Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 266 (2019) 630–652

Journal of Differential Equations

www.elsevier.com/locate/jde

The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation

Wanrong Yang [a], Quansen Jiu [b], Jiahong Wu [c,*]

[a] School of Mathematics and Information Sciences, North Minzu University, Yinchuan, Ningxia 750021, PR China
[b] School of Mathematics, Capital Normal University, Beijing 100037, PR China
[c] Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences, Stillwater, OK 74078, USA

Received 31 October 2017; revised 15 June 2018
Available online 20 July 2018

Abstract

The magnetohydrodynamic (MHD) equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering. The fundamental problem of whether or not classical solutions of the 3D MHD equations can develop finite-time singularities remains an outstanding open problem. Mathematically this problem is supercritical in the sense that the 3D MHD equations do not have enough dissipation. If we replace the standard velocity dissipation Δu and the magnetic diffusion Δb by $-(-\Delta)^\alpha u$ and $-(-\Delta)^\beta b$, respectively, the resulting equations with $\alpha \ge \frac{5}{4}$ and $\alpha + \beta \ge \frac{5}{2}$ then always have global classical solutions. An immediate issue is whether or not the hyperdissipation can be further reduced. This paper shows that the global regularity still holds even if there is only directional velocity dissipation and horizontal magnetic diffusion $-(-\Delta_h)^{\frac{5}{4}} b$, where $\Delta_h = \partial_1^2 + \partial_2^2$.

MSC: 35Q35; 76D03

Keywords: 3D magnetohydrodynamic equations; Fractional partial dissipation; Global regularity

* Corresponding author.
E-mail addresses: yangwanrong1618@163.com (W. Yang), jiuqs@mail.cnu.edu.cn (Q. Jiu), jiahong.wu@okstate.edu (J. Wu).

https://doi.org/10.1016/j.jde.2018.07.046
0022-0396/

1. Introduction

Whether or not smooth solutions of the 3D incompressible Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{cases} \tag{1.1}$$

can develop finite-time singularities remains an outstanding open problem. This Millennium prize problem is supercritical in the sense that the standard Laplacian dissipation in (1.1) may not provide sufficient regularization. If we consider the generalized Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu(-\Delta)^\alpha \mathbf{u}, \quad x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{cases} \tag{1.2}$$

with $\alpha \geq \frac{5}{4}$, then any smooth initial data $\mathbf{u}_0$ with finite energy leads to a unique global-in-time solution. Here the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

This result can be found in several works (see, e.g., [36,30,44]). A natural and difficult issue is whether or not we can establish the global existence and regularity for any $\alpha < \frac{5}{4}$. T. Tao [40] was able to improve the classical result by a logarithm and obtained the global regularity for

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu \frac{(-\Delta)^{\frac{5}{4}}}{\log^{\frac{1}{2}}(2-\Delta)} \mathbf{u}, \quad x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1.3}$$

A further improvement was obtained by [2], which removed the $\frac{1}{2}$-power of the logarithm. These results indicate that it is extremely difficult to reduce α below $\frac{5}{4}$. One reason may be that $\frac{5}{4}$ is the critical index of the natural energy functional associated with (1.2). More precisely, if we set

$$E(\mathbf{u})(t) = \frac{1}{2}\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^2}^2 \, d\tau, \qquad \mathbf{u}_\lambda(x,t) = \lambda^{2\alpha-1} \mathbf{u}(\lambda x, \lambda^{2\alpha} t),$$

then

$$E(\mathbf{u}_\lambda)(t) = \lambda^{4\alpha-5} E(\mathbf{u})(\lambda^{2\alpha} t)$$

and the natural energy functional is invariant when $\lambda = \frac{5}{4}$.

Very recently Yang, Jiu and Wu [54] was able to reduce the hyperdissipation from a different perspective. [54] examined the global regularity problem on the 3D Navier–Stokes equations with partial hyperdissipation,

$$
\begin{cases}
\partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{\frac{5}{2}} u_1 - \nu \Lambda_2^{\frac{5}{2}} u_1 \\
\partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_2^{\frac{5}{2}} u_2 - \nu \Lambda_3^{\frac{5}{2}} u_2, \\
\partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{\frac{5}{2}} u_3 - \nu \Lambda_3^{\frac{5}{2}} u_3, \\
\nabla \cdot \mathbf{u} = 0, \\
\mathbf{u}(x, 0) = \mathbf{u}_0(x),
\end{cases}
\tag{1.4}
$$

where Λ_k^γ with $\gamma > 0$ and $k = 1, 2, 3$ denote directional fractional operators defined via the Fourier transform

$$
\widehat{\Lambda_k^\gamma f}(\xi) = |\xi_k|^\gamma \, \widehat{f}(\xi), \quad k = 1, 2, 3,
$$

where $\xi = (\xi_1, \xi_2, \xi_3)$. In addition, we will also use $\Lambda = (-\Delta)^{\frac{1}{2}}$ to denote the Zygmund operator. [54] established the global existence and uniqueness of strong solutions to (1.4) when $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$. The hyperdissipation in (1.4) is only partial, weaker than that in (1.2). In comparison with (1.2) with $\alpha = \frac{5}{4}$, the first component equation of (1.4) does not involve the fractional dissipation in the x_3-direction, the second does not involve the dissipation in the x_1-direction and the third does not involve the dissipation in the x_2-direction. Other partially dissipated Navier–Stokes systems that can be converted to (1.4) are also globally well-posedness in the H^1-setting, for example,

$$
\begin{cases}
\partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{\frac{5}{2}} u_1 - \nu \Lambda_3^{\frac{5}{2}} u_1 \\
\partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_1^{\frac{5}{2}} u_2 - \nu \Lambda_2^{\frac{5}{2}} u_2, \\
\partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_2^{\frac{5}{2}} u_3 - \nu \Lambda_3^{\frac{5}{2}} u_3, \\
\nabla \cdot \mathbf{u} = 0
\end{cases}
\tag{1.5}
$$

is also globally well-posed. However, when all the component equations do not have any dissipation in the same direction, say

$$
\begin{cases}
\partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{\frac{5}{2}} u_1 - \nu \Lambda_2^{\frac{5}{2}} u_1 \\
\partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_1^{\frac{5}{2}} u_2 - \nu \Lambda_2^{\frac{5}{2}} u_2, \\
\partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{\frac{5}{2}} u_3 - \nu \Lambda_2^{\frac{5}{2}} u_3, \\
\nabla \cdot \mathbf{u} = 0
\end{cases}
\tag{1.6}
$$

with no dissipation in the x_3-direction, the global well-posedness remains an open problem.

Partially motivated by the open well-posedness problem on (1.6), we study the global existence and regularity problem on the following 3D incompressible MHD equations with partial hyperdissipation,

$$
\begin{cases}
\partial_t u_1 + \mathbf{u}\cdot\nabla u_1 = -\partial_1 p - \nu\Lambda^{\frac{5}{2}} u_1 + \mathbf{b}\cdot\nabla b_1,\\
\partial_t u_2 + \mathbf{u}\cdot\nabla u_2 = -\partial_2 p - \nu\Lambda_2^{\frac{5}{2}} u_2 - \nu\Lambda_3^{\frac{5}{2}} u_2 + \mathbf{b}\cdot\nabla b_2,\\
\partial_t u_3 + \mathbf{u}\cdot\nabla u_3 = -\partial_3 p - \nu\Lambda_1^{\frac{5}{2}} u_3 - \nu\Lambda_3^{\frac{5}{2}} u_3 + \mathbf{b}\cdot\nabla b_3,\\
\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} - \eta\Lambda_1^{\frac{5}{2}}\mathbf{b} - \eta\Lambda_2^{\frac{5}{2}}\mathbf{b},\\
\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0,\\
\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{b}(x,0) = \mathbf{b}_0(x),
\end{cases}
\tag{1.7}
$$

where $\mathbf{u} = (u_1, u_2, u_3)$ represents the velocity field and $\mathbf{b} = (b_1, b_2, b_3)$ the magnetic field. The equation of the magnetic field does not have dissipation in the x_3 direction. In addition, the second and the third velocity equations do not have full fractional dissipation. This paper proves the following global existence and uniqueness of H^1-solutions to (1.7).

Theorem 1.1. *Assume* $(\mathbf{u}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^3)$. *Then (1.7) has a unique global solution* $(\mathbf{u}, \mathbf{b})$ *satisfying,*

$$
\begin{aligned}
&\mathbf{u} \in L^\infty(0,\infty; H^1), \quad \Lambda^{\frac{5}{4}} u_1,\ \nabla\Lambda^{\frac{5}{4}} u_1 \in L^2(0,\infty; L^2),\\
&\Lambda_2^{\frac{5}{4}} u_2,\ \Lambda_3^{\frac{5}{4}} u_2,\ \nabla\Lambda_2^{\frac{5}{4}} u_2,\ \nabla\Lambda_3^{\frac{5}{4}} u_2 \in L^2(0,\infty; L^2),\\
&\Lambda_1^{\frac{5}{4}} u_3,\ \Lambda_3^{\frac{5}{4}} u_3,\ \nabla\Lambda_1^{\frac{5}{4}} u_3,\ \nabla\Lambda_3^{\frac{5}{4}} u_3 \in L^2(0,\infty; L^2),\\
&\mathbf{b} \in L^\infty(0,\infty; H^1), \quad \Lambda_1^{\frac{5}{4}}\mathbf{b},\ \Lambda_2^{\frac{5}{4}}\mathbf{b},\ \nabla\Lambda_1^{\frac{5}{4}}\mathbf{b},\ \nabla\Lambda_2^{\frac{5}{4}}\mathbf{b} \in L^2(0,\infty; L^2).
\end{aligned}
\tag{1.8}
$$

The bound of $(\mathbf{u}, \mathbf{b})$ *in the functional setting in (1.8) is uniform in time.*

The uniqueness result we prove here is actually stronger than stated in Theorem 1.1. It asserts that any two solutions with one in the regularity class (1.8) and the other in the natural energy setting must be identical.

