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The 2D Boussinesq equations with vertical viscosity and vertical diffusivity

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A B S T R A C T

This paper aims at the global regularity of classical solutions to the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. We prove that the L^r-norm of the vertical velocity v for any $1 < r < \infty$ is globally bounded and that the L^∞-norm of v controls any possible breakdown of classical solutions. In addition, we show that an extra thermal diffusion given by the fractional Laplace $(-\Delta)^\delta$ for $\delta > 0$ would guarantee the global regularity of classical solutions.

1. Introduction

We consider the initial value problem for the 2D Boussinesq equations with vertical viscosity and vertical diffusivity

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = \kappa\theta_{yy}, \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \quad \theta(x, y, 0) = \theta_0(x, y), \end{cases} \tag{1.1}$$

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where u, v, p and θ are scalar functions of $(x, y) \in \mathbf{R}^2$ and $t \geqslant 0$. Physically, (u, v) denotes the 2D velocity field, p the pressure, θ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, ν the viscosity and κ the thermal diffusivity. (1.1) may be useful in modeling dynamics of geophysical flows for which the vertical dissipation dominates such as in the large-time dynamics of certain strongly stratified flows (see [13] and the references therein).

This paper aims at the issue of whether (1.1) possesses a global solution for every reasonably smooth initial data (u_0, v_0, θ_0). We first provide some background and review closely related results. (1.1) is a very important special case of the general 2D Boussinesq equations

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu_1 v_{xx} + \nu_2 v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy}, \end{cases} \tag{1.2}$$

which also include the horizontal dissipation $\nu_1 u_{xx}$ and $\nu_1 v_{xx}$, and the horizontal diffusivity $\kappa_1 \theta_{xx}$. The Boussinesq equations model buoyancy-driven flows such as atmospheric fronts and oceanic circulation (see e.g. [14,16]). One fundamental issue concerning the Boussinesq equations is whether or not their classical solutions are always global in time. When all parameters ν_1, ν_2, κ_1 and κ_2 are positive, this issue has long been resolved (see e.g. [2]). When all four parameters are zero, the global regularity problem is currently open.

Important progress has recently been made on the cases when some of the parameters are zero. In [4], Chae established the global regularity for the cases when $\kappa_1 = \kappa_2 = 0$ or when $\nu_1 = \nu_2 = 0$. In [12] Hou and Li obtained the global regularity for the case when $\kappa_1 = \kappa_2 = 0$. Very recently Danchin and Paicu [7] successfully settled the global regularity issue for the cases when $\nu_1 > 0$ and $\nu_2 = \kappa_1 = \kappa_2 = 0$ or when $\kappa_1 > 0$ and $\nu_1 = \nu_2 = \kappa_2 = 0$. When $\nu_1 > 0$ and $\nu_2 = \kappa_1 = \kappa_2 = 0$, the full Boussinesq equations reduce to

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu_1 u_{xx}, \\ v_t + uv_x + vv_y = -p_y + \nu_1 v_{xx} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = 0 \end{cases} \tag{1.3}$$

and the vorticity $\omega = v_x - u_y$ satisfies

$$\omega_t + u\omega_x + v\omega_y = \nu_1 \omega_{xx} + \theta_x.$$

Since the partial derivative ω_{xx} matches that of θ_x, the derivative in θ_x can be shifted to ω through integration by parts in the process of energy estimates. Therefore, one can avoid bounding θ_x and still get a global bound for ω. This convenience plays a crucial role in establishing the global regularity for the case $\nu_1 > 0$ and $\nu_2 = \kappa_1 = \kappa_2 = 0$.

However, the vorticity equation associated with (1.1) is given by

$$\omega_t + u\omega_x + v\omega_y = \nu_1 \omega_{yy} + \theta_x$$

and the mismatch of the derivatives in ω_{yy} and θ_x makes it much harder to derive a global bound for the vorticity. Therefore, it appears to be necessary to estimate ω (or $(\nabla u, \nabla v)$) and $\nabla\theta$ simultaneously. We then have to bound the term

$$\int_{\mathbf{R}^2} u_x (\theta_x)^2 \, dx \, dy,$$

which is hard to handle due to the lack of dissipation and diffusivity in the horizontal direction. If we make the assumption that the vertical velocity v satisfies

$$\int_0^T \left\| v(\cdot,t) \right\|_\infty^2 dt < \infty, \tag{1.4}$$

then an H^1-bound can be established for (u, v, θ) on the time interval $[0, T]$. In addition, we can further show that (u, v, θ) is actually a classical solution on $[0, T]$ if the initial data (u_0, v_0, θ_0) is sufficiently smooth, say in H^2. We remark that the condition in (1.4) is a regularity criterion (or blowup criterion). We leave the details to Section 3.

