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Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion

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ABSTRACT

We prove the global existence and the decay estimates of small smooth solution for the 2-D MHD equations without magnetic diffusion. This confirms the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity.

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1. Introduction

In this paper, we consider the 2-D MHD equations without magnetic diffusion

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla \pi = 0, \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ \operatorname{div} u = \operatorname{div} b = 0. \end{cases} \tag{1.1}$$

Here u is the velocity field, b is the magnetic field and π is the pressure. Eqs. (1.1) can be applied to plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small [2].

It is well-known that the 2-D MHD equations have the global smooth solution when the magnetic diffusion is included. In the case without magnetic diffusion, the question of whether smooth solution of the 2-D MHD equations develops singularity in finite time is open [13,7]. Recent important progress has been obtained by Cao, Regmi and Wu [5,4], where the authors studied the global regularity of the 2-D MHD equations with partial dissipation and magnetic diffusion. Lei [9] studied the global regularity for the axially symmetric MHD equations with nontrivial magnetic fields.

Due to $\operatorname{div} b = 0$, there exists a potential function ϕ such that

$$b = (\partial_{x_2}\phi, -\partial_{x_1}\phi).$$

In terms of ϕ, the MHD equations (1.1) can be rewritten as

$$\begin{cases} \partial_t \phi + u \cdot \nabla \phi = 0, \\ \partial_t u - \Delta u + u \cdot \nabla u + \operatorname{div}(\nabla\phi \otimes \nabla\phi) + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1.2}$$

where $p = \pi - |\nabla\phi|^2$. In a recent remarkable paper [10], Lin, Xu and Zhang proved the global existence of smooth solution of the system (1.2) around the trivial solution $(x_2, 0)$ (see [11] for 3-D case). By using the Lagrangian coordinates system, they first transform (1.2) into a damped wave system with partial dissipation like

$$Y_{tt} - \Delta Y_t - \partial_{x_1}^2 Y = f(Y, Y_t, \nabla Y).$$

Due to the partial dissipation, the solution has weak dissipation. Roughly speaking, the solution Y behaves as

$$\widehat{Y}(t,\xi) \sim a(\xi)e^{t\lambda_+(\xi)} + b(\xi)e^{t\lambda_-(\xi)},$$

where as $|\xi| \to +\infty$,

$$\lambda_-(\xi) \to -\frac{\xi_1^2}{|\xi|^2} \sim \begin{cases} -1, & |\xi| \sim |\xi_1|, \\ 0, & |\xi| \gg |\xi_1|. \end{cases}$$

Hence, the dissipation is very weak in the case of $|\xi| \gg |\xi_1|$. This simple analysis reveals the anisotropy of the eigenvalues $\lambda_+(\xi)$ and $\lambda_-(\xi)$, which signals the dependence of the time decay rates on the spatial directions.

This anisotropy has prompted [10] to use the anisotropic Littlewood–Paley decomposition as well as anisotropic Besov spaces in order to capture the weak dissipation. The anisotropic Besov type spaces appear to be essential in handling the anisotropic regularity in transport equations. Due to the use of the Lagrangian coordinates, [10] imposed an admissible condition on the initial data ψ_0 in the sense that

$$\int_{\mathbb{R}} (\partial_{x_2}\psi_0)\big(X(t,x)\big)dt = 0 \quad \text{for } x \in \mathbb{R},$$

where $X(t,x)$ is the integral curve of the vector field $(1+\partial_{x_2}\psi_0, -\partial_{x_1}\psi_0)$. It seems unclear what the physical interpretation of the admissible condition is.

The aim of this paper is two-fold: first, to establish the global existence of small solutions to (1.2) without imposing the admissible condition on the initial data, and second, to rigorously confirm the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity [3]. Our second goal is achieved by providing explicit time decay rates for various Sobolev norms of the solutions. These rates are identical to those for the linearized equations. Our main result can be stated as follows.

Theorem 1.1. *Let $\psi = \phi - x_2$. Assume that the initial data (ψ_0, u_0) satisfies $(\nabla\psi_0, u_0) \in H^8(\mathbb{R}^2)$. Then, there exist two small positive constants ϵ, ε such that if $(\nabla\psi_0, u_0) \in H^{-s,-s} \cap H^{-s,8}(\mathbb{R}^2)$ with $s = \frac{1}{2} - \epsilon$ and*

$$\big\|(\nabla\psi_0, u_0)\big\|_{H^8} + \big\|(\nabla\psi_0, u_0)\big\|_{H^{-s,-s}} + \big\|(\nabla\psi_0, u_0)\big\|_{H^{-s,8}} \le \varepsilon,$$

then the MHD system (1.2) has a unique global solution (ψ, u) satisfying

$$(\nabla\psi, u) \in C\big([0,+\infty); H^8\big(\mathbb{R}^2\big)\big).$$

Moreover, it holds that

$$\big\|\partial_{x_1}^k \nabla\psi\big\|_{L^2} + \big\|\partial_{x_1}^k u\big\|_{L^2} \le C\varepsilon(1+t)^{-\frac{s+k}{2}} \tag{1.3}$$

for any $t \in [0,+\infty)$ and $k = 0,1,2$. Here $H^{\sigma,s}(\mathbb{R}^2)$ is the homogeneous anisotropic Sobolev space, whose norm is defined by

$$\|f\|_{H^{\sigma,s}} = \big\||D|^s|D_1|^\sigma f\big\|_{L^2}.$$

Remark 1.1. (1) The decay estimates for the nonlinear equation are consistent with those of the linear equation obtained in Proposition 2.1, where the anisotropic Sobolev space plays the essential role.

(2) Due to $s < \frac{1}{2}$, $(\nabla\psi_0, u_0) \in H^{-s,-s}(\mathbb{R}^2)$ if $(\nabla\psi_0, u_0) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

In order to prove Theorem 1.1, we first analyze the solutions of the linearized equations. Especially we obtain explicit time decay rates for the energy and various Sobolev norms. The estimates for the nonlinear equations are more complex. The main difficulty is that the system has a weak dissipation so that it is difficult to control the growth of nonlinear terms. For example, the linear decay estimates do not insure that $\int_0^{+\infty}\|\nabla u(t)\|_{L^\infty}dt$ is finite, which is important to control the growth of ψ, since it satisfies a transport equation and its H^s norm is bounded as

$$\big\|\nabla\psi(t)\big\|_{H^s}\le\|\nabla\psi_0\|_{H^s}\exp\left(\int_0^t\big\|\nabla u(\tau)\big\|_{L^\infty}d\tau\right)+\cdots.$$

To control the growth of nonlinear terms, we will use in a crucial way the anisotropic Sobolev space and the special structure of nonlinear terms. First of all, under the assumption that the solution is bounded in the anisotropic Sobolev spaces, one can show that the energy of the solution has some kind of decay in time like (1.3). Then we use the obtained decay estimates to prove that the solution is indeed bounded in the anisotropic Sobolev spaces with a refined bound. Thus, the theorem follows from a continuous argument.

2. Decay estimate of the linear equation

2.1. Decay of the linearized MHD equations

Introduce the perturbation $\psi=\phi-x_2$. The system (1.2) is transformed into the form

$$\begin{cases}\partial_t\psi+u^2=-u\cdot\nabla\psi,\\ \partial_t u^1-\Delta u^1-\partial_{x_1}\partial_{x_2}\psi=-u\cdot\nabla u^1-\operatorname{div}(\nabla\psi\partial_{x_1}\psi)-\partial_{x_1}p,\\ \partial_t u^2-\Delta u^2+\partial_{x_1}^2\psi=-u\cdot\nabla u^2-\operatorname{div}(\nabla\psi\partial_{x_2}\psi)-\partial_{x_2}p,\\ \operatorname{div}u=0,\end{cases}\tag{2.1}$$

where the pressure p is determined by

$$p=(-\Delta)^{-1}\partial_{x_i}\partial_{x_j}\big(u^iu^j\big)+(-\Delta)^{-1}\partial_{x_i}\partial_{x_j}(\partial_{x_i}\psi\partial_{x_j}\psi)\triangleq p_1+p_2.\tag{2.2}$$

Specifically, the linearized MHD equation takes the form

$$\begin{cases}\partial_t\psi+u^2=0,\\ \partial_t u^1-\Delta u^1-\partial_{x_1}\partial_{x_2}\psi=0,\\ \partial_t u^2-\Delta u^2+\partial_{x_1}^2\psi=0,\\ \psi|_{t=0}=\psi_0,\quad u|_{t=0}=u_0.\end{cases}\tag{2.3}$$

Now let us study the energy decay of the linearized equation by using the energy method, which will be used to analyze the nonlinear system. The idea is partially motivated by the paper [8] on the viscous surface waves.

Proposition 2.1. *Assume that the initial data* $(\nabla\psi_0, u_0) \in H^1(\mathbb{R}^2)$ *and* $(|D_1|^{-s}\nabla\psi_0, |D_1|^{-s}u_0) \in H^{1+s}(\mathbb{R}^2)$ *for* $s > 0$. *Then it holds that*

$$\|u\|_{L^2} + \|\nabla\psi\|_{L^2} \le C(1+t)^{-\frac{s}{2}}.$$

Proof. We introduce

$$\begin{aligned}
E(t) &\triangleq \|u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2\psi\|_{L^2}^2 + 2\epsilon_1\langle u^2, \Delta\psi\rangle, \\
D(t) &\triangleq \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \epsilon_1\|\nabla\partial_{x_1}\psi\|_{L^2}^2 - \epsilon_1\|\nabla u^2\|_{L^2}^2 - \epsilon_1\langle \Delta u^2, \Delta\psi\rangle, \\
E_s(t) &\triangleq \||D_1|^{-s}u\|_{L^2}^2 + \||D_1|^{-s}\nabla\psi\|_{L^2}^2 + \||D|^{1+s}|D_1|^{-s}u\|_{L^2}^2 + \||D|^{1+s}|D_1|^{-s}\nabla\psi\|_{L^2}^2,
\end{aligned}$$

where $\epsilon_1 > 0$ is taken such that for some $c > 0$,

$$\begin{aligned}
E(t) &\ge c\big(\|u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2\psi\|_{L^2}^2\big), \\
D(t) &\ge c\big(\|\nabla u\|_{L^2}^2 + \|\nabla\partial_{x_1}\psi\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\big).
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
\langle \Delta u^2, \Delta\psi\rangle &= \langle \partial_{x_1}^2 u^2, \Delta\psi\rangle - \langle \partial_{x_2}\partial_{x_1} u^1, \Delta\psi\rangle \\
&\le \|\nabla^2 u\|_{L^2}\|\nabla\partial_{x_1}\psi\|_{L^2}.
\end{aligned}$$

Direct energy estimates yield that

$$\frac{1}{2}\frac{d}{dt}E(t) + D(t) \le 0, \qquad \frac{d}{dt}E_s(t) \le 0.$$

By the interpolation, we get

$$\begin{aligned}
\|\nabla\psi\|_{L^2}^2 &\le \||D_1|^{-s}\nabla\psi\|_{L^2}^{\frac{2}{s+1}}\||D_1|\nabla\psi\|_{L^2}^{\frac{2s}{s+1}} \le E_s(t)^{\frac{1}{s+1}}D(t)^{\frac{s}{s+1}}, \\
\|\nabla^2\psi\|_{L^2}^2 &\le \||D|^{1+s}|D_1|^{-s}\nabla\psi\|_{L^2}^{\frac{2}{s+1}}\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac{2s}{s+1}} \le E_s(t)^{\frac{1}{s+1}}D(t)^{\frac{s}{s+1}}.
\end{aligned}$$

This gives

$$E(t) \le CE_s(0)^{\frac{1}{s+1}}D(t)^{\frac{s}{s+1}},$$

that is,

$$D(t) \ge C^{-1}E_s(0)^{-\frac{1}{s}}E(t)^{1+\frac{1}{s}}.$$

This yields that

$$\frac{d}{dt}E(t) + C^{-1}E_s(0)^{-\frac{1}{s}}E(t)^{1+\frac{1}{s}} \le 0,$$

which implies that

$$E(t) \le E(0)\left(1 + \frac{1}{Cs}t\right)^{-s}.$$

The proof is completed. $\square$

Remark 2.1. More general, one can show that for any $k \ge 0$,

$$\left\|\partial_{x_1}^k u\right\|_{L^2} + \left\|\partial_{x_1}^k \nabla\psi\right\|_{L^2} \le C(1+t)^{-\frac{s+k}{2}}.$$

Proposition 2.1 can be directly proved by using the representation formula of the solution. Indeed, it is easy to see that u satisfies a damped wave equation

$$\partial_t^2 u - \Delta u_t - \partial_{x_1}^2 u = 0, \qquad u(0) = u_0, \qquad \partial_t u(0) = u_1.$$

Let $\lambda_\pm(\xi) = -\frac{|\xi|^2}{2}(1 \pm \sqrt{1 - \frac{4|\xi_1|^2}{|\xi|^4}})$. Then we get

$$\widehat{u}(t,\xi) = a(\xi)e^{t\lambda_+(\xi)} + b(\xi)e^{t\lambda_-(\xi)},$$

with

$$a(\xi) = \frac{(\lambda_-(\xi)\widehat{u}_0(\xi) - \widehat{u}_1(\xi))}{|\xi|^2\sqrt{1 - \frac{4|\xi_1|^2}{|\xi|^4}}},$$

$$b(\xi) = -\frac{(\lambda_+(\xi)\widehat{u}_0(\xi) - \widehat{u}_1(\xi))}{|\xi|^2\sqrt{1 - \frac{4|\xi_1|^2}{|\xi|^4}}}.$$

We find that as $|\xi| \to +\infty$,

$$\lambda_-(\xi) \to -\frac{\xi_1^2}{|\xi|^2} \sim \begin{cases} -1, & |\xi| \sim |\xi_1|, \\ 0, & |\xi| \gg |\xi_1|. \end{cases}$$

So, the dissipation is very weak in the case of $|\xi| \gg |\xi_1|$.

We find from Proposition 2.1 that the solution has the decay estimate when the initial data is bounded in the anisotropic Sobolev space. This motivates us to use the anisotropic Sobolev space to capture the dissipation of the solution. The role of the anisotropic Sobolev (Besov) spaces in the study of the Navier–Stokes system with anisotropic vertical viscosity appears in [6,11,12].

2.2. Anisotropic Littlewood–Paley analysis

Choose a nonnegative even function $\varphi \in \mathcal{S}(\mathbb{R})$ supported in $\mathcal{C} = \{\xi \in \mathbb{R},\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\xi\big) = 1 \quad \text{for any } \xi \in \mathbb{R}\setminus\{0\}.$$

We need to introduce two classes of the frequency localization operators. The localization operators Δ_j and S_j in the full direction are defined by

$$\Delta_j f = \mathcal{F}^{-1}\big(\varphi\big(2^{-j}|\xi|\big)\widehat{f}\big), \qquad S_j f = \sum_{j'\leq j-1} \Delta_{j'} f \quad \text{for } j \in \mathbb{Z}.$$

The localization operators Δ_j^h and S_j^h in the horizontal direction are defined by

$$\Delta_j^h f = \mathcal{F}^{-1}\big(\varphi\big(2^{-j}|\xi_1|\big)\widehat{f}\big), \qquad S_j^h f = \sum_{j'\leq j-1} \Delta_{j'}^h f \quad \text{for } j \in \mathbb{Z}.$$

We denote by $S_h'(\mathbb{R}^2)$ the set of the tempered distribution f satisfying

$$\lim_{\lambda\to+\infty} \big\|\chi(\lambda D)f\big\|_{L^\infty} = 0,$$

for some $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\chi(0) \neq 0$.

Definition 2.1. Let $\sigma, s \in \mathbb{R}$. The anisotropic Sobolev space $H^{\sigma,s}(\mathbb{R}^2)$ is defined by

$$H^{\sigma,s}\big(\mathbb{R}^2\big) \overset{\text{def}}{=} \big\{f \in S_h'\big(\mathbb{R}^2\big):\ \|f\|_{H^{\sigma,s}} < +\infty\big\},$$

where

$$\|f\|_{H^{\sigma,s}} \triangleq \big\|\big\{2^{js}2^{\sigma k}\big\|\Delta_j\Delta_k^h f\big\|_{L^2}\big\}_{j,k}\big\|_{\ell^2}.$$

It is easy to see that $\|f\|_{H^{\sigma,s}} \sim \||D|^s|D_1|^\sigma f\|_{L^2}$.