Theorem 1.1 appears to indicate that the MHD equations are not completely parallel to the Navier–Stokes equations. In addition, Theorem 1.1 improves some of the previous global regularity results on the MHD equations with fractional Laplacian. We first provide some background information and then explain the improvement. The MHD equations in (1.7) generalizes the standard MHD equations

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b},\\
\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \eta\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u},\\
\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0,
\end{cases}
\tag{1.9}
$$

by replacing the standard Laplacian by fractional Laplacian. (1.9) is the centerpiece of the magnetohydrodynamics, a field initiated by the Nobel Laureate H. Alfvén in 1942 [1]. (1.9) has played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [3,14,37,38]). As a coupled system of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism, the MHD equations contain much richer structures than the Navier–Stokes equations. They are not merely a combination of two parallel Navier–Stokes type equations but an interactive and integrated system. The global regularity problem on the MHD equations have recently garnered considerable interests and many important results have been obtained on this problem and related issues (see, e.g., [4–13,15–29, 31,32,34,35,39,41–53,55–58]).

The 3D MHD equations with standard Laplacian dissipation are considered to be supercritical in the same sense as for the standard Navier–Stokes equations. How much dissipation do we really need in order to ensure the global regularity of the MHD equations? As is well-known (see, e.g., [44,46,53]), the generalized MHD equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu(-\Delta)^{\alpha} \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{b}, \\ \partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} = -\eta(-\Delta)^{\beta} \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0 \end{cases} \tag{1.10}$$

with α and β satisfying

$$\alpha \geq \frac{5}{4}, \quad \alpha + \beta \geq \frac{5}{2} \tag{1.11}$$

always possess global classical solutions corresponding to any sufficiently smooth initial data. Especially $\alpha = \frac{5}{4}$ and $\beta = \frac{5}{4}$ guarantee the global existence and regularity. Theorem 1.1 asserts that the global regularity still holds even when there is only horizontal magnetic diffusion together with a reduced velocity dissipation.

The proof of Theorem 1.1 is naturally divided into two parts: the existence part and the uniqueness part. The existence part focuses on the global *a priori* bound of $(\mathbf{u}, \mathbf{b})$ in H^1. The global L^2-bound follows directly from the special structure of the MHD equations and the divergence-free condition. The proof for the global L^2-bound of the gradient $(\nabla \mathbf{u}, \nabla \mathbf{b})$ is lengthy and involves bounding a lot of triple product terms via several basic ingredients: divergence-free condition, Sobolev's embedding inequalities and Minkowski's inequality for exchanging integrals. The main idea of the estimates is to fully exploit the available directional dissipation and avoid the derivatives in the directions along which the dissipation is lacking. The uniqueness part estimates the difference of two solutions with one being in the regularity class in (1.8) and the other in the natural L^2 energy class. The rest of this paper is divided into two Sections. Section 2 proves the global regularity while Section 3 proves a strong version of the uniqueness part of Theorem 1.1.

2. Proof for the global regularity part

This section proves the global existence and regularity part of Theorem 1.1. The crucial piece is the global H^1 bound for $(\mathbf{u}, \mathbf{b})$. We first list several tools to be used in the proof.

We start with a Sobolev embedding inequality.

Lemma 2.1. *Let $2 \leq p \leq \infty$. Let $s > d(\frac{1}{2} - \frac{1}{p})$. Then, there exists a constant $C = C(d, p, s)$ such that, for any d-dimensional functions $f \in H^s(\mathbb{R}^d)$,*

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \, \|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{s}\left(\frac{1}{2}-\frac{1}{p}\right)} \, \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}. \tag{2.1}$$

When $p \neq \infty$, (2.1) also holds for $s = d(\frac{1}{2} - \frac{1}{p})$.

The next tool lemma states one version of the Minkowski inequality, which is the foundation for exchanging two Lebesgue norms (see, e.g., [33]).

Lemma 2.2. *Let $f = f(x, y)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be a measurable function on $\mathbb{R}^m \times \mathbb{R}^n$. Let $1 \le q \le p \le \infty$. Then*

$$\|\|f\|_{L^q_y(\mathbb{R}^n)}\|_{L^p_x(\mathbb{R}^m)} \le \|\|f\|_{L^p_x(\mathbb{R}^m)}\|_{L^q_y(\mathbb{R}^n)}.$$

The following Hölder type inequality will be useful as well.

Lemma 2.3. *Let $f_1 \ge 0$, $f_2 \ge 0$, and $f_1, f_2 \in L^p$. Let $a_1, a_2 \in [0, 1]$ and $a_1 + a_2 = 1$. Let $p \in [1, \infty]$. Then*

$$\|f_1^{a_1} f_2^{a_2}\|_{L^p} \le \|f_1\|_{L^p}^{a_1} \|f_2\|_{L^p}^{a_2}.$$

Throughout the rest of this paper, we use $\|f\|_{L^2}$ to denote $\|f\|_{L^2(\mathbb{R}^3)}$ and $\|f\|_{L^2_{x_i}}$ to denote the one-dimensional L^2-norm (in terms of x_i), and $\|f\|_{L^2_{x_i x_j}}$ to denote the two-dimensional L^2-norm (in terms of x_i and x_j). In addition, we also use the notion

$$\|f\|_{L^r_{x_i} L^q_{x_j} L^p_{x_k}} := \|\|\|f\|_{L^p_{x_k}}\|_{L^q_{x_j}}\|_{L^r_{x_i}}.$$

We now prove the global existence and regularity part of Theorem 1.1.

Proof of the global regularity of Theorem 1.1. Taking the inner product of (1.7) with $(\mathbf{u}, \mathbf{b})$, integrating by parts and using $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$, we obtain

$$\begin{aligned}
&\|(\mathbf{u}, \mathbf{b})(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 \, d\tau + 2\nu \int_0^t \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_2\|_{L^2}^2 \, d\tau \\
&\quad + 2\nu \int_0^t \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_3\|_{L^2}^2 \, d\tau + 2\eta \int_0^t \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \mathbf{b}\|_{L^2}^2 \, d\tau = \|(\mathbf{u}_0, \mathbf{b}_0)\|_{L^2}^2,
\end{aligned}$$

where we have used the notation

$$\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) u_2\|_{L^2}^2 := \|\Lambda_2^{\frac{5}{4}} u_2\|_{L^2}^2 + \|\Lambda_3^{\frac{5}{4}} u_2\|_{L^2}^2.$$

Next we prove the global bound for $\|(\nabla \mathbf{u}, \nabla \mathbf{b})\|_{L^2}$. This process is more complex. Applying ∇ to (1.7) and then taking the inner product of the resulting equations with $(\nabla \mathbf{u}, \nabla \mathbf{b})$, we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(\nabla \mathbf{u}, \nabla \mathbf{b})\|_{L^2}^2 + \nu \|\nabla \Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 + \nu \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_2\|_{L^2}^2 \\
&\quad + \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_3\|_{L^2}^2 + \eta \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla \mathbf{b}\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{2.2}$$

where

$$I_1 = -\int \nabla(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{u}\,dx,$$

$$I_2 = \int \nabla(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\nabla\mathbf{u}\,dx,$$

$$I_3 = \int \nabla(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{b}\,dx,$$

$$I_4 = -\int \nabla(\mathbf{u}\cdot\nabla\mathbf{b})\cdot\nabla\mathbf{b}\,dx.$$

After integrating by parts and applying the divergence-free conditions $\nabla\cdot\mathbf{u}=0$ and $\nabla\cdot\mathbf{b}=0$, we write the terms above as

$$I_1 = -\int \partial_i u_j\,\partial_j u_k\,\partial_i u_k\,dx, \tag{2.3}$$

$$I_2 + I_3 = \int \partial_i b_j\,\partial_j b_k \partial_i u_k\,dx + \int \partial_i b_j\,\partial_j u_k\,\partial_i b_k\,dx \equiv \widetilde{I}_2 + \widetilde{I}_3, \tag{2.4}$$

$$I_4 = -\int \partial_i u_j \partial_j b_k\,\partial_i b_k\,dx, \tag{2.5}$$

where we have used Einstein's summation convention (repeated indices are summed). For the sake of clarity, we divided the rest of this section into four subsections. $\square$

2.1. The estimates of I_1

The estimates of I_1 are similar to those in [54]. We provide the estimates of some of the terms for reader's convenience. We consider the first nine terms in I_1,

$$\begin{aligned} I_{11} := -\int \Big(&(\partial_1 u_1)^3 + \partial_1 u_1\,\partial_1 u_2\,\partial_2 u_1 + \partial_1 u_1\,\partial_1 u_3\,\partial_3 u_1 \\ &+ (\partial_2 u_1)^2\,\partial_1 u_1 + (\partial_2 u_1)^2\,\partial_2 u_2 + \partial_2 u_1\,\partial_2 u_3\,\partial_3 u_1 \\ &+ (\partial_3 u_1)^2\,\partial_1 u_1 + \partial_3 u_1\,\partial_3 u_2\,\partial_2 u_1 + (\partial_3 u_1)^2\,\partial_3 u_3\Big)\,dx. \end{aligned} \tag{2.6}$$

The terms in (2.6) will be labelled as I_{111}, I_{112}, $\cdots$ according to the order they appear in (2.6). We first deal with I_{112}, the second term in (2.6). We will return to I_{111} later. Integration by parts yields

$$\begin{aligned} I_{112} &= -\int \partial_1 u_1\,\partial_1 u_2\,\partial_2 u_1\,dx = \int u_2\,\partial_1\partial_2 u_1\,\partial_1 u_1\,dx + \int u_2\,\partial_1\partial_1 u_1\,\partial_2 u_1\,dx \\ &:= I_{1121} + I_{1122}. \end{aligned}$$

By Hölder's inequality and Lemma 2.2,

$$|I_{1121}| \le \|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3}L^4_{x_2}} \|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \|u_2\|_{L^2_{x_3}L^4_{x_1x_2}}.$$