Invoking the logarithmic Sobolev inequality (see [3,7])

$$\|f\|_{L^\infty(\mathbf{R}^2)} \leqslant C \sup_{r \geqslant 2} \frac{\|f\|_r}{\sqrt{r}} \left(\ln\left(e + \|f\|_{H^2(\mathbf{R}^2)}\right)\right)^{\frac{1}{2}}, \tag{1.5}$$

we can replace the assumption in (1.4) by

$$\int_0^T \sup_{r \geqslant 2} \frac{\|v(\cdot,t)\|_r^2}{r} dt < \infty. \tag{1.6}$$

We do not know if (1.6) holds at this moment. What we are able to show is that, for any $r \geqslant 1$ and $t \leqslant T$,

$$\left\| v(\cdot,t) \right\|_{2r} < C(r,T) < \infty$$

where $C(r, T)$ is an exponential function of r and T. This bound is proven in Section 2.

If we add to the equation for θ an extra dissipative term $\epsilon(-\Delta)^\delta\theta$ with $\epsilon > 0$ and $\delta > 0$, then the resulting equations can be shown to have a global classical solution for any sufficiently smooth initial data. That is, the following system of equations

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = \kappa\theta_{yy} + \epsilon(-\Delta)^\delta\theta, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y), \end{cases} \tag{1.7}$$

is globally well-posed for smooth (u_0, v_0, θ_0). This is established in Section 4. We take this opportunity to mention a few recent papers on the 2D Boussinesq equations with fractional dissipation. In [10] and [11] Hmidi, Keraani and Rousset showed the global well-posedness of the Euler–Boussinesq system with critical dissipation, namely (1.7) with $\nu = \kappa = 0$, $\epsilon = 1$ and $\delta = 1/2$ and of the Boussinesq–Navier–Stokes system with critical dissipation. In [15] Miao and Xue established the global regularity of the 2D Boussinesq equations with fractional dissipation and thermal diffusion whose total fractional power is greater than or equal to 1. Some other interesting recent results on the 2D Boussinesq equations can be found in [1,5,6,8,9].

2. A bound for the vertical velocity in Lebesgue spaces

This section establishes a global bound for the vertical velocity v of (1.1) in Lebesgue spaces. For notational convenience, we omit $dx\,dy$ in the integrals over $(x, y) \in \mathbf{R}^2$.

Theorem 2.1. *Let $r \geqslant 1$. Then, for any smooth solution (u, v, θ) of (1.1),*

$$\big\| v(\cdot, t)\big\|_{2r} \leqslant e^{C_1 r^3 (\|(u_0, v_0)\|_2 + \|\theta_0\|_2 t)^2} \big(\| v_0\|_{2r} + C_2\big(r^3 \|\theta_0\|^2_{\frac{2r}{r+1}} + \|\theta_0\|^2_{2r}\big)t\big), \tag{2.1}$$

where C_1 and C_2 are constants independent of r and t.

To prove this theorem, we first state the following basic *a priori* bounds.

Proposition 2.2. *Let (u, v, θ) be a smooth solution of (1.1). Then*

$$\big\|\big(u(t), v(t)\big)\big\|_2^2 + 2\nu \int_0^t \big\|\big(u_y(\tau), v_y(\tau)\big)\big\|_2^2 \, d\tau = \big(\big\|(u_0, v_0)\big\|_2 + t\|\theta_0\|_2\big)^2 \tag{2.2}$$

and, for any $q \geqslant 2$,

$$\big\|\theta(t)\big\|_q^q + \kappa q(q-1) \int_0^t \big\|\theta_y |\theta|^{\frac{q-2}{2}}(\tau)\big\|_2^2 \, d\tau = \|\theta_0\|_q^q. \tag{2.3}$$

