1(\partial_x u_2 + \partial_y u_1) - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\big] j\,dx\,dy\right|.$$

Applying Lemma 1, we can bound the terms on the right as follows. C's in these estimates denote either pure constants or constants depending on ν and η only

$$\begin{aligned}
\int |\partial_x b_1||\partial_x u_2||j|\,dx\,dy &\leqslant C\|\partial_x u_2\|_2^{1/2}\|\partial_{xy}u_2\|_2^{1/2}\|j\|_2^{1/2}\|j_x\|_2^{1/2}\|\partial_x b_1\|_2 \\
&\leqslant \frac{\nu}{4}\|\partial_{xy}u_2\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\partial_x u_2\|_2\|\partial_x b_1\|^2\|j\|_2 \\
&\leqslant \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\omega\|_2\|\partial_x b_1\|_2^2\|j\|_2 \\
&\leqslant \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\partial_x b_1\|_2^2 X(t), \\
\int |\partial_x b_1||\partial_y u_1||j|\,dx\,dy &\leqslant C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xx}b_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_{yy}u_1\|_2^{\frac12}\|j\| \\
&\leqslant \frac{\nu}{4}\|\partial_{yy}u_1\|_2^2 + \frac{\eta}{8}\|\partial_{xx}b_1\|_2^2 + C\|\partial_x b_1\|\|\partial_y u_1\|_2\|j\|_2^2 \\
&\leqslant \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\big(\|\partial_x b_1\|_2^2 + \|\partial_y u_1\|_2^2\big)\|j\|_2^2, \\
\left|\int \partial_x u_1\partial_x b_2 j\,dx\,dy\right| &= \left|\int (u_1\partial_{xx}b_2 j + u_1\partial_x b_2 j_x)\,dx\,dy\right| \\
&\leqslant C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|j\|_2^{\frac12}\|j_x\|_2^{\frac12}\|\partial_{xx}b_2\|_2 \\
&\quad + C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx}b_2\|_2^{\frac12}\|j_x\|_2 \\
&\leqslant C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|j\|_2^{\frac12}\|j_x\|_2^{\frac32} + C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|j_x\|_2^{\frac32} \\
&\leqslant \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\partial_x b_2\|_2^2 \\
&\leqslant \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2,
\end{aligned}$$

$$\begin{aligned}
\left|\int \partial_x u_1 \partial_y b_1 j\,dx\,dy\right| &\leqslant \left|\int (u_1 \partial_{xy} b_1 j + u_1 \partial_y b_1 j_x)\,dx\,dy\right| \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|j_x\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2 \\
&\quad + C\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|j_x\|_2 \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|j_x\|_2^{\frac{3}{2}} + C\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|j_x\|_2^{\frac{3}{2}} \\
&\leqslant \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\partial_y b_1\|_2^2 \\
&\leqslant \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2.
\end{aligned}$$

Combining these estimates, we have

$$\frac{dX(t)}{dt} + \nu\|\omega_y\|_2^2 + \eta\|j_x\|_2^2 \leqslant C\big(\|\partial_y u_1\|_2^2 + \|\partial_x b_1\|_2^2\big)X(t),$$

which, together with (15), yields (14). □

2.3. A priori *bounds for* $\|\nabla\omega\|_2$ *and* $\|\nabla j\|_2$

This subsection provides global *a priori* bounds for $\|\nabla\omega\|_2$ and $\|\nabla j\|_2$.

Proposition 3. *If* (u,b) *solves* (1)–(4) *with* $\nu_1 = 0$, $\nu_2 = \nu > 0$, $\eta_1 = \eta > 0$ *and* $\eta_2 = 0$, *then the vorticity* ω *and the current density* j *satisfy*

$$\begin{aligned}
&\big\|\nabla\omega(t)\big\|_2^2 + \big\|\nabla j(t)\big\|_2^2 + \nu\int_0^t \big\|\nabla\omega_y(\tau)\big\|_2^2\,d\tau + \eta\int_0^t \big\|\nabla j_x(\tau)\big\|_2^2\,d\tau \\
&\quad \leqslant C\big(\|\nabla\omega_0\|_2^2 + \|\nabla j_0\|_2^2\big)
\end{aligned} \tag{18}$$

where the constant C *depends on* ν, η, $\|u_0\|_{H^1}$ *and* $\|b_0\|_{H^1}$ *only.*

Proof. Taking the inner products of (16) with $\Delta\omega$ lead to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_2^2 + \nu\|\nabla\omega_y\|_2^2 = &-\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx\,dy + \int \nabla\omega\cdot\nabla b\cdot\nabla j\,dx\,dy \\
&+ \int b\cdot\nabla(\nabla j)\cdot\nabla\omega\,dx\,dy.
\end{aligned}$$

Similarly, taking the inner product of (17) with Δj yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla j\|_2^2 + \eta\|\nabla j_x\|_2^2 = &-\int \nabla j\cdot\nabla u\cdot\nabla j\,dx\,dy + \int \nabla j\cdot\nabla b\cdot\nabla\omega\,dx\,dy \\
&+ \int b\cdot\nabla(\nabla\omega)\cdot\nabla j\,dx\,dy
\end{aligned}$$

$$+2\int \nabla\big[\partial_x b_1(\partial_x u_2+\partial_y u_1)\big]\cdot\nabla j\,dx\,dy$$
$$-2\int \nabla\big[\partial_x u_1(\partial_x b_2+\partial_y b_1)\big]\cdot\nabla j\,dx\,dy.$$