By Sobolev's inequality,

$$\|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3}L^4_{x_2}} \le C\,\|\Lambda_2^{\frac14}\partial_1\partial_2 u_1\|_{L^2}$$

and

$$\|u_2\|_{L^2_{x_3}L^4_{x_1x_2}} \le C\,\|u_2\|_{L^2}^{\frac12}\,\|\nabla_h u_2\|_{L^2}^{\frac12},$$

where $\nabla_h = (\partial_1, \partial_2)$. Applying Gagliardo–Nirenberg's inequality and Lemma 2.1 with $p=\infty$ and $s=1$, we have

$$\begin{aligned}\|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} &\le C\,\|\Lambda_1^{\frac14}\partial_1 u_1\|_{L^2_{x_1x_2}L^\infty_{x_3}}\\ &\le C\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}.\end{aligned}$$

Combining the estimates above and applying Young's inequality yield

$$\begin{aligned}|I_{1121}| &\le C\,\|\Lambda_2^{\frac14}\partial_1\partial_2 u_1\|_{L^2}\,\|u_2\|_{L^2}^{\frac12}\,\|\nabla_h u_2\|_{L^2}^{\frac12}\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}\\ &\le \frac{\nu}{2048}\|\Lambda_2^{\frac54}\partial_1 u_1\|_{L^2}^2 + \frac{\nu}{2048}\|\Lambda_1^{\frac54}\partial_3 u_1\|_{L^2}^2 + C\,\|u_2\|_{L^2}^2\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^2\,\|\nabla_h u_2\|_{L^2}^2.\end{aligned}$$

Similarly,

$$|I_{1122}| \le C\,\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\|_{L^2}\,\|u_2\|_{L^2}^{\frac12}\,\|\nabla_h u_2\|_{L^2}^{\frac12}\,\|\Lambda_1^{\frac14}\partial_2 u_1\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_1^{\frac14}\partial_2 u_1\|_{L^2}^{\frac12}.$$

Due to the elementary inequalities

$$|\xi_2|^{\frac14}\,|\xi_1| \le \frac45|\xi_1|^{\frac54} + \frac15|\xi_2|^{\frac54}, \qquad |\xi_1|^{\frac14}\,|\xi_2| \le \frac15|\xi_1|^{\frac54} + \frac45|\xi_2|^{\frac54}$$

and Plancherel's theorem, we have

$$\begin{aligned}\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\|_{L^2}^2 &\le C\,(\|\Lambda_1^{\frac54}\partial_1 u_1\|_{L^2}^2 + \|\Lambda_2^{\frac54}\partial_1 u_1\|_{L^2}^2),\\ \|\Lambda_1^{\frac14}\partial_2 u_1\|_{L^2}^2 &\le C\,(\|\Lambda_1^{\frac54}u_1\|_{L^2}^2 + \|\Lambda_2^{\frac54}u_1\|_{L^2}^2).\end{aligned}$$

Therefore, by Young's inequality,

$$\begin{aligned}|I_{1122}| \le{}& \frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_1 u_1\|_{L^2}^2 + \frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2\\ &+ C\,\|u_2\|_{L^2}^2\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^2\,\|\nabla_h u_2\|_{L^2}^2.\end{aligned}$$

We return to estimate I_{111}, the first term in (2.6). I_{111} can be handled similarly as I_{112}. Integrating by parts and then bounding it as I_{1121}, we have

$$
\begin{aligned}
I_{111} &= 2\int u_1\,\partial_1\partial_1 u_1\,\partial_1 u_1\,dx \\
&\le C\|\partial_1\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\
&\le \frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_1 u_1\|_{L^2}^2+\frac{\nu}{2048}\|\Lambda_1^{\frac54}\partial_3 u_1\|_{L^2}^2 \\
&\quad + C\,\|u_1\|_{L^2}^2\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^2\,\|\nabla_h u_1\|_{L^2}^2.
\end{aligned}
$$

We now estimate I_{113}. Integrating by parts, we have

$$
\begin{aligned}
I_{113} &= -\int \partial_1 u_1\,\partial_1 u_3\,\partial_3 u_1\,dx = \int u_1\,\partial_3\partial_1 u_1\,\partial_1 u_3\,dx + \int u_1\,\partial_1 u_1\,\partial_1\partial_3 u_3\,dx \\
&:= I_{1131}+I_{1132}.
\end{aligned}
$$

In fact, as in the estimates of I_{1121},

$$
\begin{aligned}
I_{1131} &\le C\,\|\partial_3\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|\partial_1 u_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\
&\le C\,\|\Lambda_2^{\frac14}\partial_3\partial_1 u_1\|_{L^2}\,\|\Lambda_1^{\frac14}\partial_1 u_3\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_1^{\frac14}\partial_1 u_3\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\frac12}\,\|\nabla_h u_1\|_{L^2}^{\frac12} \\
&\le C\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}\,\|\Lambda_1^{\frac54}u_3\|_{L^2}^{\frac12}\,\|\Lambda_1^{\frac54}\partial_3 u_3\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\frac12}\,\|\nabla_h u_1\|_{L^2}^{\frac12} \\
&\le \frac{\nu}{2048}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2+\frac{\nu}{2048}\,\|\Lambda_1^{\frac54}\partial_3 u_3\|_{L^2}^2 \\
&\quad + C\,\|u_1\|_{L^2}^2\,\|\Lambda_1^{\frac54}u_3\|_{L^2}^2\,\|\nabla_h u_1\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{1132} &\le C\,\|\partial_1\partial_3 u_3\|_{L^2_{x_1x_2}L^4_{x_3}}\|\partial_1 u_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|u_1\|_{L^2_{x_2}L^4_{x_1x_3}} \\
&\le C\,\|\Lambda_3^{\frac54}\partial_1 u_3\|_{L^2}\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}\,\|\Lambda_2\Lambda_1^{\frac54}u_1\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\frac12}\,\|(\partial_1,\partial_3)u_1\|_{L^2}^{\frac12} \\
&\le \frac{\nu}{2048}\,\|\Lambda_3^{\frac54}\partial_1 u_3\|_{L^2}^2+\frac{\nu}{2048}\,\|\Lambda_1^{\frac54}\partial_2 u_1\|_{L^2}^2 \\
&\quad + C\,\|u_1\|_{L^2}^2\,\|\Lambda_1^{\frac54}u_1\|_{L^2}^2\,\|(\partial_1,\partial_3)u_1\|_{L^2}^2.
\end{aligned}
$$

This settles the estimate of I_{113}. We turn to I_{114} and I_{115}. Integrating by parts and invoking the divergence-free condition $\nabla\cdot\mathbf{u}=0$, we have

$$
\begin{aligned}
I_{114}+I_{115} &= -\int(\partial_2 u_1)^2\,\partial_1 u_1\,dx-\int(\partial_2 u_1)^2\,\partial_2 u_2\,dx \\
&= \int(\partial_2 u_1)^2\,\partial_3 u_3\,dx \\
&= -2\int u_3\,\partial_3\partial_2 u_1\,\partial_2 u_1\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\|\partial_3\partial_2 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|\partial_2 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_3\|_{L^2_{x_3}L^4_{x_1x_2}}\\
&\le C\,\|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^{\frac12}\|\Lambda_3(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^{\frac12}\|u_3\|_{L^2}^{\frac12}\|\nabla_h u_3\|_{L^2}^{\frac12}\\
&\le \frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2+C\,\|u_3\|_{L^2}^2\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^2\|\nabla_h u_3\|_{L^2}^2.
\end{aligned}
$$

To estimate I_{116}, we integrate by parts to obtain

$$
\begin{aligned}
I_{116} &= \int u_1\,\partial_3\partial_2 u_3\,\partial_2 u_1\,dx+\int u_1\,\partial_2 u_3\,\partial_3\partial_2 u_1\,dx\\
&:= I_{1161}+I_{1162}.
\end{aligned}
$$

The first term I_{1161} can be estimated similarly as I_{112}. In fact,

$$
\begin{aligned}
I_{1161} &\le \|\partial_3\partial_2 u_3\|_{L^2_{x_1x_2}L^4_{x_3}}\|\partial_2 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}\|u_1\|_{L^2_{x_1}L^4_{x_2x_3}}\\
&\le \frac{\nu}{2048}\|\Lambda_3^{\frac54}\partial_2 u_3\|_{L^2}^2+\frac{\nu}{2048}\|\Lambda_2^{\frac54}\partial_1 u_1\|_{L^2}^2\\
&\quad +C\,\|u_1\|_{L^2}^2\|\Lambda_2^{\frac54}u_1\|_{L^2}^2\|(\partial_2,\partial_3)u_1\|_{L^2}^2.
\end{aligned}
$$

The estimate of I_{1162} is different.

$$
\begin{aligned}
I_{1162} &\le \|\partial_3\partial_2 u_1\|_{L^2_{x_2x_3}L^4_{x_1}}\|\partial_2 u_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\\
&\le C\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}\|\Lambda_1^{\frac14}\partial_2 u_3\|_{L^2}^{\frac12}\|\Lambda_3\Lambda_1^{\frac14}\partial_2 u_3\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac35}\|\Lambda_2^{\frac54}u_1\|_{L^2}^{\frac25}\\
&\le C\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}\|\partial_2 u_3\|_{L^2}^{\frac25}\|\Lambda_1^{\frac54}\partial_2 u_3\|_{L^2}^{\frac1{10}}\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\partial_2 u_3\|_{L^2}^{\frac12}\\
&\quad\times\|u_1\|_{L^2}^{\frac35}\|\Lambda_2^{\frac54}u_1\|_{L^2}^{\frac25}\\
&\le \frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2+\frac{\nu}{2048}\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\partial_2 u_3\|_{L^2}^2\\
&\quad +C\,\|u_1\|_{L^2}^3\|\Lambda_2^{\frac54}u_1\|_{L^2}^2\|\partial_2 u_3\|_{L^2}^2.
\end{aligned}
$$

Due to $\nabla\cdot\mathbf{u}=0$, the last three terms in (2.6) can be regrouped into two terms,

$$
-\int \partial_3 u_1\,\partial_3 u_2\,\partial_2 u_1\,dx+\int \partial_2 u_2(\partial_3 u_1)^2\,dx := I_{117}+I_{118}.
$$