In particular, for $2 \leqslant q \leqslant \infty$,

$$\big\|\theta(t)\big\|_q \leqslant \|\theta_0\|_q. \tag{2.4}$$

Proof of Theorem 2.1. Taking the inner product of the second equation in (1.1) with $v|v|^{2r-2}$ and integrating by parts, we obtain

$$\frac{1}{2r}\frac{d}{dt} \int |v|^{2r} + \nu(2r-1) \int v_y^2 |v|^{2r-2} = (2r-1) \int p v_y |v|^{2r-2} + \int \theta v |v|^{2r-2}. \tag{2.5}$$

By Hölder's inequality,

$$\int \theta v |v|^{2r-2} \leqslant \|\theta\|_{2r} \|v\|_{2r}^{2r-1}, \tag{2.6}$$

$$\int p v_y |v|^{2r-2} \leqslant \|p\|_{2r} \big\| v_y |v|^{r-1}\big\|_2 \big\| |v|^{r-1}\big\|_{\frac{2r}{r-1}}. \tag{2.7}$$

Obviously,

$$\big\| |v|^{r-1}\big\|_{\frac{2r}{r-1}} = \|v\|_{2r}^{r-1}. \tag{2.8}$$

By Sobolev's inequality, for a constant C independent of r,

$$\|p\|_{2r} \leqslant Cr \|\nabla p\|_{\frac{2r}{r+1}}. \tag{2.9}$$

To further the estimate for p, we take the divergence of the first two equations in (1.1) to get

$$\begin{aligned}\Delta p &= -(uu_x + vu_y)_x - (uv_x + vv_y)_y + \theta_y \\ &= -2(vu_y)_x - 2(vv_y)_y + \theta_y.\end{aligned}$$

Since Riesz transforms are bounded on $L^{\frac{2r}{r+1}}$, we have

$$\begin{aligned}\|\nabla p\|_{\frac{2r}{r+1}} &\leqslant 2\big(\|vu_y\|_{\frac{2r}{r+1}} + \|vv_y\|_{\frac{2r}{r+1}}\big) + \|\theta\|_{\frac{2r}{r+1}} \\ &\leqslant 2\big(\|u_y\|_2 + \|v_y\|_2\big)\|v\|_{2r} + \|\theta\|_{\frac{2r}{r+1}}.\end{aligned} \tag{2.10}$$

Combining (2.7)–(2.9) and (2.10) and by Young's inequality, we have

$$\begin{aligned}(2r-1)\int pv_y|v|^{2r-2} \leqslant{}& \frac{\nu(2r-1)}{2}\big\|v_y|v|^{r-1}\big\|_2^2 + C(\nu)r^3\big(\|u_y\|_2^2 + \|v_y\|_2^2\big)\|v\|_{2r}^{2r} \\ &+ C(\nu)r^3\|v\|_{2r}^{2r-2}\|\theta\|_{\frac{2r}{r+1}}^2,\end{aligned} \tag{2.11}$$

where $C(\nu)$ is constant depending on ν only. Now, (2.5), (2.6) and (2.11) yield

$$\begin{aligned}&\frac{d}{dt}\|v\|_{2r}^{2r} + 2r(2r-1)\nu\int v_y^2|v|^{2r-2} \\ &\quad\leqslant C(\nu)r^4\big(\|u_y\|_2^2 + \|v_y\|_2^2\big)\|v\|_{2r}^{2r} + C(\nu)r^4\|v\|_{2r}^{2r-2}\|\theta\|_{\frac{2r}{r+1}}^2 + 2r\|\theta\|_{2r}\|v\|_{2r}^{2r-1}.\end{aligned} \tag{2.12}$$

(2.1) then follows from Gronwall's inequality and Proposition 2.2. In fact, by ignoring the second term on the left and then dividing each side by $\|v\|_{2r}^{2r-2}$, we have

$$\begin{aligned}\frac{d}{dt}\|v\|_{2r}^2 &\leqslant C(\nu)r^3\big(\|u_y\|_2^2 + \|v_y\|_2^2\big)\|v\|_{2r}^2 + C(\nu)r^3\|\theta_0\|_{\frac{2r}{r+1}}^2 + \|\theta_0\|_{2r}\|v\|_{2r} \\ &\leqslant \big(C(\nu)r^3\big(\|u_y\|_2^2 + \|v_y\|_2^2\big) + 1\big)\|v\|_{2r}^2 + C(\nu)r^3\|\theta_0\|_{\frac{2r}{r+1}}^2 + \|\theta_0\|_{2r}^2.\end{aligned}$$

Applying Gronwall's inequality and recalling the L^2-bound in (2.2), we obtain the desired inequality in (2.1). □

3. Conditional global regularity for (1.1)

This section establishes the following global regularity result.