Adding the above equations and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2\big)+\nu\|\nabla\omega_y\|_2^2+\eta\|\nabla j_x\|_2^2=I_1+I_2+I_3+I_4+I_5,$$

where

$$I_1=-\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx\,dy,$$
$$I_2=-\int \nabla j\cdot\nabla u\cdot\nabla j\,dx\,dy,$$
$$I_3=2\int \nabla\omega\cdot\nabla b\cdot\nabla j\,dx\,dy,$$
$$I_4=2\int \nabla\big[\partial_x b_1(\partial_x u_2+\partial_y u_1)\big]\cdot\nabla j\,dx\,dy,$$
$$I_5=-2\int \nabla\big[\partial_x u_1(\partial_x b_2+\partial_y b_1)\big]\cdot\nabla j\,dx\,dy.$$

To bound I_1, we write the integrand explicitly and further divide it into four terms

$$\begin{aligned}I_1&=\int \big(\partial_x u_1\omega_x^2+\partial_x u_2\omega_x\omega_y+\partial_y u_1\omega_x\omega_y+\partial_y u_2\omega_y^2\big)\,dx\,dy\\&=I_{11}+I_{12}+I_{13}+I_{14}.\end{aligned}$$

By the divergence-free condition $\partial_x u_1+\partial_y u_2=0$ and Lemma 1,

$$\begin{aligned}I_{11}&=-\int \partial_y u_2\omega_x^2\,dx\,dy\\&\leqslant C\|\partial_y u_2\|_2^{\frac12}\|\partial_{xy}u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}\|\omega_x\|\\&\leqslant C\|\omega\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\nabla\omega_y\|_2^{\frac12}\|\nabla\omega\|_2^{\frac32}\\&\leqslant \frac{\nu}{10}\|\nabla\omega_y\|_2^2+C\|\omega\|_2^{\frac23}\|\omega_y\|_2^{\frac23}\|\nabla\omega\|_2^2.\end{aligned}$$

By Lemma 1,

$$\begin{aligned}I_{12}&\leqslant C\|\partial_x u_2\|_2^{\frac12}\|\partial_{xy}u_2\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\|\omega_x\|_2\\&\leqslant C\|\omega\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\nabla\omega_y\|_2^{\frac12}\|\nabla\omega\|_2^{\frac32}\\&\leqslant \frac{\nu}{10}\|\nabla\omega_y\|_2^2+C\|\omega\|_2^{\frac23}\|\omega_y\|_2^{\frac23}\|\nabla\omega\|_2^2.\end{aligned}$$

I_{13} and I_{14} can be similarly bounded,

$$I_{13}, I_{14} \leqslant \frac{\nu}{10}\|\nabla\omega_y\|_2^2 + C\|\omega\|_2^{\frac{2}{3}}\|\omega_y\|_2^{\frac{2}{3}}\|\nabla\omega\|_2^2.$$

I_2 and I_3 can be bounded by applying Lemma 1

$$\begin{aligned}
I_2 &\leqslant C\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u_y\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla j_x\|_2^{\frac{1}{2}}\|\nabla j\|_2 \\
&\leqslant C\|\omega\|_2^{\frac{1}{2}}\|\omega_y\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{3}{2}}\|\nabla j_x\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2^{\frac{2}{3}}\|\omega_y\|_2^{\frac{2}{3}}\|\nabla j\|_2^2, \\
I_3 &\leqslant C\|\nabla b\|_2\|\nabla\omega\|_2^{\frac{1}{2}}\|\nabla\omega_y\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla j_x\|_2^{\frac{1}{2}} \\
&\leqslant C\|j\|_2\|\nabla\omega\|_2^{\frac{1}{2}}\|\nabla\omega_y\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla j_x\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\nu}{10}\|\nabla\omega_y\|_2^2 + \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|j\|_2^2\|\nabla\omega\|_2\|\nabla j\|_2 \\
&\leqslant \frac{\nu}{10}\|\nabla\omega_y\|_2^2 + \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|j\|_2^2\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big).
\end{aligned}$$

To bound I_4, we split it into two parts:

$$\begin{aligned}
I_4 &= 2\int \partial_x\big[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\big] j_x\,dx\,dy + 2\int \partial_y\big[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\big] j_y\,dx\,dy \\
&\equiv I_{41} + I_{42}.
\end{aligned}$$

Integrating by parts in I_{41} and applying Lemma 1, we have

$$\begin{aligned}
I_{41} &= -2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1) j_{xx}\,dx\,dy \\
&\leqslant C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2^{\frac{1}{2}}\|j_{xx}\|_2 \\
&\quad + C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{yy} u_1\|_2^{\frac{1}{2}}\|j_{xx}\|_2 \\
&\leqslant C\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\omega\|_2^{\frac{1}{2}}\|\omega_y\|_2^{\frac{1}{2}}\|\nabla j_x\|_2 \\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2\|j\|_2\|\nabla\omega\|_2\|\nabla j\|_2 \\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2\|j\|_2\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big).
\end{aligned}$$

I_{42} can be further decomposed into two parts:

$$\begin{aligned}
I_{42} &= 2\int \partial_{xy} b_1(\partial_x u_2 + \partial_y u_1) j_y\,dx\,dy + 2\int \partial_x b_1(\partial_{xy} u_2 + \partial_{yy} u_1) j_y\,dx\,dy \\
&\equiv I_{421} + I_{422}
\end{aligned}$$

and these two terms can be bounded as follows

$$\begin{aligned}
I_{421} &\leqslant C\|\partial_{xy}b_1\|_2\|\partial_x u_2\|_2^{\frac12}\|\partial_{xy}u_2\|_2^{\frac12}\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\quad + C\|\partial_{xy}b_1\|_2\|\partial_y u_1\|_2^{\frac12}\|\partial_{yy}u_1\|_2^{\frac12}\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\leqslant C\|\omega\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\nabla j\|_2^{\frac32}\|\nabla j_x\|_2^{\frac12}\\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2^{\frac23}\|\omega_y\|_2^{\frac23}\|\nabla j\|_2^2,\\
I_{422} &\leqslant C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\|\partial_{xy}u_2\|_2\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\quad + C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\|\partial_{yy}u_1\|_2\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\leqslant C\|j\|_2^{\frac12}\|j_x\|_2^{\frac12}\|\omega_y\|_2\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|j\|_2^{\frac23}\|j_x\|_2^{\frac23}\|\nabla\omega\|_2^{\frac43}\|\nabla j\|_2^{\frac23}\\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|j\|_2^{\frac23}\|j_x\|_2^{\frac23}\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big).
\end{aligned}$$