I_{117} can be estimated as I_{112}. By integration by parts,

$$
I_{117}=\int u_1\,\partial_3\partial_3 u_2\,\partial_2 u_1\,dx+\int u_1\,\partial_3\partial_2 u_1\,\partial_3 u_2\,dx := I_{1171}+I_{1172}.
$$

I_{1171} can be estimated similarly as I_{112} and we have

$$
\begin{aligned}
|I_{1171}| &\le \|\partial_3\partial_3 u_2\|_{L^2_{x_1x_3}L^4_{x_2}} \|\partial_2 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\
&\le C\,\|\Lambda_2^{\frac14}\,\partial_3\partial_3 u_2\|_{L^2}\,\|\Lambda_1^{\frac14}\partial_2 u_1\|_{L^2}^{\frac12}\,\|\Lambda_1^{\frac14}\partial_2\partial_3 u_1\|_{L^2}^{\frac12}\,\|u_1\|_{L^2}^{\frac12}\,\|(\partial_1,\partial_2)u_1\|_{L^2}^{\frac12} \\
&\le \frac{\nu}{2048}\,\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2 + \frac{\nu}{2048}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|\,|_{L^2}^2 \\
&\quad + C\,\|u_1\|_{L^2}^2\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^2\,\|(\partial_1,\partial_2)u_1\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_{1172}| \le \frac{\nu}{2048}\,&\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2 + \frac{\nu}{2048}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|\,|_{L^2}^2 \\
&+ C\,\|u_1\|_{L^2}^2\,\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2\,\|(\partial_1,\partial_2)u_1\|_{L^2}^2.
\end{aligned}
$$

We deal with I_{118}. By integration by parts,

$$
I_{118} = \int \partial_2 u_2 (\partial_3 u_1)^2\,dx = -2\int u_2\,\partial_3 u_1\,\partial_2\partial_3 u_1\,dx.
$$

By Hölder's inequality and by Minkowski's inequality,

$$
|I_{118}| \le 2\,\|\partial_2\partial_3 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\,\|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}}\,\|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}. \tag{2.7}
$$

By Lemma 2.1 and an interpolation inequality,

$$
\begin{aligned}
\|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}} &\le C\,\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2_{x_2x_3}L^\infty_{x_1}} \\
&\le C\,\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2}^{\frac12}\,\|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\|_{L^2}^{\frac12} \\
&\le C\,\|\partial_3 u_1\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2}^{\frac1{10}}\,\|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\|_{L^2}^{\frac12} \\
&\le C\,\|\partial_3 u_1\|_{L^2}^{\frac25}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^{\frac35},
\end{aligned} \tag{2.8}
$$

where we have invoked the interpolation inequality

$$
\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2_{x_2}} \le C\,\|\partial_3 u_1\|_{L^2_{x_2}}^{\frac45}\,\|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2_{x_2}}^{\frac15}.
$$

In addition,

$$
\|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}} \le C\,\|u_2\|_{L^2}^{\frac35}\,\|\Lambda_3^{\frac54}u_2\|_{L^2}^{\frac25}. \tag{2.9}
$$

Inserting (2.8) and (2.9) in (2.7) yields

$$|I_{118}| \le C\,\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1\|_{L^2}\,\|u_2\|_{L^2}^{\frac{3}{5}}\,\|\Lambda_3^{\frac{5}{4}}u_2\|_{L^2}^{\frac{2}{5}}\,\|\partial_3 u_1\|_{L^2}^{\frac{2}{5}}\,\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_3 u_1\|_{L^2}^{\frac{3}{5}}$$
$$\le \frac{\nu}{2048}\,\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_3 u_1\|_{L^2}^2 + C\,\|u_2\|_{L^2}^3\,\|\Lambda_3^{\frac{5}{4}}u_2\|_{L^2}^2\,\|\partial_3 u_1\|_{L^2}^2.$$

Combining the estimating above yields the bound

$$\begin{aligned}|I_1| \le\ & \frac{\nu}{64}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\nabla u_1\|_{L^2}^2 + \frac{\nu}{64}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_2\|_{L^2}^2\\ & + \frac{\nu}{64}\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_3\|_{L^2}^2 + C\,(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^3)(\|\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2\\ & + \|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_2\|_{L^2}^2 + \|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_3\|_{L^2}^2)\,\|\nabla \mathbf{u}\|_{L^2}^2.\end{aligned}$$

2.2. *The estimates of* $\widetilde{I}_2$

We divide the terms in $\widetilde{I}_2$ into three parts,

$$\begin{aligned}\widetilde{I}_2 &= \int \partial_i b_j\,\partial_j b_k\,\partial_i u_k\,dx \qquad (2.10)\\ &= \int \partial_i b_j\,\partial_j b_1\,\partial_i u_1\,dx + \int \partial_i b_j\,\partial_j b_2\,\partial_i u_2\,dx + \int \partial_i b_j\,\partial_j b_3\,\partial_i u_3\,dx\\ &:= I_{21} + I_{22} + I_{23}.\end{aligned}$$

We write out the nine terms in I_{21} explicitly,

$$\begin{aligned}I_{21} =& \int \partial_1 b_1\,\partial_1 b_1\,\partial_1 u_1\,dx + \int \partial_1 b_2\,\partial_2 b_1\,\partial_1 u_1\,dx + \int \partial_1 b_3\,\partial_3 b_1\,\partial_1 u_1\,dx\\ &+ \int \partial_2 b_1\,\partial_1 b_1\,\partial_2 u_1\,dx + \int \partial_2 b_2\,\partial_2 b_1\,\partial_2 u_1\,dx + \int \partial_2 b_3\,\partial_3 b_1\,\partial_2 u_1\,dx\\ &+ \int \partial_3 b_1\,\partial_1 b_1\,\partial_3 u_1\,dx + \int \partial_3 b_2\,\partial_2 b_1\,\partial_3 u_1\,dx + \int \partial_3 b_3\,\partial_3 b_1\,\partial_3 u_1\,dx\\ :=& I_{211} + I_{212} + \cdots + I_{219}.\end{aligned}$$

We estimate them one by one. By Lemma 2.1,

$$\begin{aligned}|I_{211}| &\le \|\partial_1 b_1\|_{L^2}\,\|\partial_1 b_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}\,\|\partial_1 u_1\|_{L^\infty_{x_3}L^2_{x_1}L^4_{x_2}}\\ &\le \|\partial_1 b_1\|_{L^2}\,\|\Lambda_2^{\frac{1}{4}}\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_1\Lambda_2^{\frac{1}{4}}\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\\ &\quad\times \|\Lambda_2^{\frac{1}{4}}\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_3\Lambda_2^{\frac{1}{4}}\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\\ &\le \frac{\nu}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2 + \frac{\eta}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_1 b_1\|_{L^2}^2\\ &\quad + C\,\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}\,\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})b_1\|_{L^2}\,\|\partial_1 b_1\|_{L^2}^2\end{aligned}$$

$$\leq \frac{\nu}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_1 b_1\|_{L^2}^2$$
$$+C\left(\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2+\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})b_1\|_{L^2}^2\right)\|\partial_1 b_1\|_{L^2}^2.$$

The second and the third terms in I_{21} are bounded similarly. In fact,

$$|I_{212}|\leq \frac{\nu}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_1 b_2\|_{L^2}^2$$
$$+C\left(\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2+\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})b_2\|_{L^2}^2\right)\|\partial_2 b_1\|_{L^2}^2$$

and

$$|I_{213}|\leq \frac{\nu}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_1 b_3\|_{L^2}^2$$
$$+C\left(\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2+\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})b_3\|_{L^2}^2\right)\|\partial_3 b_1\|_{L^2}^2.$$

I_{214}, I_{215} and I_{216} can all be bounded similarly.

$$|I_{214}|+|I_{215}|+|I_{216}|\leq \frac{\nu}{2048}\|\Lambda_2^{\frac{5}{4}}\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|\Lambda_2^{\frac{5}{4}}\Lambda_1 \mathbf{b}\|_{L^2}^2$$
$$+C\left(\|\Lambda_2^{\frac{5}{4}}u_1\|_{L^2}^2+\|\Lambda_2^{\frac{5}{4}}\mathbf{b}\|_{L^2}^2)\right)\|\nabla \mathbf{b}\|_{L^2}^2.$$

I_{217} and I_{218} can also be bounded similarly and we have

$$|I_{217}|\leq \frac{\nu}{2048}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\Lambda_1 b_1\|_{L^2}^2$$
$$+C\left(\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_1\|_{L^2}^2+\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})b_1\|_{L^2}^2\right)\|\partial_3 b_1\|_{L^2}^2.$$
$$|I_{218}|\leq \frac{\nu}{2048}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|\Lambda_2^{\frac{5}{4}}\Lambda_1 b_1\|_{L^2}^2$$
$$+C\left(\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_1\|_{L^2}^2+\|\Lambda_2^{\frac{5}{4}}b_1\|_{L^2}^2\right)\|\partial_3 b_2\|_{L^2}^2.$$

The last term I_{219} is slightly different since the integrand contains $\partial_3 b_3$ and $\partial_3 b_1$ and there is no magnetic diffusion in the x_3-direction. By $\nabla\cdot\mathbf{b}=0$,

$$\partial_3 b_3=-\partial_1 b_1-\partial_2 b_2$$

and I_{219} is then split into two terms, I_{2191} and I_{2192}. I_{2191} is the same as I_{217} and I_{2192} can be bounded as I_{211}. In fact,

$$
\begin{aligned}
|I_{2192}| &\le C\,\|\partial_3 b_1\|_{L^2}\,\|\partial_2 b_2\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}\,\|\partial_3 u_1\|_{L^\infty_{x_3}L^2_{x_1}L^4_{x_2}}\\
&\le \frac{\nu}{2048}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\Lambda_3 u_1\|_{L^2}^2+\frac{\eta}{2048}\|\Lambda_2^{\frac54}\Lambda_1 b_2\|_{L^2}^2\\
&\quad +C\left(\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_1\|_{L^2}^2+\|\Lambda_2^{\frac54}b_2\|_{L^2}^2\right)\|\partial_3 b_1\|_{L^2}^2.
\end{aligned}
$$