Theorem 3.1. *Assume* $(u_0, v_0, \theta_0) \in H^2(\mathbf{R}^2)$ *and let* (u, v, θ) *be the corresponding solution of* (1.1). *Suppose* v *satisfies*

$$\int_0^T \big\|v(t)\big\|_\infty^2\,dt < \infty, \tag{3.1}$$

then (u, v, θ) *remains regular on* $[0, T]$, *namely* $(u, v, \theta) \in C([0, T]; H^2)$.

The proof of this theorem is divided into two major parts. The first part establishes the H^1-bound and the second part provides higher-order estimates. We will need the following lemma from [3].

Lemma 3.2. *Assume that f, g, g_y, h and h_x are all in $L^2(\mathbf{R}^2)$. Then,*

$$\int_{\mathbf{R}^2} |fgh|\,dx\,dy \leqslant C\|f\|_2\|g\|_2^{1/2}\|g_y\|_2^{1/2}\|h\|_2^{1/2}\|h_x\|_2^{1/2}. \tag{3.2}$$

3.1. H^1-bound

Proposition 3.3. *Assume $(u_0, v_0, \theta_0) \in H^1$. Let (u, v, θ) be the corresponding solution of (1.1). If ν satisfies (3.1), then (u, v, θ) obeys*

$$(u, v, \theta) \in C\big([0, T]; H^1\big).$$

Proof. Adding the inner products of the first equation in (1.1) with Δu and of the second equation with Δv and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\big\|(\nabla u, \nabla v)\big\|_2^2 + \nu\big\|(\nabla u_y, \nabla v_y)\big\|_2^2 = I_1 + I_2 + I_3, \tag{3.3}$$

where

$$I_1 = -\int u_x^3, \qquad I_2 = -\int v_y^3, \qquad I_3 = \int (\theta_x v_x + \theta_y v_y).$$

To estimate I_1, we apply Lemma 3.2 and Young's inequality to obtain

$$\begin{aligned} I_1 &= -\int u_x v_y^2 \\ &\leqslant C\|u_x\|_2\|v_y\|_2^{\frac{1}{2}}\|v_{xy}\|_2^{\frac{1}{2}}\|v_y\|_2^{\frac{1}{2}}\|v_{yy}\|_2^{\frac{1}{2}} \\ &\leqslant \frac{\nu}{4}\|v_{xy}\|_2^2 + \frac{\nu}{4}\|v_{yy}\|_2^2 + C\|v_y\|_2^2\|u_x\|_2^2. \end{aligned} \tag{3.4}$$

The estimate for I_2 is similar and

$$I_2 \leqslant \frac{\nu}{4}\|v_{xy}\|_2^2 + \frac{\nu}{4}\|v_{yy}\|_2^2 + C\|v_y\|_2^4. \tag{3.5}$$

By Hölder's and Young's inequality

$$I_3 \leqslant \|\nabla\theta\|_2\|\nabla v\|_2 \leqslant \frac{1}{2}\|\nabla\theta\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2. \tag{3.6}$$

Taking the inner product of the third equations in (1.1) with $\Delta\theta$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \kappa\|\nabla\theta_y\|_2^2 = J_1 + J_2 + J_3 + J_4, \tag{3.7}$$

where

$$J_1 = -\int u_x\theta_x^2, \qquad J_2 = -\int v_x\theta_x\theta_y, \qquad J_3 = -\int u_y\theta_x\theta_y, \qquad J_4 = -\int v_y\theta_y^2.$$

By $u_x + v_y = 0$, integration by parts and basic inequalities,

$$\begin{aligned} J_1 = \int v_y\theta_x^2 &= -2\int v\theta_x\theta_{xy} \\ &\leqslant 2\|v\|_\infty\|\theta_x\|_2\|\theta_{xy}\|_2 \\ &\leqslant \frac{\kappa}{4}\|\theta_{xy}\|_2^2 + C\|v\|_\infty^2\|\theta_x\|_2^2. \end{aligned} \tag{3.8}$$