To bound I_5, we first write it into three terms,

$$\begin{aligned}
I_5 &= -2\int \partial_x\big[\partial_x u_1(\partial_x b_2 + \partial_y b_1)\big] j_x\,dx\,dy - 2\int \partial_y\big[\partial_x u_1(\partial_x b_2 + \partial_y b_1)\big] j_y\,dx\,dy\\
&= 2\int \partial_x u_1(\partial_x b_2 + \partial_y b_1) j_{xx}\,dx\,dy - 2\int \partial_{xy}u_1(\partial_x b_2 + \partial_y b_1) j_y\,dx\,dy\\
&\quad - 2\int \partial_x u_1(\partial_{xy}b_2 + \partial_{yy}b_1) j_y\,dx\,dy\\
&\equiv I_{51} + I_{52} + I_{53}.
\end{aligned}$$

We bound these terms as follows

$$\begin{aligned}
I_{51} &\leqslant C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx}b_2\|_2^{\frac12}\|j_{xx}\|_2\\
&\quad + C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\|j_{xx}\|_2\\
&\leqslant C\|\omega\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\nabla j_x\|_2\\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2\|j\|_2\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big),\\
I_{52} &\leqslant C\|\partial_{xy}u_1\|_2\|\partial_x b_2\|_2^{\frac12}\|\partial_{xy}b_2\|_2^{\frac12}\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\quad + C\|\partial_{xy}u_1\|_2\|\partial_y b_1\|_2^{\frac12}\|\partial_{yy}b_1\|_2^{\frac12}\|j_y\|_2^{\frac12}\|j_{xy}\|_2^{\frac12}\\
&\leqslant C\|\omega_y\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\nabla j\|_2\|\nabla j_x\|_2^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega_y\|_2^{\frac{2}{3}}\|j\|_2^{\frac{2}{3}}\|\nabla\omega\|_2^{\frac{2}{3}}\|\nabla j\|_2^{\frac{4}{3}} \\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega_y\|_2^{\frac{2}{3}}\|j\|_2^{\frac{2}{3}}\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big), \\
I_{53} &\leqslant C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_2\|_2\|j_y\|_2^{\frac{1}{2}}\|j_{xy}\|_2^{\frac{1}{2}} \\
&\quad + C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{yy}b_1\|_2\|j_y\|_2^{\frac{1}{2}}\|j_{xy}\|_2^{\frac{1}{2}} \\
&\leqslant C\|\omega\|_2^{\frac{1}{2}}\|\omega_y\|_2^{\frac{1}{2}}\|\nabla j\|_2\|j_y\|_2^{\frac{1}{2}}\|j_{xy}\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\eta}{16}\|\nabla j_x\|_2^2 + C\|\omega\|_2^{\frac{2}{3}}\|\omega_y\|_2^{\frac{2}{3}}\|\nabla j\|_2^2.
\end{aligned}
$$

Collecting the above estimates, we finally obtain

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big) + \nu\|\nabla\omega_y\|_2^2 + \eta\|\nabla j_x\|_2^2 \\
&\quad\leqslant C\Big(\big(\|\omega_y\|_2^{\frac{2}{3}} + \|j_x\|_2^{\frac{2}{3}}\big)\big(\|\omega\|_2^{\frac{2}{3}} + \|j\|_2^{\frac{2}{3}}\big) + \|j\|_2\big(\|\omega\|_2 + \|j\|_2\big)\Big)\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big).
\end{aligned}
$$

Applying the bound from Proposition 2, we find

$$
\begin{aligned}
&\big\|\nabla\omega(t)\big\|_2^2 + \big\|\nabla j(t)\big\|_2^2 + \nu\int_0^t \big\|\nabla\omega_y(\tau)\big\|_2^2\,d\tau + \eta\int_0^t \big\|\nabla j_x(\tau)\big\|_2^2\,d\tau \\
&\quad\leqslant C(\nu,\eta)\big(\|\nabla\omega_0\|_2^2 + \|\nabla j_0\|_2^2\big).
\end{aligned}
$$

This completes the proof of Proposition 3. □

2.4. *Proof of Theorem 1*

This subsection presents the proof of Theorem 1.

Proof of Theorem 1. With the *a priori* bounds of Propositions 2 and 3 at our disposal, the proof of this theorem can be achieved through a parabolic regularization process. Let $\epsilon > 0$ be a small parameter and consider a family of solutions (u_ϵ, b_ϵ) satisfying the regularized system of equations

$$\partial_t u_\epsilon + u_\epsilon\cdot\nabla u_\epsilon = -\nabla p_\epsilon + \nu\partial_{yy}u_\epsilon + b_\epsilon\cdot\nabla b_\epsilon + \epsilon\Delta u_\epsilon, \tag{19}$$

$$\partial_t b_\epsilon + u_\epsilon\cdot\nabla b_\epsilon = \eta\partial_{xx}b_\epsilon + b_\epsilon\cdot\nabla u_\epsilon + \epsilon\Delta b_\epsilon, \tag{20}$$

$$\nabla\cdot u_\epsilon = 0, \tag{21}$$

$$\nabla\cdot b_\epsilon = 0, \tag{22}$$

$$u_\epsilon(x,0) = \psi_\epsilon * u_0, \qquad b_\epsilon(x,0) = \psi_\epsilon * b_0, \tag{23}$$

where $\psi_\epsilon(x) = \epsilon^{-2}\psi(x/\epsilon)$ with ψ satisfying

$$\psi \geqslant 0, \quad \psi \in C_0^\infty(\mathbf{R}^2) \quad \text{and} \quad \|\psi\|_1 = 1.$$