Definition 2.2. Let $\sigma, s \in \mathbb{R}$. The anisotropic Besov space $B^{\sigma,s}(\mathbb{R}^2)$ is defined by

$$B^{\sigma,s}\big(\mathbb{R}^2\big) \overset{\text{def}}{=} \big\{f \in S_h'\big(\mathbb{R}^2\big):\ \|f\|_{B^{\sigma,s}} < +\infty\big\},$$

where

$$\|f\|_{B^{\sigma,s}} \triangleq \big\|\big\{2^{js}2^{\sigma k}\big\|\Delta_j\Delta_k^h f\big\|_{L^2}\big\}_{j,k}\big\|_{\ell^1}.$$

The norm of the anisotropic Lebesgue space $L^p_{x_1}L^q_{x_2}(\mathbb{R}^2)$ is given by

$$\|f\|_{L^p_{x_1}L^q_{x_2}} \stackrel{\text{def}}{=} \big\|\big\|f(x_1,x_2)\big\|_{L^q_{x_2}}\big\|_{L^p_{x_1}}.$$

In the sequel, we will constantly use the following anisotropic Bony's decomposition that

$$\begin{aligned}
fg = {} & \sum_{j,k} S_{j-1}S^h_{k-1}f\Delta_j\Delta^h_k g + \sum_{j,k} S_{j-1}\Delta^h_k f\Delta_j S^h_{k-1}g \\
& + \sum_j \sum_{|k'-k''|\le 1} S_{j-1}\Delta^h_{k'}f\Delta_j\Delta^h_{k''}g + \sum_{j,k}\Delta_j S^h_{k-1}fS_{j-1}\Delta^h_k g \\
& + \sum_{j,k}\Delta_j\Delta^h_k fS_{j-1}S^h_{k-1}g + \sum_j \sum_{|k'-k''|\le 1}\Delta_j\Delta^h_{k'}fS_{j-1}\Delta^h_{k''}g \\
& + \sum_{|j-j'|\le 1}\sum_k \Delta_j S^h_{k-1}f\Delta_{j'}\Delta^h_k g + \sum_{|j-j'|\le 1}\sum_k \Delta_j\Delta^h_k f\Delta_{j'}S^h_{k-1}g \\
& + \sum_{|j-j'|\le 1}\sum_{|k'-k''|\le 1}\Delta_j\Delta^h_{k'}f\Delta_{j'}\Delta^h_{k''}g.
\end{aligned}$$

Due to the choice of φ, there exists $N_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\Delta_j\Delta^h_k(fg) = {} & \sum_{|j-j'|\le 4,|k-k'|\le 4}\Delta_j\Delta^h_k\big(S_{j'-1}S^h_{k'-1}f\Delta_{j'}\Delta^h_{k'}g\big) \\
& + \sum_{|j-j'|\le 4,|k-k'|\le 4}\Delta_j\Delta^h_k\big(S_{j'-1}\Delta^h_{k'}f\Delta_{j'}S^h_{k'-1}g\big) \\
& + \sum_{|j-j'|\le 4}\sum_{|k'-k''|\le 1,k',k''>k-N_0}\Delta_j\Delta^h_k\big(S_{j'-1}\Delta^h_{k'}f\Delta_{j'}\Delta^h_{k''}g\big) \\
& + \sum_{|j-j'|\le 4,|k-k'|\le 4}\Delta_j\Delta^h_k\big(\Delta_{j'}S^h_{k'-1}fS_{j'-1}\Delta^h_{k'}g\big) \\
& + \sum_{|j-j'|\le 4,|k-k'|\le 4}\Delta_j\Delta^h_k\big(\Delta_{j'}\Delta^h_{k'}fS_{j'-1}S^h_{k'-1}g\big) \\
& + \sum_{|j-j'|\le 4}\sum_{|k'-k''|\le 1,k',k''>k-N_0}\Delta_j\Delta^h_k\big(\Delta_{j'}\Delta^h_{k'}fS_{j'-1}\Delta^h_{k''}g\big) \\
& + \sum_{|j'-j''|\le 1,j',j''>j-N_0}\sum_{|k-k'|\le 4}\Delta_j\Delta^h_k\big(\Delta_{j'}S^h_{k'-1}f\Delta_{j''}\Delta^h_{k'}g\big) \\
& + \sum_{|j'-j''|\le 1,j',j''>j-N_0}\sum_{|k-k'|\le 4}\Delta_j\Delta^h_k\big(\Delta_{j'}\Delta^h_{k'}f\Delta_{j''}S^h_{k'-1}g\big)
\end{aligned}$$

$$+\sum_{|j'-j''|\le 1, j',j''>j-N_0}\ \sum_{|k'-k''|\le 1, k',k''>k-N_0} \Delta_j\Delta_k^h\big(\Delta_{j'}\Delta_{k'}^h f\Delta_{j''}\Delta_{k''}^h g\big)$$
$$\triangleq \mathcal{B}^1_{j,k}(f,g)+\cdots+\mathcal{B}^9_{j,k}(f,g). \tag{2.4}$$

The following Bernstein's lemma will be constantly used [1].

Lemma 2.1. *Let $1 \le p \le q \le \infty$ and $R > 0$, $c \in (0,1)$. Assume that $f \in L^p(\mathbb{R}^d)$, then there exists a constant C independent of f, R such that*

$$\operatorname{supp}\hat{f} \subset \{|\xi| \leqslant R\} \quad\Rightarrow\quad \|\partial^\alpha f\|_{L^q} \le CR^{|\alpha|+d(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$
$$\operatorname{supp}\hat{f} \subset \{cR \leqslant |\xi| \leqslant R\} \quad\Rightarrow\quad \|f\|_{L^p} \le CR^{-|\alpha|}\sup_{|\beta|=|\alpha|}\|\partial^\beta f\|_{L^p}.$$

It is easy to infer from Lemma 2.1 that for $q_1, q_2 \ge 2$,

$$\|\Delta_j\Delta_k^h f\|_{L^{q_1}_{x_2}L^{q_2}_{x_1}} \le C2^{j(\frac{1}{2}-\frac{1}{q_1})}2^{k(\frac{1}{2}-\frac{1}{q_2})}\|\Delta_j\Delta_k^h f\|_{L^2}. \tag{2.5}$$

We refer to the book [6] for more details about the Littlewood–Paley analysis.

3. Decay estimates for the nonlinear system

Motivated by the analysis of the linear system, we introduce the following energy

$$D_0(t) \triangleq \|u\|^2_{L^2}+\|\nabla\psi\|^2_{L^2}+\|\nabla u\|^2_{L^2}+\|\nabla^2\psi\|^2_{L^2}+2\epsilon_1\langle u^2,\Delta\psi\rangle,$$
$$H_0(t) \triangleq \|\nabla u\|^2_{L^2}+\|\nabla^2 u\|^2_{L^2}+\epsilon_1\|\partial_{x_1}\nabla\psi\|^2_{L^2}-\epsilon_1\|\nabla u^2\|^2_{L^2}-\epsilon_1\langle\Delta u^2,\Delta\psi\rangle,$$

and for $\ell = 1, 2$,

$$D_\ell(t) \triangleq \sum_{j,k}2^{2k\ell}\big(\|\Delta_j\Delta_k^h u\|^2_{L^2}+\|\Delta_j\Delta_k^h\nabla\psi\|^2_{L^2}+\|\Delta_j\Delta_k^h\nabla u\|^2_{L^2}+\|\Delta_j\Delta_k^h\nabla^2\psi\|^2_{L^2}$$
$$+2\epsilon_1\langle\Delta_j\Delta_k^h u^2,\Delta_j\Delta_k^h\Delta\psi\rangle\big),$$
$$H_\ell(t) \triangleq \sum_{j,k}2^{2k\ell}\big(\|\Delta_j\Delta_k^h\nabla u\|^2_{L^2}+\|\Delta_j\Delta_k^h\nabla^2 u\|^2_{L^2}+\epsilon_1\|\Delta_j\Delta_k^h\nabla\partial_{x_1}\psi\|^2_{L^2}$$
$$-\epsilon_1\|\Delta_j\Delta_k^h\nabla u^2\|^2_{L^2}-\epsilon_1\langle\Delta_j\Delta_k^h\Delta u^2,\Delta_j\Delta_k^h\Delta\psi\rangle\big).$$

Throughout this paper, $\epsilon_1 > 0$ is taken such that for some $c > 0$,

$$D_\ell(t) \ge c\big(\|\partial^\ell_{x_1}\nabla\psi\|^2_{L^2}+\|\partial^\ell_{x_1}\nabla^2\psi\|^2_{L^2}+\|\partial^\ell_{x_1}u\|^2_{L^2}+\|\partial^\ell_{x_1}\nabla u\|^2_{L^2}\big), \tag{3.1}$$
$$H_\ell(t) \ge c\big(\|\partial^\ell_{x_1}\nabla u\|^2_{L^2}+\|\partial^\ell_{x_1}\nabla^2 u\|^2_{L^2}+\|\partial^{\ell+1}_{x_1}\nabla\psi\|^2_{L^2}\big), \tag{3.2}$$

for $\ell = 0, 1, 2$.

Proposition 3.1. *Assume that the solution* (ψ, u) *of the system* (2.1) *satisfies*

$$\|u(t)\|_{H^4} + \|\nabla\psi(t)\|_{H^4} \le c_0, \tag{3.3}$$

for $t \in [0, T]$. *If* c_0 *is suitable small, then for some* $c > 0$,

$$\frac{d}{dt} D_0(t) + cH_0(t) \le 0,$$

for any $t \in [0, T]$.

Proof. From the proof of Proposition 2.1, we know that

$$\frac{1}{2}\frac{d}{dt} D_0(t) + H_0(t) = F,$$

where F is given by

$$\begin{aligned} F = {} & \langle u\cdot\nabla\psi, \Delta\psi\rangle - \langle u\cdot\nabla\psi, \Delta^2\psi\rangle - \epsilon_1\langle u\cdot\nabla\psi, \Delta u^2\rangle - \epsilon_1\langle u\cdot\nabla u^2, \Delta\psi\rangle \\ & - \langle u\cdot\nabla u, u\rangle + \langle u\cdot\nabla u, \Delta u\rangle - \langle \operatorname{div}(\nabla\psi\otimes\nabla\psi), u\rangle \\ & + \langle \operatorname{div}(\nabla\psi\otimes\nabla\psi), \Delta u\rangle - \epsilon_1\langle \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \Delta\psi\rangle - \epsilon_1\langle \partial_{x_2}p, \Delta\psi\rangle. \end{aligned}$$

By integration by parts and $\operatorname{div} u = 0$, it is easy to see that

$$\langle u\cdot\nabla\psi, \Delta\psi\rangle - \langle \operatorname{div}(\nabla\psi\otimes\nabla\psi), u\rangle - \langle u\cdot\nabla u, u\rangle = 0.$$

By Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned} \langle u\cdot\nabla u, \Delta u\rangle & \le \|u\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2} \\ & \le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{3}{2}} \le Cc_0 H_0(t). \end{aligned}$$

Using the following Sobolev inequality in $\mathbb{R}$

$$\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_x f\|_{L^2}^{\frac{1}{2}},$$

we infer that

$$\begin{aligned} \langle u\cdot\nabla\psi, \Delta u^2\rangle & \le \|u\|_{L^2_{x_1}L^\infty_{x_2}}\|\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}}\|\Delta u\|_{L^2} \\ & \le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\psi\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2} \\ & \le Cc_0 H_0(t), \end{aligned}$$

and

$$\begin{aligned}\big\langle u\cdot\nabla u^2,\Delta\psi\big\rangle &= -\big\langle \nabla u\cdot\nabla u^2,\nabla\psi\big\rangle - \big\langle u\cdot\nabla\nabla u^2,\nabla\psi\big\rangle\\ &\le \|\nabla u\|_{L^4}^2\|\nabla\psi\|_{L^2} + \|u\|_{L^2_{x_1}L^\infty_{x_2}}\|\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}}\|\Delta u\|_{L^2}\\ &\le Cc_0H_0(t). \end{aligned} \tag{3.4}$$

By integration by parts and $\operatorname{div} u=0$, we have

$$\begin{aligned}\big\langle u\cdot\nabla\psi,\Delta^2\psi\big\rangle &= \langle \Delta u\cdot\nabla\psi,\Delta\psi\rangle + \langle u\cdot\nabla\Delta\psi,\Delta\psi\rangle + 2\langle\partial_{x_i}u\cdot\nabla\partial_{x_i}\psi,\Delta\psi\rangle\\ &= \langle \Delta u\cdot\nabla\psi,\Delta\psi\rangle + 2\langle\partial_{x_i}u\cdot\nabla\partial_{x_i}\psi,\Delta\psi\rangle.\end{aligned}$$

We just check the most troubled two terms

$$\begin{aligned}\big\langle\partial_{x_2}^2u^2\partial_{x_2}\psi,\partial_{x_2}^2\psi\big\rangle &= -\big\langle\partial_{x_2}\partial_{x_1}u^1\partial_{x_2}\psi,\partial_{x_2}^2\psi\big\rangle\\ &= \big\langle\partial_{x_2}u^1\partial_{x_2}\partial_{x_1}\psi,\partial_{x_2}^2\psi\big\rangle + \big\langle\partial_{x_2}u^1\partial_{x_2}\psi,\partial_{x_2}^2\partial_{x_1}\psi\big\rangle\\ &= \big\langle\partial_{x_2}u^1\partial_{x_2}\partial_{x_1}\psi,\partial_{x_2}^2\psi\big\rangle - \big\langle\partial_{x_2}^2u^1\partial_{x_2}\psi,\partial_{x_2}\partial_{x_1}\psi\big\rangle\\ &\quad - \big\langle\partial_{x_2}u^1\partial_{x_2}^2\psi,\partial_{x_2}\partial_{x_1}\psi\big\rangle,\end{aligned}$$

and

$$\begin{aligned}\big\langle\partial_{x_2}u^2\partial_{x_2}^2\psi,\partial_{x_2}^2\psi\big\rangle &= -\big\langle\partial_{x_1}u^1\partial_{x_2}^2\psi,\partial_{x_2}^2\psi\big\rangle = 2\big\langle u^1\partial_{x_2}^2\psi,\partial_{x_1}\partial_{x_2}^2\psi\big\rangle\\ &= -2\big\langle\partial_{x_2}u^1\partial_{x_2}^2\psi,\partial_{x_1}\partial_{x_2}\psi\big\rangle - 2\big\langle u^1\partial_{x_2}^3\psi,\partial_{x_1}\partial_{x_2}\psi\big\rangle.\end{aligned}$$

So, we get

$$\begin{aligned}\big\langle\partial_{x_2}^2u^2\partial_{x_2}\psi,\partial_{x_2}^2\psi\big\rangle &\le 2\|\nabla u\|_{L^4}\|\nabla\partial_{x_1}\psi\|_{L^2}\big\|\nabla^2\psi\big\|_{L^4} + \big\|\nabla^2u\big\|_{L^2}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{L^\infty}\\ &\le Cc_0H_0(t),\end{aligned}$$

and

$$\begin{aligned}\big\langle\partial_{x_2}u^2\partial_{x_2}^2\psi,\partial_{x_2}^2\psi\big\rangle &\le c_0H_0(t) + \|u\|_{L^2_{x_1}L^\infty_{x_2}}\big\|\partial_{x_2}^3\psi\big\|_{L^\infty_{x_1}L^2_{x_2}}\|\nabla\partial_{x_1}\psi\|_{L^2}\\ &\le c_0H_0(t) + \|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_{x_2}\partial_{x_1}\psi\|_{L^2}^{\frac12}\big\|\partial_{x_2}^5\psi\big\|_{L^2}^{\frac12}\|\nabla\partial_{x_1}\psi\|_{L^2}\\ &\le c_0H_0(t).\end{aligned}$$

Here we use the following interpolation argument

$$\begin{aligned}\big\|\partial_{x_2}^3\psi(x_1,\cdot)\big\|_{L^2_{x_2}}^2 &= \int_{\mathbb{R}}\big|\partial_{x_2}^3\psi(x_1,x_2)\big|^2dx_2 = 2\int_{\mathbb{R}}\int_{-\infty}^{x_1}\partial_{x_2}^3\partial_{x_1}\psi\partial_{x_2}^3\psi dx_1dx_2\\ &= 2\int_{\mathbb{R}}\int_{-\infty}^{x_1}\partial_{x_2}\partial_{x_1}\psi\partial_{x_2}^5\psi dx_1dx_2 \le 2\|\partial_{x_2}\partial_{x_1}\psi\|_{L^2}\big\|\partial_{x_2}^5\psi\big\|_{L^2}.\end{aligned}$$