By integration by parts,

$$\begin{aligned} J_2 &= \int (\theta v_{xy}\theta_x + \theta v_x\theta_{xy}) \\ &\leqslant \|\theta\|_\infty\|v_{xy}\|_2\|\theta_x\|_2 + \|\theta\|_\infty\|\theta_{xy}\|_2\|v_x\|_2 \\ &\leqslant \frac{\nu}{4}\|v_{xy}\|_2^2 + \frac{\kappa}{4}\|\theta_{xy}\|_2^2 + \|\theta\|_\infty^2\big(\|v_x\|_2^2 + \|\theta_x\|_2^2\big). \end{aligned} \tag{3.9}$$

By Lemma 3.2,

$$\begin{aligned} J_3 &\leqslant C\|u_y\|_2\|\theta_x\|_2^{\frac12}\|\theta_{xy}\|_2^{\frac12}\|\theta_y\|_2^{\frac12}\|\theta_{xy}\|_2^{\frac12} \\ &\leqslant \frac{\kappa}{4}\|\theta_{xy}\|_2^2 + C\|u_y\|_2^2\|\nabla\theta\|_2^2. \end{aligned} \tag{3.10}$$

Similarly,

$$J_4 \leqslant \frac{\kappa}{4}\|\theta_{xy}\|_2^2 + C\|v_y\|_2^2\|\nabla\theta\|_2^2. \tag{3.11}$$

Combining (3.3)–(3.10) and (3.11), we find

$$\begin{aligned} &\frac{d}{dt}\big\|(\nabla u, \nabla v, \nabla\theta)\big\|_2^2 + \nu\big\|(\nabla u_y, \nabla v_y)\big\|_2^2 + \kappa\|\nabla\theta_y\|_2^2 \\ &\quad \leqslant C\big(\big\|(u_y, v_y)\big\|_2^2 + \|\theta\|_\infty^2 + 1\big)\big\|(\nabla u, \nabla v, \nabla\theta)\big\|_2^2 + C\|v\|_\infty^2\|\theta_x\|_2^2 \end{aligned}$$

Gronwall's inequality then yields the desired result. □

3.2. Higher-order bounds

Proposition 3.4. *Assume $(u_0, v_0, \theta_0) \in H^2(\mathbf{R}^2)$ and let (u, v, θ) be the corresponding solution of* (1.1)*. Suppose v satisfies* (3.1)*, then $(u, v, \theta) \in C([0, T]; H^2)$.*

Proof. Adding the inner products of the first three equations in (1.1) with $\Delta^2 u$, $\Delta^2 v$ and $\Delta^2\theta$, respectively, and integrating by parts, we have

$$\begin{aligned} &\frac12\frac{d}{dt}\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\Delta\theta\|_2^2\big) + \nu\|\Delta u_y\|_2^2 + \nu\|\Delta v_y\|_2^2 + \kappa\|\Delta\theta_y\|_2^2 \\ &\quad = -\int \Delta(uu_x + vu_y)\Delta u + \Delta(uv_x + vv_y)\Delta v + \Delta(u\theta_x + v\theta_y)\Delta\theta - \Delta\theta\Delta v. \end{aligned}$$

We split the right-hand side into several terms and estimate each of them separately.

$$\begin{aligned} I_1 &\equiv \int \Delta(uu_x + vu_y)\Delta u \\ &= \int \big(u_x(\Delta u)^2 + u_y \Delta v \Delta u + 2\nabla u \cdot \nabla u_x \Delta u + 2\nabla v \cdot \nabla u_y \Delta u\big) \\ &= I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned}$$

By Lemma 3.2, Young's inequality and $u_x + v_y = 0$,

$$\begin{aligned} I_{11} &\leqslant C\|\Delta u\|_2 \|\Delta u\|_2^{\frac{1}{2}} \|\Delta u_y\|_2^{\frac{1}{2}} \|u_x\|_2^{\frac{1}{2}} \|u_{xx}\|_2^{\frac{1}{2}} \\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + C\|u_x\|_2^{\frac{2}{3}} \|u_{xx}\|_2^{\frac{2}{3}} \|\Delta u\|_2^2 \\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + C\|\nabla u\|_2^{\frac{2}{3}} \|v_{xy}\|_2^{\frac{2}{3}} \|\Delta u\|_2^2. \end{aligned}$$