Since $u_\epsilon(x,0)$ and $b_\epsilon(x,0)$ are smooth, the standard theory on the 2D viscous MHD equations (see e.g. [9]) guarantees that (19)–(23) has a unique global smooth solution (u_ϵ, b_ϵ). It is easy to see that (u_ϵ, b_ϵ) obeys the *a priori* bounds in Propositions 2 and 3 uniformly in ϵ. The solution (u,b) of (1)–(4) is then obtained as a limit of (u_ϵ, b_ϵ) and obeys the bounds in Propositions 2 and 3.

The uniqueness of the solutions follows from the elementary inequalities (see Lemma 14 of [3])

$$\|f\|_\infty \leqslant C\big(\|f\|_2 + \|f_x\|_2 + \|f_{yy}\|_2\big) \quad \text{and} \quad \|f\|_\infty \leqslant C\big(\|f\|_2 + \|f_y\|_2 + \|f_{xx}\|_2\big).$$

In fact, applying these inequalities, we have

$$\begin{aligned}\int_0^t \big(\big\|\omega(\tau)\big\|_\infty + \big\|j(\tau)\big\|_\infty\big)\,d\tau &\leqslant \int_0^t \big(\big\|\omega(\tau)\big\|_2 + \big\|\omega_y(\tau)\big\|_2 + \big\|\nabla\omega_y(\tau)\big\|_2\big)\,d\tau \\ &\quad + \int_0^t \big(\big\|j(\tau)\big\|_2 + \big\|j_x(\tau)\big\|_2 + \big\|\nabla j_x(\tau)\big\|_2\big)\,d\tau < \infty\end{aligned}$$

for any $t > 0$. It is well known (see e.g. [1,10]) that this bound yields the uniqueness. □

2.5. (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$

A global regularity result similar to Theorem 1 can be established for the 2D MHD equations (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$.

Theorem 4. *Consider the 2D MHD equations* (1)–(4) *with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. Assume $u_0 \in H^2(\mathbf{R}^2)$ and $b_0 \in H^2(\mathbf{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then* (1)–(4) *has a unique global classical solution* (u,b). *In addition,* (u,b) *satisfies*

$$(u,b) \in L^\infty\big([0,\infty); H^2\big), \qquad \omega_x \in L^2\big([0,\infty); H^1\big), \qquad j_y \in L^2\big([0,\infty); H^1\big), \tag{24}$$

where $\omega = \nabla \times u$ and $j = \nabla \times b$ represent the vorticity and the current density, respectively.

Proof. Although this theorem can be proven in a similar fashion as that of Theorem 1, we provide an alternative proof. The idea is to convert (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$ into a form dealt with by Theorem 1. Assume that (u,b) solves (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. Set

$$U_1(x,y,t) = u_2(y,x,t), \qquad U_2(x,y,t) = u_1(y,x,t), \qquad P(x,y,t) = p(y,x,t),$$
$$B_1(x,y,t) = b_2(y,x,t), \qquad B_2(x,y,t) = b_1(y,x,t).$$

Then $U = (U_1, U_2)$, P and $B = (B_1, B_2)$ satisfy

$$U_t + U \cdot \nabla U = -\nabla P + \nu U_{yy} + B \cdot \nabla B, \tag{25}$$

$$B_t + U \cdot \nabla B = \eta B_{xx} + B \cdot \nabla U, \tag{26}$$

$$\nabla \cdot U = 0, \tag{27}$$

$$\nabla \cdot B = 0. \tag{28}$$

The global regularity of (25)–(28) guaranteed by Theorem 1 allows us to obtain the global regularity for (1)–(4) with $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$. This completes the proof of Theorem 4. □

3. The MHD with magnetic diffusion

This section focuses on (1)–(4) with $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 = \eta > 0$. Two major results are established. The first is the global existence of a weak solution and the second assesses the global regularity and uniqueness of the weak solution under a suitable condition.

Theorem 5. *Consider* (1)–(4) *with* $\nu_1 = \nu_2 = 0$ *and* $\eta_1 = \eta_2 = \eta > 0$. *Assume that* $(u_0, b_0) \in H^1$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then* (1)–(4) *has a global weak solution* (u, b) *satisfying*

$$u \in C\big([0,\infty); H^1\big), \qquad b \in C\big([0,\infty); H^1\big) \cap L^2\big([0,\infty); H^2\big). \tag{29}$$

The proof of this result relies on a global *a priori* bound for $\omega = \nabla \times u$ and $j = \nabla \times b$.

Theorem 6. *Assume the initial data* $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Let* (u, b) *be the corresponding solution of* (1)–(4) *with* $\nu_1 = \nu_2 = 0$ *and* $\eta_1 = \eta_2 = \eta > 0$. *If, for some* $T > 0$,

$$\sup_{q \geqslant 2} \frac{1}{q} \int_0^T \big\|\nabla u(t)\big\|_q \, dt < \infty, \tag{30}$$

then (u, b) *is regular on* $[0, T]$, *namely*

$$(u, b) \in C\big([0, T]; H^3\big).$$

In addition, two weak solutions (u, b) *and* $(\tilde{u}, \tilde{b})$ *in the regularity class* (29) *must be identical on the time interval* $[0, T]$ *if* u *satisfies* (30).