We now turn to the terms in I_{22}. The nine terms in I_{22} are

$$
\begin{aligned}
I_{22} = &\int \partial_1 b_1\,\partial_1 b_2\,\partial_1 u_2\,dx+\int \partial_1 b_2\,\partial_2 b_2\,\partial_1 u_2\,dx+\int \partial_1 b_3\,\partial_3 b_2\,\partial_1 u_2\,dx\\
&+\int \partial_2 b_1\,\partial_1 b_2\,\partial_2 u_2\,dx+\int \partial_2 b_2\,\partial_2 b_2\,\partial_2 u_2\,dx+\int \partial_2 b_3\,\partial_3 b_2\,\partial_2 u_2\,dx\\
&+\int \partial_3 b_1\,\partial_1 b_2\,\partial_3 u_2\,dx+\int \partial_3 b_2\,\partial_2 b_2\,\partial_3 u_2\,dx+\int \partial_3 b_3\,\partial_3 b_2\,\partial_3 u_2\,dx.
\end{aligned}
$$

Most of the terms can be handled as before. It suffices to examine some of the most difficult terms and one of them is

$$
I_{223} := \int \partial_1 b_3\,\partial_3 b_2\,\partial_1 u_2\,dx.
$$

The integrand contains $\partial_3 b_2$ and $\partial_1 u_2$ and the lack of the regularization in the x_3-direction in the equation of b_2 and in the x_1-direction of u_2. To handle this term, we shift the undesirable derivatives by integrating by parts,

$$
\begin{aligned}
I_{223} &= -\int \partial_{11} b_3\,\partial_3 b_2\,u_2\,dx-\int \partial_1 b_3\,\partial_3\partial_1 b_2\,u_2\,dx\\
&:= I_{2231}+I_{2232}.
\end{aligned}
$$

By Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
|I_{2231}| &\le \|\partial_{11} b_3\|_{L^4_{x_1}L^2_{x_2x_3}}\,\|\partial_3 b_2\|_{L^4_{x_1}L^2_{x_3}L^4_{x_2}}\,\|u_2\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}\\
&\le C\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}\,\|\partial_3 b_2\|_{L^2}^{\frac35}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_2\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}.
\end{aligned}
$$

To complete the estimates, we invoke the following interpolation inequality

$$
\|\Lambda_2^{\frac14}u_2\|_{L^2}\le \|u_2\|_{L^2}^{\frac45}\,\|\Lambda_2^{\frac54}u_2\|_{L^2}^{\frac15}.
$$

Therefore,

$$
\begin{aligned}
|I_{2231}| &\le C\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}\,\|\partial_3 b_2\|_{L^2}^{\frac35}\,\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_2\|_{L^2}^{\frac25}\,\|u_2\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac54}u_2\|_{L^2}^{\frac1{10}}\,\|\Lambda_3\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}\\
&\le \frac{\eta}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla b_3\|_{L^2}^2+\frac{\eta}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla b_2\|_{L^2}^2\\
&\quad +C\,\|u_2\|_{L^2}^{\frac43}\,\|\partial_3 b_2\|_{L^2}^2\,\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2.
\end{aligned}
$$

I_{2232} can be bounded as follows. By Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
|I_{2232}| &\le C\,\|\partial_3\partial_1 b_2\|_{L^2_{x_1x_3}L^4_{x_2}}\,\|\partial_1 b_3\|_{L^\infty_{x_1}L^2_{x_2x_3}}\,\|u_2\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}\\
&\le C\,\|\Lambda_2^{\frac14}\partial_3\partial_1 b_2\|_{L^2}\,\|\partial_1 b_3\|_{L^2}^{\frac35}\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac14}u_2\|_{L^\infty_{x_3}L^2_{x_1x_2}}\\
&\le C\,\|\Lambda_2^{\frac14}\partial_3\partial_1 b_2\|_{L^2}\,\|\partial_1 b_3\|_{L^2}^{\frac35}\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}\,\|\Lambda_3\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}\\
&\le C\,\|\Lambda_2^{\frac14}\partial_3\partial_1 b_2\|_{L^2}\,\|\partial_1 b_3\|_{L^2}^{\frac35}\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}^{\frac25}\,\|u_2\|_{L^2}^{\frac25}\,\|\Lambda_2^{\frac54}u_2\|_{L^2}^{\frac1{10}}\,\|\Lambda_3\Lambda_2^{\frac14}u_2\|_{L^2}^{\frac12}\\
&\le C\,\|\Lambda_2^{\frac14}\partial_3\partial_1 b_2\|_{L^2}\,\|\partial_1 b_3\|_{L^2}^{\frac35}\,\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}^{\frac25}\,\|u_2\|_{L^2}^{\frac25}\,\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^{\frac35}\\
&\le \frac{\eta}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_2\|_{L^2}^2+\frac{\eta}{2048}\|\Lambda_1^{\frac54}\partial_1 b_3\|_{L^2}^2\\
&\quad +C\,\|\partial_1 b_3\|_{L^2}^2\,\|u_2\|_{L^2}^{\frac43}\,\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2.
\end{aligned}
\tag{2.11}
$$

The terms in I_{23} can be estimated pretty much the same way as for the terms in I_{22}. We shall omit the details for the sake of conciseness. To summarize, the terms in $\widetilde{I}_2$ are bounded by

$$
\begin{aligned}
|\widetilde{I}_2| \le{}& \frac{\nu}{64}\|\Lambda^{\frac54}\nabla u_1\|_{L^2}^2+\frac{\nu}{64}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2+\frac{\eta}{64}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla\mathbf{b}\|_{L^2}^2\\
&+C\Big(\|\Lambda^{\frac54}u_1\|_{L^2}^2+\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2+\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})u_3\|_{L^2}^2+\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\mathbf{b}\|_{L^2}^2\Big)\|\nabla\mathbf{b}\|_{L^2}^2.
\end{aligned}
$$

2.3. *The estimates of $\widetilde{I}_3$ and I_4*

We now turn to the terms in $\widetilde{I}_3$ in (2.4),

$$
\widetilde{I}_3=\int \partial_i b_j\,\partial_j u_k\,\partial_i b_k\,dx. \tag{2.12}
$$

Some of the terms in (2.12) are the corresponding ones in (2.10), but (2.12) also contains terms that are different. To bound the terms in (2.12), we split $\widetilde{I}_3$ into three parts,

$$
\begin{aligned}
\widetilde{I}_3 &= \int \partial_i b_j\,\partial_j u_1\,\partial_i b_1\,dx+\int \partial_i b_j\,\partial_j u_2\,\partial_i b_2\,dx+\int \partial_i b_j\,\partial_j u_3\,\partial_i b_3\,dx\\
&:= I_{31}+I_{32}+I_{33}.
\end{aligned}
$$

The terms in I_{31} are

$$
\begin{aligned}
I_{31}={}&\int \partial_1 b_1\,\partial_1 u_1\,\partial_1 b_1\,dx+\int \partial_1 b_2\,\partial_2 u_1\,\partial_1 b_1\,dx+\int \partial_1 b_3\,\partial_3 u_1\,\partial_1 b_1\,dx\\
&+\int \partial_2 b_1\,\partial_1 u_1\,\partial_2 b_1\,dx+\int \partial_2 b_2\,\partial_2 u_1\,\partial_2 b_1\,dx+\int \partial_2 b_3\,\partial_3 u_1\,\partial_2 b_1\,dx\\
&+\int \partial_3 b_1\,\partial_1 u_1\,\partial_3 b_1\,dx+\int \partial_3 b_2\,\partial_2 u_1\,\partial_3 b_1\,dx+\int \partial_3 b_3\,\partial_3 u_1\,\partial_3 b_1\,dx.
\end{aligned}
$$

Since most of the terms can be handled as before, we shall just provide the estimates for some of the most difficult terms. The integrands of these terms involve partial derivatives for which the equations in (1.7) have no dissipation. One of them is I_{317},

$$I_{317} = \int \partial_3 b_1\, \partial_1 u_1\, \partial_3 b_1\, dx = -2\int \partial_3 b_1\, \partial_1\partial_3 b_1\, u_1\, dx.$$

As in the estimates of (2.11),

$$\begin{aligned}
I_{317} &\le C\, \|\partial_1\partial_3 b_1\|_{L^2_{x_2x_3}L^4_{x_1}}\, \|\partial_3 b_1\|_{L^2_{x_3}L^4_{x_1x_2}}\, \|u_1\|_{L^\infty_{x_3}L^4_{x_2}L^2_{x_1}} \\
&\le C\, \|\partial_1\partial_3 b_1\|_{L^2_{x_2x_3}L^4_{x_1}}\, \|\partial_3 b_1\|_{L^2_{x_3}L^4_{x_1x_2}}\, \|u_1\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}} \\
&\le C\, \|\Lambda_1^{\frac14}\partial_1\partial_3 b_1\|_{L^2}\, \|\partial_3 b_1\|_{L^2}^{\frac35}\, \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_1\|_{L^2}^{\frac25}\, \|u_1\|_{L^2}^{\frac25} \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_1\|_{L^2}^{\frac35} \\
&\le \frac{\eta}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_1\|_{L^2}^2 + C\, \|\partial_3 b_1\|_{L^2}^2\, \|u_1\|_{L^2}^{\frac43}\, \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_1\|_{L^2}^2.
\end{aligned}$$

The terms in I_{32} and I_{33} can be handled similarly. As an illustration, we take out one of the most difficult terms and integrate by parts,

$$\begin{aligned}
I_{327} &= \int \partial_3 b_1\, \partial_1 u_2\, \partial_3 b_2\, dx \\
&= -\int u_2\, \partial_3 b_1\, \partial_1\partial_3 b_2\, dx - \int u_2\, \partial_3 b_2 \partial_1\partial_3 b_1\, dx \\
&:= I_{3271} + I_{3272}.
\end{aligned}$$