By integration by parts and $\operatorname{div} u = 0$, we have

$$\big\langle \operatorname{div}(\nabla\psi\otimes\nabla\psi), \Delta u\big\rangle = \big\langle \operatorname{div}(\nabla\psi\partial_{x_1}\psi), \Delta u^1\big\rangle + \big\langle \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \partial_{x_1}^2 u^2\big\rangle \\ - \big\langle \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \partial_{x_1}\partial_{x_2} u^1\big\rangle,$$

where the most troubled term is

$$\begin{aligned}
&\big\langle \operatorname{div}(\nabla\psi\partial_{x_1}\psi), \Delta u^1\big\rangle \\
&\quad = \big\langle \Delta\psi\partial_{x_1}\psi, \Delta u^1\big\rangle + \big\langle \nabla\psi\cdot\nabla\partial_{x_1}\psi, \Delta u^1\big\rangle \\
&\quad \le \|\Delta\psi\|_{L^\infty_{x_1}L^2_{x_2}}\|\partial_{x_1}\psi\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta u\|_{L^2} + \|\nabla\psi\|_{L^\infty}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\Delta u\|_{L^2} \\
&\quad \le C\big(\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac12}\big\|\nabla^3\psi\big\|_{L^2}^{\frac12}\|\partial_{x_1}\psi\|_{L^2}^{\frac12}\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac12} + \|\nabla\psi\|_{L^\infty}\|\nabla\partial_{x_1}\psi\|_{L^2}\big)\|\Delta u\|_{L^2} \\
&\quad \le Cc_0H_0(t).
\end{aligned}$$

Recall that $p_2 = -\frac{\operatorname{div}\operatorname{div}}{\Delta}(\nabla\psi\otimes\nabla\psi)$,

$$\begin{aligned}
&\big\langle \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \Delta\psi\big\rangle + \langle\partial_{x_2}p_2, \Delta\psi\rangle \\
&\quad = \big\langle \partial_{x_1}(\partial_{x_1}\psi\partial_{x_2}\psi), \Delta\psi\big\rangle + \big\langle \partial_{x_2}(\partial_{x_2}\psi\partial_{x_2}\psi), \partial_{x_1}^2\psi\big\rangle \\
&\qquad - \left\langle \frac{\partial_{x_2}\partial_{x_1}\operatorname{div}}{\Delta}(\partial_{x_1}\psi\nabla\psi), \Delta\psi\right\rangle - \left\langle \frac{\partial_{x_2}\partial_{x_2}\partial_{x_1}}{\Delta}(\partial_{x_2}\psi\partial_{x_1}\psi), \Delta\psi\right\rangle,
\end{aligned} \tag{3.5}$$

from which, it is easy to get

$$\big\langle \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \Delta\psi\big\rangle + \langle\partial_{x_2}p_2, \Delta\psi\rangle \le Cc_0H_0(t).$$

Similar to (3.4), we have

$$\langle\partial_{x_2}p_1, \Delta\psi\rangle \le Cc_0H_0(t).$$

Summing up, we conclude the proof. □

The proof of Proposition 3.1 does not work for the case of $\ell = 1, 2$. Indeed, if we use the energy method of Proposition 3.1, then we have to bound

$$\big\langle u\cdot\nabla\psi, \partial_{x_1}^2\Delta\psi\big\rangle \le CH_1(t),$$

which seems impossible. Instead, we use the anisotropic Littlewood–Paley decomposition. We denote

$$e(t) \triangleq \big\|(u,\nabla\psi)\big\|_{H^{-s,-s}} + \big\|(u,\nabla\psi)\big\|_{H^{-s,8}} + \big\|(u,\nabla\psi)\big\|_{H^8}.$$

Proposition 3.2. *Assume that the solution (ψ, u) of the system (2.1) satisfies*

$$\sup_{t\in[0,T]} e(t) \le c_0.$$

If c_0 is suitably small, then for some $c > 0$,

$$\frac{d}{dt} D_\ell(t) + cH_\ell(t) \le 0,$$

for any $t \in [0,T]$ and $\ell = 1, 2$.

Proof. Direct calculations yield that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\big\|\Delta_j\Delta_k^h u\big\|_{L^2}^2 + \big\|\Delta_j\Delta_k^h \nabla\psi\big\|_{L^2}^2 + \big\|\Delta_j\Delta_k^h \nabla u\big\|_{L^2}^2 + \big\|\Delta_j\Delta_k^h \nabla^2\psi\big\|_{L^2}^2 \\
&\qquad + 2\epsilon_1\big\langle\Delta_j\Delta_k^h u^2, \Delta_j\Delta_k^h\Delta\psi\big\rangle\big) + \big(\big\|\Delta_j\Delta_k^h \nabla u\big\|_{L^2}^2 + \big\|\Delta_j\Delta_k^h \nabla^2 u\big\|_{L^2}^2 \\
&\qquad + \epsilon_1\big\|\Delta_j\Delta_k^h \nabla\partial_{x_1}\psi\big\|_{L^2}^2 - \epsilon_1\big\|\Delta_j\Delta_k^h \nabla u^2\big\|_{L^2}^2 - \epsilon_1\big\langle\Delta_j\Delta_k^h \Delta u^2, \Delta_j\Delta_k^h\Delta\psi\big\rangle\big) \\
&\triangleq I_1 + \cdots + I_{10},
\end{aligned} \tag{3.6}$$

where I_i is given by

$$\begin{aligned}
&I_1 = \big\langle\Delta_j\Delta_k^h(u\cdot\nabla\psi), \Delta_j\Delta_k^h\Delta\psi\big\rangle, \qquad I_2 = -\big\langle\Delta_j\Delta_k^h(u\cdot\nabla u), \Delta_j\Delta_k^h u\big\rangle, \\
&I_3 = -\big\langle\Delta_j\Delta_k^h \operatorname{div}(\nabla\psi\otimes\nabla\psi), \Delta_j\Delta_k^h u\big\rangle, \qquad I_4 = -\big\langle\Delta_j\Delta_k^h(u\cdot\nabla\psi), \Delta_j\Delta_k^h\Delta^2\psi\big\rangle, \\
&I_5 = \big\langle\Delta_j\Delta_k^h(u\cdot\nabla u), \Delta_j\Delta_k^h\Delta u\big\rangle, \qquad I_6 = \big\langle\Delta_j\Delta_k^h \operatorname{div}(\nabla\psi\otimes\nabla\psi), \Delta_j\Delta_k^h\Delta u\big\rangle, \\
&I_7 = -\epsilon_1\big\langle\Delta_j\Delta_k^h(u\cdot\nabla\psi), \Delta_j\Delta_k^h\Delta u^2\big\rangle, \qquad I_8 = -\epsilon_1\big\langle\Delta_j\Delta_k^h\big(u\cdot\nabla u^2\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle, \\
&I_9 = -\epsilon_1\big\langle\Delta_j\Delta_k^h \operatorname{div}(\nabla\psi\partial_{x_2}\psi), \Delta_j\Delta_k^h\Delta\psi\big\rangle, \qquad I_{10} = -\epsilon_1\big\langle\Delta_j\Delta_k^h\partial_{x_2}p, \Delta_j\Delta_k^h\Delta\psi\big\rangle.
\end{aligned}$$

The equality (3.6) together with Lemma A.1–Lemma A.10 implies that

$$\frac{d}{dt} D_\ell(t) + H_\ell(t) \le Cc_0 H_\ell(t).$$

The proposition follows by taking c_0 small. □

Now we are in a position to derive the decay estimate of the solution. We set

$$\mathcal{E}_{s,k}(t) = E_{s,0}(t) + E_{s,k+s}(t),$$

where E_{s,s_1} is defined by (4.1).

Proposition 3.3. *Assume that the solution* (ψ, u) *of the system* (2.1) *satisfies*

$$\sup_{t\in[0,T]} e(t) \le c_0, \qquad \sup_{t\in[0,T],k=0,1,2} \mathcal{E}_{s,k}(t) \le C\varepsilon^2.$$

If c_0 *is suitably small, then it holds that*

$$\begin{aligned}
&\left\|\partial_{x_1}^k \nabla\psi\right\|_{L^2} + \left\|\partial_{x_1}^k \nabla^2\psi\right\|_{L^2} + \left\|\partial_{x_1}^k u\right\|_{L^2} + \left\|\partial_{x_1}^k \nabla u\right\|_{L^2} \le C\varepsilon(1+t)^{-\frac{s+k}{2}},\\
&\int_0^t \left(\left\|\partial_{x_1}^k \nabla u(\tau)\right\|_{L^2}^2 + \left\|\partial_{x_1}^k \nabla^2 u(\tau)\right\|_{L^2}^2 + \left\|\partial_{x_1}^{k+1} \nabla\psi(\tau)\right\|_{L^2}^2\right) d\tau \le C\varepsilon^2,
\end{aligned}$$

for any $t \in [0,T]$ *and* $k = 0, 1, 2$*, where* C *is a constant independent of* ε, t.

Proof. Proposition 3.1 and Proposition 3.2 yield that

$$\frac{1}{2}\frac{d}{dt} D_k(t) + cH_k(t) \le 0. \tag{3.7}$$

This gives the second inequality by (3.2). From the interpolation argument of Proposition 2.1, we find that

$$D_k(t) \le C\mathcal{E}_{s,k}(t)^{\frac{1}{s+k+1}} H_k(t)^{\frac{s+k}{s+k+1}} \le C\varepsilon^{\frac{2}{s+k+1}} H_k(t)^{\frac{s+k}{s+k+1}},$$

which along with (3.7) gives

$$\frac{d}{dt} D_k(t) + c\varepsilon^{-\frac{2}{s+k}} D_k(t)^{1+\frac{1}{s+k}} \le 0.$$

This implies the first inequality by (3.1). $\square$

4. High order energy estimates

The high order energy $E_{s,s_1}(t)$ is defined by

$$E_{s,s_1}(t) \stackrel{\text{def}}{=} \left\|u(t)\right\|_{H^{-s,s_1}}^2 + \left\|\nabla\psi(t)\right\|_{H^{-s,s_1}}^2 + \left\|u(t)\right\|_{H^{-s,s_1+1}}^2 + \left\|\nabla\psi(t)\right\|_{H^{-s,s_1+1}}^2. \tag{4.1}$$

We denote

$$\begin{aligned}
g_1(t) &\triangleq \left\|\psi(t)\right\|_{B^{\frac12,\frac32}} + \left\|\psi(t)\right\|_{B^{\frac12,\frac12}} + \left\|u(t)\right\|_{B^{\frac12,\frac12}} + \left\|u(t)\right\|_{L^2},\\
g_2(t) &\triangleq \left\|u(t)\right\|_{B^{\frac32,\frac12}},\\
g_3(t) &\triangleq \left\|u(t)\right\|_{B^{\frac12,\frac32}} + \left\|u(t)\right\|_{B^{\frac12,\frac12}} + \left\|\psi(t)\right\|_{B^{\frac32,\frac12}}^2 + \left\|\psi(t)\right\|_{B^{\frac12,1}}^2.
\end{aligned}$$

Proposition 4.1. *Let $s \in [0, \frac{1}{2})$, $s_1 > -\frac{1}{2}$. Assume that (u, ψ) is a solution of the system (2.1) on $[0, T]$. There exists $c_1 > 0$ such that if $g_1(t) \le c_1$ for $t \in [0, T]$, then*

$$E_{s,s_1}(t) \le CE_{s,s_1}(0) + C\int_0^t \big(g_2(\tau) + g_3(\tau)^2\big)E_{s,s_1}(\tau)d\tau,$$

for any $t \in [0, T]$.

Proof. The energy equality (3.6) implies that

$$E_{s,s_1}(t) + c\big(\|\nabla u\|^2_{L^2_t(H^{-s,s_1})} + \|\nabla u\|^2_{L^2_t(H^{-s,s_1+1})} + \|\nabla\partial_{x_1}\psi\|^2_{L^2_t(H^{-s,s_1})}\big)$$
$$\le E_{s,s_1}(0) + \sum_{j,k} 2^{2s_1 j}2^{-2ks}\int_0^t \big(I_1(\tau) + \cdots + I_{10}(\tau)\big)d\tau.$$

We infer from Lemma A.11–Lemma A.15 that

$$\sum_{j,k} 2^{2s_1 j}2^{-2ks}\int_0^t \big(I_1(\tau) + \cdots + I_{10}(\tau)\big)d\tau$$
$$\le \left(C\sup_{\tau\in[0,t]} g_1(\tau) + \frac{c}{2}\right)\big(\|\nabla u\|^2_{L^2_t(H^{-s,s_1})} + \|\nabla u\|^2_{L^2_t(H^{-s,s_1+1})} + \|\nabla\partial_{x_1}\psi\|^2_{L^2_t(H^{-s,s_1})}\big)$$
$$+ C\int_0^t \big(g_2(\tau) + g_3(\tau)^2\big)E_{s,s_1}(\tau)d\tau.$$

Then the proposition follows by taking c_1 suitably small. □

5. Proof of Theorem 1.1

The following local well-posedness can be proved by using the standard energy method. So we omit its proof.

Theorem 5.1. *Let $\psi = \phi - x_2$. Assume that the initial data (ψ_0, u_0) satisfies $(\nabla\psi_0, u_0) \in H^8(\mathbb{R}^2)$ and $(\nabla\psi_0, u_0) \in H^{-s,-s} \cap H^{-s,8}(\mathbb{R}^2)$ for $s \in (0, \frac{1}{2})$. Then there exists $T > 0$ such that the MHD system (1.2) has a unique solution (ψ, u) on $[0, T]$ satisfying*

$$(\nabla\psi, u) \in C\big([0, T]; H^8(\mathbb{R}^2)\big), \qquad (\nabla\psi, u) \in C\big([0, T]; H^{-s,-s} \cap H^{-s,8}(\mathbb{R}^2)\big).$$

Now let us assume that the solution (ψ, u) satisfies

$$\mathcal{E}_s(t) \triangleq E_{0,7}(t) + E_{s,-s}(t) + E_{s,7}(t) \le C_1\varepsilon^2, \quad t \in [0, T]. \tag{5.1}$$

In order to use Proposition 3.3, we take ε small enough such that $C_1\varepsilon \le c_0$.

Lemma 5.1. *Let $g_i(t)$ $(i=1,2,3)$ be given by Section 4. If $s>\frac{1}{3}$, then we have*

$$g_1(t)\le C\varepsilon,\qquad \int_0^t g_2(\tau)d\tau\le C\varepsilon,\qquad \int_0^t g_3(\tau)^2 d\tau\le C\varepsilon^2$$

for any $t\in[0,T]$, where C is a constant independent of ε,t.