Similarly,

$$\begin{aligned} I_{12} &\leqslant C\|\Delta u\|_2 \|\Delta v\|_2^{\frac{1}{2}} \|\Delta v_y\|_2^{\frac{1}{2}} \|u_y\|_2^{\frac{1}{2}} \|u_{xy}\|_2^{\frac{1}{2}} \\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|u_y\|_2^{\frac{2}{3}} \|u_{xy}\|_2^{\frac{2}{3}} \big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big) \\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|\nabla u\|_2^{\frac{2}{3}} \|u_{xy}\|_2^{\frac{2}{3}} \big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big). \end{aligned}$$

$$\begin{aligned} I_{13} &\leqslant C\|\nabla u\|_2 \|\nabla u_x\|_2^{\frac{1}{2}} \|\nabla u_{xx}\|_2^{\frac{1}{2}} \|\Delta u\|_2^{\frac{1}{2}} \|\Delta u_y\|_2^{\frac{1}{2}} \\ &= C\|\nabla u\|_2 \|\nabla u_x\|_2^{\frac{1}{2}} \|\nabla v_{xy}\|_2^{\frac{1}{2}} \|\Delta u\|_2^{\frac{1}{2}} \|\Delta u_y\|_2^{\frac{1}{2}} \\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|\nabla u\|_2^2 \|\Delta u\|_2^2. \end{aligned}$$

$$\begin{aligned} I_{14} &\leqslant C\|\nabla v\|_2 \|\nabla u_y\|_2^{\frac{1}{2}} \|\nabla u_{xy}\|_2^{\frac{1}{2}} \|\Delta u\|_2^{\frac{1}{2}} \|\Delta u_y\|_2^{\frac{1}{2}} \\ &\leqslant C\|\nabla v\|_2 \|\Delta u\|_2 \|\Delta u_y\|_2 \\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + C\|\nabla v\|_2^2 \|\Delta u\|_2^2. \end{aligned}$$

Collecting the estimates for I_1, we have

$$\begin{aligned} I_1 \leqslant{}& \frac{3\nu}{16}\|\Delta u_y\|_2^2 + \frac{3\nu}{16}\|\Delta v_y\|_2^2 \\ &+ C\Big(\big\|(\nabla u, \nabla v)\big\|_2^2 + \|\nabla u\|_2^{\frac{2}{3}} \big\|(\nabla u_y, \nabla v_y)\big\|_2^{\frac{2}{3}}\Big)\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big). \end{aligned}$$

In a similar fashion, we can also show that

$$\begin{aligned}I_2 &\equiv \int \Delta(uv_x + vv_y)\Delta v\\ &\leqslant \frac{\nu}{8}\|\Delta u_y\|_2^2 + \frac{\nu}{8}\|\Delta v_y\|_2^2\\ &\quad + C\big(\|\nabla v\|_2^2 + \big\|(\nabla u, \nabla v)\big\|_2^{\frac{2}{3}}\|\nabla v_y\|_2^{\frac{2}{3}}\big)\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big).\end{aligned}$$

In fact,

$$\begin{aligned}I_2 &\equiv \int \Delta(uv_x + vv_y)\Delta v\\ &= \int \big(v_x\Delta u\Delta v + v_y(\Delta v)^2 + 2\nabla u\cdot\nabla v_x\Delta v + 2\nabla v\cdot\nabla v_y\Delta v\big)\\ &= I_{21} + I_{22} + I_{23} + I_{24}.\end{aligned}$$

These terms can be bounded as follows.