The rest of this section is divided into four subsections. The first subsection presents a global *a priori* bound for $\|u\|_{H^1}$ and $\|b\|_{H^1}$ and the second proves Theorem 5. The third subsection establishes a logarithmic Sobolev inequality, which serves as a preparation for the proof of Theorem 6. The last subsection proves Theorem 6.

3.1. *An* a priori *bound for* $\|\nabla u\|_2$ *and* $\|\nabla b\|_2$

Proposition 7. *If* (u,b) *solves the 2D MHD equations* (1)–(4) *with* $\nu_1=\nu_2=0$ *and* $\eta_1=\eta_2=\eta>0$*, then, for any* $t>0$*,*

$$\|\omega(t)\|_2^2+\|j(t)\|_2^2+\eta\int_0^t\|\nabla j\|_2^2\,d\tau\leqslant C(\eta)\big(\|\nabla u_0\|_2^2+\|\nabla b_0\|_2^2\big), \tag{31}$$

where $C(\eta)$ *is a constant depending on* η *only. Therefore,*

$$\|u(t)\|_{H^1}^2+\|b(t)\|_{H^1}^2+\eta\int_0^t\|b\|_{H^2}^2\,d\tau\leqslant C(\eta)\big(\|u_0\|_{H^1}^2+\|b_0\|_{H^1}^2\big). \tag{32}$$

Proof. It follows easily from (1) and (2) that, for any $t>0$,

$$\|u(t)\|_2^2+\|b(t)\|_2^2+2\eta\int_0^t\|\nabla b(\tau)\|_2^2\,d\tau=\|u(0)\|_2^2+\|b(0)\|_2^2. \tag{33}$$

To prove (31), we employ the equations of the vorticity ω and the current density j,

$$\omega_t+u\cdot\nabla\omega=b\cdot\nabla j, \tag{34}$$

$$j_t+u\cdot\nabla j=\eta\Delta j+b\cdot\nabla\omega+2\partial_x b_1(\partial_x u_2+\partial_y u_1)-2\partial_x u_1(\partial_x b_2+\partial_y b_1). \tag{35}$$

Taking the inner products of (34) with ω and of (35) with j, we find

$$\begin{aligned}\frac{1}{2}\frac{d\|\omega\|_2^2}{dt}&=\int b\cdot\nabla j\omega\,dx\,dy,\\ \frac{1}{2}\frac{d\|j\|_2^2}{dt}+\eta\|\nabla j\|_2^2&=\int b\cdot\nabla\omega j\,dx\,dy\\ &\quad+2\int\big(\partial_x b_1(\partial_x u_2+\partial_y u_1)-\partial_x u_1(\partial_x b_2+\partial_y b_1)\big)j\,dx\,dy.\end{aligned}$$

Since

$$\int b\cdot\nabla j\omega\,dx\,dy+\int b\cdot\nabla\omega j\,dx\,dy=0,$$

we have, for $X(t)=\|\omega(t)\|_2^2+\|j(t)\|_2^2$,

$$\frac{dX(t)}{dt}+2\eta\|\nabla j\|_2^2\leqslant 8\|\nabla u\|_2\|\nabla b\|_4\|j\|_4,$$

where we have applied the Hölder inequality. Applying the inequalities

$$\|\nabla u\|_2 \leqslant \|\omega\|_2, \qquad \|\nabla b\|_4 \leqslant \|j\|_4, \qquad \|j\|_4^2 \leqslant \|j\|_2 \|\nabla j\|_2$$

and Young's inequality, we find

$$\frac{dX(t)}{dt} + 2\eta\|\nabla j\|_2^2 \leqslant \frac{16}{\eta}\|\omega\|_2^2\|j\|_2^2 + \eta\|\nabla j\|_2^2.$$

In particular,

$$\frac{dX(t)}{dt} + \eta\|\nabla j\|_2^2 \leqslant \frac{16}{\eta}\|j\|_2^2 X(t).$$

By Gronwall's inequality,

$$X(t) + \eta\int_0^t \big\|\nabla j(\tau)\big\|_2^2\,d\tau \leqslant X(0)\exp\left(\frac{16}{\eta}\int_0^t \|j\|_2^2\,d\tau\right),$$

which, together with (33), yields (31) and (32). □

3.2. *Proof of Theorem 5*

Let $\epsilon > 0$ be a small parameter and consider the regularized system of equations

$$\begin{aligned}
\partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon &= -\nabla p_\epsilon + \epsilon\Delta u_\epsilon + b_\epsilon \cdot \nabla b_\epsilon, \\
\partial_t b_\epsilon + u_\epsilon \cdot \nabla b_\epsilon &= \eta\Delta b_\epsilon + b_\epsilon \cdot \nabla u_\epsilon, \\
\nabla \cdot u_\epsilon &= 0, \\
\nabla \cdot b_\epsilon &= 0.
\end{aligned}$$

This system of equations admits a unique global solution (u_ϵ, b_ϵ) that satisfies the global *a priori* bound stated in Proposition 7 uniformly in terms of ϵ. By going through a standard limit process, we conclude that (u_ϵ, b_ϵ) converge to a weak solution of (1)–(4) with $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 = \eta$. This completes the proof of Theorem 5.

3.3. *A logarithmic inequality*

This subsection presents a logarithmic Sobolev inequality, which plays an important role in the proof of Theorem 6. A similar inequality was previously obtained by P. Zhang [11] and by Danchin and Paicu [3] and their proofs involve tools from Fourier analysis such as the Littlewood–Paley decomposition. The proof presented here is different and remains valid for a general domain other than the whole plane.