Proof. By the definition of B^{s,s_1}, we have

$$\begin{aligned}\|\psi\|_{B^{\frac{1}{2},\frac{3}{2}}}&\le\sum_{k\le j}2^{\frac{3}{2}j}2^{\frac{1}{2}k}\big\|\Delta_j\Delta_k^h\psi\big\|_{L^2}\le\sum_j 2^{2j}\big\|\Delta_j\Delta_k^h\psi\big\|_{L^2}\\&\le C\big(\|\nabla\psi\|_{L^2}+\big\|\nabla^3\psi\big\|_{L^2}\big).\end{aligned}$$

Similarly, we have

$$\|u\|_{B^{\frac{1}{2},\frac{1}{2}}}\le C\big(\|u\|_{L^2}+\big\|\nabla^2 u\big\|_{L^2}\big),$$

and

$$\begin{aligned}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}&\le\sum_{k\le j}2^{\frac{1}{2}j}2^{\frac{1}{2}k}\big\|\Delta_j\Delta_k^h\psi\big\|_{L^2}\le\sum_k\big\|\Delta_k^h\nabla\psi\big\|_{L^2}\\&\le C\big(\big\||D_1|^{-s}\nabla\psi\big\|_{L^2}+\|\partial_{x_1}\nabla\psi\|_{L^2}\big).\end{aligned}$$

So, we conclude by (5.1) that

$$g_1(t)\le C\big(E_{0,3}(t)+E_{s,0}(t)\big)^{\frac{1}{2}}\le C\varepsilon.$$

Next we consider $g_2(t)$. We have

$$\begin{aligned}\|u\|_{B^{\frac{3}{2},\frac{1}{2}}}&\le\sum_{k\le j}2^{\frac{1}{2}j}2^{\frac{3}{2}k}\big\|\Delta_j\Delta_k^h u\big\|_{L^2}\\&\le\sum_j 2^{\frac{1}{2}j}\|\Delta_j u\|_{L^2}^{\frac{1}{4}}\big\|\Delta_j\partial_{x_1}^2 u\big\|_{L^2}^{\frac{3}{4}}\\&\le C\big\|\partial_{x_1}^2 u\big\|_{L^2}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{1}{4}}+C\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{1}{4}}.\end{aligned}$$

So, we get by Proposition 3.3 and $s>\frac{1}{3}$ that

$$\begin{aligned}\int_0^t g_2(\tau)d\tau&\le C\bigg(\int_0^t\big\|\partial_{x_1}^2 u(\tau)\big\|_{H^1}^{\frac{6}{7}}d\tau\bigg)^{\frac{7}{8}}\bigg(\int_0^t\big\|\nabla u(\tau)\big\|_{H^1}^2 d\tau\bigg)^{\frac{1}{8}}\\&\le C\varepsilon\bigg(\int_0^t(1+\tau)^{-\frac{6+3s}{7}}d\tau\bigg)^{\frac{7}{8}}\le C\varepsilon.\end{aligned}$$

Finally, let us consider $g_3(t)$. We have

$$\begin{aligned}
\|u\|_{B^{\frac{1}{2},\frac{3}{2}}} &\le \sum_{k\le j} 2^{\frac{3}{2}j}2^{\frac{1}{2}k}\big\|\Delta_j\Delta_k^h u\big\|_{L^2} \\
&\le \sum_{j} 2^{\frac{3}{2}j}\|\Delta_j u\|_{L^2}^{\frac{1}{2}}\|\Delta_j\partial_{x_1}u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_{x_1}u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + C\|\nabla\partial_{x_1}u\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2 u\big\|_{L^2}^{\frac{1}{2}-\delta}\big\|\nabla^3 u\big\|_{L^2}^{\delta},
\end{aligned}$$

for any $\delta \in (0, \frac{1}{2})$. Taking $\delta < \frac{s}{2}$, we infer from Proposition 3.3 and (5.1) that

$$\int_0^t \big\|u(\tau)\big\|_{B^{\frac{1}{2},\frac{3}{2}}}^2 d\tau \le C\varepsilon^2\left(\int_0^t (1+t)^{-(1+s)}d\tau\right)^{\frac{1}{2}} + C\varepsilon^2\left(\int_0^t (1+t)^{-\frac{1+s}{1+2\delta}}d\tau\right)^{\frac{1+2\delta}{2}} \le C\varepsilon^2.$$

Similarly, we have

$$\begin{aligned}
\|u\|_{B^{\frac{1}{2},\frac{1}{2}}} &\le \sum_{k\le j} 2^{\frac{1}{2}j}2^{\frac{1}{2}k}\big\|\Delta_j\Delta_k^h u\big\|_{L^2} \\
&\le \sum_{j} 2^{\frac{1}{2}j}\|\Delta_j u\|_{L^2}^{\frac{1}{2}}\|\Delta_j\partial_{x_1}u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla\partial_{x_1}u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + C\|\partial_{x_1}u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}-\delta}\|u\|_{L^2}^{\delta}.
\end{aligned}$$

Thus, we also have

$$\int_0^t \big\|u(\tau)\big\|_{B^{\frac{1}{2},\frac{1}{2}}}^2 d\tau \le C\varepsilon^2.$$

We have

$$\begin{aligned}
\|\psi\|_{B^{\frac{1}{2},1}} &\le \sum_{k\le j} 2^{j}2^{\frac{1}{2}k}\big\|\Delta_j\Delta_k^h \psi\big\|_{L^2} \\
&\le \sum_{j} 2^{\frac{1}{2}j}\|\Delta_j \psi\|_{L^2}^{\frac{1}{2}}\|\Delta_j\nabla\partial_{x_1}\psi\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac{1}{2}}\big(\big\||D_1|^{-s}\nabla\psi\big\|_{L^2} + \big\|\nabla^2\psi\big\|_{L^2}\big)^{\frac{1}{2}}.
\end{aligned}$$

So, we get

$$\int_0^t \big\|\psi(\tau)\big\|_{B^{\frac{1}{2},1}}^4 d\tau \le C\varepsilon^2\int_0^t \big\|\nabla\partial_{x_1}\psi(\tau)\big\|_{L^2}^2 d\tau$$

$$\leq C\varepsilon^4 \int_0^t (1+t)^{-1-s} d\tau \leq C\varepsilon^4.$$

Similarly, we have

$$\|\psi\|_{B^{\frac{1}{2},\frac{3}{2}}} \leq C\|\nabla\partial_{x_1}\psi\|_{L^2}^{\frac{1}{2}} \left(\|\nabla\psi\|_{L^2} + \left\|\nabla^3\psi\right\|_{L^2}\right)^{\frac{1}{2}}.$$

Hence,

$$\int_0^t \left\|\psi(\tau)\right\|_{B^{\frac{1}{2},\frac{3}{2}}}^4 d\tau \leq C\varepsilon^4.$$

Summing up, we conclude that

$$\int_0^t g_3(\tau)^2 d\tau \leq C\varepsilon^2.$$

The proof is finished. □

Now we are in a position to complete the proof of Theorem 1.1. Take ε small such that $C\varepsilon \leq c_1$. Then we infer from Proposition 4.1 and Lemma 5.1 that

$$\begin{aligned}\mathcal{E}_s(t) &\leq C\mathcal{E}_s(0) + C\int_0^t \left(g_2(\tau) + g_3(\tau)^2\right)\mathcal{E}_s(\tau)d\tau \\ &\leq C\mathcal{E}_s(0) + C\varepsilon \sup_{\tau\in[0,T]} \mathcal{E}_s(\tau).\end{aligned}$$

Taking $\varepsilon > 0$ small such that $C\varepsilon \leq \frac{1}{2}$, we get

$$\mathcal{E}_s(t) \leq 2C\mathcal{E}_s(0) \leq 2C\varepsilon^2.$$

Hence, we conclude Theorem 1.1 by a continuous argument.

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Appendix A

This appendix is devoted to proving nonlinear estimates for the energy estimates in Section 3 and Section 4. Throughout this section, we denote by $\{c_{j,k}\}_{j,k\in\mathbb{Z}}$ a sequence satisfying $\|\{c_{j,k}\}_{j,k\in\mathbb{Z}}\|_{\ell^1} \le 1$.

A.1. Nonlinear estimates for the decay energy $D_1(t)$

Recall that

$$H_1(t) \ge c\big(\|\partial_{x_1}\nabla u\|_{L^2}^2 + \big\|\partial_{x_1}\nabla^2 u\big\|_{L^2}^2 + \big\|\partial_{x_1}^2\nabla\psi\big\|_{L^2}^2\big),$$

and

$$e(t) = \big\|(u,\nabla\psi)\big\|_{H^{-s,-s}} + \big\|(u,\nabla\psi)\big\|_{H^{-s,8}} + \big\|(u,\nabla\psi)\big\|_{H^8}.$$

In what follows, we take s_1 such that

$$0 < s_1 \le s, \qquad s_2 = \frac{3}{2\theta} - 2 \le s \quad \text{with } \theta = \frac{1+2s_1}{2+2s_1},$$

which is possible by recalling $s = \frac{1}{2} - \epsilon$ and taking ϵ small enough.

Lemma A.1.

$$I_1 + I_4 \le Cc_{j,k}2^{-2k}e(t)H_1(t).$$

Proof. By integration by parts, we rewrite I_1 as

$$\begin{aligned} I_1 &= -\big\langle\Delta_j\Delta_k^h(\nabla u\cdot\nabla\psi), \Delta_j\Delta_k^h\nabla\psi\big\rangle - \big\langle\Delta_j\Delta_k^h\big(u^1\partial_{x_1}\nabla\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\ &\quad - \big\langle\big[\Delta_j\Delta_k^h, u^2\big]\partial_{x_2}\nabla\psi, \Delta_j\Delta_k^h\nabla\psi\big\rangle - \big\langle u^2\partial_{x_2}\Delta_j\Delta_k^h\nabla\psi, \Delta_j\Delta_k^h\nabla\psi\big\rangle \\ &\triangleq I_1^1 + I_1^2 + I_1^3 + I_1^4. \end{aligned} \tag{A.1}$$

Step 1. Estimate of I_1^1.

Using Bony's decomposition (2.4) and $\operatorname{div} u = 0$, we write

$$\begin{aligned} \big\langle\Delta_j\Delta_k^h(\nabla u\cdot\nabla\psi), \Delta_j\Delta_k^h\nabla\psi\big\rangle = \sum_{m=1}^{9} &\big\langle\mathcal{B}_{j,k}^m\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\ &+ \big\langle\mathcal{B}_{j,k}^m\big(\partial_{x_1}u^2, \partial_{x_2}\psi\big), \Delta_j\Delta_k^h\partial_{x_1}\psi\big\rangle \\ &- \big\langle\mathcal{B}_{j,k}^m\big(\partial_{x_1}u^1, \partial_{x_2}\psi\big), \Delta_j\Delta_k^h\partial_{x_2}\psi\big\rangle. \end{aligned} \tag{A.2}$$

We get by (2.5) that

$$\begin{aligned}
&\big\langle \mathcal{B}^1_{j,k}\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\quad\le \sum_{|j-j'|\le 4, |k-k'|\le 4} \big\|S_{j'-1}S^h_{k'-1}u^1\big\|_{L^\infty}\big\|\Delta_{j'}\Delta^h_{k'}\nabla\partial_{x_1}\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-2k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{H^{2,0}} \\
&\quad\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\partial_{x_1}u\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac32}_{H^{2,0}}.
\end{aligned}$$

Here we used the inequality

$$\begin{aligned}
\|u\|_{B^{\frac12,\frac12}} &\le \sum_j 2^{\frac{j}{2}}\|\Delta_j u\|^{\frac12}_{L^2}\|\Delta_j\partial_{x_1}u\|^{\frac12}_{L^2} \\
&\le C\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\partial_{x_1}u\|^{\frac12}_{L^2}. \qquad \text{(A.3)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\big\langle \mathcal{B}^1_{j,k}\big(\partial_{x_1}u^1, \partial_{x_2}\psi\big), \Delta_j\Delta^h_k\partial_{x_2}\psi\big\rangle + \big\langle \mathcal{B}^1_{j,k}\big(\partial_{x_1}u^2, \partial_{x_2}\psi\big), \Delta_j\Delta^h_k\partial_{x_1}\psi\big\rangle \\
&\quad\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{-s,0}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\psi\|^{\frac12}_{L^2}H_1(t),
\end{aligned}$$

and

$$\begin{aligned}
&\big\langle \mathcal{B}^2_{j,k}\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta^h_k\nabla\psi\big\rangle + \big\langle \mathcal{B}^3_{j,k}\big(\nabla u^i, \partial_{x_i}\psi\big), \Delta_j\Delta^h_k\nabla\psi\big\rangle \\
&\quad\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{-s,0}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\psi\|^{\frac12}_{L^2}H_1(t).
\end{aligned}$$

For the other two terms of $m=2$, we get by Sobolev's inequality and the interpolation that

$$\begin{aligned}
&\big\langle \mathcal{B}^2_{j,k}\big(\partial_{x_1}u^1, \partial_{x_2}\psi\big), \Delta_j\Delta^h_k\partial_{x_2}\psi\big\rangle + \big\langle \mathcal{B}^2_{j,k}\big(\partial_{x_1}u^2, \partial_{x_2}\psi\big), \Delta_j\Delta^h_k\partial_{x_1}\psi\big\rangle \\
&\quad\le \sum_{|j-j'|\le 4, |k-k'|\le 4} \big\|\Delta^h_{k'}\partial_{x_1}u\big\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\partial_{x_2}\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-2k}\big\|\partial_{x_1}|D|^{-1}u\big\|^{\frac14}_{L^2}\|\partial_{x_1}\nabla u\|^{\frac34}_{L^2}\|\nabla\psi\|^{\frac34}_{L^2}\big\|\nabla\partial^2_{x_1}\psi\big\|^{\frac14}_{L^2}\|\nabla\psi\|_{H^{2,0}} \\
&\quad\le Cc_{j,k}2^{-2k}\|u\|^{\frac14}_{L^2}\|\nabla\psi\|^{\frac34}_{L^2}H_1(t).
\end{aligned}$$

By (2.5) and the interpolation, we get

$$\begin{aligned}
&\big\langle \mathcal{B}^4_{j,k}\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta^h_k\nabla\psi\big\rangle \\
&\quad\le C\sum_{|j-j'|\le 4, |k-k'|\le 4} \big\|\partial_{x_1}\Delta^h_{k'}\psi\big\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\nabla u\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2}
\end{aligned}$$

$$\le Cc_{j,k}2^{-2k}\|\nabla\psi\|^{\theta}_{H^{2-\frac{1}{\theta},-\frac{1}{2\theta}}}\big\|\nabla\partial^2_{x_1}\psi\big\|^{1-\theta}\big\||D_1|^{-s_1}\nabla u\big\|^{1-\theta}_{L^2}\|\nabla\partial_{x_1}u\|^{\theta}_{L^2}\|\nabla\psi\|_{H^{2,0}}$$
$$\le Cc_{j,k}2^{-2k}\|\nabla\psi\|^{\theta}_{H^{-s_2,0}}\|u\|^{1-\theta}_{H^{-s_1,1}}H_1(t).$$

While for $m=5,6,8,9$, it is easy to see that

$$\big\langle\mathcal{B}^m_{j,k}\big(\nabla u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\nabla\psi\big\rangle\le Cc_{j,k}2^{-2k}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}H_1(t),$$

and for $m=4,7$, it holds

$$\big\langle\mathcal{B}^m_{j,k}\big(\partial_{x_1}u^2,\partial_{x_2}\psi\big),\Delta_j\Delta^h_k\partial_{x_1}\psi\big\rangle+\big\langle\mathcal{B}^m_{j,k}\big(\partial_{x_1}u^1,\partial_{x_2}\psi\big),\Delta_j\Delta^h_k\partial_{x_2}\psi\big\rangle$$
$$\le Cc_{j,k}2^{-2k}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}H_1(t).$$

We infer from (2.5) and (A.3) that

$$\big\langle\mathcal{B}^7_{j,k}\big(\nabla u^1,\partial_{x_1}\psi\big),\Delta_j\Delta^h_k\nabla\psi\big\rangle$$
$$\le C\sum_{|j'-j''|\le1,j',j''>j-N_0}\sum_{|k-k'|\le4}2^{\frac{j}{2}}\big\|\Delta_{j'}\Delta^h_{k'}\nabla\partial_{x_1}\psi\big\|_{L^2}\big\|\Delta_{j''}u^1\big\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2}$$
$$\le Cc_{j,k}2^{-2k}\|u\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{H^{2,0}}$$
$$\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}_{L^2}H_1(t).$$

Step 2. Estimate of I_1^2 and I_1^4.
Using Bony's decomposition (2.4), (2.5) and (A.3), it is easy to show that

$$I_1^2\le Cc_{j,k}2^{-2k}\|u\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{H^{2,0}}$$
$$\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}_{L^2}H_1(t),$$

and by integration by parts,

$$I_1^4=-2\big\langle u^1\partial_{x_1}\Delta_j\Delta^h_k\nabla\psi,\Delta_j\Delta^h_k\nabla\psi\big\rangle$$
$$\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}_{L^2}H_1(t).$$

Step 3. Estimate of I_1^3.
Using Bony's decomposition (2.4), we write

$$\big[\Delta_j\Delta^h_k,u^2\big]\partial_{x_2}\nabla\psi$$
$$=\sum_{|j-j'|\le4,|k-k'|\le4}\big[\Delta_j\Delta^h_k,S_{j'-1}S^h_{k'-1}u^2\big]\big(\Delta_{j'}\Delta^h_{k'}\partial_{x_2}\nabla\psi\big)$$
$$+\sum_{|j-j'|\le4,k'\ge k-N_0}\big[\Delta_j\Delta^h_k,S_{j'-1}\Delta^h_{k'}u^2\big]\big(\Delta_{j'}S^h_{k'-1}\partial_{x_2}\nabla\psi\big)$$

$$
\begin{aligned}
&+\sum_{|j-j'|\le 4}\ \sum_{|k'-k''|\le 1,k',k''>k-N_0}\big[\Delta_j\Delta_k^h, S_{j'-1}\Delta_{k'}^h u^2\big]\big(\Delta_{j'}\Delta_{k''}^h\partial_{x_2}\nabla\psi\big)\\
&+\mathcal{B}_{j,k}^4\big(u^2,\partial_{x_2}\nabla\psi\big)+\cdots+\mathcal{B}_{j,k}^9\big(u^2,\partial_{x_2}\nabla\psi\big)\\
&+\widetilde{\mathcal{B}}_{j,k}^4\big(u^2,\partial_{x_2}\nabla\psi\big)+\cdots+\widetilde{\mathcal{B}}_{j,k}^9\big(u^2,\partial_{x_2}\nabla\psi\big),
\end{aligned}
$$

where $\widetilde{\mathcal{B}}_{j,k}^i$ for $i=4,\cdots,9$ is similar to $\mathcal{B}_{j,k}^i$ like

$$
\widetilde{\mathcal{B}}_{j,k}^4(f,g)=\sum_{j'\ge j-N_0,|k-k'|\le 4}\Delta_{j'}S_{k'-1}^h f S_{j'-1}\Delta_{k'}^h\Delta_j\Delta_k^h g.
$$