$$\begin{aligned}I_{21} &\leqslant C\|\Delta v\|_2\|\Delta u\|_2^{\frac{1}{2}}\|\Delta u_x\|_2^{\frac{1}{2}}\|v_x\|_2^{\frac{1}{2}}\|v_{xy}\|_2^{\frac{1}{2}}\\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|v_x\|_2^{\frac{2}{3}}\|v_{xy}\|_2^{\frac{2}{3}}\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big)\\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + C\|\nabla v\|_2^{\frac{2}{3}}\|\nabla v_y\|_2^{\frac{2}{3}}\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\big).\end{aligned}$$

$$\begin{aligned}I_{22} &\leqslant C\|\Delta v\|_2\|\Delta v\|_2^{\frac{1}{2}}\|\Delta v_y\|_2^{\frac{1}{2}}\|v_y\|_2^{\frac{1}{2}}\|v_{xy}\|_2^{\frac{1}{2}}\\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|v_y\|_2^{\frac{2}{3}}\|v_{xy}\|_2^{\frac{2}{3}}\|\Delta v\|_2^2\\ &\leqslant \frac{\nu}{16}\|\Delta u_y\|_2^2 + C\|\nabla v\|_2^{\frac{2}{3}}\|\nabla v_y\|_2^{\frac{2}{3}}\|\Delta v\|_2^2.\end{aligned}$$

$$\begin{aligned}I_{23} &\leqslant C\|\nabla v_x\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u_x\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|\Delta v_y\|_2^{\frac{1}{2}}\\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|\nabla u\|_2^{\frac{2}{3}}\|\nabla v_y\|_2^{\frac{2}{3}}\|\Delta v\|_2^2.\end{aligned}$$

$$\begin{aligned}I_{24} &\leqslant C\|\nabla v\|_2\|\nabla v_y\|_2^{\frac{1}{2}}\|\nabla v_{xy}\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|\Delta v_y\|_2^{\frac{1}{2}}\\ &\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + C\|\nabla v\|_2^2\|\Delta v\|_2^2.\end{aligned}$$

We now deal with the third term.

$$\begin{aligned}I_3 &\equiv \int \Delta(u\theta_x + v\theta_y)\Delta\theta\\ &= \int (\Delta u\theta_x\Delta\theta + 2\nabla u\cdot\nabla\theta_x\Delta\theta + \Delta v\theta_y\Delta\theta + 2\nabla v\cdot\nabla\theta_y\Delta\theta)\\ &= I_{31} + I_{32} + I_{33} + I_{34}.\end{aligned}$$

By $u_x + u_y = 0$ and Lemma 3.2, we have the following estimates.

$$
\begin{aligned}
I_{31} &\leqslant C\|\theta_x\|_2\|\Delta u\|_2^{\frac{1}{2}}\|\Delta u_x\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{1}{2}}\|\Delta\theta_y\|_2^{\frac{1}{2}} \\
&\leqslant C\|\theta_x\|_2\|\Delta u\|_2^{\frac{1}{2}}\|\Delta v_y\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{1}{2}}\|\Delta\theta_y\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\nu}{16}\|\Delta v_y\|_2^2 + \frac{\kappa}{16}\|\Delta\theta_y\|_2^2 + C\|\theta_x\|_2^2\big(\|\Delta u\|_2^2 + \|\Delta\theta\|_2^2\big).
\end{aligned}
$$

$$
\begin{aligned}
I_{32} &\leqslant C\|\Delta\theta\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u_x\|_2^{\frac{1}{2}}\|\nabla\theta_x\|_2^{\frac{1}{2}}\|\nabla\theta_{xy}\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\kappa}{16}\|\Delta\theta_y\|_2^2 + C\|\nabla u\|_2^{\frac{2}{3}}\|\nabla v_y\|_2^{\frac{2}{3}}\|\Delta\theta\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
I_{33} &\leqslant C\|\Delta v\|_2\|\theta_y\|_2^{\frac{1}{2}}\|\theta_{xy}\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{1}{2}}\|\Delta\theta_y\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\kappa}{16}\|\Delta\theta_y\|_2^2 + C\|\theta_y\|_2^{\frac{2}{3}}\|\theta_{xy}\|_2^{\frac{2}{3}}\|\Delta v\|_2^{\frac{4}{3}}\|\Delta\theta\|_2^{\frac{2}{3}} \\
&\leqslant \frac{\kappa}{16}\|\Delta\theta_y\|_2^2 + C\|\theta_y\|_2^{\frac{2}{3}}\|\theta_{xy}\|_2^{\frac{2}{3}}\big(\|\Delta v\|_2^2 + \|\Delta\theta\|_2^2\big).
\end{aligned}
$$

$$
\begin{aligned}
I_{34} &\leqslant C\|\nabla v\|_2\|\nabla\theta_y\|_2^{\frac{1}{2}}\|\nabla\theta_{xy}\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{1}{2}}\|\Delta\theta_y\|_2^{\frac{1}{2}} \\
&\leqslant \frac{\kappa}{16}\|\Delta\theta_y\|_2^2 + C\|\nabla v\|_2^2\|\Delta\theta\|_2^2.
\end{aligned}
$$