Lemma 8. *Let $f \in H^2(\mathbf{R}^2)$ and let $a > 0$ be real. Then the following logarithmic inequality holds*

$$\|f\|_{L^\infty} \leqslant C \sup_{q\geqslant 2} \frac{\|f\|_q}{q^a}\big[\ln\big(e + \|f\|_{H^2}\big)\big]^a.$$

Proof. We follow the approach of Hou and Li [7]. Denote by B_r the disk centered at the origin with radius r. Let $\phi \in C^\infty(\mathbf{R}^2)$ be a smooth cutoff function satisfying

$$\phi(0) = 1, \qquad |\nabla\phi| \leqslant C, \qquad |\Delta\phi| \leqslant C, \qquad \operatorname{supp}\phi \subset B_1.$$

Set $w = f\phi$. According to the solution formula of the 2D Laplace equation, we have, for any $p \geqslant 2$,

$$\begin{aligned} w^p(0) &= \frac{1}{2\pi}\int_{B_\epsilon}\big(\ln|y| - \ln\epsilon\big)\Delta w^p(y)\,dy + \frac{1}{2\pi}\int_{B_1\setminus B_\epsilon}\big(\ln|y| - \ln\epsilon\big)\Delta w^p(y)\,dy \\ &= I + II. \end{aligned}$$

Since

$$\Delta w^p = p w^{p-1}\Delta w + p(p-1)w^{p-2}|\nabla w|^2,$$

we obtain by applying Hölder's inequality

$$|I| \leqslant \frac{p}{2\pi}\epsilon^{\frac{2}{3}}\big[\|\Delta w\|_2\|w\|_{6(p-1)}^{p-1} + (p-1)\|\nabla w\|_4^2\|w\|_{6(p-2)}^{p-2}\big].$$

By the embedding inequality

$$\|\nabla w\|_4 \leqslant C\|w\|_2^{\frac{1}{4}}\|\Delta w\|_2^{\frac{3}{4}},$$

we have, for C independent of p,

$$|I| \leqslant Cp\epsilon^{\frac{2}{3}}\|\Delta w\|_2\|w\|_{6(p-1)}^{p-1} + Cp(p-1)\epsilon^{\frac{2}{3}}\|w\|_2^{\frac{1}{2}}\|\Delta w\|_2^{\frac{3}{2}}\|w\|_{6(p-2)}^{p-2}.$$

Integrating by parts in II yields

$$II = \frac{p}{2\pi}\int_{B_1\setminus B_\epsilon} w^{p-1}\frac{y\cdot\nabla w}{|y|^2}\,dy.$$

By Hölder's inequality,

$$|II| \leqslant Cp\left(\ln\frac{1}{\epsilon}\right)^{\frac{1}{2}}\|\nabla w\|_4\|w\|_{4(p-1)}^{p-1} \leqslant Cp\left(\ln\frac{1}{\epsilon}\right)^{\frac{1}{2}}\|w\|_2^{\frac{1}{4}}\|\Delta w\|_2^{\frac{3}{4}}\|w\|_{4(p-1)}^{p-1}.$$

Now, set

$$\epsilon^{\frac{2}{3}}\|\Delta w\|_2 = 1 \quad \text{or} \quad \epsilon = \|\Delta w\|_2^{-\frac{3}{2}} \quad \text{and} \quad p = \ln\frac{1}{\epsilon}.$$

In the case when $\|\Delta w\|_2 \leqslant 1$, it suffices to set $0 < \epsilon < 1$.

We then have

$$\begin{aligned}\left|w(0)\right| &\leqslant Cp^{\frac{1}{p}}\|w\|_{6(p-1)}^{1-\frac{1}{p}} + C\big(p(p-1)\big)^{\frac{1}{p}}\|w\|_2^{\frac{1}{2p}}\|\Delta w\|_2^{\frac{1}{2p}}\|w\|_{6(p-2)}^{1-\frac{2}{p}} \\ &\quad + Cp^{\frac{3}{2p}}\|w\|_2^{\frac{1}{4p}}\|\Delta w\|_2^{\frac{3}{4p}}\|w\|_{4(p-1)}^{1-\frac{1}{p}}.\end{aligned}$$

Use the fact that $p^{\frac{1}{p}} < C$, $(p(p-1))^{\frac{1}{p}} < C$, and

$$\|\Delta w\|_2^{\frac{1}{2p}} = \epsilon^{\frac{1}{3\ln\epsilon}} = e^{\frac{1}{3}}, \qquad \|w\|_q \leqslant p^a \sup_{q\geqslant 2}\frac{\|w\|_q}{q^a},$$

we obtain that

$$\left|w(0)\right| \leqslant C\sup_{q\geqslant 2}\frac{\|w\|_q}{q^a}\big(\ln\big(e+\|\Delta w\|_2\big)\big)^a.$$

Noticing that

$$\left|f(0)\right| = \left|w(0)\right| \quad \text{and} \quad \|\Delta w\|_2 \leqslant C\big(\|f\|_2 + \|\Delta f\|_2\big) \leqslant C\|f\|_{H^2},$$

we conclude the proof of Lemma 8. □

3.4. *Proof of Theorem 6*

To show the regularity, we bound $\|(u,b)\|_{H^3}$. According to Proposition 7, $\|(u,b)\|_{H^1}$ admits a global uniform bound. Now, consider $\nabla\omega$ and ∇j, which satisfy

$$\begin{aligned}\partial_t\nabla\omega + u\cdot\nabla(\nabla\omega) &= -(\nabla u)\nabla\omega + b\cdot\nabla(\nabla j) + (\nabla b)\nabla j, \\ \partial_t\nabla j + u\cdot\nabla(\nabla j) &= \eta\Delta(\nabla j) - (\nabla u)\nabla j + b\cdot\nabla(\nabla\omega) + (\nabla b)\nabla\omega \\ &\quad + 2\nabla\big[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\big] - 2\nabla\big[\partial_x u_1(\partial_x b_2 + \partial_y b_1)\big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big) + \eta\|\Delta j\|_2^2 \\ &\quad = -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega - \int\nabla u\cdot\nabla j\cdot\nabla j \\ &\qquad + 2\int\nabla b\cdot\nabla j\cdot\nabla\omega + 2\int\nabla\big[\partial_x b_1(\partial_y u_1 + \partial_x u_2)\big]\cdot\nabla j \\ &\qquad - 2\int\nabla\big[\partial_x u_1(\partial_y b_1 + \partial_x b_2)\big]\cdot\nabla j \\ &\quad \equiv K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned}$$