Using the fact $\nabla u^2=(\partial_{x_1}u^2,-\partial_{x_1}u^1)$, it is easy to get

$$
\sum_{m=4}^9\big\langle\mathcal{B}_{j,k}^m\big(u^2,\partial_{x_2}\nabla\psi\big),\Delta_j\Delta_k^h\nabla\psi\big\rangle\le Cc_{j,k}2^{-2k}\|\psi\|_{B^{\frac12,\frac12}}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}},
$$

and

$$
\begin{aligned}
&\big\langle\widetilde{\mathcal{B}}_{j,k}^4\big(u^2,\partial_{x_2}\nabla\psi\big),\Delta_j\Delta_k^h\nabla\psi\big\rangle\\
&\quad\le C\sum_{j'\ge j-N_0,|k-k'|\le 4}2^j\big\|\Delta_{j'}\nabla u^2\big\|_{L^2}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2}\\
&\quad\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{L^2}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\langle\widetilde{\mathcal{B}}_{j,k}^5\big(u^2,\partial_{x_2}\nabla\psi\big),\Delta_j\Delta_k^h\nabla\psi\big\rangle\\
&\quad\le C\sum_{j'\ge j-N_0,k'\ge k-N_0}2^{\frac{j}{2}}2^{\frac{k}{2}}\big\|\Delta_{j'}\Delta_{k'}\nabla u^2\big\|_{L^2}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2}\\
&\quad\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{L^2}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}}.
\end{aligned}
$$

Similarly, for $m=6,7,8,9$, we have

$$
\big\langle\widetilde{\mathcal{B}}_{j,k}^m\big(u^2,\partial_{x_2}\nabla\psi\big),\Delta_j\Delta_k^h\nabla\psi\big\rangle\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{L^2}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}}.
$$

Now we consider the commutator terms. Let $\breve{\varphi}$ be the inverse Fourier transform of φ, and $\breve{\varphi}_{2^j}(x)=2^{2j}\breve{\varphi}(2^jx),\breve{\varphi}_{2^j}(x_1)=2^j\breve{\varphi}(2^jx_1)$. Then we have

$$
\begin{aligned}
&\big[\Delta_j\Delta_k^h,S_{j'-1}\Delta_{k'}^h u^2\big]\big(\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi\big)\\
&\quad=\int_{\mathbb{R}^2}\breve{\varphi}_{2^j}(x-y)\breve{\varphi}_{2^k}(x_1-y_1)\big(S_{j'-1}\Delta_{k'}^h u^2(y)-S_{j'-1}\Delta_{k'}^h u^2(x)\big)\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi(y)dy
\end{aligned}
$$

$$= \int_{\mathbb{R}^2} \breve{\varphi}_{2^j}(x-y)\breve{\varphi}_{2^k}(x_1-y_1)(x-y)$$

$$\cdot \nabla \int_0^1 S_{j'-1}\Delta_{k'}^h u^2\big(\tau y+(1-\tau)x\big)d\tau \Delta_{j'} S_{k'-1}^h \partial_{x_2}\nabla\psi(y)dy,$$

from which and Young's inequality, we infer that

$$\begin{aligned}
&\sum_{|j-j'|\le 4, k'\ge k-N_0} \big\| \big[\Delta_j\Delta_k^h, S_{j'-1}\Delta_{k'}^h u^2\big]\big(\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi\big)\big\|_{L^2} \\
&\quad \le C \sum_{|j-j'|\le 4, k'\ge k-N_0} 2^{-j}\big\|\nabla S_{j'-1}\Delta_{k'}^h u^2\big\|_{L^2_{x_1}L^\infty_{x_2}} \big\|\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi\big\|_{L^2_{x_2}L^\infty_{x_1}} \\
&\quad \le C \sum_{|j-j'|\le 4, k'\ge k-N_0} 2^{k'}\big\|\Delta_{k'}^h u\big\|_{L^2_{x_1}L^\infty_{x_2}} \|\Delta_{j'}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}},
\end{aligned}$$

which along with (2.5) implies that

$$\begin{aligned}
&\sum_{|j-j'|\le 4, |k-k'|\le 4} \big\langle \big[\Delta_j\Delta_k^h, S_{j'-1}\Delta_{k'}^h u^2\big]\big(\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\quad \le Cc_{j,k}2^{-2k}\big\|\partial_{x_1}^3|D|^{-3}u\big\|_{L^2}^{\frac18}\|\partial_{x_1}\nabla u\|_{L^2}^{\frac78}\|\nabla\psi\|_{L^2}^{\frac34}\big\|\nabla\partial_{x_1}^2\psi\big\|_{L^2}^{\frac14}\|\nabla\psi\|_{L^2}^{\frac18}\|\nabla\psi\|_{H^{2,0}}^{\frac78} \\
&\quad \le Cc_{j,k}2^{-2k}\|u\|_{L^2}^{\frac18}\|\nabla\psi\|_{L^2}^{\frac78}H_1(t).
\end{aligned}$$

The other two commutator terms are bounded by

$$\begin{aligned}
&Cc_{j,k}2^{-2k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{H^{2,0}} \\
&\quad \le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{-s,0}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\psi\|_{L^2}^{\frac12}H_1(t).
\end{aligned}$$

Step 4. Estimate of I_4.

By integration by parts, we get

$$\begin{aligned}
I_4 = &-\big\langle\Delta_j\Delta_k^h(\Delta u\cdot\nabla\psi), \Delta_j\Delta_k^h\Delta\psi\big\rangle - 2\big\langle\Delta_j\Delta_k^h(\nabla u\cdot\nabla\nabla\psi), \Delta_j\Delta_k^h\Delta\psi\big\rangle \\
&- \big\langle\Delta_j\Delta_k^h\big(u^1\cdot\partial_{x_1}\Delta\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle - \big\langle\big[\Delta_j\Delta_k^h, u^2\big]\partial_{x_2}\Delta\psi, \Delta_j\Delta_k^h\Delta\psi\big\rangle \\
&- \big\langle u^2\partial_{x_2}\Delta_j\Delta_k^h\Delta\psi, \Delta_j\Delta_k^h\Delta\psi\big\rangle \triangleq I_4^1+\cdots+I_4^5.
\end{aligned} \tag{A.4}$$

Since the estimates for I_4^i are very similar to those of Step 1–Step 3, we only record some main estimates. Following the proof of Step 1, we deduce that for $m=1,2,3,7,8,9$,

$$\begin{aligned}
\big\langle\mathcal{B}_{j,k}^m\big(\Delta u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle &\le Cc_{j,k}2^{-2k}\|\Delta u\|_{B^{\frac12,\frac12}}\|\nabla\partial_{x_1}\psi\|_{L^2}\|\nabla\psi\|_{H^{2,0}} \\
&\le Cc_{j,k}2^{-2k}\|u\|_{H^4}^{\frac12}\big\|\nabla^2\partial_{x_1}u\big\|_{L^2}^{\frac12}\|\nabla\psi\|_{L^2}^{\frac12}\|\nabla\psi\|_{H^{2,0}}^{\frac32},
\end{aligned}$$

and for $m = 4, 5, 6$,

$$\begin{aligned}
&\big\langle \mathcal{B}^m_{j,k}\big(\Delta u^1, \partial_{x_1}\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla\psi\|^{\theta}_{H^{2-\frac{1}{\theta}, -\frac{1}{2\theta}}} \big\|\nabla \partial^2_{x_1}\psi\big\|^{1-\theta}_{L^2} \big\||D_1|^{-s_1}\nabla^2 u\big\|^{1-\theta}_{H^{\frac{1}{1-\theta}}} \big\|\nabla^2 \partial_{x_1} u\big\|^{\theta}_{L^2} \|\nabla\psi\|_{H^{2,0}} \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla\psi\|^{\theta}_{H^{-s_2,0}} \|u\|^{1-\theta}_{H^{-s_1, \frac{3-2\theta}{1-\theta}}} H_1(t).
\end{aligned}$$

Using $\Delta u^2 = \partial^2_{x_1} u^2 - \partial_{x_2}\partial_{x_1} u^1$, we deduce that for $m = 1, 2, 3, 7, 8, 9$,

$$\big\langle \mathcal{B}^m_{j,k}\big(\Delta u^2, \partial_{x_2}\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle \le C c_{j,k} 2^{-2k} \|\nabla \partial_{x_1} u\|_{L^2} \big\|\nabla^2\psi\big\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}},$$

and for $m = 4, 5, 6$,

$$\big\langle \mathcal{B}^m_{j,k}\big(\Delta u^2, \partial_{x_2}\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle \le C c_{j,k} 2^{-2k} \big\|\nabla^2 \partial_{x_1} u\big\|_{L^2} \|\nabla\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}}.$$

So, we obtain

$$I^1_4 \le C c_{j,k} 2^{-2k} \big(\|u\|_{H^4} + \|\nabla\psi\|_{H^3} + \|u\|_{H^{-s_1, \frac{3-2\theta}{1-\theta}}} + \|\nabla\psi\|_{H^{-s_2,0}}\big) H_1(t).$$

Now we consider I^2_4. For $m = 1, 2, 3, 7, 8, 9$, we have

$$\begin{aligned}
\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^1, \partial_{x_1}\nabla\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle &\le C c_{j,k} 2^{-2k} \|\nabla u\|_{B^{\frac{1}{2},\frac{1}{2}}} \big\|\nabla^2 \partial_{x_1}\psi\big\|_{L^2} \|\nabla\psi\|_{H^{2,0}} \\
&\le C c_{j,k} 2^{-2k} \|u\|^{\frac{1}{2}}_{H^3} \|\nabla \partial_{x_1} u\|^{\frac{1}{2}}_{L^2} \|\nabla\psi\|^{\frac{1}{2}}_{H^2} \|\nabla\psi\|^{\frac{3}{2}}_{H^{2,0}},
\end{aligned}$$

and for $m = 4, 5, 6$,

$$\begin{aligned}
\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^1, \partial_{x_1}\nabla\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle &\le C c_{j,k} 2^{-2k} \big\|\nabla^2 u\big\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla \partial_{x_1}\psi\|_{L^2} \|\nabla\psi\|_{H^{2,0}} \\
&\le C c_{j,k} 2^{-2k} \|u\|^{\frac{1}{2}}_{H^4} \big\|\nabla^2 \partial_{x_1} u\big\|^{\frac{1}{2}}_{L^2} \|\nabla\psi\|^{\frac{1}{2}}_{L^2} \|\nabla\psi\|^{\frac{3}{2}}_{H^{2,0}}.
\end{aligned}$$

Using $\nabla u^2 = (\partial_{x_1} u^2, -\partial_{x_1} u^1)$, we deduce that for $m = 1, 2, 3, 7, 8, 9$,

$$\begin{aligned}
&\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^2, \partial_{x_2}\nabla\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle \\
&\quad \le C c_{j,k} 2^{-2k} \|u\|^{\frac{1}{4}}_{L^2} \|\nabla \partial_{x_1} u\|^{\frac{3}{4}}_{L^2} \|\nabla\psi\|^{\frac{3}{4}}_{H^4} \|\nabla\psi\|^{\frac{1}{4}}_{H^{2,0}} \|\nabla\psi\|_{H^{2,0}},
\end{aligned}$$

and for $m = 4, 5, 6$,

$$\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^2, \partial_{x_2}\nabla\psi\big), \Delta_j \Delta^h_k \Delta\psi\big\rangle \le C c_{j,k} 2^{-2k} \big\|\nabla^2 \partial_{x_1} u\big\|_{L^2} \|\nabla\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}}.$$

So, we obtain

$$I^2_4 \le C c_{j,k} 2^{-2k} \big(\|u\|_{H^4} + \|\nabla\psi\|_{H^4}\big) H_1(t).$$

For I_4^3 and I_4^5, we have

$$I_4^3 + I_4^5 \le C c_{j,k} 2^{-2k} \big(\|u\|_{H^{0,-s}} + \|u\|_{H^1} + \|\nabla\psi\|_{H^4} \big) H_1(t).$$

Following the proof of Step 3, we get

$$I_4^4 \le C c_{j,k} 2^{-2k} \big(\|u\|_{L^2} + \|\nabla\psi\|_{H^4} \big) H_1(t).$$

The proof of the lemma is finished. $\square$

Lemma A.2.

$$I_2 + I_5 \le C c_{j,k} 2^{-2k} \big\|u(t)\big\|_{H^1} H_1(t).$$

Proof. We only check the most troubled terms

$$\begin{aligned}
&\big\langle \partial_{x_\ell} \mathcal{B}_{j,k}^2 \big(u^\ell, u^i\big), \Delta_j \Delta_k^h u^i \big\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} 2^{\frac{3}{2}j} \big\|\Delta_{k'}^h u\big\|_{L^2} \|\Delta_{j'} u\|_{L^2_{x_2} L^\infty_{x_1}} \big\|\Delta_j \Delta_k^h u\big\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2k} \|u\|_{L^2} \|\nabla \partial_{x_1} u\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\big\langle \partial_{x_\ell} \mathcal{B}_{j,k}^2 \big(u^\ell, u^i\big), \Delta_j \Delta_k^h \Delta u^i \big\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} 2^{j} \big\|\Delta_{k'}^h u\big\|_{L^2_{x_1} L^\infty_{x_2}} \|\Delta_{j'} u\|_{L^2_{x_2} L^\infty_{x_1}} \big\|\Delta_j \Delta_k^h \Delta u\big\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla u\|_{L^2} \|\nabla \partial_{x_1} u\|_{L^2} \big\|\nabla^2 \partial_{x_1} u\big\|_{L^2}. \qquad \square
\end{aligned}$$

Lemma A.3.

$$I_3 + I_6 \le C c_{j,k} 2^{-2k} e(t) H_1(t).$$

Proof. Due to $\operatorname{div} u = 0$, we write

$$\begin{aligned}
&\big\langle \Delta_j \Delta_k^h \operatorname{div}(\nabla\psi \otimes \nabla\psi), \Delta_j \Delta_k^h \nabla^\alpha u \big\rangle \\
&\quad = \big\langle \Delta_j \Delta_k^h \operatorname{div}(\nabla\psi \partial_{x_1}\psi), \Delta_j \Delta_k^h \nabla^\alpha u^1 \big\rangle + \big\langle \Delta_j \Delta_k^h \partial_{x_1} (\partial_{x_1}\psi \partial_{x_2}\psi), \Delta_j \Delta_k^h \nabla^\alpha u^2 \big\rangle \\
&\qquad + \big\langle \Delta_j \Delta_k^h \partial_{x_1} (\partial_{x_2}\psi \partial_{x_2}\psi), \Delta_j \Delta_k^h \nabla^\alpha u^1 \big\rangle. \qquad \text{(A.5)}
\end{aligned}$$

Case 1. $\alpha = 0$.

It follows from (2.5) that for $m = 1, 2, 3, 5, 6, 7, 8, 9$,

$$\big\langle \operatorname{div} \mathcal{B}_{j,k}^m (\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta_k^h u^1 \big\rangle \le C c_{j,k} 2^{-2k} \|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}} \|\nabla \partial_{x_1} u\|_{L^2},$$

and by (2.5),

$$\begin{aligned}
&\left\langle \operatorname{div} \mathcal{B}^4_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta^h_k u^1 \right\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} \|\Delta_{j'} \nabla\psi\|_{L^2_{x_2} L^\infty_{x_1}} \left\|\Delta^h_{k'} \partial_{x_1}\psi\right\|_{L^2_{x_1} L^\infty_{x_2}} \left\|\Delta_j \Delta^h_k \nabla u\right\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla\psi\|_{L^2} \|\nabla\psi\|_{H^{2,0}} \|\nabla\partial_{x_1} u\|_{L^2},
\end{aligned}$$

and for $m = 1, \cdots, 6$,

$$\left\langle \partial_{x_1} \mathcal{B}^m_{j,k}(\partial_{x_2}\psi, \partial_{x_2}\psi), \Delta_j \Delta^h_k u^1 \right\rangle \le C c_{j,k} 2^{-2k} \|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}} \|\nabla\partial_{x_1} u\|_{L^2},$$

and for $m = 7$,

$$\begin{aligned}
&\left\langle \partial_{x_1} \mathcal{B}^m_{j,k}(\partial_{x_2}\psi, \partial_{x_2}\psi), \Delta_j \Delta^h_k u^1 \right\rangle \\
&\quad \le C \sum_{|j'-j''|\le 1, j',j''>j-N_0} \sum_{|k-k'|\le 4} 2^{\frac{j}{2}} 2^k \|\Delta_{j'} \partial_{x_2}\psi\|_{L^2_{x_2} L^\infty_{x_1}} \left\|\Delta_{j''} \Delta^h_{k'} \partial_{x_2}\psi\right\|_{L^2} \left\|\Delta_j \Delta^h_k u\right\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla\psi\|^{\frac{3}{4}}_{H^{\frac{s}{3}}} \|\nabla\psi\|^{\frac{5}{4}}_{H^{2,0}} \|u\|^{\frac{1}{4}}_{H^{0,-s}} \|\nabla\partial_{x_1} u\|^{\frac{3}{4}}_{L^2}.
\end{aligned}$$

The estimate for $m = 8, 9$ is similar.