Collecting these estimates yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\Delta(u,v,\theta)\big\|_2^2 + \nu\|\Delta u_y\|_2^2 + \nu\|\Delta v_y\|_2^2 + \kappa\|\Delta\theta_y\|_2^2 \\
&\quad\leqslant C\Big(\big\|\nabla(u,v,\theta)\big\|_2^2 + \big\|\nabla(u,v,\theta)\big\|_2^{\frac{2}{3}}\big\|\nabla(u_y,v_y,\theta_y)\big\|_2^{\frac{2}{3}}\Big)\big\|\Delta(u,v,\theta)\big\|_2^2
\end{aligned}
$$

Gronwall's inequality, together with Proposition 3.3, then leads to the desired bound. □

4. Global regularity for (1.7)

This section establishes the global regularity of (1.7). We first state it as a rigorous theorem.

Theorem 4.1. *Let* $(u_0, v_0, \theta_0) \in H^2(\mathbf{R}^2)$. *Then* (1.7) *with* $\nu > 0$, $\kappa > 0$, $\epsilon > 0$ *and* $\delta > 0$ *has a unique global classical solution* (u, v, θ).

Proof. To prove this theorem, it suffices to establish the global H^1 bound for (u, v, θ) since the H^2 bounds can be similarly obtained as in the proof of Theorem 3.1.

As in the proof of Theorem 3.1, we bound the L^2-norm of $(\nabla u, \nabla v, \nabla\theta)$ and only one term, namely J_1, is estimated differently here. By integration by parts,

$$
J_1 = -\int u_x(\theta_x)^2 = \int v_y(\theta_x)^2 = -2\int v\theta_x\theta_{xy}.
$$

Choose q such that $q\delta > 2$. By Hölder's inequality,

$$
|J_1| \leqslant 2\|v\|_q\|\theta_x\|_{\frac{2q}{q-2}}\|\theta_{xy}\|_2. \tag{4.1}
$$

By Sobolev's inequality and setting $\Lambda = (-\Delta)^{\frac{1}{2}}$, we have

$$\|\theta_x\|_{\frac{2q}{q-2}} \leqslant C\|\theta_x\|_2^{1-\frac{2}{q\delta}} \left\|\Lambda^\delta \theta_x\right\|_2^{\frac{2}{q\delta}}. \tag{4.2}$$

Inserting (4.2) in (4.1) and applying Young's inequality, we obtain

$$|J_1| \leqslant \frac{\kappa}{4}\|\theta_{xy}\|_2^2 + \frac{\epsilon}{4}\left\|\Lambda^\delta \nabla\theta\right\|_2^2 + C\|v\|_q^{\frac{2q\delta}{q\delta-2}}\|\theta_x\|_2^2.$$

Other terms can be estimated as in the proof of Theorem 3.1. Putting together these estimates yields the following closed inequality

$$\begin{aligned}\frac{d}{dt}\left\|(\nabla u, \nabla v, \nabla\theta)\right\|_2^2 + \nu\left\|(\nabla u_y, \nabla v_y)\right\|_2^2 + \kappa\|\nabla\theta_y\|_2^2 + \epsilon\left\|\Lambda^\delta\nabla\theta\right\|_2^2 \\ \leqslant C\left(\left\|(u_y, v_y)\right\|_2^2 + \|\theta\|_\infty^2 + 1\right)\left\|(\nabla u, \nabla v, \nabla\theta)\right\|_2^2 + C\|v\|_q^{\frac{2q\delta}{q\delta-2}}\|\theta_x\|_2^2.\end{aligned}$$

The boundedness of $\|(\nabla u, \nabla v, \nabla\theta)\|_2$ on any finite time interval then follows from applying Gronwall's inequality. □

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