The terms on the right can be estimated as follows

$$K_1 \leqslant \|\nabla u\|_\infty \|\nabla\omega\|_2^2,$$

$$\begin{aligned} K_2 &= -\int \nabla u \cdot \nabla j \cdot \nabla j \leqslant \|\nabla u\|_2 \|\nabla j\|_4^2 \\ &\leqslant C\|\nabla u\|_2 \|\nabla j\|_2 \|\Delta j\|_2 \\ &\leqslant \frac{\eta}{8}\|\Delta j\|_2^2 + C\|\omega\|_2^2 \|\nabla j\|_2^2, \end{aligned}$$

$$\begin{aligned} K_3 &= 2\int \nabla b \cdot \nabla\omega \cdot \nabla j \leqslant 2\|\nabla\omega\|_2 \|\nabla b\|_4 \|\nabla j\|_4 \\ &\leqslant C\|\nabla\omega\|_2 \|\nabla b\|_2^{\frac{1}{2}} \|\Delta b\|_2^{\frac{1}{2}} \|\nabla j\|_2^{\frac{1}{2}} \|\Delta j\|_2^{\frac{1}{2}} \\ &\leqslant \frac{\eta}{8}\|\Delta j\|_2^2 + C\|\nabla\omega\|_2^{\frac{4}{3}} \|\nabla b\|_2^{\frac{2}{3}} \|\nabla j\|_2^{\frac{4}{3}} \\ &\leqslant \frac{\eta}{8}\|\Delta j\|_2^2 + C\|j\|_2^{\frac{2}{3}} \|\nabla j\|_2^{\frac{2}{3}} \big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big), \end{aligned}$$

$$\begin{aligned} K_4 &= 2\int \nabla\big[\partial_x b_1(\partial_y u_1 + \partial_x u_2)\big] \cdot \nabla j \\ &= 2\int \partial_x \nabla b_1 \cdot \nabla j(\partial_y u_1 + \partial_x u_2) + 2\int \partial_x b_1(\partial_y \nabla u_1 + \partial_x \nabla u_2) \cdot \nabla j \\ &\leqslant 4\int |\nabla j|^2 |\nabla u| + 4\int |\nabla b||\nabla\omega||\nabla j| \\ &\leqslant \frac{\eta}{4}\|\Delta j\|_2^2 + C\|\omega\|_2^2 \|\nabla j\|_2^2 + C\|j\|_2^{\frac{2}{3}} \|\nabla j\|_2^{\frac{2}{3}} \big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big). \end{aligned}$$

Putting together these estimates, we have

$$\begin{aligned} \frac{d}{dt}\big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big) + \eta\|\Delta j\|_2^2 &\leqslant \|\nabla u\|_\infty \|\nabla\omega\|_2^2 + C\|\omega\|_2^2 \|\nabla j\|_2^2 \\ &\quad + C\|j\|_2^{\frac{2}{3}} \|\nabla j\|_2^{\frac{2}{3}} \big(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2\big). \end{aligned}$$

We now bound the third-order derivatives of (u, b). For any multi-index β with $|\beta| = 3$, $D^\beta u$ and $D^\beta b$ satisfy

$$\begin{aligned} \partial_t D^\beta u + u \cdot \nabla D^\beta u &= -\nabla D^\beta p + b \cdot \nabla D^\beta b - \big[D^\beta, u \cdot \nabla\big]u + \big[D^\beta, b \cdot \nabla\big]b, \\ \partial_t D^\beta b + u \cdot \nabla D^\beta b &= \eta\Delta D^\beta b + b \cdot \nabla D^\beta u - \big[D^\beta, u \cdot \nabla\big]b + \big[D^\beta, b \cdot \nabla\big], \end{aligned}$$

where $[D^\beta, f \cdot \nabla]g = D^\beta(f \cdot \nabla g) - f \cdot \nabla D^\beta g$. Taking the inner products of these equations with $D^\beta u$ and $D^\beta b$, respectively, and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\big(\big\|D^\beta u\big\|_2^2 + \big\|D^\beta b\big\|_2^2\big) + \eta\big\|\nabla D^\beta b\big\|_2^2 = L_1 + L_2 + L_3 + L_4$$

where

$$L_1 = -\big(\big[D^\beta, u\cdot\nabla\big]u, D^\beta u\big), \qquad L_2 = \big(\big[D^\beta, b\cdot\nabla\big]b, D^\beta u\big),$$
$$L_3 = -\big(\big[D^\beta, u\cdot\nabla\big]b, D^\beta b\big), \qquad L_4 = \big(\big[D^\beta, b\cdot\nabla\big]b, D^\beta b\big).$$