Case 2. $\alpha = 2$.

For $m = 1, 2, 3, 5, 6, 7, 8, 9$, we have

$$\left\langle \operatorname{div} \mathcal{B}^m_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta^h_k \Delta u^1 \right\rangle \le C c_{j,k} 2^{-2k} \|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}} \left\|\nabla^2 \partial_{x_1} u\right\|_{L^2},$$

and by (2.5),

$$\begin{aligned}
&\left\langle \operatorname{div} \mathcal{B}^4_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta^h_k \Delta u^1 \right\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} 2^j \|\Delta_{j'} \nabla\psi\|_{L^2_{x_2} L^\infty_{x_1}} \left\|\Delta^h_{k'} \partial_{x_1}\psi\right\|_{L^2_{x_1} L^\infty_{x_2}} \left\|\Delta_j \Delta^h_k \Delta u\right\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2k} \|\nabla\psi\|^{\frac{3}{4}}_{H^{\frac{4}{3}}} \|\nabla\psi\|^{\frac{1}{4}}_{L^2} \|\nabla\psi\|_{H^{2,0}} \left\|\nabla^2 \partial_{x_1} u\right\|_{L^2},
\end{aligned}$$

and

$$\left\langle \Delta_j \Delta^h_k \partial_{x_1}(\partial_{x_2}\psi \partial_{x_2}\psi), \Delta_j \Delta^h_k \nabla^2 u^1 \right\rangle \le C c_{j,k} 2^{-2k} \|\nabla\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{2,0}} \left\|\nabla^2 \partial_{x_1} u\right\|_{L^2}. \qquad \square$$

Lemma A.4.

$$I_7 + I_8 \le C c_{j,k} 2^{-2k} e(t) H_1(t).$$

Proof. We first consider I_7. We have by (2.5) that

$$\begin{aligned}&\big\langle \mathcal{B}^1_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\big\rangle\\&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}\|u\|_{B^{\frac12,\frac12}}\big\|\Delta_{j'}\Delta^h_{k'}\nabla\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k\Delta u\big\|_{L^2}\\&\quad\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\partial_{x_1}u\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac12}_{H^{2,0}}\big\|\nabla^2\partial_{x_1}u\big\|_{L^2},\end{aligned}$$

and for $m=3,4,7$, we also have

$$\begin{aligned}&\big\langle \mathcal{B}^m_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\big\rangle\\&\quad\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{0,-s}}+\|\nabla u\|_{L^2}\big)^{\frac12}\|\nabla\partial_{x_1}u\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac12}_{L^2}\|\nabla\psi\|^{\frac12}_{H^{2,0}}\big\|\nabla^2\partial_{x_1}u\big\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}&\big\langle \mathcal{B}^2_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\big\rangle\\&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}\big\|\Delta^h_{k'}u\big\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta^h_k\Delta u\big\|_{L^2}\\&\quad\le Cc_{j,k}2^{-2k}\|u\|^{\frac14}_{L^2}\|\nabla\partial_{x_1}u\|^{\frac34}_{L^2}\|\nabla\psi\|^{\frac34}_{L^2}\|\nabla\psi\|^{\frac14}_{H^{2,0}}\big\|\nabla^2\partial_{x_1}u\big\|_{L^2},\end{aligned}$$

and for $m=5,6,8,9$,

$$\big\langle \mathcal{B}^m_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\big\rangle\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{H^2}\|\nabla\partial_{x_1}u\|^2_{L^2}.$$

Now we consider I_8. Using $\nabla u^2=(\partial_{x_1}u^2,-\partial_{x_1}u^1)$, we infer that for $m=1,2,3,5,6,7,8,9$,

$$\big\langle \mathcal{B}^m_{j,k}\big(u^i,\partial_{x_i}u^2\big),\Delta_j\Delta^h_k\Delta\psi\big\rangle\le Cc_{j,k}2^{-2k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}},$$

and by (2.5),

$$\begin{aligned}&\big\langle \mathcal{B}^4_{j,k}\big(u^i,\partial_{x_i}u^2\big),\Delta_j\Delta^h_k\Delta\psi\big\rangle\\&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}2^j\|\Delta_{j'}u\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta^h_{k'}\nabla u^2\big\|_{L^2_{x_1}L^\infty_{x_2}}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2}\\&\quad\le Cc_{j,k}2^{-2k}\|\nabla u\|_{L^2}\|\nabla\partial_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{2,0}}.\qquad\square\end{aligned}$$

Lemma A.5.

$$I_9-\epsilon_1\big\langle\Delta_j\Delta^h_k\partial_{x_2}p_2,\Delta_j\Delta^h_k\Delta\psi\big\rangle\le Cc_{j,k}2^{-2k}e(t)H_1(t).$$

Proof. Recall that

$$\operatorname{div}(\nabla\psi\partial_{x_2}\psi)+\partial_{x_2}p_2=\partial_{x_1}(\partial_{x_1}\psi\partial_{x_2}\psi)-\frac{\partial_{x_2}\partial_{x_1}\operatorname{div}}{\Delta}(\nabla\psi\partial_{x_1}\psi)$$
$$-\frac{\partial_{x_2}\partial_{x_2}\partial_{x_1}}{\Delta}(\partial_{x_1}\psi\partial_{x_2}\psi)+\frac{\partial_{x_2}\partial_{x_1}\partial_{x_1}}{\Delta}(\partial_{x_2}\psi\partial_{x_2}\psi).$$

We check the terms

$$\begin{aligned}&\big\langle\partial_{x_1}\mathcal{B}^1_{j,k}(\partial_{x_1}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k\Delta\psi\big\rangle\\&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}2^k\|\partial_{x_1}\psi\|_{L^\infty}\big\|\Delta_{j'}\Delta^h_{k'}\nabla^2\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2}\\&\quad\le Cc_{j,k}2^{-2k}\big(\|\nabla\psi\|_{H^{0,-s}}+\big\|\nabla^2\psi\big\|_{L^2}\big)^{\frac12}\big\|\nabla^3\psi\big\|^{\frac12}_{L^2}\|\nabla\psi\|^2_{H^{2,0}},\end{aligned}$$

and a similar estimate for $\langle\partial_{x_1}\mathcal{B}^3_{j,k}(\partial_{x_1}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k\Delta\psi\rangle$,

$$\begin{aligned}&\big\langle\partial_{x_1}\mathcal{B}^2_{j,k}(\partial_{x_1}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k\Delta\psi\big\rangle\\&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}2^k\big\|\partial_{x_1}\Delta^h_{k'}\psi\big\|_{L^2_{x_1}L^\infty_{x_2}}\big\|\Delta_{j'}\nabla^2\psi\big\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta^h_k\nabla\psi\big\|_{L^2}\\&\quad\le Cc_{j,k}2^{-2k}\|\nabla\psi\|^{\frac14}_{L^2}\|\nabla\psi\|^{\frac34}_{H^{2,0}}\big\||D|^{\frac73}\psi\big\|^{\frac34}_{L^2}\|\nabla\psi\|^{\frac14}_{H^{2,0}}\|\nabla\psi\|_{H^{2,0}},\end{aligned}$$

and for $m=4,\cdots,9$,

$$\big\langle\partial_{x_1}\mathcal{B}^m_{j,k}(\partial_{x_1}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k\Delta\psi\big\rangle\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{H^2}\|\nabla\psi\|^2_{H^{2,0}}.$$

Obviously, we have

$$\left\langle\Delta_j\Delta^h_k\left(\frac{\partial_{x_2}\partial_{x_1}\partial_{x_1}}{\Delta}(\partial_{x_2}\psi\partial_{x_2}\psi)\right),\Delta_j\Delta^h_k\Delta\psi\right\rangle\le Cc_{j,k}2^{-2k}\|\nabla\psi\|_{H^2}\|\nabla\psi\|^2_{H^{2,0}}.\qquad\square$$

A.2. Nonlinear estimates for the decay energy $D_2(t)$

Recall that

$$H_2(t)\ge c\big(\big\|\partial^2_{x_1}\nabla u\big\|^2_{L^2}+\big\|\partial^2_{x_1}\nabla^2u\big\|^2_{L^2}+\big\|\partial^3_{x_1}\nabla\psi\big\|^2_{L^2}\big).$$

We will follow the proofs and notations of Lemma A.1–Lemma A.5 step by step. In what follows, we take s_1 such that

$$0<s_1\le s,\qquad s_2,s_3\le s,\qquad s_2+s_3=\frac{3-6\theta_1}{2\theta_1}\quad\text{with }\theta_1=\frac{2s_1+1}{2s_1+4},$$

which is also possible by taking ϵ small enough.

Lemma A.6.

$$I_1 + I_4 \le Cc_{j,k}2^{-4k}e(t)H_2(t).$$

Proof. Since the proof is similar to that of Lemma A.1, we only present a sketch.

Step 1. Estimate of I_1^1.

By Lemma 2.1, we get for $m = 1, 3$,

$$\begin{aligned}
&\big\langle \mathcal{B}_{j,k}^m\big(\nabla u^i, \partial_{x_i}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\quad\le Cc_{j,k}2^{-4k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\psi\|_{L^2}^{\frac13}\|\nabla\psi\|_{H^{3,0}}^{\frac23}\|\nabla\psi\|_{H^{3,0}} \\
&\quad\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac14,0}} + \|u\|_{H^{-\frac14,1}}\big)^{\frac23}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac13}\|\nabla\psi\|_{L^2}^{\frac13}\|\nabla\psi\|_{H^{3,0}}^{\frac53}.
\end{aligned}$$

Here we used the inequality

$$\begin{aligned}
\|u\|_{B^{\frac12,\frac12}} &\le \sum_j 2^{\frac{j}{6}}\big\||D_1|^{-\frac14}\Delta_j u\big\|_{L^2}^{\frac23}\big\|\Delta_j\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac13} \\
&\le C\big(\|u\|_{H^{-\frac14,0}} + \|u\|_{H^{-\frac14,1}}\big)^{\frac23}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac13}.
\end{aligned} \tag{A.6}$$

For $m = 2$, we have

$$\begin{aligned}
&\big\langle \mathcal{B}_{j,k}^2\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\quad\le \sum_{|j-j'|\le 4, |k-k'|\le 4}\big\|S_{j'-1}\Delta_{k'}^h\nabla u^1\big\|_{L^2}\big\|\Delta_{j'}S_{k'-1}^h\partial_{x_1}\psi\big\|_{L^\infty}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}\|\psi\|_{B^{\frac12,\frac12}}\|\nabla\psi\|_{H^{3,0}},
\end{aligned}$$

and by (2.5),

$$\begin{aligned}
&\big\langle \mathcal{B}_{j,k}^2\big(\partial_{x_1}u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle + \big\langle \mathcal{B}_{j,k}^2\big(\partial_{x_1}u^2, \partial_{x_2}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\quad\le \sum_{|j-j'|\le 4, |k-k'|\le 4}\big\|\Delta_{k'}^h\partial_{x_1}u^2\big\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\partial_{x_2}\psi\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|u\|_{L^2}^{\frac16}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac56}\|\nabla\psi\|_{L^2}^{\frac56}\|\nabla\psi\|_{H^{3,0}}^{\frac76}.
\end{aligned} \tag{A.7}$$

By (2.5), we have

$$\begin{aligned}
&\big\langle \mathcal{B}_{j,k}^4\big(\nabla u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle \\
&\le C\sum_{|j-j'|\le 4, |k-k'|\le 4}\big\|\partial_{x_1}\Delta_{k'}^h\psi\big\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\nabla u\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_j\Delta_k^h\nabla\psi\big\|_{L^2} \\
&\le Cc_{j,k}2^{-4k}\|\nabla\psi\|^{\theta_1}_{H^{3-\frac{1}{\theta_1}, -\frac{1}{2\theta_1}}}\big\|\nabla\partial_{x_1}^3\psi\big\|^{1-\theta_1}\big\||D_1|^{-s_1}\nabla u\big\|_{L^2}^{1-\theta_1}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\theta_1}\|\nabla\psi\|_{H^{3,0}} \\
&\le Cc_{j,k}2^{-4k}\|\nabla\psi\|^{\theta_1}_{H^{-s_2,-s_3}}\big\||D_1|^{-s_1}\nabla u\big\|_{L^2}^{1-\theta_1}H_2(t),
\end{aligned}$$

and similar to (A.7), for $m=4,7$,

$$\begin{aligned}&\langle \mathcal{B}^m_{j,k}(\partial_{x_1}u^2,\partial_{x_2}\psi),\Delta_j\Delta^h_k\partial_{x_1}\psi\rangle+\langle \mathcal{B}^m_{j,k}(\partial_{x_1}u^1,\partial_{x_2}\psi),\Delta_j\Delta^h_k\partial_{x_2}\psi\rangle\\&\quad\le Cc_{j,k}2^{-4k}\|u\|^{\frac16}_{L^2}\|\nabla\partial^2_{x_1}u\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac76}_{H^{3,0}}.\end{aligned}$$

While for $m=5,6,8,9$, it is easy to see that

$$\langle \mathcal{B}^5_{j,k}(\nabla u^i,\partial_{x_i}\psi),\Delta_j\Delta^h_k\nabla\psi\rangle\le Cc_{j,k}2^{-4k}\|\psi\|_{B^{\frac12,\frac12}}H_2(t).$$

We get by (2.5) that

$$\begin{aligned}&\langle \mathcal{B}^7_{j,k}(\nabla u^1,\partial_{x_1}\psi),\Delta_j\Delta^h_k\nabla\psi\rangle\\&\quad\le C\sum_{|j'-j''|\le1,j',j''>j-N_0}\sum_{|k-k'|\le4}2^{\frac{j}{2}}\|\Delta_{j'}\Delta^h_{k'}\partial_{x_1}\nabla\psi\|_{L^2}\|\Delta_{j''}u^1\|_{L^2_{x_2}L^\infty_{x_1}}\|\Delta_j\Delta^h_k\nabla\psi\|_{L^2}\\&\quad\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac14,0}}+\|u\|_{H^{-\frac14,1}}\big)^{\frac23}\|\nabla\partial^2_{x_1}u\|^{\frac13}_{L^2}\|\nabla\psi\|^{\frac13}_{H^{\frac32}}\|\nabla\psi\|^{\frac53}_{H^{3,0}}.\end{aligned}$$

Step 2. Estimate of I_1^2 and I_1^4.

Using Bony's decomposition (2.4), (2.5) and (A.6), it is easy to show that

$$\begin{aligned}I_1^2+I_1^4&\le Cc_{j,k}2^{-2k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\psi\|^{\frac13}_{L^2}\|\nabla\psi\|^{\frac53}_{H^{3,0}}\\&\le Cc_{j,k}2^{-2k}\big(\|u\|_{H^{-\frac14,0}}+\|u\|_{H^{-\frac14,1}}\big)^{\frac23}\|\nabla\psi\|^{\frac13}_{L^2}H_2(t).\end{aligned}$$

Step 3. Estimate of I_1^3.