It is easy to see that the two terms above can be handled as I_{317}. In fact,

$$\begin{aligned}
|I_{3271}| &\le \|\partial_1\partial_3 b_2\|_{L^2_{x_2x_3}L^4_{x_1}}\, \|\partial_3 b_1\|_{L^2_{x_3}L^4_{x_1x_2}}\, \|u_2\|_{L^\infty_{x_3}L^4_{x_2}L^2_{x_1}} \\
&\le \|\partial_1\partial_3 b_2\|_{L^2_{x_2x_3}L^4_{x_1}}\, \|\partial_3 b_1\|_{L^2_{x_3}L^4_{x_1x_2}}\, \|u_2\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}} \\
&\le C\, \|\Lambda_1^{\frac14}\partial_1\partial_3 b_2\|_{L^2}\, \|\partial_3 b_1\|_{L^2}^{\frac35}\, \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_1\|_{L^2}^{\frac25}\, \|u_2\|_{L^2}^{\frac25} \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^{\frac35} \\
&\le \frac{\eta}{2048}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 b_1\|_{L^2}^2 + \frac{\eta}{2048}\|\Lambda_1^{\frac54}\partial_3 b_2\|_{L^2}^2 \\
&\quad + C\, \|\partial_3 b_1\|_{L^2}^2\, \|u_2\|_{L^2}^{\frac43}\, \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2.
\end{aligned}$$

I_{3272} can be similarly bounded. In addition, the terms in I_{33} can also be similarly bounded. Combining all the estimates in $\widetilde{I}_3$ yields the following bound,

$$\begin{aligned}
|\widetilde{I}_3| &\le \frac{\nu}{64}\|\Lambda^{\frac54}\nabla u_1\|_{L^2}^2 + \frac{\nu}{64}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2 + \frac{\eta}{64}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla \mathbf{b}\|_{L^2}^2 \\
&+ C\left(\|\Lambda^{\frac54}u_1\|_{L^2}^2 + \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2 + \|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})u_3\|_{L^2}^2 + \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\mathbf{b}\|_{L^2}^2\right)\|\nabla \mathbf{b}\|_{L^2}^2.
\end{aligned}$$

The terms in I_4 can be similarly estimated as those in $\widetilde{I}_2$ and $\widetilde{I}_3$. We thus omit the details.

2.4. Completion of the proof

This subsection combines the estimates in the previous subsections to complete the proof for the global regularity of Theorem 1.1.

Proof of the global regularity of Theorem 1.1 (continued). By (2.2) and the bounds in the previous subsections, we have

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{b})\|_{L^2}^2 + \nu\|\nabla\Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 + \nu\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_2\|_{L^2}^2 + \nu\|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\nabla u_3\|_{L^2}^2 \\
&+\eta\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\nabla \mathbf{b}\|_{L^2}^2 \le C\Big(\|\Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_2\|_{L^2}^2 \\
&\qquad\qquad + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})u_3\|_{L^2}^2 + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\mathbf{b}\|_{L^2}^2\Big)\|(\nabla \mathbf{u}, \nabla \mathbf{b})\|_{L^2}^2,
\end{aligned}$$

where C may depend on $\|(\mathbf{u}_0, \mathbf{b}_0)\|_{L^2}$. Gronwall's inequality then implies the desired uniform global H^1-bound as stated in Theorem 1.1. □

3. Proof of the uniqueness

This section proves the following uniqueness result.

Proposition 3.1. *Assume* $(\mathbf{u}^{(1)}, \mathbf{b}^{(1)})$ *and* $(\mathbf{u}^{(2)}, \mathbf{b}^{(2)})$ *are two solutions of (1.7) on the time interval* $[0, T]$. *Assume* $(\mathbf{u}^{(2)}, \mathbf{b}^{(2)})$ *is in the regularity class (1.8) while* $(\mathbf{u}^{(1)}, \mathbf{b}^{(1)})$ *is in the natural energy class,*

$$\begin{aligned}
&(\mathbf{u}^{(1)}, \mathbf{b}^{(1)}) \in L^\infty(0, T; L^2), \\
&\Lambda^{\frac{5}{4}} u_1^{(1)},\ \Lambda_2^{\frac{5}{4}} u_2^{(1)},\ \Lambda_3^{\frac{5}{4}} u_2^{(1)},\ \Lambda_1^{\frac{5}{4}} u_3^{(1)},\ \Lambda_3^{\frac{5}{4}} u_3^{(1)},\ \Lambda_1^{\frac{5}{4}} \mathbf{b}^{(1)},\ \Lambda_2^{\frac{5}{4}} \mathbf{b}^{(1)} \in L^2(0, T; L^2).
\end{aligned}$$

Then $(\mathbf{u}^{(1)}, \mathbf{b}^{(1)}) = (\mathbf{u}^{(2)}, \mathbf{b}^{(2)})$.

A special consequence of Proposition 3.1 is the uniqueness stated in Theorem 1.1.

Proof of Proposition 3.1. The proof estimates the L^2-difference of these two solutions,

$$(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}}) = (\mathbf{u}^{(1)}, \mathbf{b}^{(1)}) - (\mathbf{u}^{(2)}, \mathbf{b}^{(2)}),$$

which satisfies

$$\begin{cases}
\partial_t \widetilde{u}_1 + \mathbf{u}^{(1)} \cdot \nabla \widetilde{u}_1 + \widetilde{\mathbf{u}} \cdot \nabla u_1^{(2)} = -\partial_1 \widetilde{p} - \nu\Lambda^{\frac{5}{2}} \widetilde{u}_1 + \mathbf{b}^{(1)} \cdot \nabla \widetilde{b}_1 + \widetilde{\mathbf{b}} \cdot \nabla b_1^{(2)}, \\
\partial_t \widetilde{u}_2 + \mathbf{u}^{(1)} \cdot \nabla \widetilde{u}_2 + \widetilde{\mathbf{u}} \cdot \nabla u_2^{(2)} = -\partial_2 \widetilde{p} - \nu\Lambda_2^{\frac{5}{2}} \widetilde{u}_2 - \nu\Lambda_3^{\frac{5}{2}} \widetilde{u}_2 + \mathbf{b}^{(1)} \cdot \nabla \widetilde{b}_2 + \widetilde{\mathbf{b}} \cdot \nabla b_2^{(2)}, \\
\partial_t \widetilde{u}_3 + \mathbf{u}^{(1)} \cdot \nabla \widetilde{u}_3 + \widetilde{\mathbf{u}} \cdot \nabla u_3^{(2)} = -\partial_3 \widetilde{p} - \nu\Lambda_1^{\frac{5}{2}} \widetilde{u}_3 - \nu\Lambda_3^{\frac{5}{2}} \widetilde{u}_3 + \mathbf{b}^{(1)} \cdot \nabla \widetilde{b}_3 + \widetilde{\mathbf{b}} \cdot \nabla b_3^{(2)}, \\
\partial_t \widetilde{\mathbf{b}} + \mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{b}} + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{b}^{(2)} = \mathbf{b}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{b}} \cdot \nabla \mathbf{u}^{(2)} - \eta\Lambda_1^{\frac{5}{2}} \widetilde{\mathbf{b}} - \eta\Lambda_2^{\frac{5}{2}} \widetilde{\mathbf{b}}, \\
\nabla \cdot \widetilde{\mathbf{u}} = 0, \quad \nabla \cdot \widetilde{\mathbf{b}} = 0, \\
\widetilde{\mathbf{u}}(x, 0) = 0, \quad \widetilde{\mathbf{b}}(x, 0) = 0,
\end{cases}$$

where $\widetilde{p} = p^{(1)} - p^{(2)}$ denotes the difference between the corresponding pressures. Taking the inner product with $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}})$ and integrating by parts yield

$$\frac{1}{2}\frac{d}{dt}\|(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}})\|_{L^2}^2 + \nu\|\Lambda^{\frac{5}{4}}\widetilde{u}_1\|_{L^2}^2 + \nu\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2 + \nu\|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_3\|_{L^2}^2$$
$$+\eta\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\widetilde{\mathbf{b}}\|_{L^2}^2 = L_1 + L_2 + L_3 + L_4, \tag{3.1}$$

where

$$L_1 = -\int (\widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)}) \cdot \widetilde{\mathbf{u}}\, dx,$$
$$L_2 = \int (\widetilde{\mathbf{b}} \cdot \nabla \mathbf{b}^{(2)}) \cdot \widetilde{\mathbf{u}}\, dx,$$
$$L_3 = -\int (\widetilde{\mathbf{u}} \cdot \nabla \mathbf{b}^{(2)}) \cdot \widetilde{\mathbf{b}}\, dx,$$
$$L_4 = \int (\widetilde{\mathbf{b}} \cdot \nabla \mathbf{u}^{(2)}) \cdot \widetilde{\mathbf{b}}\, dx.$$

L_1 can be bounded as in [54]. For the sake of reader's convenience, we provide the estimates for two terms in L_1,

$$L_{14} = -\int \widetilde{u}_1 \partial_1 u_2^{(2)} \widetilde{u}_2\, dx, \qquad L_{15} = -\int \widetilde{u}_2 \partial_2 u_2^{(2)} \widetilde{u}_2\, dx.$$

By Hölder's inequality and Lemma 2.1,

$$\begin{aligned}
|L_{14}| &\le \|\partial_1 u_2^{(2)}\|_{L^2}\, \|\widetilde{u}_2\|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}}\, \|\widetilde{u}_1\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \\
&\le C\, \|\partial_1 u_2^{(2)}\|_{L^2}\, \|\Lambda_2^{\frac{1}{4}} \widetilde{u}_2\|_{L^2_{x_1x_2} L^\infty_{x_3}}\, \|\Lambda_2^{\frac{1}{4}} \widetilde{u}_1\|_{L^2_{x_2x_3} L^\infty_{x_1}} \\
&\le C\, \|\partial_1 u_2^{(2)}\|_{L^2}\, \|\Lambda_2^{\frac{1}{4}} \widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\, \|\Lambda_2^{\frac{1}{4}} \partial_3 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\, \|\Lambda_2^{\frac{1}{4}} \widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\, \|\Lambda_2^{\frac{1}{4}} \partial_1 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Invoking the interpolation inequalities as (3.2), we have

$$\begin{aligned}
|L_{14}| &\le C\, \|\partial_1 u_2^{(2)}\|_{L^2}\, \|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}}\, \|\widetilde{u}_2\|_{L^2}^{\frac{2}{5}}\, \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^{\frac{3}{5}}\, \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^{\frac{3}{5}} \\
&\le \frac{\nu}{128}\, \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^2 + \frac{\nu}{128}\, \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2 \\
&\quad + C\, \|\partial_1 u_2^{(2)}\|_{L^2}^{\frac{5}{2}}\, \|\widetilde{u}_1\|_{L^2}\, \|\widetilde{u}_2\|_{L^2}.
\end{aligned}$$