To bound L_1, L_2, L_3 and L_4, we recall the commutator estimate (see [8, p. 334])

$$\big\|\big[D^\beta, f\cdot\nabla\big]g\big\|_p \leqslant C\big(\|\nabla f\|_{p_1}\|\nabla g\|_{W^{2,p_2}} + \|f\|_{W^{3,p_3}}\|\nabla g\|_{p_4}\big) \tag{36}$$

valid for any $p, p_2, p_3 \in (1,\infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Applying this inequality, we obtain

$$|L_1| \leqslant \big\|\big[D^\beta, u\cdot\nabla\big]u\big\|_2\big\|D^\beta u\big\|_2 \leqslant C\|\nabla u\|_\infty\|u\|_{H^3}\big\|D^\beta u\big\|_2,$$
$$|L_2| \leqslant \big\|\big[D^\beta, b\cdot\nabla\big]b\big\|_2\big\|D^\beta u\big\|_2 \leqslant C\big(\|\nabla b\|_4\|\nabla b\|_{W^{2,4}} + \|b\|_{W^{3,4}}\|\nabla b\|_4\big)\big\|D^\beta u\big\|_2.$$

By the basic calculus inequality, for any $f \in H^1(\mathbf{R}^2)$,

$$\|f\|_4 \leqslant C\|f\|_2^{\frac{1}{2}}\|\nabla f\|_2^{\frac{1}{2}}, \tag{37}$$

we have

$$|L_2| \leqslant C\|\nabla b\|_2^{\frac{1}{2}}\|\Delta b\|_2^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}\|\nabla b\|_{H^3}^{\frac{1}{2}}\big\|D^\beta u\big\|_2.$$

By Young's inequality,

$$\begin{aligned}|L_2| &\leqslant \frac{\eta}{4}\|\nabla b\|_{H^3}^2 + C\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\|b\|_{H^3}^{\frac{2}{3}}\big\|D^\beta u\big\|_2^{\frac{4}{3}} \\ &\leqslant \frac{\eta}{4}\|\nabla b\|_{H^3}^2 + C\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\big(\|b\|_{H^3}^2 + \big\|D^\beta u\big\|_2^2\big).\end{aligned}$$

By (36) again,

$$|L_3| \leqslant \big\|\big[D^\beta, u\cdot\nabla\big]b\big\|_{\frac{4}{3}}\big\|D^\beta b\big\|_4 \leqslant C\big(\|\nabla u\|_2\|\nabla b\|_{W^{2,4}} + \|u\|_{H^3}\|\nabla b\|_4\big)\big\|D^\beta b\big\|_4.$$

Therefore,

$$\begin{aligned}|L_3| &\leqslant C\|\omega\|_2\|b\|_{H^3}\|\nabla b\|_{H^3} + C\|\nabla b\|_2^{\frac{1}{2}}\|\Delta b\|_2^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}\|\nabla b\|_{H^3}^{\frac{1}{2}}\|u\|_{H^3} \\ &\leqslant \frac{\eta}{4}\|\nabla b\|_{H^3}^2 + C\|\omega\|_2^2\|b\|_{H^3}^2 + C\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\big(\|b\|_{H^3}^2 + \|u\|_{H^3}^2\big).\end{aligned}$$

Similarly, L_4 is bounded as follows

$$|L_4| \leqslant \frac{\eta}{4}\|\nabla b\|_{H^3}^2 + C\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\|b\|_{H^3}^2.$$

Combining all these estimates, we obtain

$$\frac{d}{dt}\big(\|u\|_{H^3}^2+\|b\|_{H^3}^2\big)+\eta\|\nabla b\|_{H^3}^2\leqslant C\|\nabla u\|_\infty\|u\|_{H^3}^2+\|\omega\|_2^2\|b\|_{H^3}^2 + C\|j\|_2^{\frac{2}{3}}\|\nabla j\|_2^{\frac{2}{3}}\|b\|_{H^3}^2.$$

Applying Lemma 8 with $a=1$ to bound $\|\nabla u\|_\infty$, we obtain the regularity part of Theorem 6.

To prove the uniqueness, we consider the difference

$$(W,B)=(\tilde{u},\tilde{b})-(u,b),$$

which satisfies the equations

$$W_t+\tilde{u}\cdot\nabla W+W\cdot\nabla u=-\nabla P+\tilde{b}\cdot\nabla B+B\cdot\nabla b, \tag{38}$$

$$B_t+\tilde{u}\cdot\nabla B+W\cdot\nabla b=\eta\Delta B+\tilde{b}\cdot\nabla W+B\cdot\nabla u, \tag{39}$$

where P is the difference between the corresponding pressures. Adding the inner products of (38) with W and of (39) with B and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|W\|_2^2+\|B\|_2^2\big)+\eta\|\nabla B\|_2^2&\leqslant\int|W\cdot\nabla u\cdot W|+\int|B\cdot\nabla u\cdot B|+2\int|W||\nabla b||B|\\&\leqslant\|\nabla u\|_\infty\big(\|W\|_2^2+\|B\|_2^2\big)+2\|W\|_2\|B\|_4\|\nabla b\|_4.\end{aligned} \tag{40}$$

By (37), we have

$$\begin{aligned}2\|W\|_2\|B\|_4\|\nabla b\|_4&\leqslant C\|W\|_2\|B\|_2^{\frac{1}{2}}\|\nabla B\|_2^{\frac{1}{2}}\|\nabla b\|_2^{\frac{1}{2}}\|\Delta b\|_2^{\frac{1}{2}}\\&\leqslant\frac{\eta}{2}\|\nabla B\|_2^2+C\|W\|_2^{\frac{4}{3}}\|B\|_2^{\frac{2}{3}}\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\\&\leqslant\frac{\eta}{2}\|\nabla B\|_2^2+C\|\nabla b\|_2^{\frac{2}{3}}\|\Delta b\|_2^{\frac{2}{3}}\big(\|W\|_2^2+\|B\|_2^2\big).\end{aligned}$$

Inserting the above estimate in (40) and applying Lemma 8 to bound $\|\nabla u\|_\infty$, we obtain the desired uniqueness. This completes the proof of Theorem 6.

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