Using the fact $\nabla u^2=(\partial_{x_1}u^2,-\partial_{x_1}u^1)$, it is easy to get for $m=5,6,8,9$,

$$\begin{aligned}&\langle \mathcal{B}^m_{j,k}(u^2,\partial_{x_2}\nabla\psi),\Delta_j\Delta^h_k\nabla\psi\rangle+\langle \widetilde{\mathcal{B}}^m_{j,k}(u^2,\partial_{x_2}\nabla\psi),\Delta_j\Delta^h_k\nabla\psi\rangle\\&\quad\le Cc_{j,k}2^{-4k}\|\psi\|_{B^{\frac12,\frac12}}\|\nabla\partial^2_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{3,0}},\end{aligned}$$

and for $m=4,7$,

$$\begin{aligned}&\langle \mathcal{B}^m_{j,k}(u^2,\partial_{x_2}\nabla\psi),\Delta_j\Delta^h_k\nabla\psi\rangle+\langle \widetilde{\mathcal{B}}^m_{j,k}(u^2,\partial_{x_2}\nabla\psi),\Delta_j\Delta^h_k\nabla\psi\rangle\\&\quad\le Cc_{j,k}2^{-4k}\|u\|^{\frac16}_{L^2}\|\nabla\partial^2_{x_1}u\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac76}_{H^{3,0}}.\end{aligned}$$

For the commutator terms, we only show the following term

$$\begin{aligned}&\sum_{|j-j'|\le4,k'\ge k-N_0}\big\|\big[\Delta_j\Delta^h_k,S_{j'-1}\Delta^h_{k'}u^2\big]\big(\Delta_{j'}S^h_{k'-1}\partial_{x_2}\nabla\psi\big)\big\|_{L^2}\\&\quad\le C\sum_{|j-j'|\le4,k'\ge k-N_0}2^{-j}\|\nabla S_{j'-1}\Delta^h_{k'}u^2\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}S^h_{k'-1}\partial_{x_2}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}}\\&\quad\le C\sum_{|j-j'|\le4,k'\ge k-N_0}2^{k'}\|\Delta^h_{k'}u^2\|_{L^2_{x_1}L^\infty_{x_2}}\|\Delta_{j'}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}},\end{aligned}$$

from which, it is easy to get

$$\sum_{|j-j'|\le 4,|k-k'|\le 4} \big\langle \big[\Delta_j\Delta_k^h, S_{j'-1}\Delta_{k'}^h u^2\big]\big(\Delta_{j'}S_{k'-1}^h\partial_{x_2}\nabla\psi\big), \Delta_j\Delta_k^h\nabla\psi\big\rangle$$
$$\le Cc_{j,k}2^{-4k}\|u\|_{L^2}^{\frac{1}{6}}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac{5}{6}}\|\nabla\psi\|_{L^2}^{\frac{5}{6}}\|\nabla\psi\|_{H^{3,0}}^{\frac{7}{6}}.$$

Step 4. Estimate of I_4.

First of all, we consider I_4^1. Following the proof of Step 1, we deduce that for $m = 1, 7$,

$$\begin{aligned}\big\langle \mathcal{B}_{j,k}^1\big(\Delta u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle &\le Cc_{j,k}2^{-4k}\|\Delta u\|_{B^{\frac{1}{2},\frac{1}{2}}}\big\|\nabla\partial_{x_1}^2\psi\big\|_{L^2}\|\nabla\psi\|_{H^{2,0}}\\ &\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac{1}{4},0}} + \|u\|_{H^{-\frac{1}{4},3}}\big)^{\frac{2}{3}}\|\nabla\psi\|_{L^2}^{\frac{1}{3}}H_2(t),\end{aligned}$$

and for $m = 1, 4, 7$,

$$\big\langle \mathcal{B}_{j,k}^m\big(\Delta u^2, \partial_{x_2}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle \le Cc_{j,k}2^{-4k}\|u\|_{H^1}^{\frac{1}{3}}\|\nabla\psi\|_{H^3}^{\frac{2}{3}}\big\|\nabla^2\partial_{x_1}^2 u\big\|_{L^2}^{\frac{2}{3}}\|\nabla\psi\|_{H^{3,0}}^{\frac{4}{3}}.$$

For $m = 2, 3, 8, 9$, we have

$$\big\langle \mathcal{B}_{j,k}^m\big(\Delta u^i, \partial_{x_i}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^3}\big\|\nabla\partial_{x_1}^2 u\big\|_{H^1}\|\nabla\psi\|_{H^{3,0}},$$

and for $m = 5, 6$,

$$\big\langle \mathcal{B}_{j,k}^m\big(\Delta u^2, \partial_{x_2}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^3}\big\|\nabla^2\partial_{x_1}^2 u\big\|_{L^2}\|\nabla\psi\|_{H^{3,0}}.$$

For $m = 4, 5, 6$, we have

$$\big\langle \mathcal{B}_{j,k}^m\big(\Delta u^1, \partial_{x_1}\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^{-s_2,-s_3}}^{\theta_1}\|u\|_{H^{-s_1,\frac{3-2\theta_1}{1-\theta_1}}}^{1-\theta_1}H_2(t).$$

Now we consider I_4^2. For the term $\langle \mathcal{B}_{j,k}^m(\nabla u^i, \partial_{x_i}\nabla\psi), \Delta_j\Delta_k^h\Delta\psi\rangle$ with $m = 4, \cdots, 9$, the estimate is similar to that of I_4^1. For $m = 1, 2, 3$, it holds that

$$\begin{aligned}\big\langle \mathcal{B}_{j,k}^m\big(\nabla u^1, \partial_{x_1}\nabla\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle &\le Cc_{j,k}2^{-4k}\|\nabla u\|_{B^{\frac{1}{2},\frac{1}{2}}}\big\|\nabla^2\partial_{x_1}^2\psi\big\|_{L^2}\|\nabla\psi\|_{H^{3,0}}\\ &\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac{1}{4},0}} + \|u\|_{H^{-\frac{1}{4},2}}\big)^{\frac{2}{3}}\|\nabla\psi\|_{H^3}^{\frac{1}{3}}H_2(t),\\ \big\langle \mathcal{B}_{j,k}^m\big(\nabla u^2, \partial_{x_2}\nabla\psi\big), \Delta_j\Delta_k^h\Delta\psi\big\rangle &\le Cc_{j,k}2^{-4k}\|u\|_{H^2}^{\frac{2}{3}}\big\|\nabla\partial_{x_1}^2 u\big\|_{L^2}^{\frac{1}{3}}\|\nabla\psi\|_{H^3}^{\frac{1}{3}}\|\nabla\psi\|_{H^{3,0}}^{\frac{5}{3}}.\end{aligned}$$

The estimate of I_4^4 is the same as that of I_4^2. For I_4^3 and I_4^5, we have

$$I_4^3 + I_4^5 \le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac{1}{4},0}} + \|u\|_{H^{-\frac{1}{4},2}}\big)^{\frac{2}{3}}\|\nabla\psi\|_{H^3}^{\frac{1}{3}}H_2(t). \qquad \square$$

Lemma A.7.

$$I_2 + I_5 \le C c_{j,k} 2^{-4k} e(t) H_2(t).$$

Proof. We only show the most troubled terms

$$\begin{aligned}
&\big\langle \mathcal{B}^2_{j,k}\partial_{x_\ell}\big(u^\ell, u^i\big), \Delta_j \Delta^h_k u^i\big\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} 2^j \big\|\Delta^h_{k'} u\big\|_{L^2_{x_1} L^\infty_{x_2}} \|\Delta_{j'} u\|_{L^2_{x_2} L^\infty_{x_1}} \big\|\Delta_j \Delta^h_k u\big\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-4k} \|u\|_{L^2} \big\|\nabla \partial^2_{x_1} u\big\|^2_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
&\big\langle \partial_{x_\ell}\mathcal{B}^2_{j,k}\big(u^\ell, u^i\big), \Delta_j \Delta^h_k \Delta u^i\big\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} 2^j \big\|\Delta^h_{k'} u\big\|_{L^2_{x_1} L^\infty_{x_2}} \|\Delta_{j'} u\|_{L^2_{x_2} L^\infty_{x_1}} \big\|\Delta_j \Delta^h_k \Delta u\big\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-4k} \|\nabla u\|_{L^2} \big\|\nabla \partial^2_{x_1} u\big\|_{L^2} \big\|\nabla^2 \partial^2_{x_1} u\big\|_{L^2}. \qquad \square
\end{aligned}$$

Lemma A.8.

$$I_3 + I_6 \le C c_{j,k} 2^{-4k} e(t) H_2(t).$$

Proof. Using (A.5), we check two cases.

Case 1. $\alpha = 0$.

For $m = 1, 2, 3, 5, 6, 7, 8, 9$, we have

$$\big\langle \operatorname{div} \mathcal{B}^m_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta^h_k u^1\big\rangle \le C c_{j,k} 2^{-4k} \|\psi\|_{B^{\frac12,\frac12}} \|\nabla\psi\|_{H^{3,0}} \big\|\nabla \partial^2_{x_1} u\big\|_{L^2},$$

and by (2.5),

$$\begin{aligned}
&\big\langle \operatorname{div} \mathcal{B}^4_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j \Delta^h_k u^1\big\rangle \\
&\quad \le C \sum_{|j-j'|\le 4, |k-k'|\le 4} \|\Delta_{j'} \nabla\psi\|_{L^2_{x_2} L^\infty_{x_1}} \big\|\Delta^h_{k'} \partial_{x_1}\psi\big\|_{L^2_{x_1} L^\infty_{x_2}} \big\|\Delta_j \Delta^h_k \nabla u\big\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-4k} \|\nabla\psi\|_{L^2} \|\nabla\psi\|_{H^{3,0}} \big\|\nabla \partial^2_{x_1} u\big\|_{L^2}.
\end{aligned}$$

For $m = 1, \cdots, 6$, we have

$$\big\langle \partial_{x_1}\mathcal{B}^m_{j,k}(\partial_{x_2}\psi, \partial_{x_2}\psi), \Delta_j \Delta^h_k u^1\big\rangle \le C c_{j,k} 2^{-4k} \|\psi\|_{B^{\frac12,\frac12}} \|\nabla\psi\|_{H^{3,0}} \big\|\nabla \partial^2_{x_1} u\big\|_{L^2},$$

and by (2.5), we get

$$\begin{aligned}
&\left\langle \partial_{x_1}\mathcal{B}^7_{j,k}(\partial_{x_2}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k u^1\right\rangle \\
&\quad\le C\sum_{|j'-j''|\le 1,j',j''>j-N_0}\ \sum_{|k-k'|\le 4} 2^{\frac{j}{2}}2^k\|\Delta_{j'}\partial_{x_2}\psi\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta_{j''}\Delta^h_{k'}\partial_{x_2}\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k u\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|\nabla\psi\|^{\frac{2}{3}}_{H^{\frac{3}{4}}}\|\nabla\psi\|^{\frac{4}{3}}_{H^{3,0}}\|u\|^{\frac{1}{3}}_{L^2}\big\|\nabla\partial^2_{x_1}u\big\|^{\frac{2}{3}}_{L^2}.
\end{aligned}$$

The estimate for $m=8,9$ is similar.

Case 2. $\alpha=2$.

For $m=1,2,3,5,6,7,8,9$, we have

$$\left\langle \operatorname{div}\mathcal{B}^m_{j,k}(\nabla\psi,\partial_{x_1}\psi),\Delta_j\Delta^h_k\Delta u^1\right\rangle\le Cc_{j,k}2^{-4k}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\psi\|_{H^{3,0}}\big\|\nabla^2\partial^2_{x_1}u\big\|_{L^2},$$

and by (2.5),

$$\begin{aligned}
&\left\langle \operatorname{div}\mathcal{B}^4_{j,k}(\nabla\psi,\partial_{x_1}\psi),\Delta_j\Delta^h_k\Delta u^1\right\rangle \\
&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4} 2^j\|\Delta_{j'}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}}\big\|\Delta^h_{k'}\partial_{x_1}\psi\big\|_{L^2_{x_1}L^\infty_{x_2}}\big\|\Delta_j\Delta^h_k\Delta u\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|\nabla\psi\|^{\frac{5}{6}}_{H^{\frac{6}{5}}}\|\nabla\psi\|^{\frac{1}{6}}_{L^2}\|\nabla\psi\|_{H^{3,0}}\big\|\nabla^2\partial^2_{x_1}u\big\|_{L^2}.
\end{aligned}$$

Obviously, we have

$$\left\langle \Delta_j\Delta^h_k\partial_{x_1}(\partial_{x_2}\psi\partial_{x_2}\psi),\Delta_j\Delta^h_k\nabla^2u^1\right\rangle\le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\psi\|_{H^{3,0}}\big\|\nabla^2\partial^2_{x_1}u\big\|_{L^2}. \qquad \square$$

Lemma A.9.

$$I_7+I_8\le Cc_{j,k}2^{-4k}e(t)H_2(t).$$

Proof. We first consider I_7. It follows from (2.5) that

$$\begin{aligned}
&\left\langle \mathcal{B}^1_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\right\rangle \\
&\quad\le C\sum_{|j-j'|\le 4,|k-k'|\le 4}\|u\|_{B^{\frac{1}{2},\frac{1}{2}}}\big\|\Delta_{j'}\Delta^h_{k'}\nabla\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k\Delta u\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac{1}{4},0}}+\|u\|_{H^{-\frac{1}{4},1}}\big)^{\frac{2}{3}}\|\nabla\psi\|^{\frac{1}{3}}_{L^2}H_2(t),
\end{aligned}$$

and for $m=3,4,7$, we have

$$\left\langle \mathcal{B}^m_{j,k}\big(u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta u^2\right\rangle\le Cc_{j,k}2^{-4k}\big(\|u\|_{H^{-\frac{1}{4},0}}+\|u\|_{H^{-\frac{1}{4},1}}\big)^{\frac{2}{3}}\|\nabla\psi\|^{\frac{1}{2}}_{L^2}H_2(t),$$

and

$$\begin{aligned}
&\langle \mathcal{B}^2_{j,k}(u^i, \partial_{x_i}\psi), \Delta_j\Delta^h_k\Delta u^2\rangle \\
&\quad\le C\sum_{|j-j'|\le 4, |k-k'|\le 4} \|\Delta^h_{k'}u\|_{L^2_{x_1}L^\infty_{x_2}} \|\Delta_{j'}\nabla\psi\|_{L^2_{x_2}L^\infty_{x_1}} \|\Delta_j\Delta^h_k\Delta u\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|u\|^{\frac16}_{L^2}\|\nabla\partial^2_{x_1}u\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac16}_{H^{3,0}}\|\nabla^2\partial^2_{x_1}u\|_{L^2},
\end{aligned}$$

and for $m = 5, 6, 8, 9$,

$$\langle \mathcal{B}^m_{j,k}(u^i, \partial_{x_i}\psi), \Delta_j\Delta^h_k\Delta u^2\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^2}\|\nabla\partial^2_{x_1}u\|^2_{L^2}.$$

Next we consider I_8. Using $\nabla u^2 = (\partial_{x_1}u^2, -\partial_{x_1}u^1)$, we infer that for $m = 1, 3, 5, 6, 7, 8, 9$,

$$\langle \mathcal{B}^m_{j,k}(u^i, \partial_{x_i}u^2), \Delta_j\Delta^h_k\Delta\psi\rangle \le Cc_{j,k}2^{-4k}\|u\|_{B^{\frac12,\frac12}}\|\nabla\partial^2_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{3,0}},$$

and for $m = 2, 4$, by (2.5),

$$\begin{aligned}
&\langle \mathcal{B}^m_{j,k}(u^i, \partial_{x_i}u^2), \Delta_j\Delta^h_k\Delta\psi\rangle \\
&\quad\le C\sum_{|j-j'|\le 4, |k-k'|\le 4} 2^j2^k\|\Delta_{j'}u\|_{L^2_{x_2}L^\infty_{x_1}} \|\Delta^h_{k'}u\|_{L^2_{x_1}L^\infty_{x_2}} \|\Delta_j\Delta^h_k\nabla\psi\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|\nabla u\|_{L^2}\|\nabla\partial^2_{x_1}u\|_{L^2}\|\nabla\psi\|_{H^{3,0}}. \qquad \square
\end{aligned}$$

Lemma A.10.

$$I_9 - \epsilon_1\langle \Delta_j\Delta^h_k\partial_{x_2}p_2, \Delta_j\Delta^h_k\Delta\psi\rangle \le Cc_{j,k}2^{-4k}e(t)H_2(t).$$