By Hölder's inequality and Lemma 2.1,

$$
\begin{aligned}
|L_{15}| &\leq \|\widetilde{u}_2\|_{L^2} \,\|\widetilde{u}_2\|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \,\|\partial_2 u_2^{(2)}\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \\
&\leq C\,\|\widetilde{u}_2\|_{L^2} \,\|\Lambda_2^{\frac{1}{4}} \widetilde{u}_2\|_{L^2_{x_1 x_2} L^\infty_{x_3}} \,\|\Lambda_2^{\frac{1}{4}} \partial_2 u_2^{(2)}\|_{L^2_{x_3 x_2} L^\infty_{x_1}} \\
&\leq C\,\|\widetilde{u}_2\|_{L^2} \,\|\Lambda_2^{\frac{1}{4}} \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \,\|\Lambda_2^{\frac{1}{4}} \Lambda_3 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \,\|\Lambda_2^{\frac{1}{4}} \partial_2 u_2^{(2)}\|_{L^2}^{\frac{1}{2}} \,\|\Lambda_2^{\frac{1}{4}} \Lambda_1 \partial_2 u_2^{(2)}\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Inserting the interpolation inequality above,

$$
\|\Lambda_2^{\frac{1}{4}} \widetilde{u}_2\|_{L^2_{x_2}} \leq C\,\|\widetilde{u}_2\|_{L^2_{x_2}}^{\frac{4}{5}} \,\|\Lambda_2^{\frac{5}{4}} \widetilde{u}_2\|_{L^2_{x_2}}^{\frac{1}{5}}, \tag{3.2}
$$

we obtain

$$
\begin{aligned}
|L_{15}| &\leq C\,\|\widetilde{u}_2\|_{L^2}^{\frac{7}{5}} \,\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^{\frac{3}{5}} \,\|\Lambda_2^{\frac{5}{4}} u_2^{(2)}\|_{L^2}^{\frac{1}{2}} \,\|\Lambda_2^{\frac{5}{4}} \Lambda_1 u_2^{(2)}\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\nu}{128} \,\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2 + C\,\|\widetilde{u}_2\|_{L^2}^2 \,\|\Lambda_2^{\frac{5}{4}} u_2^{(2)}\|_{L^2}^{\frac{5}{7}} \,\|\Lambda_2^{\frac{5}{4}} \Lambda_1 u_2^{(2)}\|_{L^2}^{\frac{5}{7}}.
\end{aligned}
$$

To estimate L_2, we write L_2 in terms of the components,

$$
L_2 = \int \widetilde{b}_k \partial_k b_m^{(2)} \widetilde{u}_m \, dx,
$$

where repeated indices are summed. One of the most difficult terms is

$$
L_{26} = \int \widetilde{b}_3 \partial_3 b_2^{(2)} \widetilde{u}_2 \, dx.
$$

This term is more difficult due to $\partial_3 b_2^{(2)}$ and the lack of magnetic diffusion in the x_3-direction. By Hölder's inequality and Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
|L_{26}| &\leq \|\widetilde{b}_3\|_{L^\infty_{x_1} L^2_{x_2 x_3}} \,\|\partial_3 b_2^{(2)}\|_{L^2_{x_1 x_3} L^\infty_{x_2}} \,\|\widetilde{u}_2\|_{L^2_{x_1} L^\infty_{x_3} L^2_{x_2}} \\
&\leq \|\widetilde{b}_3\|_{L^2_{x_2 x_3} L^\infty_{x_1}} \,\|\partial_3 b_2^{(2)}\|_{L^2_{x_1 x_3} L^\infty_{x_2}} \,\|\widetilde{u}_2\|_{L^2_{x_1} L^2_{x_2} L^\infty_{x_3}} \\
&\leq C\,\|\widetilde{b}_3\|_{L^2}^{\frac{3}{5}} \,\|\Lambda_1^{\frac{5}{4}} \widetilde{b}_3\|_{L^2}^{\frac{2}{5}} \,\|\partial_3 b_2^{(2)}\|_{L^2}^{\frac{3}{5}} \,\|\Lambda_2^{\frac{5}{4}} \partial_3 b_2^{(2)}\|_{L^2}^{\frac{2}{5}} \,\|\widetilde{u}_2\|_{L^2}^{\frac{3}{5}} \,\|\Lambda_3^{\frac{5}{4}} \widetilde{u}_2\|_{L^2}^{\frac{2}{5}} \\
&\leq \frac{\nu}{32} \|\Lambda_3^{\frac{5}{4}} \widetilde{u}_2\|_{L^2}^2 + \frac{\eta}{32} \|\Lambda_1^{\frac{5}{4}} \widetilde{b}_3\|_{L^2}^2 \\
&\quad + C\,\|\widetilde{b}_3\|_{L^2} \,\|\widetilde{u}_2\|_{L^2} \,\|\partial_3 b_2^{(2)}\|_{L^2} \,\|\Lambda_2^{\frac{5}{4}} \partial_3 b_2^{(2)}\|_{L^2}^{\frac{2}{3}}.
\end{aligned}
$$

Other terms in L_2 can be handled similarly. We now turn to L_3 and write out the component terms explicitly,

$$
L_3 = -\int \widetilde{u}_k \, \partial_k b_m^{(2)} \, \widetilde{b}_m \, dx.
$$

It suffices to look at a typical term,

$$L_{33} = \int \widetilde{u}_3 \, \partial_3 b_1^{(2)} \, \widetilde{b}_1 \, dx.$$

It can be bounded similarly as $\widetilde{L}_2$. In fact,

$$\begin{aligned}
|L_{33}| &\le \|\widetilde{u}_3\|_{L^\infty_{x_3} L^2_{x_1x_2}} \|\partial_3 b_1^{(2)}\|_{L^2_{x_1x_3} L^\infty_{x_2}} \|\widetilde{b}_1\|_{L^2_{x_1} L^\infty_{x_1} L^2_{x_2}} \\
&\le \frac{\nu}{32} \|\Lambda_3^{\frac{5}{4}} \widetilde{u}_3\|_{L^2}^2 + \frac{\eta}{32} \|\Lambda_1^{\frac{5}{4}} \widetilde{b}_1\|_{L^2}^2 \\
&\quad + C \, \|\widetilde{b}_1\|_{L^2} \, \|\widetilde{u}_3\|_{L^2} \, \|\partial_3 b_1^{(2)}\|_{L^2} \, \|\Lambda_2^{\frac{5}{4}} \partial_3 b_1^{(2)}\|_{L^2}^{\frac{2}{3}}.
\end{aligned}$$

The estimate of L_4 is similar. Again it suffices to look at a typical term,

$$L_{44} = \int \widetilde{b}_1 \, \partial_1 u_2^{(2)} \, \widetilde{b}_2 \, dx.$$

By Hölder's inequality, Lemma 2.2 and Lemma 2.1,

$$\begin{aligned}
|L_{44}| &\le \|\widetilde{b}_1\|_{L^\infty_{x_1} L^2_{x_2x_3}} \|\partial_1 u_2^{(2)}\|_{L^2_{x_1} L^\infty_{x_3} L^2_{x_2}} \|\widetilde{b}_2\|_{L^2_{x_1x_3} L^\infty_{x_2}} \\
&\le \frac{\eta}{32} \|\Lambda_1^{\frac{5}{4}} \widetilde{b}_1\|_{L^2}^2 + \frac{\eta}{32} \|\Lambda_2^{\frac{5}{4}} \widetilde{b}_2\|_{L^2}^2 + C \, \|\widetilde{b}_1\|_{L^2} \, \|\widetilde{b}_2\|_{L^2} \, \|\partial_1 u_2^{(2)}\|_{L^2} \, \|\Lambda_3^{\frac{5}{4}} \partial_1 u_2^{(2)}\|_{L^2}^{\frac{2}{3}}.
\end{aligned}$$

Inserting the estimates for L_1 through L_4 in (3.1), we obtain

$$\begin{aligned}
&\frac{d}{dt} \|(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}})\|_{L^2}^2 + \nu \|\Lambda^{\frac{5}{4}} \widetilde{u}_1\|_{L^2}^2 + \nu \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \widetilde{u}_2\|_{L^2}^2 + \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \widetilde{u}_3\|_{L^2}^2 \\
&\qquad + \eta \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \widetilde{\mathbf{b}}\|_{L^2}^2 \le A(t) \, \|(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}})\|_{L^2}^2,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
A(t) &= C \, \|\nabla \mathbf{b}^{(2)}\|_{L^2} \, \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla \mathbf{b}^{(2)}\|_{L^2}^{\frac{2}{3}} + C \, \|\nabla \mathbf{u}^{(2)}\|_{L^2} \Big(\|\Lambda^{\frac{5}{4}} \nabla u_1^{(2)}\|_{L^2}^{\frac{2}{3}} \\
&\quad + \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_2^{(2)}\|_{L^2}^{\frac{2}{3}} + \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla u_3^{(2)}\|_{L^2}^{\frac{2}{3}} \Big) \\
&\quad + C \, \|\Lambda_1^{\frac{5}{4}} u_1^{(2)}\|_{L^2}^{\frac{5}{7}} \|\Lambda_1^{\frac{5}{4}} \nabla u_1^{(2)}\|_{L^2}^{\frac{5}{7}} + C \, \|\Lambda_2^{\frac{5}{4}} u_2^{(2)}\|_{L^2}^{\frac{5}{7}} \|\Lambda_2^{\frac{5}{4}} \nabla u_2^{(2)}\|_{L^2}^{\frac{5}{7}} \\
&\quad + C \, \|\Lambda_3^{\frac{5}{4}} u_3^{(2)}\|_{L^2}^{\frac{5}{7}} \|\Lambda_3^{\frac{5}{4}} \nabla u_3^{(2)}\|_{L^2}^{\frac{5}{7}} + C \, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^{\frac{5}{2}}.
\end{aligned}$$