Proof. It follows from (2.5) that

$$\begin{aligned}
&\langle \partial_{x_1}\mathcal{B}^1_{j,k}(\partial_{x_1}\psi, \partial_{x_2}\psi), \Delta_j\Delta^h_k\Delta\psi\rangle \\
&\quad\le C\sum_{|j-j'|\le 4, |k-k'|\le 4} 2^k\|\partial_{x_1}\psi\|_{L^\infty}\|\Delta_{j'}\Delta^h_{k'}\nabla^2\psi\|_{L^2}\|\Delta_j\Delta^h_k\nabla\psi\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\left(\|\nabla\psi\|_{H^{-s,0}} + \|\nabla^2\psi\|_{L^2}\right)^{\frac23}\|\nabla^4\psi\|^{\frac13}_{L^2}\|\nabla\psi\|^2_{H^{3,0}},
\end{aligned}$$

and

$$\begin{aligned}
&\langle \partial_{x_1}\mathcal{B}^2_{j,k}(\partial_{x_1}\psi, \partial_{x_2}\psi), \Delta_j\Delta^h_k\Delta\psi\rangle \\
&\quad\le C\sum_{|j-j'|\le 4, |k-k'|\le 4} 2^k\|\partial_{x_1}\Delta^h_{k'}\psi\|_{L^2_{x_1}L^\infty_{x_2}} \|\Delta_{j'}\nabla^2\psi\|_{L^2_{x_2}L^\infty_{x_1}} \|\Delta_j\Delta^h_k\nabla\psi\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-4k}\|\nabla\psi\|^{\frac16}_{L^2}\|\nabla\psi\|^{\frac56}_{H^{3,0}}\||D|^{\frac{11}{5}}\psi\|^{\frac56}_{L^2}\|\nabla\psi\|^{\frac16}_{H^{3,0}}\|\nabla\psi\|_{H^{3,0}}.
\end{aligned}$$

For $m = 4, \cdots, 9$, we have

$$\langle \partial_{x_1}\mathcal{B}^m_{j,k}(\partial_{x_1}\psi,\partial_{x_2}\psi),\Delta_j\Delta^h_k\Delta\psi\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^2}\|\nabla\psi\|^2_{H^{3,0}}.$$

It is obvious that

$$\left\langle \Delta_j\Delta^h_k\left(\frac{\partial_{x_2}\partial_{x_1}\partial_{x_1}}{\Delta}(\partial_{x_2}\psi\partial_{x_2}\psi)\right),\Delta_j\Delta^h_k\Delta\psi\right\rangle \le Cc_{j,k}2^{-4k}\|\nabla\psi\|_{H^2}\|\nabla\psi\|^2_{H^{3,0}}. \qquad \square$$

A.3. Nonlinear estimates for the high order energy

Recall that

$$\begin{aligned} g_1(t) &= \|\psi\|_{B^{\frac12,\frac32}} + \|\psi\|_{B^{\frac12,\frac12}} + \|u\|_{B^{\frac12,\frac12}} + \|u\|_{L^2}, \qquad g_2(t) = \|u\|_{B^{\frac32,\frac12}}, \\ g_3(t) &= \|u\|_{B^{\frac12,\frac32}} + \|u\|_{B^{\frac12,\frac12}} + \|\psi\|_{B^{\frac32,\frac12}} + \|\psi\|^2_{B^{\frac12,1}}. \end{aligned}$$

We will use the notations of Lemma A.1–Lemma A.5 and always assume that $s \in [0,\frac12)$ and $s_1 > -\frac12$.

Lemma A.11.

$$\begin{aligned} I_1 \le Cc_{j,k}2^{-2s_1j}2^{2ks}\big(&g_1(t)\|\nabla u\|_{H^{-s,s_1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} + g_2(t)\|\nabla\psi\|^2_{H^{-s,s_1+1}} \\ &+ g_3(t)\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla\psi\|_{H^{-s,s_1}}\big), \end{aligned}$$

and

$$\begin{aligned} I_4 \le Cc_{j,k}2^{-2s_1j}2^{2ks}\big(&g_1(t)\|\nabla u\|_{H^{-s,s_1+1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} + g_2(t)\|\nabla\psi\|^2_{H^{-s,s_1+1}} \\ &+ g_3(t)\big(\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} + \|\nabla u\|_{H^{-s,s_1+1}}\big)\|\nabla\psi\|_{H^{-s,s_1+1}}\big). \end{aligned}$$

Proof. We write I_4 as

$$\begin{aligned} I_4 = &-\langle\Delta_j\Delta^h_k\big(\Delta u^i,\partial_{x_i}\psi\big),\Delta_j\Delta^h_k\Delta\psi\rangle - 2\langle\Delta_j\Delta^h_k(\nabla u\cdot\nabla\nabla\psi),\Delta_j\Delta^h_k\Delta\psi\rangle \\ &- \langle\big[\Delta_j\Delta^h_k,u\big]\nabla\Delta\psi,\Delta_j\Delta^h_k\Delta\psi\rangle \triangleq I^1_4 + I^2_4 + I^3_4. \end{aligned}$$

Step 1. Estimate of I^1_4.
It is easy to infer from (2.5) that

$$\begin{aligned} &\langle\mathcal{B}^1_{j,k}(\Delta u\cdot\nabla\psi),\Delta_j\Delta^h_k\Delta\psi\rangle + \langle\mathcal{B}^2_{j,k}\big(\Delta u^1,\partial_{x_1}\psi\big),\Delta_j\Delta^h_k\Delta\psi\rangle \\ &\quad\le Cc_{j,k}2^{-2s_1j}2^{2ks}\|u\|_{B^{\frac12,\frac32}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla\psi\|_{H^{-s,s_1+1}}, \end{aligned}$$

and using $\Delta u^2 = \partial^2_{x_1}u^2 - \partial_{x_2}\partial_{x_1}u^1$,

$$\langle\mathcal{B}^2_{j,k}\big(\Delta u^2,\partial_{x_2}\psi\big),\Delta_j\Delta^h_k\Delta\psi\rangle \le Cc_{j,k}2^{-2s_1j}2^{2ks}\|u\|_{B^{\frac32,\frac12}}\|\nabla\psi\|^2_{H^{-s,s_1+1}}.$$

By (2.5) and $s < \frac{1}{2}$, we get

$$\begin{aligned}
&\big\langle \mathcal{B}^3_{j,k}\big(\Delta u^i, \partial_{x_i}\psi\big), \Delta_j\Delta^h_k\Delta\psi\big\rangle \\
&\quad\le C\sum_{|j-j'|\le 4}\ \sum_{|k-k'|\le 1, k',k''>k-N_0} 2^{\frac{k}{2}}2^{-(\frac{1}{2}-s)k''}2^{\frac{k'}{2}}\big\|\Delta^h_{k'}\nabla u\big\|_{L^2_{x_1}L^\infty_{x_2}} \\
&\qquad\times 2^{-k''s}\big\|\Delta_{j'}\Delta^h_{k''}\nabla\partial_{x_1}\psi\big\|_{L^2}\big\|\Delta_j\Delta^h_k\Delta\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|u\|_{B^{\frac{1}{2},\frac{3}{2}}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla\psi\|_{H^{-s,s_1+1}}.
\end{aligned}$$

For $m = 4, 5$, we have

$$\begin{aligned}
&\big\langle \mathcal{B}^m_{j,k}\big(\Delta u^1, \partial_{x_1}\psi\big), \Delta_j\Delta^h_k\Delta\psi\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|\psi\|_{B^{\frac{3}{2},\frac{1}{2}}}\|\nabla u\|_{H^{-s,s_1+1}}\|\nabla\psi\|_{H^{-s,s_1+1}}, \\
&\big\langle \mathcal{B}^m_{j,k}\big(\Delta u^2, \partial_{x_2}\psi\big), \Delta_j\Delta^h_k\Delta\psi\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|\psi\|_{B^{\frac{1}{2},\frac{3}{2}}}\|\nabla u\|_{H^{-s,s_1+1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}},
\end{aligned}$$

and by (2.5),

$$\begin{aligned}
&\big\langle \mathcal{B}^6_{j,k}\big(\Delta u^i, \partial_{x_i}\psi\big), \Delta_j\Delta^h_k\Delta\psi\big\rangle \\
&\quad\le C\sum_{|j-j'|\le 4}\ \sum_{|k-k'|\le 1, k',k''>k-N_0} 2^{\frac{k}{2}}2^{-(\frac{1}{2}-s)k''}2^{-k's}\big\|\Delta_{j'}\Delta^h_{k'}\nabla^2 u\big\|_{L^2} \\
&\qquad\times 2^{\frac{k''}{2}}\big\|\Delta^h_{k''}\partial_{x_1}\psi\big\|_{L^2_{x_1}L^\infty_{x_2}}\big\|\Delta_j\Delta^h_k\Delta\psi\big\|_{L^2} \\
&\quad\le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|\psi\|_{B^{\frac{3}{2},\frac{1}{2}}}\|\nabla u\|_{H^{-s,s_1+1}}\|\nabla\psi\|_{H^{-s,s_1+1}}.
\end{aligned}$$

Noting that $s_1 > -\frac{1}{2}$, we can get the similar estimates as in $m = 4$ for $m = 7, 8, 9$.

Step 2. Estimate of I^2_4 and I^3_4.

The estimates are almost the same as those of Step 1. We have

$$\begin{aligned}
I^2_4 + I^3_4 \le{}& Cc_{j,k}2^{-2s_1 j}2^{2ks}\big(g_1(t)\|\nabla u\|_{H^{-s,s_1+1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} + g_2(t)\|\nabla\psi\|^2_{H^{-s,s_1+1}} \\
&+ g_3(t)\big(\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} + \|\nabla u\|_{H^{-s,s_1+1}}\big)\|\nabla\psi\|_{H^{-s,s_1+1}}\big).
\end{aligned}$$

Step 3. Estimate of I_1.

We write I_1 as

$$I_1 = -\big\langle \Delta_j\Delta^h_k(\nabla u\cdot\nabla\psi), \Delta_j\Delta^h_k\nabla\psi\big\rangle - \big\langle\big[\Delta_j\Delta^h_k, u\big]\nabla\nabla\psi, \Delta_j\Delta^h_k\nabla\psi\big\rangle \triangleq I^1_1 + I^2_1.$$

For $m = 1, 2, 3, 7, 8, 9$, we have

$$\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^i, \partial_{x_i}\psi\big), \Delta_j\Delta^h_k\nabla\psi\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|u\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla\psi\|_{H^{-s,s_1}}.$$

While for $m = 4, 5, 6$, we have

$$\big\langle \mathcal{B}^m_{j,k}\big(\nabla u^i, \partial_{x_i}\psi\big), \Delta_j\Delta^h_k\nabla\psi\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2ks}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla u\|_{H^{-s,s_1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}.$$

This gives the estimate of I_1^1. The estimate of I_1^2 is similar. The proof of the lemma is finished. $\square$

In a similar way, the following lemmas can be proved by using Bony's decomposition (2.4) and Lemma 2.1. So, we just present a sketch.

Lemma A.12.

$$I_2 + I_5 \le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|u\|_{L^2}\big(\|\nabla u\|^2_{H^{-s,s_1}} + \|\nabla u\|^2_{H^{-s,s_1+1}}\big).$$

Proof. Due to $s < \frac{1}{2}$ and $s_1 > -\frac{1}{2}$, it is easy to show that

$$\begin{aligned} I_2 &\le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|u\|_{L^2}\|\nabla u\|^2_{H^{-s,s_1}}, \\ I_5 &\le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|u\|_{L^2}\|\nabla u\|^2_{H^{-s,s_1+1}}. \qquad \square \end{aligned}$$

Lemma A.13.

$$\begin{aligned} I_3 \le Cc_{j,k}2^{-2s_1 j}2^{2sk}\big(&g_1(t)\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}} + g_3(t)\|\nabla\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}} \\ &+ \|\psi\|_{B^{\frac{1}{2},1}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|u\|_{H^{-s,s_1+\frac{1}{2}}}\big), \end{aligned}$$

and

$$\begin{aligned} I_6 \le Cc_{j,k}2^{-2s_1 j}2^{2sk}\big(&g_1(t)\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1+1}} \\ &+ g_3(t)\|\nabla\psi\|_{H^{-s,s_1+1}}\|\nabla u\|_{H^{-s,s_1+1}}\big). \end{aligned}$$

Proof. For $m = 1, 2, 3, 5, 6, 7, 8, 9$, we have

$$\big\langle \operatorname{div}\mathcal{B}^m_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j\Delta^h_k u^1\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}},$$

and

$$\big\langle \operatorname{div}\mathcal{B}^4_{j,k}(\nabla\psi, \partial_{x_1}\psi), \Delta_j\Delta^h_k u^1\big\rangle \le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|\psi\|_{B^{\frac{3}{2},\frac{1}{2}}}\|\nabla\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}},$$

and for $m = 1, \cdots, 6$,

$$\begin{aligned} \big\langle \partial_{x_1}\mathcal{B}^m_{j,k}(\partial_{x_2}\psi, \partial_{x_2}\psi), \Delta_j\Delta^h_k u^1\big\rangle &\le Cc_{j,k}2^{-2s_1 j}2^{2sk}\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}} \\ &\quad + Cc_{j,k}2^{-2s_1 j}2^{2sk}\|\psi\|_{B^{\frac{3}{2},\frac{1}{2}}}\|\nabla\psi\|_{H^{-s,s_1}}\|\nabla u\|_{H^{-s,s_1}}, \end{aligned}$$

and by (2.5),

$$\begin{aligned}
&\left\langle \partial_{x_1}\mathcal{B}^m_{j,k}(\partial_{x_2}\psi, \partial_{x_2}\psi), \Delta_j\Delta^h_k u^1\right\rangle \\
&\quad \le C \sum_{|j'-j''|\le 1, j', j''>j-N_0} \sum_{|k-k'|\le 4} 2^{\frac{j}{2}} 2^k \|\Delta_{j'}\partial_{x_2}\psi\|_{L^2_{x_2}L^\infty_{x_1}} \left\|\Delta_{j''}\Delta^h_{k'}\partial_{x_2}\psi\right\|_{L^2} \left\|\Delta_j\Delta^h_k u\right\|_{L^2} \\
&\quad \le C c_{j,k} 2^{-2s_1 j} 2^{2sk} \|\psi\|_{B^{\frac{1}{2},1}} \|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} \|u\|_{H^{-s,s_1+\frac{1}{2}}}.
\end{aligned}$$

The estimate for $m = 8, 9$ is similar. This gives the estimate of I_3. The estimate of I_6 is similar. Here we omit the details. □

Lemma A.14.

$$\begin{aligned}
I_7 + I_8 \le C c_{j,k} 2^{-2s_1 j} 2^{2sk} \big(& g_1(t) \|\nabla u\|_{H^{-s,s_1}} \|\nabla u\|_{H^{-s,s_1+1}} \\
& + g_3(t) \|\nabla\psi\|_{H^{-s,s_1}} \|\nabla u\|_{H^{-s,s_1+1}} \big).
\end{aligned}$$

Proof. It is easy to show that

$$\begin{aligned}
I_7 \le{}& C c_{j,k} 2^{-2s_1 j} 2^{2sk} \big(\|\psi\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla u\|_{H^{-s,s_1}} \|\nabla u\|_{H^{-s,s_1+1}} \\
& + \|u\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{-s,s_1}} \|\nabla u\|_{H^{-s,s_1+1}} \big), \\
I_8 \le{}& C c_{j,k} 2^{-2s_1 j} 2^{2sk} \|u\|_{B^{\frac{1}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{-s,s_1}} \|\nabla u\|_{H^{-s,s_1+1}}. \qquad \square
\end{aligned}$$

Lemma A.15.

$$\begin{aligned}
& I_9 - \epsilon_1 \left\langle \Delta_j\Delta^h_k \partial_{x_2} p_2, \Delta_j\Delta^h_k \Delta\psi \right\rangle \\
&\quad \le C c_{j,k} 2^{-2s_1 j} 2^{2sk} \big(g_1(t) \|\nabla\partial_{x_1}\psi\|^2_{H^{-s,s_1}} + g_3(t) \|\nabla\psi\|_{H^{-s,s_1+1}} \|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} \big).
\end{aligned}$$

Proof. Using (3.5), it is easy to deduce that

$$\begin{aligned}
& I_9 - \epsilon_1 \left\langle \Delta_j\Delta^h_k \partial_{x_2} p_2, \Delta_j\Delta^h_k \Delta\psi \right\rangle \\
&\quad \le C c_{j,k} 2^{-2s_1 j} 2^{2sk} \big(\|\psi\|_{B^{\frac{1}{2},\frac{3}{2}}} \|\nabla\partial_{x_1}\psi\|^2_{H^{-s,s_1}} \\
&\qquad + \|\psi\|_{B^{\frac{3}{2},\frac{1}{2}}} \|\nabla\psi\|_{H^{-s,s_1+1}} \|\nabla\partial_{x_1}\psi\|_{H^{-s,s_1}} \big). \qquad \square
\end{aligned}$$

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