The regularity assumption on $\mathbf{u}^{(2)}$ and $\mathbf{b}^{(2)}$ in (1.8) ensures the integrability of $A(t)$, for any $T > 0$,

$$\int_0^T A(t) \, dt < \infty.$$

Applying Growall's inequality to (3.3) yields the desired uniqueness, $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}}) \equiv 0$. This completes the proof of Proposition 3.1. □

Acknowledgments

Yang was supported by NNSFC (No. 11601011), NSF of Ningxia (No. NZ16092), and the Higher Education Specialized Research Fund of Ningxia (No. NGY2015140). Q. Jiu is supported by NNSFC (No. 11671273 and No. 11231006). J. Wu is supported by NSF grant DMS 1614246, by the AT&T Foundation at Oklahoma State University, and by NNSFC (No. 11471103, a grant awarded to B. Yuan). J. Wu would also like to thank Academy of Mathematics & Systems Science (Chinese Academy of Sciences, Beijing) and North Minzu University for their hospitality.

References

[1] H. Alfvén, Existence of electromagnetic-hydrodynamic waves, Nature 150 (1942) 405–406.
[2] D. Barbato, F. Morandin, M. Romito, Global regularity for a slightly supercritical hyperdissipative Navier–Stokes system, Anal. PDE 7 (2014) 2009–2027.
[3] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, Cambridge, 1993.
[4] Y. Cai, Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, arXiv:1605.00439 [math.AP], 2 May 2016.
[5] C. Cao, D. Regmi, J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, J. Differential Equations 254 (2013) 2661–2681.
[6] C. Cao, D. Regmi, J. Wu, X. Zheng, Global regularity for the 2D magnetohydrodynamics equations with horizontal dissipation and horizontal magnetic diffusion, preprint.
[7] C. Cao, J. Wu, Two regularity criteria for the 3D MHD equations, J. Differential Equations 248 (2010) 2263–2274.
[8] C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math. 226 (2011) 1803–1822.
[9] C. Cao, J. Wu, B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, SIAM J. Math. Anal. 46 (2014) 588–602.
[10] J. Chemin, D. McCormick, J. Robinson, J. Rodrigo, Local existence for the non-resistive MHD equations in Besov spaces, Adv. Math. 286 (2016) 1–31.
[11] Q. Chen, C. Miao, Z. Zhang, The Beale–Kato–Majda criterion for the 3D magneto-hydrodynamics equations, Comm. Math. Phys. 275 (2007) 861–872.
[12] Q. Chen, C. Miao, Z. Zhang, On the well-posedness of the ideal MHD equations in the Triebel–Lizorkin spaces, Arch. Ration. Mech. Anal. 195 (2010) 561–578.
[13] P. Constantin, Lagrangian–Eulerian methods for uniqueness in hydrodynamic systems, Adv. Math. 278 (2015) 67–102.
[14] P. Davidson, An Introduction to Magnetohydrodynamics, Cambridge University Press, Cambridge, England, 2001.
[15] W. Deng, P. Zhang, Large time behavior of solutions to 3-D MHD system with initial data near equilibrium, arXiv:1702.05260 [math.AP], 17 Feb 2017.
[16] B. Dong, J. Li, J. Wu, Global regularity for the 2D MHD equations with partial hyperresistivity, Int. Math. Res. Not. (2018) rnx240, https://doi.org/10.1093/imrn/rnx240, in press.
[17] B. Dong, Y. Jia, J. Li, J. Wu, Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion, J. Math. Fluid Mech. (2018), https://doi.org/10.1007/s00021-018-0376-3, in press.
[18] L. Du, D. Zhou, Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion, SIAM J. Math. Anal. 47 (2015) 1562–1589.
[19] J. Fan, H. Malaikah, S. Monaquel, G. Nakamura, Y. Zhou, Global Cauchy problem of 2D generalized MHD equations, Monatsh. Math. 175 (2014) 127–131.
[20] C. Fefferman, D. McCormick, J. Robinson, J. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, J. Funct. Anal. 267 (2014) 1035–1056.
[21] C.L. Fefferman, D. McCormick, J. Robinson, J. Rodrigo, Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces, Arch. Ration. Mech. Anal. 223 (2017) 677–691.

[22] C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, J. Funct. Anal. 227 (2005) 113–152.
[23] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations 213 (2005) 235–254.
[24] L. He, L. Xu, P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, arXiv:1603.08205 [math.AP], 27 Mar 2016.
[25] X. Hu, F. Lin, Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity, arXiv:1405.0082v1 [math.AP], 1 May 2014.
[26] Q. Jiu, D. Niu, Mathematical results related to a two-dimensional magneto-hydrodynamic equations, Acta Math. Sci. Ser. B Engl. Ed. 26 (2006) 744–756.
[27] Q. Jiu, D. Niu, J. Wu, X. Xu, H. Yu, The 2D magnetohydrodynamic equations with magnetic diffusion, Nonlinearity 28 (2015) 3935–3955.
[28] Q. Jiu, J. Zhao, A remark on global regularity of 2D generalized magnetohydrodynamic equations, J. Math. Anal. Appl. 412 (2014) 478–484.
[29] Q. Jiu, J. Zhao, Global regularity of 2D generalized MHD equations with magnetic diffusion, Z. Angew. Math. Phys. 66 (2015) 677–687.
[30] N. Katz, N. Pavlovic, A cheap Caffarelli–Kohn–Nirenberg inequality for the Navier–Stokes equation with hyper-dissipation, Geom. Funct. Anal. 12 (2002) 355–379.
[31] Z. Lei, On axially symmetric incompressible magnetohydrodynamics in three dimensions, J. Differential Equations 259 (2015) 3202–3215.
[32] Z. Lei, Y. Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, Discrete Contin. Dyn. Syst. 25 (2009) 575–583.
[33] E. Lieb, M. Loss, Analysis, American Mathematical Society, Providence, 2001.
[34] F. Lin, L. Xu, P. Zhang, Global small solutions to 2-D incompressible MHD system, J. Differential Equations 259 (2015) 5440–5485.
[35] F. Lin, P. Zhang, Global small solutions to MHD type system (I): 3-D case, Comm. Pure Appl. Math. 67 (2014) 531–580.
[36] J. Lions, Méthodes de résolution des problémes aux limites non linéeaires, vol. 1, Dunod, Paris, 1969.
[37] S. Pai, Magnetogasdynamics and Plasma Dynamics, Springer-Verlag, Vienna, 1962.
[38] E. Priest, T. Forbes, Magnetic Reconnection, MHD Theory and Applications, Cambridge University Press, Cambridge, 2000.
[39] X. Ren, J. Wu, Z. Xiang, Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, J. Funct. Anal. 267 (2014) 503–541.
[40] T. Tao, Global regularity for a logarithmically supercritical hyperdissipative Navier–Stokes equation, Anal. PDE 2 (2009) 361–366.
[41] C. Tran, X. Yu, Z. Zhai, On global regularity of 2D generalized magnetohydrodynamic equations, J. Differential Equations 254 (2013) 4194–4216.
[42] W. Wang, Z. Zhang, Limiting case for the regularity criterion to the 3-D magneto-hydrodynamics equations, J. Differential Equations 252 (2012) 5751–5762.
[43] D. Wei, Z. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, arXiv:1607.04397 [math.AP], 15 Jul 2016.
[44] J. Wu, Generalized MHD equations, J. Differential Equations 195 (2003) 284–312.
[45] J. Wu, Regularity criteria for the generalized MHD equations, Comm. Partial Differential Equations 33 (2008) 285–306.
[46] J. Wu, Global regularity for a class of generalized magnetohydrodynamic equations, J. Math. Fluid Mech. 13 (2011) 295–305.
[47] J. Wu, Y. Wu, X. Xu, Global small solution to the 2D MHD system with a velocity damping term, SIAM J. Math. Anal. 47 (2015) 2630–2656.
[48] J. Wu, P. Zhang, The global regularity problem on the 2D magnetohydrodynamic equations with magnetic diffusion only, work in progress.
[49] L. Xu, P. Zhang, Global small solutions to three-dimensional incompressible MHD system, SIAM J. Math. Anal. 47 (2015) 26–65.
[50] K. Yamazaki, On the global well-posedness of N-dimensional generalized MHD system in anisotropic spaces, Adv. Differential Equations 19 (2014) 201–224.
[51] K. Yamazaki, Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation, Nonlinear Anal. 94 (2014) 194–205.

[52] K. Yamazaki, On the global regularity of two-dimensional generalized magnetohydrodynamics system, J. Math. Anal. Appl. 416 (2014) 99–111.
[53] K. Yamazaki, Global regularity of logarithmically supercritical MHD system with zero diffusivity, Appl. Math. Lett. 29 (2014) 46–51.
[54] W. Yang, Q. Jiu, J. Wu, The 3D incompressible Navier–Stokes equations with partial hyperdissipation, submitted for publication.
[55] Z. Ye, Global regularity for the 2D MHD equations with partial hyperresistivity, preprint.
[56] B. Yuan, L. Bai, Remarks on global regularity of 2D generalized MHD equations, J. Math. Anal. Appl. 413 (2014) 633–640.
[57] T. Zhang, An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, J. Differential Equations 260 (2016) 5450–5480.
[58] Y. Zhou, J. Fan, A regularity criterion for the 2D MHD system with zero magnetic diffusivity, J. Math. Anal. Appl. 378 (2011) 169–172.